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Honours Thesis

Asset Pricing Model with Adaptive Learning and Heterogeneity

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Declaration

I declare that this thesis is my own work and that, to the best of my knowledge, it contains no material that has been written by another person or persons, except where acknowledgement has been made. This thesis has not been submitted for the award of any degree or diploma at the University of New South Wales, or at any other institute of higher education.

Timothy Tze-Hymn Lo
6 June, 2016

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Abstract

This thesis examines how investment decisions made by two groups of agents influence the dynamics of a stock market. Both groups of agents are risk-averse and form their demand based on expected returns and risk of a stock. They form expectations on future price and forecasts of variance of a stock's return. One group learns the coefficients of their expectation rule when new information is available. Through recursively updating forecasts, learning agents slowly improve the knowledge of the stock price. The other group does not learn and has constant parameters in their expectation rule. The existence of non-learning agents creates additional disturbance to the market and changes its dynamics. Real-time simulations under decreasing-gain learning illustrate that learning agents may fail to reach the stable stationary equilibrium and often fall into a random walk equilibrium. I further investigate the sensitivity of the model to selected exogenous parameters using bifurcation diagrams. I also show that under constant-gain learning, the model can generate recurrent bubbles and crashes due to occasional large endogenous shocks.

Chapter 1
Introduction

Contemporary macroeconomics modelling emphasizes the role of expectations. In a typical macroeconomic model, agents form expectations on some relevant variables in an economy based on information available to them. Their expectations also influence the time path of the variables. A central aspect of expectations is that this two-way relationship determines the dynamics of the economy (Evans and Honkapohja, 2001). Numerous examples demonstrate the importance of expectations. In a stock market, prices depend on expected future prices and dividends. In business cycles theory, expectations play an important role in determining output and inflation.

The standard methodology for modelling expectations is to assume rational expectations (RE) which has been well documented in the literature. This is a strong assumption on information of agents since agents are expected to know the true structure of the economy and all information is perfectly available to them. RE was first formulated by Muth (1961) in the context of a well-known cobweb model and made its explicit appearance in the paper of Lucas (1972). Recent models relax rational expectations and include imperfect knowledge, heterogeneous expectations and learning which describes the evolution of expectation rules over time. All these models are widely applied in different macroeconomic theories.

The main motivation of this thesis is to investigate how expectations affect market outcomes. This thesis links adaptive learning, a common learning mechanism for expectations, to dynamics of a stock market. Financial crises have always been of interest to economists due to its ripple effect from one facet to the entire economy. Kindleberger and Aliber (2011) presuppose the formation of bubbles starts off from an unanticipated economic event which triggers abrupt shifts in expectations. This view has been supported by historical financial crises. Branch and Evans (2011) adopt their view and develop a simple asset pricing model replacing rational expectations with adaptive learning in order to identify channels through which learning leads to recurrent bubbles and crashes in an economy where agents are concerned about risk and returns of a stock. Occasional endogenous shocks lead agents to adjust their expectations on future price and risk estimates. It thus results in deviations from the fundamental value of the stock. However, their model only considers a single type of agents in the stock market. The main contribution of this thesis is an extension of this model to two types of agents. Both types of agents are concerned about risk and returns of a stock. They form expectations on future price and measure risk by the variance of the stock's return. Learning agents actively update their beliefs whereas non-learning agents have persistent beliefs. Through recursively updating forecasts, learning agents slowly improve their knowledge of the stock market and realise the existence of non-learning agents. This setting attempts to bring the model closer to different behaviours of investors in a real stock market where numerous investors forecast prices differently. Their forecasts determine the time path of the prices. Some investors, who have the knowledge, resources and time, may study the trends and patterns of the stock market using technical analysis whereas other investors may not. Therefore, models with heterogeneous beliefs may provide more insight into the dynamics of a real stock market. The main result of this thesis is that the presence of non-learning agents creates additional disturbance in the stock market and drives stock prices down comparing to the market with representative learning agents.

I first analyse the asset pricing model with adaptive learning introduced by Branch and Evans (2011). I refer to their model as the Basic Model. Then I extend the Basic Model to include heterogeneous beliefs where both learning and non-learning agents exist in the stock market. I consider two cases. In the Homogeneous Variance Case, learning and non-learning agents have different methods of forming expectations on future stock price but have identical risk estimate. In the Heterogeneous Variance Case, each group of agents has their individual method of forming expectations on future stock price and has different risk estimates. For both cases, I run multiple numerical simulations in order to validate my analytical findings and demonstrate the dynamics of the model. I use bifurcation diagrams to examine the behaviour of endogenous variables when changing the values of certain exogenous parameters. Lastly I demonstrate the model can generate recurrent bubbles and crashes under constant-gain learning when occasional endogenous shocks increase volatility in the market.

Chapter 2
Literature Review

This chapter provides an overview of the relevant literature on possible sources of bubbles and the role of expectations. Section 2.1 explains the formation of a bubble via credit cycles. Section 2.2 examines rational models which assume agents know the true structure of the economy and have perfect information. Other models can be viewed as deviations from the rational models in different directions. Section 2.3 reviews explicit models of associated learning processes. Section 2.4 discusses heterogeneous-beliefs models which consider various types of agents in the economy instead of only rational agents.

2.1 Bubbles as Economic Investment Cycles

Hyman Minsky's model provides an early characterization of bubbles and crashes (Kindleberger and Aliber, 2011). The Minsky model focuses on explaining the procyclical changes in money supply. When the economy is booming, the supply is high whereas it decreases during economic downturns. His argument that fluctuations in credit supply increase vulnerability of banking system explains earlier financial crises in U.S., Europe and Japan from the 1980s to the end of 1990s.

Minsky distinguishes five phases of a classic bubble: 1) displacement phase where a technological innovation or financial liberalization is introduced into the economy. These shocks lead to expectation of economic growth. Individuals and businesses anticipate profits in multiple sectors of the economy and would borrow to take advantage of these opportunities. The economy grows rapidly and it thus results in an even more optimistic feedback. 2) boom phase where it involves credit expansion and increase in investment. Asset prices rise first at a slower rate but then accelerate. Asset prices eventually exceed the actual fundamental improvement from the displacement shocks. A bubble is thus formed in the background. 3) euphoria phase where investors trade the mispriced asset rapidly and push the prices even higher. 4) profit taking phase where sophisticated investors start selling their assets in order to realise profits and 5) panic phase where investors dump the assets as they recognise the mispricing. Most models in the literature only explain some phases of the Minsky model but not all five phases.

2.2 Rational Models

The literature on rational models draws an important conclusion that for an infinitely-lived asset, a bubble can occur even when all agents are perfectly rational (Brunnermeier and Oehmke, 2012). A rational bubble can be sustained if the bubble's rate of growth is equal to the discounted rate. In other words, all agents expect the bubble to grow or at least not burst in the future. By definition of net return of an asset, $r_{t+1} = (p_{t+1} + d_{t+1})/p_t - 1$ where p_t is the price and d_t is the dividend payment at time t. Rearranging and taking rational expectations on both sides,

$$p_t = E_t \left[\frac{p_{t+1} + d_{t+1}}{1 + r_{t+1}} \right] \tag{2.1}$$

it shows that the current price is the discounted value of expected future price and dividend payment in the next period. For simplicity, constant expected return (i.e. $E_t(r_{t+1}) = r$) is assumed. Iterating equation (2.1) forward and using the law of total expectation,

$$p_t = E_t \left[\sum_{\tau=1}^{T-t} \frac{1}{(1+r)^\tau} d_{t+\tau} \right] + E_t \left[\frac{1}{(1+r)^{T-t}} p_T \right] \tag{2.2}$$

the equilibrium price is governed by the expected discounted future dividend stream from $t+1$ to T and the expected discounted price at time T where T is the time at maturity. Suppose the asset is infinitely-lived (i.e. $T \to \infty$) and if the following transveraslity condition holds,

$$\lim_{T \to \infty} E_t \left[\frac{1}{(1+r)^{T-t}} p_T \right] = 0$$

the equilibrium price is only governed by the dividend stream. It is commonly referred as the fundamental value. Without the transveraslity condition, the fundamental value is no longer the unique solution for equation (2.2). Instead, an alternative solution for equation (2.2) can be decomposed into $p_t = v_t + b_t$ where v_t is the fundamental value and b_t is the bubble component such that

$$b_t = E_t \left[\frac{1}{(1+r)} b_{t+1} \right]$$

This class of solution highlights that under rational expectations, a bubble only persists if the asset price increases at rate r. Blanchard and Watson (1982) illustrate this solution in their model where a bubble remains with probability π and crashes with probability $1 - \pi$ each period. Hence the bubble only survives if it grows at rate $(1 + r)/\pi$. This specific condition imposes an upper bound for the asset price. For instance, an ever-growing asset would eventually be too expensive for investors who will substitute it with other similar assets. In addition, if an asset's required rate of return r is higher than the growth rate of the economy, a bubble in this asset cannot exist since it would outgrow the aggregate wealth of the entire economy. Therefore, a rational bubble only exists when it grows at the rate of r which has to be lower than or equal to the growth rate of the economy.

If an asset is finitely-lived, the bubble will surely burst at the end of the asset's life T since no one would be willing to buy the asset at $T-1$, $T-2$ and so on given that all agents have common knowledge of the asset's life. By backward-induction argument, a bubble cannot exist for a finitely-lived asset. However, Allen, Morris, and Postlewaite (1993) show that a bubble could exist for a finitely-lived asset if there is a lack of common knowledge that all agents know an asset is overvalued but they do not know other agents know it as well. Morris, Postlewaite, and Shin (1995) further argue for a finitely-lived asset, the size of a bubble depends on how far away agents in the economy are from perfect common knowledge.

2.3 Learning Models

Learning models depart from the rational expectations (RE) which conjectures agents know more than typical economists who in general do not know the true stochastic structure of the economy but estimate its parameters. Basic theories of learning associated with expectation formation were developed in the 1980s and 1990s. This class of models suggests a more plausible view of rationality. It assumes that agents act as econometricians. They have limited knowledge of the true stochastic structure of the economy and revise their forecast rules in response to new information available in order to gradually improve their knowledge on the economy over time (Evans and Honkapohja, 2001). Since agents form their forecasts rationally given limited information and econometric tools, this viewpoint introduces a specific form of "bounded rationality" discussed in Sargent (1993).

Perhaps the most common learning mechanism is adaptive learning where agents are assumed to estimate economic variables in their model, which is called perceived of law motion (PLM). It has same functional form as the true stochastic process of the economy. The parameters in PLM are updated by recursive least squares (RLS), a common estimation method in econometrics. The forecast from the PLM is mapped to the true stochastic process of the economy in order to illustrate economic agents improving their knowledge of the economy. It eventually leads to a temporary equilibrium which is commonly known as actual law of motion (ALM). Adaptive learning operates in real time. An alternative approach is eductive learning which assumes common knowledge of rationality. Agents engage in a process of reasoning about the possible outcomes knowing that other agents engage in the same process. Eductive learning instead operates in mental time.

In numerous models, adaptive learning provides an asymptotic justification for RE since when economic agents improving their knowledge of the true stochastic process, it will theoretically converge over time to the RE hypothesis under certain conditions on model parameters. This distinct relationship is formally known as rational expectations equilibrium (REE). Thus, the REE can be learned even though economic agents initially have limited knowledge and are boundedly rational. More importantly, the time path of convergence is of interest to economists since it provides insight into the dynamics of the economic variables in a stochastic model. In addition, adaptive learning serves as an important role in the selection criterion to determining the plausibility of a particular REE. Since models under RE hypothesis may have multiple stationary equilibria, the learning approach establishes conditions of stability for REE.

Branch and Evans (2011) develop an asset pricing model with least-squares learning. They demonstrate recurrent bubbles and crashes are endogenous responses to fundamental shocks. Agents in the model are concerned about risk and returns of a stock. They form expectations on future prices and estimate the variance of excess returns. The paper identifies two channels through which their forecasts influence stock prices. Occasional fundamental shocks may lead investors to lower their risk estimates and increase their expected returns. Both effects drive the stock fundamentals up quickly. Estimates of risk under adaptive learning also explain how bubbles crash suddenly. Along a bubble path, risk estimates increase until the perceived risk is so high that asset demand collapses and the bubble eventually bursts. These findings provide an intuition that rational learning by agents may still cause large fluctuations in asset prices. Branch and Evans (2013) extend their paper in 2011 and consider agents, who are concerned about risk and returns, perceive stock prices to follow a random walk process. It shows random walk beliefs increase market volatility. When agents update their risk estimates in real time under constant-gain learning, recurrent bubbles and crashes can arise. Moreover, when agents update their risk estimates using ARCH, it strengthens the risk effect has in generating bubbles and crashes.

Adam, Marcet, and Nicolini (2008) also link learning to asset pricing. They introduce bounded rationality in a consumption-based asset pricing model. The model deviates from rational expectations in the model of Lucas Jr (1978) and assumes investors have subjective beliefs about stock price but rationally learn from past movements in prices. The adoption of adaptive learning leads to a significant improvement in the fit of the stock price time series data. Marcet and Nicolini (2003) use a bounded rational learning model to explain recurrent hyper-inflation which has similar properties of a bubble in the sense that hyper-inflationary paths are unstable under least-squares learning. It shows that a small departure from full rational expectations still explains recurrent hyper-inflations well. The falsifiability of the model is remarkably solid and it can be further applied to policy evaluation.

2.4 Heterogeneous-Beliefs Models

Heterogeneous-beliefs models depart from the assumption of a representative agent and introduce heterogeneity with different types of agents. In general, there is at least one group of near rational or irrational agents in the model.

Miller (1977) provides a simple static model for overpricing with short-sale constraints and investor disagreement. Harrison and Kreps (1978) extend Miller's model with dynamic setting which allows even higher levels of over-pricing. In their dynamic model, asset price may exceed the valuation of the most optimistic investor since current owners of the asset optimistically believe they can resell the asset in the future at a higher price. Other traders attracted by this option are willing to buy the asset and thus push the asset price up. It only reverts back to the fundamental value when the market converges to a common belief that the asset is over-priced or when the short-sale constraints are relaxed. Hong, Scheinkman, and Xiong (2006) further investigate the relationship between speculative bubble and heterogeneous beliefs with short-sales constraints. Xiong (2013) explains bubbles and crises in financial markets through literature reviews on this class of models and concludes that heterogeneous-beliefs lead to speculative investor behaviour and thus endogenous price fluctuations. Empirical studies confirm the validity of this class of models. Diether, Malloy, and Scherbina (2002) use dispersion in analysts' earnings forecast as a proxy for heterogeneous beliefs and show that stocks with high forecast dispersion seem to be overpriced. Chen, Hong, and Stein (2002) follow a similar approach and use dispersion of mutual fund ownership as a proxy. They show stocks owned by a large cross-section of mutual funds tend to agree in price comparing to those owned by a few funds.

Heterogeneous agent models typically consider two important types of investors, fundamentalists and chartists, in a market. Fundamentalists base their trading strategies upon market fundamentals and major economic factors (Hommes, 2006; LeBaron, 2006). They invest in undervalued assets (i.e. below the fundamental value) and sell overvalued assets (i.e. above the fundamental value). In contrast, chartists do not take market fundamentals into account but make investment decisions based on observed historical patterns in past prices and attempt to extrapolate price trends using technical analysis. Brock and Hommes (1998) consider combinations of multiple belief types of agents, including fundamentalists and chartists, in an asset pricing model derived from mean-variance maximization. They also allow agents to switch between different belief types and investigate possible bifurcation routes to complicated chaotic asset price dynamics. Brock, Hommes, and Wagener (2005) further develop a theoretical framework to study the behaviour of many different belief types of investors in evolutionary markets. Hommes (2013) summarizes behavioural theory of heterogeneous expectations in complex economic systems and discusses empirical validations through financial time series data. A more recent paper by Hommes and In't Veld (2015) estimates an asset pricing model with fundamentalists and chartists. It suggests behavioural regime switching has significant impact on explaining the dot-com bubble and the financial crisis in 2008.

De Long, Shleifer, Summers, and Waldmann (1990a) consider a model with two types of traders: noise traders who demand asset based on non-fundamental considerations such as shifts in market sentiment and rational traders who are fully rational about stock returns. They show noise traders may survive in the long run. The model of De Long, Shleifer, Summers, and Waldmann (1990b) replaces noise traders with feedback traders who do not consider the signals about fundamentals but the latest observed price change only. The presence of rational investors in the model is expected to stabilize the price. However, they draw an opposite conclusion that the rational speculators ride the bubble and overshoot the asset's fundamental value even though they fully understand the overpricing will eventually collapse. Abreu and Brunnermeier (2003) also present a similar conclusion. Hong and Stein (1997) link heterogeneous-beliefs to formation of bubble and consider feedback traders and news watchers who observe private news about the asset's fundamental value. If information diffuses slowly, prices under-react initially and feedback traders profit from it. Eventually news watchers catch up with the market but cannot distinguish whether they are trading early or late in the news cycle. Prices end up overshooting the fundamentals. To sum up, modelling heterogeneity is an important direction in explaining some features of real markets. In this thesis, I combine adaptive learning with heterogeneous agent model.

Chapter 3
Asset Pricing Model with Representative Learning Agents

The main feature of the asset pricing model introduced by Branch and Evans (2011) (hereafter referred to BE) is that learning agents are concerned about risk and returns. They form expectations on future price of a risky asset by a simple forecast rule based on information available to them. Since the true stochastic process of risky asset price is unknown to agents, they update the parameters in their forecast rule when new price is realized each period. Therefore, agents are viewed as econometricians and learn recursively over time. It is thus interesting to investigate how the parameters in the forecast rule behave and evolve over time as well as whether agents eventually reach the true stochastic process through learning. I use this model as a basis for my extensions and refer to it as the Basic Model.

3.1 Simple Linear Asset Pricing Model

The true stochastic process of risky asset price which I derive further takes a simple linear form.

$$p_t = \beta \hat{E}_t(p_{t+1} + d_{t+1}) - \beta a \sigma_t^2 z_{st} \tag{3.1}$$

where

d_t: the dividend stream at time t,

p_t: the price of the risky asset at time t,

β: the discount factor,

$\hat{E}_t$: the subjective expectation formed at time t conditional on variables and information up to time t by agents,

a: the risk aversion factor,

σ_t^2: the variance of excess returns at time t conditional on variables and information up to time t,

z_{st}: the aggregate supply of the risky asset which follows a stochastic process.

Equation (3.1) has the following conditions on the variables and exogenous parameters:

$$0 \le p_t, \sigma_t^2 \tag{3.2}$$

$$0 < \beta, a < 1 \tag{3.3}$$

The conditions can be interpreted as follows. Stock price p_t is always non-negative. Discount factor β is the inverse of R where $R = 1 + r$. r is the interest rate. $\beta = 0$ implies interest rate is infinite and $\beta = 1$ implies interest rate is 0. Therefore β is always between 0 and 1. Risk aversion factor $a = 0$ implies risk-neutral since at $a = 0$, the term $\beta a \sigma_t^2 z_{st}$ becomes 0 and therefore agents do not consider risk. When a gets closer to 1, $\beta a \sigma_t^2 z_{st}$ becomes larger and therefore agents are more risk averse. The variance of excess returns σ_t^2 is always non-negative. Considering risk-neutral agents (i.e. $a = 0$), equation (3.1) reduces to the Lucas asset pricing model with risk neutrality.

The simple linear asset pricing model in equation (3.1) with $a > 0$ can be derived from the basic wealth maximization principle (see e.g. Brock and Hommes (1998)). Consider an economy with one risky asset and one risk free asset. Let p_t denote ex-dividend price per share of the risky asset at time t and d_t denote dividend per share of the risky asset. Let W_t denote total wealth of agents at time t and z_{dt} denote aggregate demand for the risky asset. The risk free asset yields constant return $R = \beta^{-1} > 1$ and the risky asset yields dividend d_t. Therefore,

$$W_{t+1} = R(W_t - p_t z_{dt}) + (p_{t+1} + d_{t+1}) z_{dt} \tag{3.4}$$

Equation (3.4) shows that wealth at time $t + 1$ (W_{t+1}) is determined by the return from risk free asset and from risky asset at time $t + 1$. At time t, agents purchase z_{dt} amount of risky asset which yields return ($p_{t+1} + d_{t+1}$) and invest the remaining portion of wealth ($W_t - p_t z_{dt}$) into risk free asset which yields return R.

Each and every agent follows mean-variance preference. It simply implies that the utility function of agents includes two arguments, mean and standard deviation of wealth. The utility function is strictly increasing in mean and decreasing in standard deviation in order to show agents seek a portfolio with the highest expected return for a given variance of excess returns or the lowest variance of excess returns for a given expected return. Let E_t and Var_t denote the conditional expectation and variance operators at time t, based on information set consisting of past prices up to time t. As agents, who are concerned about risk, revise their estimate on variance of excess returns each period, σ^2 is endogenous and time-varying (i.e. $Var_t(p_{t+1} + d_{t+1} - Rp_t) \equiv \sigma_t^2$). Then, from equation (3.4),

$$\begin{aligned} E_t(W_{t+1}) &= E_t(R(W_t - p_t z_{dt}) + (p_{t+1} + d_{t+1})z_{dt}) \\ &= R(W_t - p_t z_{dt}) + z_{dt} E_t(p_{t+1} + d_{t+1}) \\ Var_t(W_{t+1}) &= Var_t(R(W_t - p_t z_{dt}) + (p_{t+1} + d_{t+1})z_{dt}) \\ &= z_{dt}^2 Var_t(p_{t+1} + d_{t+1} - Rp_t) \\ &= z_{dt}^2 \sigma_t^2 \end{aligned}$$

The demand z_{dt} solves by maximizing the mean-variance utility function

$$\begin{aligned} z_{dt} &= \max_{z_{dt}} \{E_t(W_{t+1}) - (a/2) Var_t(W_{t+1})\} \\ &= \max_{z_{dt}} \{R(W_t - p_t z_{dt}) + z_{dt} E_t(p_{t+1} + d_{t+1}) - (a/2) z_{dt}^2 \sigma_t^2\} \end{aligned}$$

From the first order condition, the optimal z_{dt} yields

$$\begin{aligned} 0 &= -Rp_t + E_t(p_{t+1} + d_{t+1}) - a z_{dt} \sigma_t^2 \\ z_{dt} &= E_t(p_{t+1} + d_{t+1} - Rp_t)/a\sigma_t^2 \end{aligned}$$

At market clearing condition (i.e. $z_{dt} = z_{st}$), the equilibrium price follows

$$p_t = \beta E_t(p_{t+1} + d_{t+1}) - \beta a \sigma_t^2 z_{st}$$

Adaptive learning, one of the main features in the Basic Model, moves beyond rational expectations and assumes agents, who face limitations on knowledge about the economy, are bounded rational. Hence expectation E_t in equation (3.1) is replaced by $\hat{E}_t$. The risky asset equilibrium price yields

$$p_t = \beta \hat{E}_t(p_{t+1} + d_{t+1}) - \beta a \sigma_t^2 z_{st}$$

3.2 Share Supply and Dividends

BE assume the dividend follows an exogenous process which yields

$$d_t = d_0 + u_t \tag{3.5}$$

where u_t is a white noise with variance σ_u^2 where $0 < \sigma_u^2$ and $0 < d_0$. In this context, the term white noise WN has the following properties:

1. Normally distributed

2. Mean and autocovariance are zero

3. Constant and finite variance

BE assume share supply follows a multiplicative process which yields

$$z_{st} = \{\min(s_0, \Phi p_t)\} \cdot \left(1 + \frac{v_t}{s_0}\right) \tag{3.6}$$

where v_t is a white noise with variance σ_v^2 where $0 < \sigma_v^2$ and $0 < s_0$. v_t is uncorrelated with u_t. Define Φ as

$$\Phi = \frac{s_0}{\bar{p}\xi}$$

where $\bar{p}$ is mean stock price in the fundamental solution for rational expectation equilibrium (REE) derived in Section 3.3 and ξ is a small value where $0 < \xi < 1$. BE assign $\xi = 0.1$ for simulation purpose. Equation (3.6) shows share supply is exogenous and equal to $s_0 + v_t$ except at very low prices. Endogenous share supply is meant to capture any publicly traded shares which perform poorly in the market. This ensures that prices remain non-negative. The variances of dividend noise σ_u^2 and the share supply noise σ_v^2 serve an important role in determining volatility in the market.

3.3 Rational Expectations Equilibrium

Rational expectations equilibrium (REE) is an equilibrium in the model where representative agents have rational expectations. This concept is discussed in numerous literature related to macroeconomics. Section 2.3 defines REE and explains its role in adaptive learning. As a result, before examining the dynamics of the Basic Model, BE solve REE in the simple asset pricing model in equation (3.1). Blanchard and Watson (1982) show two solutions for REE, the fundamental solution and the bubble solution, in a simple asset pricing model similar to equation (3.1). They assume rational expectations (RE) and agents are concerned about expected returns only. The fundamental solution represents a stable equilibrium or the fundamental value of the risky asset whereas the bubble solution represents an unstable equilibrium or the explosive price of the risky asset. The Basic Model also yields two similar solutions for REE. I derive the fundamental solution denoted by p_t^f. First replace $\hat{E}_t$ with E_t and assume $\sigma_t^2 = \sigma^2$. Thus, equation (3.1) yields

$$\begin{aligned}
p_t &= \beta E_t(p_{t+1} + d_{t+1}) - \beta a \sigma^2 z_{st} \\
p_{t+1} &= \beta E_{t+1}(p_{t+2} + d_{t+2}) - \beta a \sigma^2 z_{st+1} \\
p_{t+2} &= \beta E_{t+2}(p_{t+3} + d_{t+3}) - \beta a \sigma^2 z_{st+2} \\
&\vdots
\end{aligned}$$

Iterating forward and substituting all p_{t+j} for $j = 1, 2, \ldots \infty$ to p_t and using the law of total expectation (i.e. $E(y|x) = E(E(y|z)|x)$). p_t^f yields

$$p_t^f = \sum_{j=1}^{\infty} \beta^j E_t(d_{t+j}) - \beta \sum_{j=0}^{\infty} \beta^j a\sigma^2 E_t z_{st+j}$$

It shows that the fundamental value of price which consists of the present value of expected dividends stream declines with variance of excess returns. Equation (3.5) shows dividend is independent to price. Provided that share supply is exogenous and therefore $z_{st} = s_0 + v_t$, equation (3.1) can be expressed in an alternative form.

$$\begin{aligned} p_t &= \beta E_t(p_{t+1} + d_{t+1}) - \beta a\sigma^2 z_{st} \\ &= \beta E_t(p_{t+1}) + \beta d_0 - \beta a\sigma^2(s_0 + v_t) \\ &= \beta(d_0 - a\sigma^2 s_0) + \beta E_t(p_{t+1}) - \beta a\sigma^2 v_t \end{aligned}$$

Iterating forward, the fundamental solution can be expressed into the following form:

$$\begin{aligned} p_t^f &= \sum_{j=1}^{\infty} \beta^j (d_0 - a\sigma^2 s_0) - \beta a\sigma^2 v_t \\ &= \frac{\beta(d_0 - a\sigma^2 s_0)}{1 - \beta} - \beta a\sigma^2 v_t \end{aligned} \tag{3.7}$$

The fundamental solution for REE in the Basic Model is different from that in Blanchard and Watson (1982) since BE assume agents learn in real-time and are concerned about risk and returns whereas Blanchard and Watson (1982) do not consider adaptive learning but assume rational expectations and agents are concerned about returns only. Later analysis justifies the fundamental solution is attainable and stable under adaptive learning.

3.4 Adaptive Learning

Evans and Honkapohja (2001) summarize the literature on adaptive learning. BE follow this framework in their model. Agents forecast future stock price based on a perceived law of motion (PLM) which has same functional form as the true stochastic process of the economy which is the actual law of motion (ALM). The mechanism of adaptive learning is that agents rationally estimate and revise the parameters in their PLM based on information available to them in order to learn the parameter values in the ALM. Section 3.4.1 examines the PLM and ALM in the Basic Model. Section 3.4.3 - 3.4.4 solve for equilibrium in the model and its stability. First BE propose a crucial assumption on information available to agents.

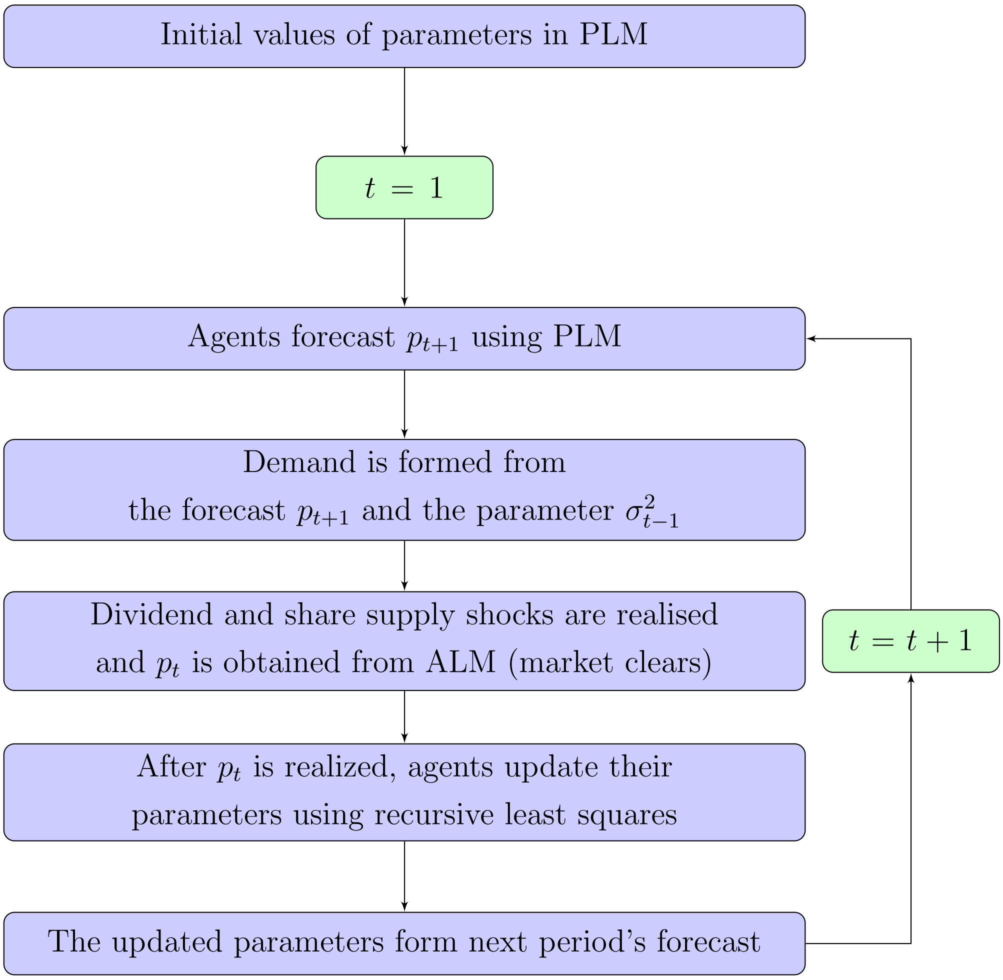

Figure 3.1: Flowchart of adaptive learning process in the Basic Model

Assumption 3.4.1

1. *At time t agents estimate parameters in the PLM based on information through time $t-1$, and that their forecasts are conditioned on variables dated $t-1$ or earlier*

2. *No contemporaneous variables including z_{st} are observable to agents at time t when they make forecasts.*

This assumption restricts information available to agents. When agents forecast future price p_{t+1} at time t, they do not observe p_t, d_t and other contemporaneous variables including σ^2_t. They only observe variables and information up to time $t-1$. I introduce $\hat{E}_t(\cdot|\Omega_{t-1})$ which denote the subjective expectation formed by agents at time t conditional on variables and information Ω_{t-1} up to time $t-1$. Under Assumption 3.4.1, the best available forecast of price and variance of excess returns for agents in equation (3.1) is $\hat{E}_t(p_{t+1}+d_{t+1}|\Omega_{t-1})$ and σ^2_{t-1} instead of $\hat{E}_t(p_{t+1}+d_{t+1})$ and σ^2_t. Since dividend is independent to price, the true stochastic process of the risky asset price becomes

$$\begin{aligned} p_t &= \beta\{\hat{E}_t(p_{t+1}|\Omega_{t-1}) + \hat{E}_t(d_{t+1}|\Omega_{t-1})\} - \beta a \sigma_{t-1}^2 z_{st} \\ &= \beta\{\hat{E}_t(p_{t+1}|\Omega_{t-1}) + d_0\} - \beta a \sigma_{t-1}^2 z_{st} \end{aligned} \tag{3.8}$$

Figure 3.1 illustrates the entire adaptive learning process in the Basic Model. Agents form expectations on future price p_{t+1} given their PLM and information available up to time $t-1$. The variance of excess returns from last period σ_{t-1}^2 is the best available forecast for the current period (i.e. time t). Forecast of future price and variance form the aggregate demand and price is realized at market clearing condition. Agents update all their parameters from the last period. Then the updated parameters become the best available forecast for the next period.

3.4.1 Perceived and Actual Law of Motion

Recall that agents demand the risky asset depending on expected price p_{t+1}. BE assume PLM of agents is a simple AR(1) process

$$p_t = k_{t-1} + c_{t-1} p_{t-1} + \varepsilon_t \tag{3.9}$$

where ε_t is an unobserved white noise. The parameters k_{t-1} and c_{t-1} are time-varying in order to illustrate the adoption of adaptive learning. Under Assumption 3.4.1, only time-varying parameters k_{t-1} and c_{t-1} are available to agents rather than k_t and c_t and agents forecast p_{t+1} at time t based on information without p_t. The expectation on future price thus yields

$$\hat{E}_t(p_{t+1}|\Omega_{t-1}) = k_{t-1}(1 + c_{t-1}) + c_{t-1}^2 p_{t-1} \tag{3.10}$$

Agents demand the risky asset also depending on risk. The variance of excess returns σ_{t-1}^2 in equation (3.8) is defined as

$$\sigma_{t-1}^2 = Var_t(p_t + d_t - R p_{t-1}|\Omega_{t-1}) \tag{3.11}$$

Stock price p_{t-1} is realized at time t. Section 3.2 shows dividend is independent to price. Hence it becomes

$$\begin{aligned} \sigma_{t-1}^2 &= Var_t(p_t + d_t|\Omega_{t-1}) \\ &= E_t[(p_t - \hat{E}_t(p_t|\Omega_{t-1}) + d_t - \hat{E}_t(p_t|\Omega_{t-1}))^2] \end{aligned} \tag{3.12}$$

From the PLM in equation (3.9) and the dividend process in equation (3.5), then

$$\hat{E}_t(p_t|\Omega_{t-1}) = k_{t-1} + c_{t-1}p_{t-1}$$
$$\hat{E}_t(d_t|\Omega_{t-1}) = d_0$$

Then equation (3.12) becomes

$$\sigma^2_{t-1} = E_t[(p_t - (k_{t-1} + c_{t-1}p_{t-1}) + u_t)^2] \tag{3.13}$$

Intuitively, $p_t - (k_{t-1} + c_{t-1}p_{t-1})$ is the forecast error between price p_t realized from the ALM and the price forecast from the PLM and u_t is the realised dividend noise. Equation (3.13) is the true variance of excess returns each period. Now I derive the actual law of motion (ALM), which is derived from market clearing as I showed before. Substituting equation (3.10) into equation (3.8) and assuming exogenous stochastic supply of the risky asset $z_{st} = s_0 + v_t$,

$$\begin{aligned} p_t &= \beta(\hat{E}_t(p_{t+1}|\Omega_{t-1}) + d_0) - \beta a\sigma^2_{t-1}z_{st} \\ &= \beta(k_{t-1}(1 + c_{t-1}) + d_0) + \beta c^2_{t-1}p_{t-1} - \beta a\sigma^2_{t-1}(s_0 + v_t) \\ &= \beta(d_0 + k_{t-1}(1 + c_{t-1}) - a\sigma^2_{t-1}s_0) + \beta c^2_{t-1}p_{t-1} - \beta a\sigma^2_{t-1}v_t \end{aligned}$$

Therefore, the ALM is the following process

$$p_t = \beta(d_0 + k_{t-1}(1 + c_{t-1}) - a\sigma^2_{t-1}s_0) + \beta c^2_{t-1}p_{t-1} - \beta a\sigma^2_{t-1}v_t \tag{3.14}$$

3.4.2 Recursive Least Squares

BE assume agents use least squares to estimate k_{t-1} and c_{t-1} in the PLM and the average forecast error to estimate the variance of excess returns σ^2_{t-1} using all previous observations. The estimation is equivalent to expanding window as new information is included in subsequent estimates. This can be compactly represented by recursive least squares (RLS). The RLS equations are listed as follows

$$\theta_t = \theta_{t-1} + \gamma_{1,t}S_t^{-1}X_t(p_t - \theta'_{t-1}X_t) \tag{3.15}$$
$$S_t = S_{t-1} + \gamma_{1,t}(X_tX'_t - S_{t-1}) \tag{3.16}$$
$$\sigma^2_t = \sigma^2_{t-1} + \gamma_{2,t}\left(\left(p_t - \theta'_{t-1}X_t + u_t\right)^2 - \sigma^2_{t-1}\right) \tag{3.17}$$

where $\theta_t = (k_t, c_t)'$ and $X_t = (1, p_{t-1})'$. S_t is an estimate of $EX_tX'_t$. $\gamma_{i\in(1,2),t}$ is a sequence of gains. Gain determines the impact of a new observation at time t on the estimate. The LS estimator is equivalent to decreasing gain (i.e. $\gamma_t = 1/t$). Constant gain is equivalent to the LS estimator where higher weights are put on the most recent observations. These three equations (3.15) - (3.17) describe the real-time learning algorithm. The first two equations (3.15) and (3.16) are the typical recursive updating equations under adaptive learning introduced in Marcet and Sargent (1989) and Evans and Honkapohja (2001). The third equation (3.17) is similar to an error correction model. It describes how σ_t^2 returns to the true variance of excess returns through learning. Appendix B shows the proof of RLS equation (3.15) and (3.16).

3.4.3 Equilibrium

Agents update k_{t-1} and c_{t-1} in equation (3.9) based on new information available to them each period. The ALM in equation (3.14) governs the true stochastic process of the price. Agents update k_{t-1}, c_{t-1} and σ_{t-1}^2 in equation (3.15), (3.16) and (3.17).

The issue becomes whether agents eventually learn the true stochastic process of the price using their PLM over time. It happens when the parameters in the PLM coincide with the parameters in the ALM. To investigate under which condition it is possible, it is useful to introduce the concept of T-map. T-map translates the parameters of the PLM to the parameters of the ALM. Evans and Honkapohja (2001) refer the solution of mapping from PLM to ALM as an equilibrium under learning. Therefore, an equilibrium is the fixed point of T-map. The ALM in this model has four state variables p_t, k_t, c_t and σ_t^2. Each of them has a fixed point and therefore all four fixed points constitute an equilibrium. Let $\bar{p}$, $\bar{k}$, $\bar{c}$ and $\bar{\sigma}^2$ denote the fixed point for p_t, k_t, c_t and σ_t^2. T-map is defined as

$$T\begin{pmatrix} k \\ c \\ \sigma^2 \end{pmatrix} = \begin{pmatrix} \beta(d_0 + k(1+c) - a\sigma^2 s_0) \\ \beta c^2 \\ \beta^2 a^2 (\sigma^2)^2 \sigma_v^2 + \sigma_u^2 \end{pmatrix} \tag{3.18}$$

Hence p_t can be written as

$$p_t = T\begin{pmatrix} k_{t-1} \\ c_{t-1} \\ \sigma_{t-1}^2 \end{pmatrix}' (1, p_{t-1}, 0)' - \beta a \sigma_{t-1}^2 v_t \tag{3.19}$$

Note that σ^2 is the conditional variance of excess returns. Equation (3.11) can be expressed in an alternative form.

$$\begin{aligned} \sigma_{t-1}^2 &= Var_t(p_t + d_t - Rp_{t-1}|\Omega_{t-1}) \\ &= Var_t(p_t|\Omega_{t-1}) + Var_t(d_t|\Omega_{t-1}) \\ &= Var_t(p_t|\Omega_{t-1}) + \sigma_u^2 \\ &= \sigma_{p,t-1}^2 + \sigma_u^2 \end{aligned} \tag{3.20}$$

where $\sigma^2_{p,t-1} = Var_t(p_t|\Omega_{t-1})$. Hence $\sigma^2 = \sigma^2_p + \sigma^2_u$ where σ^2_p can be solved by taking conditional variance of the ALM (i.e. $\sigma^2_p = \beta^2 a^2 \sigma^2 \sigma^2_v$). The T-map which captures the time-varying parameters in the ALM can be interpreted as follows. If agents have some initial non-equilibrium values for k_{t-1} and c_{t-1} in their PLM in equation (3.9) and their expectation on future price would be equation (3.10). Then through updating the parameters by recursive least squares equations, agents eventually take the same functional form as equation (3.9) but with parameters $T(k, c, \sigma^2)$ instead of the initial non-equilibrium values since the ALM and the PLM are equivalent at equilibrium.

Solving for fixed points of the T-map,

$$T \begin{pmatrix} \bar{k} \\ \bar{c} \\ \bar{\sigma}^2 \end{pmatrix} = \begin{pmatrix} \bar{k} \\ \bar{c} \\ \bar{\sigma}^2 \end{pmatrix}$$

Then,

$$\begin{aligned} \beta\bar{c}^2 &= \bar{c} \\ \bar{c} &= 0 \text{ or } \beta^{-1} \end{aligned}$$

Each $\bar{c}$ corresponds to a fixed point for k_t. For $\bar{c} = 0$,

$$\begin{aligned} \beta(d_0 + \bar{k} - a\bar{\sigma}^2 s_0) &= \bar{k} \\ \bar{k} &= \frac{\beta(d_0 - a\bar{\sigma}^2 s_0)}{1-\beta} \end{aligned}$$

For $\bar{c} = \beta^{-1}$,

$$\begin{aligned} \beta(d_0 + \bar{k}(1+\beta^{-1}) - a\bar{\sigma}^2 s_0) &= \bar{k} \\ -\beta(a\bar{\sigma}^2 s_0 - d_0) &= -\beta\bar{k} \\ \bar{k} &= a\bar{\sigma}^2 s_0 - d_0 \end{aligned}$$

Therefore, the two fixed points of T-map are $(\beta(d_0 - a\bar{\sigma}^2 s_0)/(1-\beta), 0, \bar{\sigma}^2)$ and $(a\bar{\sigma}^2 s_0 - d_0, \beta^{-1}, \bar{\sigma}^2)$. Then I solve for $\bar{\sigma}^2$.

$$\begin{aligned} \bar{\sigma}^2 &= \beta^2 a^2 (\bar{\sigma}^2)^2 \sigma^2_v + \sigma^2_u \\ &= \frac{1 \pm \sqrt{1 - 4a^2\beta^2\sigma^2_v\sigma^2_u}}{2a^2\beta^2\sigma^2_v} \end{aligned}$$

As p_t is stochastic, the fixed point is defined by its expectations $E(p_t) = E(p_{t-1}) = \bar{p}$. From equation (3.14),

$$E(p_t) = \beta(d_0 + k_{t-1}(1 + c_{t-1}) - a\sigma_{t-1}^2 s_0) + \beta c_{t-1}^2 E(p_{t-1})$$

At equilibrium, k_{t-1}, c_{t-1} and σ_{t-1}^2 converge to $\bar{k}_1$, $\bar{c}_1$ and $\bar{\sigma}^2$ respectively. Therefore,

$$\bar{p} = \frac{\beta(d_0 + \bar{k}(1 + \bar{c}) - a\bar{\sigma}^2 s_0)}{1 - \beta\bar{c}^2}$$

For $(\beta(d_0 - a\bar{\sigma}^2 s_0)/(1 - \beta), 0, \bar{\sigma}^2)$

$$\begin{aligned}\bar{p} &= \beta\left(d_0 + \frac{\beta(d_0 - a\bar{\sigma}^2 s_0)}{1 - \beta} - a\bar{\sigma}^2 s_0\right) \\ &= \frac{\beta(d_0 - a\bar{\sigma}^2 s_0)}{1 - \beta}\end{aligned}$$

For $(a\bar{\sigma}^2 s_0 - d_0, \beta^{-1}, \bar{\sigma}^2)$

$$\begin{aligned}\bar{p} &= \frac{\beta(d_0 + (a\bar{\sigma}^2 s_0 - d_0)(1 + \beta^{-1}) - a\bar{\sigma}^2 s_0)}{1 - \beta^{-1}} \\ &= \frac{\beta(d_0 - a\bar{\sigma}^2 s_0)}{1 - \beta}\end{aligned}$$

Therefore, $\bar{p}$ has the identical form for both fixed points. All possible fixed points $\bar{p}$, $\bar{k}$, $\bar{c}$ and $\bar{\sigma}^2$ are listed as follows.

$$\bar{p} = \frac{\beta(d_0 - a\bar{\sigma}^2 s_0)}{1 - \beta} \tag{3.21}$$

$$\bar{k} = \begin{cases} \frac{\beta(d_0 - a\bar{\sigma}^2 s_0)}{1-\beta} \\ a\bar{\sigma}^2 s_0 - d_0 \end{cases} \tag{3.22}$$

$$\bar{c} = \begin{cases} 0 \\ \frac{1}{\beta} \end{cases} \tag{3.23}$$

$$\bar{\sigma}^2 = \begin{cases} \frac{1+\sqrt{1-4a^2\beta^2\sigma_v^2\sigma_v^2}}{2a^2\beta^2\sigma_u^2} \to \bar{\sigma}_+^2 \\ \frac{1-\sqrt{1-4a^2\beta^2\sigma_v^2\sigma_u^2}}{2a^2\beta^2\sigma_v^2} \to \bar{\sigma}_-^2 \end{cases} \tag{3.24}$$

There are four equilibria in the Basic Model. Let $[\bar{p}, \bar{k}, \bar{c}, \bar{\sigma}^2]$ denote an equilibrium.

$$\left[\frac{\beta(d_0 - a\bar{\sigma}_+^2 s_0)}{1-\beta}, \frac{\beta(d_0 - a\bar{\sigma}_+^2 s_0)}{1-\beta}, 0, \bar{\sigma}_+^2\right]$$
$$\left[\frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0)}{1-\beta}, \frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0)}{1-\beta}, 0, \bar{\sigma}_-^2\right]$$
$$\left[\frac{\beta(d_0 - a\bar{\sigma}_+^2 s_0)}{1-\beta}, \frac{\beta(d_0 - a\bar{\sigma}_+^2 s_0)}{1-\beta}, \frac{1}{\beta}, \bar{\sigma}_+^2\right]$$
$$\left[\frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0)}{1-\beta}, \frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0)}{1-\beta}, \frac{1}{\beta}, \bar{\sigma}_-^2\right]$$

3.4.4 Stability Analysis

Once all the possible equilibria are solved, another question would be whether these solutions are stable or not. Expectational stability or E-stability is introduced in Evans and Honkapohja (2001) as an important concept in adaptive learning. If an equilibrium is (locally) stable, starting with non-equilibrium values (near the equilibrium), the dynamic system will converge to the equilibrium. On the other hand, if it is not stable, small deviations from the equilibrium will drive the system away from the equilibrium.

In order to derive E-stability conditions, it is useful to consider stochastic approximation which is substantially discussed in economics. Stochastic approximation considers convergence of recursive system similar to equation (3.15) - (3.17). This technique is first introduced by Ljung (1977). Marcet and Sargent (1989) show how this technique could be applied to the analysis of adaptive learning. Evans and Honkapohja (2001) summarize and explain the technical details of stochastic approximation in adaptive learning.

BE follow their approach and consider a stochastic recursive algorithm of the form

$$\phi_t = \phi_{t-1} + \gamma_t \mathcal{H}(t, \phi_{t-1}, D_t) \tag{3.25}$$

where ϕ_t is a vector of parameter estimates and D_t is the state vector of information or data at time t and γ_t is a sequence of gains. Define D_t and ϕ_t

$$D_t = \begin{pmatrix} p_t \\ p_{t-1} \\ u_t \\ v_t \end{pmatrix} \qquad \phi_t = \begin{pmatrix} k_t \\ c_t \\ p_t \\ p_t^2 \\ \sigma_t^2 \end{pmatrix}$$

Effectively, this defines an extended T-map which has a larger dimension (i.e. $\mathcal{T}(\phi) - \phi$). Under decreasing-gain learning where $\gamma_t = t^{-1}$, when $t \to \infty$, $\gamma_t \to 0$. Hence the system in equation (3.25) reaches its equilibrium and becomes stationary (i.e. $\phi_t = \phi_{t-1} = \bar{\phi}$). The term $t^{-1}\mathcal{H}(t, \phi_{t-1}, D_t)$ converges to 0 as $t \to \infty$.

I illustrate this concept by a simple example. Consider a system has only one parameter and assume there is an equilibrium of the parameter which is equal to 0 (i.e. $\bar{\phi} = 0$). If $\phi_{t-1} > 0$ then the term $\mathcal{H}(t, \phi_{t-1}, D_t)$ should be less than 0 in order to revert ϕ_t back to the equilibrium $\bar{\phi} = 0$ and vice versa. The term $\mathcal{H}(t, \phi_{t-1}, D_t)$ can thus be viewed as an error correction function which revert any deviation from the equilibrium back to the equilibrium.

Ljung (1977) shows that the behaviour of a stochastic recursive algorithm in equation (3.25) is well approximated by the behaviour of its ordinary differential equations (ODE) for large t. Evans and Honkapohja (2001) suggest the following lemma.

Lemma 3.4.2 *Given the recursive least squares equations expressed in the form in equation (3.25), an equilibrium is stable if the eigenvalues of the Jacobian matrix of*

$$h(\phi) = \lim_{t \to \infty} E\mathcal{H}(t, \phi_{t-1}, D_t) \tag{3.26}$$

evaluated at the equilibrium have negative real parts provided that the limit exists.

The implication of Lemma 3.4.2 is briefly explained as follows. The main objective of local stability analysis is to investigate the dynamics around the equilibrium. The limit in equation (3.26) transforms equation (3.25) into an approximately continuous-time system of equations. The dividend and share supply noise create fluctuations in the system and so it is impossible to investigate the dynamics of a static equilibrium. Expectation E in equation (3.26) gives the average of the error correction term $\mathcal{H}(t, \phi_{t-1}, D_t)$ and eliminates the mean-zero noises. Imposing $h(\phi) = 0$ in equation (3.26) yields an equilibrium and the Jacobian matrix of $h(\phi)$ gives the local stability conditions of the equilibrium. Therefore, the stability of the equilibrium solved in Section 3.4.3 can be determined by this stochastic approximation approach.

I redefine the T-map in equation (3.18) as $T(\theta; \sigma^2)$ where $\theta = (k, c)'$ due to the dependence of θ on σ^2. Hence a column vector $T(\theta; \sigma^2)$ can be viewed as a reduced form of the T-map. Equation (3.19) can be written as

$$p_t = T(\theta_{t-1}; \sigma^2_{t-1})'(1, p_{t-1})' - \beta a \sigma^2_{t-1} v_t \tag{3.27}$$

I substitute equation (3.27) into equation (3.15) to (3.17) and rewrite them in the form in equation (3.25).

$$\begin{aligned}
\theta_t &= \theta_{t-1} + t^{-1}S_t^{-1}X_t(T(\theta_{t-1};\sigma^2_{t-1})'X_t - \theta'_{t-1}X_t - \beta a\sigma^2_{t-1}v_t) \\
&= \theta_{t-1} + t^{-1}S_t^{-1}X_t(X_t'(T(\theta_{t-1};\sigma^2_{t-1}) - \theta_{t-1}) - \beta a\sigma^2_{t-1}v_t) \qquad (3.28)\\
S_t &= S_{t-1} + t^{-1}(X_tX_t' - S_{t-1}) \qquad (3.29)\\
\sigma_t^2 &= \sigma^2_{t-1} + t^{-1}\left(\left(p_t - \theta'_{t-1}X_{t-1} + u_t\right)^2 - \sigma^2_{t-1}\right)\\
&= \sigma^2_{t-1} + t^{-1}\left(z_tz_t' - \sigma^2_{t-1}\right) \qquad (3.30)
\end{aligned}$$

where

$$\begin{aligned}
z_t &= p_t - \theta'_{t-1}X_{t-1} + u_t\\
&= (T(\theta_{t-1};\sigma^2_{t-1}) - \theta_{t-1})'X_t - a\beta\sigma^2_{t-1}v_t + u_t
\end{aligned}$$

Since v_t and u_t are white noises, the term z_tz_t' yields

$$z_tz_t' = (T(\theta_{t-1};\sigma^2_{t-1}) - \theta_{t-1})'X_tX_t'(T(\theta_{t-1};\sigma^2_{t-1}) - \theta_{t-1}) + a^2\beta^2(\sigma^2_{t-1})^2v_t^2 + u_t^2$$

Then I express equation (3.28) - (3.30) as the term $\mathcal{H}(t, \phi_{t-1}, D_t)$.

$$\begin{aligned}
\mathcal{H}_\theta &= S_t^{-1}X_t\left(X_t'\left(T(\theta_{t-1};\sigma^2_{t-1}) - \theta_{t-1}\right) - \beta a\sigma^2_{t-1}v_t\right)\\
\mathcal{H}_S &= X_tX_t' - S_{t-1}\\
\mathcal{H}_{\sigma^2} &= z_tz_t' - \sigma^2_{t-1}
\end{aligned}$$

Then, apply Lemma 3.4.2, $h(\phi)$ become

$$\begin{aligned}
h_\theta &= \lim_{t\to\infty} ES^{-1}_{t-1}X_{t-1}\left(X'_{t-1}(T(\theta_{t-1};\sigma^2_{t-1}) - \theta_{t-1}) - \beta a\sigma^2_{t-1}v_t\right)\\
&= S^{-1}M(\theta,\sigma^2)(T(\theta;\sigma^2) - \theta) \qquad (3.31)\\
h_S &= \lim_{t\to\infty} E(X_tX_t' - S_{t-1})\\
&= M(\theta,\sigma^2) - S \qquad (3.32)\\
h_{\sigma^2} &= \lim_{t\to\infty} E(z_tz_t' - \sigma^2_{t-1})\\
&= \left(T(\theta;\sigma^2) - \theta\right)' M(\theta,\sigma^2)\left(T(\theta;\sigma^2) - \theta\right) + a^2\beta^2(\sigma^2)^2\sigma_v^2 + \sigma_u^2 - \sigma^2 \qquad (3.33)
\end{aligned}$$

where $M = EX(\theta,\sigma^2)X(\theta,\sigma^2)'$ which is a 2-by-2 matrix.

$$M = \begin{bmatrix} 1 & Ep \\ Ep & Ep^2 \end{bmatrix}$$

Since S_t is an estimate of EX_tX_t', equation (3.32) becomes $S \to M$ at equilibrium (i.e. $t \to \infty$). Given S is invertible, $S^{-1}M \to I$. Therefore, only h_θ and h_{σ^2} are relevant to determine the stability of the system. $h(\phi)$ is simplified to

$$\begin{aligned} h_\theta &= T(\theta;\sigma^2) - \theta \\ h_S &= M(\theta,\sigma^2) - S \\ h_{\sigma^2} &= \big(T(\theta;\sigma^2) - \theta\big)' M(\theta,\sigma^2) \big(T(\theta;\sigma^2) - \theta\big) + a^2\beta^2(\sigma^2)^2\sigma_v^2 + \sigma_u^2 - \sigma^2 \end{aligned}$$

The Jacobian matrix of the ODE evaluated at the equilibrium is

$$\begin{pmatrix} \beta(1+c)-1 & \beta k & 0 & 0 & -\beta a s_0 \\ 0 & 2\beta c - 1 & 0 & 0 & 0 \\ \dfrac{\partial M(1,2)}{\partial k} & \dfrac{\partial M(1,2)}{\partial c} & -1 & 0 & \dfrac{\partial M(1,2)}{\partial \sigma^2} \\ \dfrac{\partial M(2,2)}{\partial k} & \dfrac{\partial M(2,2)}{\partial c} & 0 & -1 & \dfrac{\partial M(2,2)}{\partial \sigma^2} \\ 0 & 0 & 0 & 0 & 2a^2\beta^2\sigma_v^2\sigma^2 - 1 \end{pmatrix} \tag{3.34}$$

The first two rows are the derivatives of h_θ with respect to ϕ_t. The third and forth rows are the derivatives of h_S with respect to ϕ_t since M is a 2-by-2 matrix with only two unique elements Ep and Ep^2. Hence there are only two rows representing the derivatives of h_S. The last row consists of derivatives of h_{σ^2} with respect to ϕ_t. At equilibrium, the derivatives of the term $(T(\theta;\sigma^2) - \theta)' M(\theta,\sigma^2) (T(\theta;\sigma^2) - \theta)$ with respect to ϕ_t except σ^2 are 0 due to its symmetric property. Hence the first four elements in the last row are 0. The eigenvalues of the matrix are on the diagonal of the matrix due to specific structure of the matrix (confirmed in Mathematica):

$$-1 + \beta + c\beta \tag{3.35}$$

$$-1 + 2c\beta \tag{3.36}$$

$$-1 + 2a^2\beta^2\sigma_v^2\sigma^2 \tag{3.37}$$

and repeated values of -1. The issue becomes which of the four equilibria derived in Section 3.4.3 has negative eigenvalues. It is evident that the only stable equilibrium corresponds to $c = 0$. The only remaining state variable is σ^2. Section 3.4.3 shows there are two roots for $\bar{\sigma}^2$: σ_-^2 and σ_+^2. Substituting each root into the eigenvalue in equation (3.37),

$$
\begin{aligned}
-1 + 2a^2\beta^2\sigma_v^2 \left(\frac{1 + \sqrt{1 - 4a^2\beta^2\sigma_v^2\sigma_v^2}}{2a^2\beta^2\sigma_v^2} \right) &= \sqrt{1 - 4a^2\beta^2\sigma_v^2\sigma_v^2} \\
-1 + 2a^2\beta^2\sigma_v^2 \left(\frac{1 - \sqrt{1 - 4a^2\beta^2\sigma_v^2\sigma_v^2}}{2a^2\beta^2\sigma_v^2} \right) &= -\sqrt{1 - 4a^2\beta^2\sigma_v^2\sigma_v^2}
\end{aligned}
$$

It can be easily verified that only the eigenvalue for $\bar{\sigma}_-^2$ is negative. Therefore, the only stable equilibrium is

$$
\left[\frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0)}{1-\beta}, \frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0)}{1-\beta}, 0, \bar{\sigma}_-^2 \right]
$$

At this equilibrium, the ALM in equation (3.14) has the following structural form:

$$
p_t = \frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0)}{1-\beta} - \beta a \bar{\sigma}_-^2 v_t \tag{3.38}
$$

The true stochastic process becomes a constant with some white noise v_t. Hence equilibrium asset price does not depend on its lag. More importantly, the fundamental solution for REE in equation (3.7) derived in Section 3.3 has the identical form of equation (3.38) and therefore is attainable and stable under adaptive learning. Intuitively, adaptive learning allows agents to reach the true stochastic process.

3.4.5 Random Walk Beliefs

The equilibrium and stability analysis so far entirely depend on the T-map. It only considers the case of stationary PLM but does not capture a possible case where non-stationary PLM may also lead to an equilibrium of the model. This case happens when agents perceive that stock prices follow a random walk. I refer this exceptional equilibrium as random walk equilibrium in order to distinguish from the stationary PLM equilibrium solved in Section 3.4.3 to which I refer it as T-map equilibrium. Let $\tilde{p}$, $\tilde{\sigma}_p^2$ and $\tilde{\sigma}^2$ denote the fixed point of p_t, $\sigma_{p,t}^2$ and σ_t^2 at random walk equilibrium respectively. The PLM in equation (3.9) under random walk beliefs becomes

$$
p_t = p_{t-1} + \varepsilon_t
$$

It can be expressed in an alternative form.

$$
\begin{aligned}
p_t &= Lp_t + \varepsilon_t \\
&= (1-L)^{-1}\varepsilon_t
\end{aligned} \tag{3.39}
$$

where L is the lag operator. Assuming σ^2_{t-1} is at its fixed point $\tilde{\sigma}^2$, the ALM in equation (3.14) under random walk beliefs becomes

$$p_t = \beta(d_0 - a\tilde{\sigma}^2 s_0) + \beta p_{t-1} - \beta a\tilde{\sigma}^2 v_t \tag{3.40}$$

It can also be expressed in the form of equation (3.39).

$$\begin{aligned} p_t &= \beta(d_0 - a\tilde{\sigma}^2 s_0) + \beta L p_t - \beta a\tilde{\sigma}^2 v_t \\ &= \frac{\beta(d_0 - a\tilde{\sigma}^2 s_0)}{1-\beta} + f(L) v_t \end{aligned}$$

where

$$f(L) = \frac{-\beta a\tilde{\sigma}^2}{1-\beta L}$$

Given $0 < \beta < 1$, express $f(L)$ into a geometric series.

$$f(L) = \frac{-\beta a\tilde{\sigma}^2}{1-\beta L} = -\beta a\tilde{\sigma}^2 \sum_{i=0}^{\infty} (\beta L)^i$$

The ALM in equation (3.40) thus becomes

$$p_t = \frac{\beta(d_0 - a\tilde{\sigma}^2 s_0)}{1-\beta} - \beta a\tilde{\sigma}^2 \sum_{i=0}^{\infty} (\beta L)^i v_t \tag{3.41}$$

Therefore, the ALM under random walk beliefs involves infinite lagged v_t. It introduces serial correlation into the model which is not serially correlated at the stable T-map equilibrium. If a sequence of random shocks leads agents to have random walk beliefs in the PLM, these beliefs will last for a substantial period of time due to its self-fulfilling behaviour. T-map could not capture the random walk equilibrium since it cannot map the ALM under random walk beliefs in equation (3.41) which does not involve p_{t-1} to the PLM which does involve p_{t-1}.

At the random walk equilibrium, the stock price (i.e. $\tilde{p}$) has the identical form to that at the stable T-map equilibrium (i.e. $\bar{p}$). Derive $\tilde{p}$ by taking unconditional expectation on both sides of the ALM in equation (3.41).

$$\begin{aligned} E(p_t) &= E\left(\frac{\beta(d_0 - a\tilde{\sigma}^2 s_0)}{1-\beta} - \beta a\tilde{\sigma}^2 \sum_{i=0}^{\infty} (\beta L)^i v_t \right) \\ \tilde{p} &= \frac{\beta(d_0 - a\tilde{\sigma}^2 s_0)}{1-\beta} \end{aligned}$$

Hence $\tilde{p}$ and $\bar{p}$ in equation (3.21) have the identical form. They are only equal if $\tilde{\sigma}^2 = \bar{\sigma}^2$. However, random walk beliefs lead to a substantial amount of excess volatility and thus the stock price at the random walk equilibrium is lower than that at the stable T-map equilibrium. It can be shown by comparing variance of price at the random walk equilibrium and at the stable T-map equilibrium. Taking unconditional variance on both sides of the ALM in equation (3.41),

$$
\begin{aligned}
Var(p_t) &= Var\left(\frac{\beta(d_0 - a\tilde{\sigma}^2 s_0)}{1-\beta} - \beta a\tilde{\sigma}^2 \sum_{i=0}^{\infty}(\beta L)^i v_t\right) \\
\tilde{\sigma}_p^2 &= \frac{\beta^2 a^2 (\tilde{\sigma}^2)^2 \sigma_v^2}{(1-\beta)^2}
\end{aligned}
$$

At the stable T-map equilibrium where $\bar{c} = 0$, taking unconditional variance on both sides of the ALM in equation (3.14),

$$
\begin{aligned}
Var(p_t) &= Var\left(\beta(d_0 + \bar{k}(1+\bar{c}) - a\bar{\sigma}^2 s_0) + \beta\bar{c}^2 p_{t-1} - \beta a\bar{\sigma}^2 v_t\right) \\
\bar{\sigma}_p^2 &= Var\left(-\beta a\bar{\sigma}^2 v_t\right) \\
\bar{\sigma}_p^2 &= \beta^2 a^2 (\bar{\sigma}^2)^2 \sigma_v^2
\end{aligned}
$$

Even for $\tilde{\sigma}^2 = \bar{\sigma}^2$, the variance of price at the random walk equilibrium is always larger than that at the stable T-map equilibrium given $0 < \beta < 1$, that is $\tilde{\sigma}_p^2 > \bar{\sigma}_p^2$. Moreover, variance of excess returns, $\sigma^2 = \sigma_p^2 + \sigma_u^2$, as $\tilde{\sigma}_p^2 > \bar{\sigma}_p^2$, in equilibrium it holds that $\tilde{\sigma}^2 > \bar{\sigma}^2$. As a result, the stock price at the random walk equilibrium is actually lower than that at the stable T-map equilibrium. Nevertheless, the stability of the random walk equilibrium cannot be verified by the Jacobian matrix in equation (3.34) since it relies on the ODE derived from the T-map which does not capture the random walk equilibrium. The only clear conclusion is that random walk beliefs in PLM may lead to an equilibrium in the ALM which increases market volatility and decreases the average stock price.

Chapter 4
Heterogeneous Beliefs

The Basic Model analysed so far considers representative forward-looking agents made investment decisions on demand for a risky asset. Agent recursively update parameters in the PLM and estimates on variance of excess returns in order to improve their knowledge of the stock market. Next I consider multiple types of agents who use different methods to forecast future price and respond differently to new information available to them. I combine adaptive learning approach with heterogeneous beliefs among agents.

The literature emphasizes the role of heterogeneous beliefs in stock markets is led by Brock and Hommes (1998). The paper investigates the dynamics of an asset pricing model considering an economy where different groups of traders have different expectations about future prices and answers questions whether heterogeneity in beliefs may lead to market instability and non-rational traders can survive in the market or be driven out of the market by rational traders. It suggests when agents who have heterogeneous beliefs in a stock market frequently switch their prediction strategies, the dynamics may lead to highly irregular, chaotic asset price fluctuations. However, this heterogeneous beliefs model does not take adaptive learning into account. Goldbaum and Panchenko (2010) consider an asset pricing model with heterogeneous beliefs, adaptive learning and switching although they have only a short-lived risky asset.

The remaining part of my thesis investigates the dynamics in a simple stock market where heterogeneous beliefs agents exist. In my model, agents also adaptively learn the true structure of the stock market and are concerned about risk and returns as in the Basic Model. Hence it can be considered as an extension of the Basic Model. I examine a simple case where there are only two groups of agents. Both groups form expectations on future prices and estimate variance of excess returns. Agents in group 1 act as econometricians. They learn over time and revise the parameters in their forecast rule or PLM whereas agents in group 2 do not learn and never update their parameters. Intuitively, learning agents have the knowledge, time and resources to analyse the stock market whereas learning agents do not. More importantly, learning agents do not realise the existence of non-learning agents but only observe prices realised from the market. The interaction between the beliefs of two groups determine the dynamics of the stock market. I refer to this model as the Two-Beliefs Model.

4.1 Two-Beliefs Model: Learning vs Non-learning Homogeneous Variance Case

I first assume learning and non-learning agents estimate the variance of excess returns identically and refer to this case as Homogeneous Variance Case. Section 4.2 relaxes this assumption and allows learning and non-learning agent have different estimate on variance of excess returns. I refer to it as Heterogeneous Variance Case.

4.1.1 Aggregate Demand

I define the environment of the Two-Beliefs Model. There is one risky asset yields a dividend stream d_t and trades at ex-dividend price p_t and also one risk free asset that pays the rate of return $R = \beta^{-1}$ in the economy. Let n_h denote the exogenous fraction of agent type h in the market.

$$\sum_{h=1}^{2} n_h = 1$$

The aggregate demand of the risky asset becomes weighted sum of the demands of each agent type, $z_{1,dt}$ and $z_{2,dt}$.

$$z_{dt} = n_1 z_{1,dt} + n_2 z_{2,dt}$$

The approach to derive demand for the risky asset in section 3.1 also applies to this environment which is similar to that in the Basic Model. Hence the demand of each agent type h is

$$z_{h,dt} = \hat{E}_{h,t}(p_{t+1} + d_{t+1} - Rp_t)/a\sigma_t^2$$

I assume risk aversion factor a is identical for all agents. At market clearing condition (i.e. $z_{st} = z_{dt}$), it yields

$$\begin{aligned}
z_{st} &= n_1 z_{1,dt} + n_2 z_{2,dt} \\
z_{st} &= n_1 \left\{ \hat{E}_{1,t}(p_{t+1} + d_{t+1} - Rp_t)/a\sigma_t^2 \right\} + n_2 \left\{ \hat{E}_{2,t}(p_{t+1} + d_{t+1} - Rp_t)/a\sigma_t^2 \right\}
\end{aligned}$$

Given $\beta = R^{-1}$, the true stochastic process of the risky asset price becomes

$$p_t = \beta \left\{ n_1 \hat{E}_{1,t}(p_{t+1} + d_{t+1}) + n_2 \hat{E}_{2,t}(p_{t+1} + d_{t+1}) - a\sigma_t^2 z_{st} \right\}$$

The Two-Beliefs Model has a functional form similar to the Basic Model in equation (3.1). It follows the conditions in equation (3.2) - (3.3) in the Basic Model and also a new condition on the fraction of learning agents in the market n_1. All the conditions on variables and exogenous parameters are listed below

$$0 \leq p_t, \sigma_t^2, \sigma_u^2, \sigma_v^2 \tag{4.1}$$

$$0 < \beta, a, n_1 < 1 \tag{4.2}$$

For share supply z_{st}, it follows the identical multiplicative process in the Basic Model in equation (3.6). The Two-Beliefs Model also follows Assumption 3.4.1 that no contemporaneous variables and information are observable to all agents. Similar to $\hat{E}_t(p_{t+1}|\Omega_{t-1})$ introduced in Section 3.4, I introduce $\hat{E}_{h,t}(\cdot|\Omega_{t-1})$ which denotes the subjective expectation formed by agent type h at time t conditional on variables and information Ω_{t-1} up to time $t-1$. Hence equation (4.3) becomes

$$p_t = \beta \left\{ n_1 \hat{E}_{1,t}(p_{t+1} + d_{t+1}|\Omega_{t-1}) + n_2 \hat{E}_{2,t}(p_{t+1} + d_{t+1}|\Omega_{t-1}) - a\sigma_{t-1}^2 z_{st} \right\}$$

Given dividend mean is constant at level d_0, the equation becomes

$$p_t = \beta \left\{ d_0 + n_1 \hat{E}_{1,t}(p_{t+1}|\Omega_{t-1}) + n_2 \hat{E}_{2,t}(p_{t+1}|\Omega_{t-1}) - a\sigma_{t-1}^2 z_{st} \right\} \tag{4.3}$$

4.1.2 Perceived and Actual Law of Motion

Agents forecast price based on a simple AR(1) process. Following the PLM in equation (3.9) in the Basic Model,

$$p_t = k_{t-1} + c_{t-1}p_{t-1} + \varepsilon_t$$

Applying the idea of heterogeneous beliefs, the parameters in the PLM of learning agents are time-varying in contrast to non-learning agents whose parameters in the PLM are fixed. Under Assumption 3.4.1, the PLM for the two types is

$$p_{1,t} = k_{1,t-1} + c_{1,t-1}p_{t-1} + \varepsilon_{1,t} \tag{4.4}$$

$$p_{2,t} = k_2 + c_2 p_{t-1} + \varepsilon_{2,t} \tag{4.5}$$

I impose a stationary condition on c_2 which yields

$$-1 < c_2 < 1 \tag{4.6}$$

Derive the expectations of future price based on the two PLM

$$\hat{E}_{1,t}(p_{1,t+1}|\Omega_{t-1}) = k_{1,t-1}(1 + c_{1,t-1}) + c_{1,t-1}^2 p_{t-1} \quad (4.7)$$
$$\hat{E}_{2,t}(p_{2,t+1}|\Omega_{t-1}) = k_2(1 + c_2) + c_2^2 p_{t-1} \quad (4.8)$$

Agents are also concerned about risk. In the Basic Model, agents are assumed to forecast σ_{t-1}^2 by measuring the forecast error between the realized p_t and their forecast from the PLM. In the Homogeneous Variance Case, I assume that both types use the same estimate of σ_{t-1}^2. Equation (3.20) shows that the variance of excess returns can be decomposed into the variance of price $\sigma_{p,t-1}^2$ and the variance of dividend noise σ_u^2. The former is an endogenous variance whereas the later is an exogenous variance. I assume both types of agents estimate $\sigma_{p,t-1}^2$ by sample variance of price based on historical prices available. Now I derive the ALM. By substituting equation (4.7) and (4.8) into equation (4.3), the ALM becomes

$$\begin{aligned} p_t &= \beta \left\{ d_0 + n_1 \hat{E}_{1,t}(p_{t+1}|\Omega_{t-1}) + n_2 \hat{E}_{2,t}(p_{t+1}|\Omega_{t-1}) - a\sigma_{t-1}^2 z_{st} \right\} \\ &= \beta(d_0 + n_1 k_{1,t-1}(1 + c_{1,t-1}) + n_1 c_{1,t-1}^2 p_{t-1} + n_2 k_2(1 + c_2) + n_2 c_2^2 p_{t-1}) - \beta a \sigma_{t-1}^2 z_{st} \\ &= \beta(d_0 - a\sigma_{t-1}^2 s_0 + n_1 k_{1,t-1}(1 + c_{1,t-1}) + n_2 k_2(1 + c_2)) + \beta(n_1 c_{1,t-1}^2 + n_2 c_2^2) p_{t-1} - \beta a \sigma_{t-1}^2 v_t \end{aligned} \quad (4.9)$$

Naturally, the ALM in the Two-Beliefs Model under the Homogeneous Variance Case has a more complicated structure than the ALM in the Basic Model in equation (3.14).

4.1.3 Recursive Least Squares

I follow the RLS equations in the Basic Model. Learning agents update parameters k_1 and c_1 in their PLM. Let $\theta_{1,t} = (k_{1,t}, c_{1,t})'$ and $X_t = (1, p_{t-1})'$. The RLS equations for $\theta_{1,t}$ and S_t which is an estimate of $X_t X_t'$ yield

$$\theta_{1,t} = \theta_{1,t-1} + \gamma_{1,t} S_t^{-1} X_t (p_t - \theta_{1,t-1}' X_t)$$
$$S_t = S_{t-1} + \gamma_{1,t}(X_t X_t' - S_{t-1})$$

The updating equation for σ_{t-1}^2 is different to the Basic Model. Agents in the Homogeneous Variance Case are assumed to use sample variance of price based on historical prices realized up to $t - 1$. When p_t is realized, agents observe new information and update their sample variance. Combined with a given σ_u^2, σ_{t-1}^2 enters the ALM at time t as the best estimate for σ_t^2. I therefore look into the RLS equations for $\sigma_{p,t}^2$ treating σ_u^2 as an exogenous variance. Noted that this is done to simplify exposition. I could have estimated it as a recursive sample variance as well.

Let $\dot{p}_t$ denote the mean of the price,

$$\dot{p}_t = \frac{1}{t}\sum_{k=1}^{t} p_k$$
$$\dot{p}_{t-1} = \frac{1}{t-1}\sum_{k=1}^{t-1} p_k \tag{4.10}$$

The sample variance of price at time $t-1$ is defined as

$$\begin{aligned}\sigma^2_{p,t-1} &= \frac{1}{t-1}\sum_{k=1}^{t-1}(p_k - \dot{p}_{t-1})^2 \\ &= \left(\frac{1}{t-1}\sum_{k=1}^{t-1}(p_k)^2\right) - \dot{p}^2_{t-1}\end{aligned} \tag{4.11}$$

Equation (4.10) and (4.11) can be rewritten in a recursive fashion. For $\dot{p}_t$,

$$\begin{aligned}\dot{p}_t &= \frac{1}{t}\sum_{k=1}^{t} p_k \\ t\dot{p}_t &= \sum_{k=1}^{t-1} p_k + p_t \\ t\dot{p}_t &= (t-1)\dot{p}_{t-1} + p_t \\ \dot{p}_t &= \dot{p}_{t-1} + t^{-1}(p_t - \dot{p}_{t-1})\end{aligned}$$

For $\sigma^2_{p,t}$,

$$\begin{aligned}\sigma^2_{p,t} &= \left(\frac{1}{t}\sum_{k=1}^{t}(p_k)^2\right) - \dot{p}^2_t \\ t(\sigma^2_{p,t} + \dot{p}^2_t) &= \sum_{k=1}^{t-1}(p_k)^2 + p^2_t \\ t(\sigma^2_{p,t} + \dot{p}^2_t) &= (t-1)(\sigma^2_{p,t-1} + \dot{p}^2_{t-1}) + p^2_t \\ t(\sigma^2_{p,t} + \dot{p}^2_t) &= t\sigma^2_{p,t-1} + t\dot{p}^2_{t-1} - \sigma^2_{p,t-1} - \dot{p}^2_{t-1} + p^2_t \\ \sigma^2_{p,t} &= \sigma^2_{p,t-1} + \dot{p}^2_{t-1} - \dot{p}^2_t + t^{-1}(p^2_t - \sigma^2_{p,t-1} - \dot{p}^2_{t-1})\end{aligned}$$

The updating equations for the system are summarized as follows

$$
\begin{aligned}
\theta_{1,t} &= \theta_{1,t-1} + \gamma_{1,t} S_t^{-1} X_t (p_t - \theta'_{1,t-1} X_t) && (4.12)\\
S_t &= S_{t-1} + \gamma_{1,t}(X_t X'_t - S_{t-1}) && (4.13)\\
\dot{p}_t &= \dot{p}_{t-1} + t^{-1}(p_t - \dot{p}_{t-1}) && (4.14)\\
\sigma^2_{p,t} &= \sigma^2_{p,t-1} + \dot{p}^2_{t-1} - \dot{p}^2_t + t^{-1}(p^2_t - \sigma^2_{p,t-1} - \dot{p}^2_{t-1}) && (4.15)\\
\sigma^2_t &= \sigma^2_{p,t} + \sigma^2_u && (4.16)
\end{aligned}
$$

4.1.4 Equilibrium

Similar by the Basic Model, T-map solves for equilibrium by mapping the PLM to the ALM. Let $\bar{p}$, $\bar{k}_1$, $\bar{c}_1$ and $\bar{\sigma}^2$ denote the fixed point for p_t, $k_{1,t}$, $c_{1,t}$ and σ^2_t. The ALM in equation (4.9) is expressed in the form of a T-map. For brevity, I do not include σ^2 in the T-map but solve for it later. T-map is defined as

$$T\begin{pmatrix} k_1 \\ c_1 \end{pmatrix} = \begin{pmatrix} \beta(d_0 - a\sigma^2 s_0 + n_1 k_1(1+c_1) + n_2 k_2(1+c_2)) \\ \beta(n_1 c_1^2 + n_2 c_2^2) \end{pmatrix} \tag{4.17}$$

Hence p_t can be written as

$$p_t = T\begin{pmatrix} k_{1,t-1} \\ c_{1,t-1} \end{pmatrix}' (1, p_{t-1})' - \beta a \sigma^2_{t-1} v_t \tag{4.18}$$

In the model, only learning agents update their parameters. Therefore, the T-map only maps the PLM of learning agents in equation (4.4) to the ALM in equation (4.9). Solving for fixed point of the T-map,

$$T\begin{pmatrix} \bar{k}_1 \\ \bar{c}_1 \end{pmatrix} = \begin{pmatrix} \bar{k}_1 \\ \bar{c}_1 \end{pmatrix}$$

Then,

$$
\begin{aligned}
\beta(n_1 \bar{c}_1^2 + n_2 c_2^2) &= \bar{c}_1 \\
\bar{c}_1 &= \frac{1 \pm \sqrt{1 - 4\beta^2 n_1 n_2 c_2^2}}{2\beta n_1}
\end{aligned}
$$

The fixed point $\bar{c}_1$ only depends on a set of exogenous parameters. Let the discriminant $1 - 4\beta^2 n_1 n_2 c_2^2$ denote Δ_{Homo}. As long as the conditions in equation (4.1), (4.2) and (4.6) are satisfied, Δ_{Homo} is always greater than 0 and hence there are always two real roots for $\bar{c}_1$. To show this, check the condition on c_2,

$$
\begin{aligned}
1 - 4\beta^2 n_1 n_2 c_2^2 &> 0 \\
|c_2| &< \frac{1}{2\beta\sqrt{n_1 n_2}}
\end{aligned}
\tag{4.19}
$$

This condition only matters if it imposes a narrower range on c_2 given the stationary condition on c_2 (i.e. $-1 < c_2 < 1$). The range of c_2 in equation (4.19) is minimized when the term $1/2\beta\sqrt{n_1 n_2}$ is minimized and thus the term $2\beta\sqrt{n_1 n_2}$ is maximized. Given $n_2 = 1 - n_1$, $n_1(1 - n_1)$ is maximized when $n_1 = n_2 = 1/2$. Equation (4.19) becomes $|c_2| < 1/\beta$ where $0 < \beta < 1$ and $1/\beta > 1$. Hence the condition c_2 in equation (4.19) does not impose any further limitation on the range of c_2. Then solve for $\bar{k}_1$,

$$
\begin{aligned}
\beta(d_0 - a\bar{\sigma}^2 s_0 + n_1 \bar{k}_1(1 + \bar{c}_1) + n_2 k_2(1 + c_2)) &= \bar{k}_1 \\
\beta(d_0 - a\bar{\sigma}^2 s_0) + \beta n_2 k_2(1 + c_2) &= \bar{k}_1(1 - \beta n_1(1 + \bar{c}_1)) \\
\bar{k}_1 &= \frac{\beta(d_0 - a\bar{\sigma}^2 s_0) + \beta n_2 k_2(1 + c_2)}{1 - \beta n_1(1 + \bar{c}_1)}
\end{aligned}
$$

Different from $\bar{c}_1$, $\bar{k}_1$ depends on $\bar{c}_1$ and $\bar{\sigma}^2$. Then I solve for $\bar{p}$. Similar to the Basic Model, taking unconditional expectation on both sides of the ALM. From the ALM in equation (4.9),

$$
E(p_t) = E\left[\beta(d_0 - a\sigma_{t-1}^2 s_0 + n_1 k_{1,t-1}(1 + c_{1,t-1}) + n_2 k_2(1 + c_2)) + \beta(n_1 c_{1,t-1}^2 + n_2 c_2^2) p_{t-1}\right]
$$

As before, $E(p_t) = E(p_{t-1}) = \bar{p}$. At equilibrium, k_{t-1}, c_{t-1} and σ_{t-1}^2 converge to $\bar{k}_1$, $\bar{c}_1$ and $\bar{\sigma}^2$ respectively. Then,

$$
\bar{p} = \frac{\beta(d_0 - a\bar{\sigma}^2 s_0 + n_1 \bar{k}_1(1 + \bar{c}_1) + n_2 k_2(1 + c_2))}{1 - \beta(n_1 \bar{c}_1^2 + n_2 c_2^2)}
$$

Hence $\bar{k}_1$ and $\bar{p}$ both depend on $\bar{c}_1$ and $\bar{\sigma}^2$. I then solve for $\bar{\sigma}^2$. From the definition of variance of excess returns in equation (3.20) where $\bar{\sigma}^2 = \bar{\sigma}_p^2 + \sigma_u^2$, I derive $\bar{\sigma}_p^2$ by taking unconditional variance of price on both sides of the ALM in equation (4.9).

$$
\begin{aligned}
Var(p_t) &= Var(\beta(d_0 - a\sigma_{t-1}^2 s_0 + n_1 k_{1,t-1}(1 + c_{1,t-1}) + n_2 k_2(1 + c_2)) \\
&\quad + \beta(n_1 c_{1,t-1}^2 + n_2 c_2^2) p_{t-1} - \beta a \sigma_{t-1}^2 v_t) \\
&= Var(\beta(n_1 c_{1,t-1}^2 + n_2 c_2^2) p_{t-1}) + Var(-\beta a \sigma_{t-1}^2 v_t) \\
&= \beta^2[(n_1 c_{1,t-1}^2 + n_2 c_2^2)^2] Var(p_{t-1}) + \beta^2 a^2 (\sigma_{t-1}^2)^2 \sigma_v^2
\end{aligned}
$$

By stationarity, $Var(p_t) = Var(p_{t-1}) = \bar{\sigma}_p^2$. At equilibrium, $c_{1,t-1}$ and σ_{t-1}^2 converge to $\bar{c}_1$ and $\bar{\sigma}^2$ respectively.

$$\begin{aligned}\bar{\sigma}_p^2 &= \beta^2(n_1\bar{c}_1^2 + n_2c_2^2)^2\bar{\sigma}_p^2 + \beta^2a^2(\bar{\sigma}^2)^2\sigma_v^2 \\ &= \frac{a^2\beta^2(\bar{\sigma}^2)^2\sigma_v^2}{1-\beta^2(n_1\bar{c}_1^2+n_2c_2^2)^2}\end{aligned} \tag{4.20}$$

Substituting $\bar{\sigma}_p^2$ into $\bar{\sigma}^2$,

$$\bar{\sigma}^2 = \frac{a^2\beta^2(\bar{\sigma}^2)^2\sigma_v^2}{1-\beta^2(n_1\bar{c}_1^2+n_2c_2^2)^2} + \sigma_u^2$$

This is a quadratic equation and can be easily solved. Let $\lambda = \dfrac{1}{1-\beta^2(n_1\bar{c}_1^2+n_2c_2^2)^2}$.

$$\begin{aligned}\bar{\sigma}^2 &= \lambda a^2\beta^2(\bar{\sigma}^2)^2\sigma_v^2 + \sigma_u^2 \\ &= \frac{1 \pm \sqrt{1-4\lambda a^2\beta^2\sigma_v^2\sigma_u^2}}{2\lambda a^2\beta^2\sigma_v^2}\end{aligned}$$

For $\bar{c}_1 = \dfrac{1-\sqrt{1-4\beta^2n_1n_2c_2^2}}{2\beta n_1}$, $\dfrac{1}{\lambda}$ yields

$$\begin{aligned}&1-\beta^2(n_1\bar{c}_1^2+n_2c_2^2)^2 \\ =&1-\beta^2\left(n_1\left(\frac{1-\sqrt{1-4\beta^2n_1n_2c_2^2}}{2\beta n_1}\right)^2+n_2c_2^2\right)^2 \\ =&1-\beta^2\left(\frac{1-2\sqrt{1-4\beta^2n_1n_2c_2^2}+1-4\beta^2n_1n_2c_2^2+4\beta^2n_1n_2c_2^2}{4\beta^2n_1}\right)^2 \\ =&1-\left(\frac{1-\sqrt{1-4\beta^2n_1n_2c_2^2}}{2\beta n_1}\right)^2 \\ =&1-\bar{c}_1^2\end{aligned}$$

Similarly, for $\bar{c}_1 = \dfrac{1+\sqrt{1-4\beta^2n_1n_2c_2^2}}{2\beta n_1}$,

$$\begin{aligned}&1-\beta^2(n_1\bar{c}_1^2+n_2c_2^2)^2 \\ &\vdots \\ =&1-\bar{c}_1^2\end{aligned}$$

Hence $\lambda = \dfrac{1}{1-\bar{c}_1^2}$. I simplify $\bar{\sigma}^2$ to

$$\bar{\sigma}^2 = \frac{(1-\bar{c}_1^2)\left[1 \pm \sqrt{1 - \dfrac{4a^2\beta^2\sigma_v^2\sigma_u^2}{1-\bar{c}_1^2}}\right]}{2a^2\beta^2\sigma_v^2}$$

It shows $\bar{\sigma}^2$ depends on $\bar{c}_1$. Since $\bar{c}_1$ and $\bar{\sigma}^2$ both have two solutions, there are four combinations of $\bar{c}_1$ and $\bar{\sigma}^2$. It indicates four equilibria in the Homogeneous Variance Case. All possible fixed points $\bar{p}$, $\bar{k}_1$, $\bar{c}_1$ and $\bar{\sigma}^2$ are listed as follows.

$$\bar{p} = \frac{\beta(d_0 - a\bar{\sigma}^2 s_0 + n_1\bar{k}_1(1+\bar{c}_1) + n_2k_2(1+c_2))}{1-\beta(n_1\bar{c}_1^2 + n_2c_2^2)} \tag{4.21}$$

$$\bar{k}_1 = \frac{\beta(d_0 - a\bar{\sigma}^2 s_0) + \beta n_2k_2(1+c_2)}{1-\beta n_1(1+\bar{c}_1)} \tag{4.22}$$

$$\bar{c}_1 = \begin{cases} \dfrac{1+\sqrt{1-4\beta^2 n_1n_2c_2^2}}{2\beta n_1} \rightarrow \bar{c}_{1,+} \\ \dfrac{1-\sqrt{1-4\beta^2 n_1n_2c_2^2}}{2\beta n_1} \rightarrow \bar{c}_{1,-} \end{cases} \tag{4.23}$$

$$\bar{\sigma}^2 = \begin{cases} \dfrac{(1-\bar{c}_1^2)\left[1 + \sqrt{1 - \dfrac{4a^2\beta^2\sigma_v^2\sigma_u^2}{1-\bar{c}_1^2}}\right]}{2a^2\beta^2\sigma_v^2} \rightarrow \bar{\sigma}_+^2 \\ \dfrac{(1-\bar{c}_1^2)\left[1 - \sqrt{1 - \dfrac{4a^2\beta^2\sigma_v^2\sigma_u^2}{1-\bar{c}_1^2}}\right]}{2a^2\beta^2\sigma_v^2} \rightarrow \bar{\sigma}_-^2 \end{cases} \tag{4.24}$$

Let $[\bar{p}, \bar{k}_1, \bar{c}_1, \bar{\sigma}^2]$ denote an equilibrium.

$$\left[\frac{\beta(d_0 - a\bar{\sigma}_+^2 s_0 + n_1\bar{k}_1(1+\bar{c}_{1,+}) + n_2k_2(1+c_2))}{1-\beta(n_1\bar{c}_{1,+}^2 + n_2c_2^2)}, \frac{\beta(d_0 - a\bar{\sigma}_+^2 s_0) + \beta n_2k_2(1+c_2)}{1-\beta n_1(1+\bar{c}_{1,+})}, \bar{c}_{1,+}, \sigma_+^2\right] \tag{4.25}$$

$$\left[\frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0 + n_1\bar{k}_1(1+\bar{c}_{1,+}) + n_2k_2(1+c_2))}{1-\beta(n_1\bar{c}_{1,+}^2 + n_2c_2^2)}, \frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0) + \beta n_2k_2(1+c_2)}{1-\beta n_1(1+\bar{c}_{1,+})}, \bar{c}_{1,+}, \sigma_-^2\right] \tag{4.26}$$

$$\left[\frac{\beta(d_0 - a\bar{\sigma}_+^2 s_0 + n_1\bar{k}_1(1+\bar{c}_{1,-}) + n_2k_2(1+c_2))}{1-\beta(n_1\bar{c}_{1,-}^2 + n_2c_2^2)}, \frac{\beta(d_0 - a\bar{\sigma}_+^2 s_0) + \beta n_2k_2(1+c_2)}{1-\beta n_1(1+\bar{c}_{1,-})}, \bar{c}_{1,-}, \sigma_+^2\right] \tag{4.27}$$

$$\left[\frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0 + n_1\bar{k}_1(1+\bar{c}_{1,-}) + n_2k_2(1+c_2))}{1-\beta(n_1\bar{c}_{1,-}^2 + n_2c_2^2)}, \frac{\beta(d_0 - a\bar{\sigma}_-^2 s_0) + \beta n_2k_2(1+c_2)}{1-\beta n_1(1+\bar{c}_{1,-})}, \bar{c}_{1,-}, \sigma_-^2\right] \tag{4.28}$$

4.1.5 Stability Analysis

The similar approach of stability analysis in Section 3.4.4 applies to the Two-Beliefs Model. To verify stability, I construct a Jacobian matrix of $h(\phi)$ in equation (3.26), Lemma 3.4.2. First, simplify $\sigma^2_{p,t}$ in equation (4.15) by substituting the updating equation for $\dot{p}_t$ in equation (4.14).

$$
\begin{aligned}
\sigma^2_{p,t} &= \sigma^2_{p,t-1} + \dot{p}^2_{t-1} - (\dot{p}_{t-1} + t^{-1}(p_t - \dot{p}_{t-1}))^2 + t^{-1}(p_t^2 - \sigma^2_{p,t-1} - \dot{p}^2_{t-1}) \\
&= \sigma^2_{p,t-1} + \dot{p}^2_{t-1} - (\dot{p}^2_{t-1} + 2t^{-1}\dot{p}_{t-1}(p_t - \dot{p}_{t-1}) + t^{-2}(p_t - \dot{p}_{t-1})^2) + t^{-1}(p_t^2 - \sigma^2_{p,t-1} - \dot{p}^2_{t-1}) \\
&= \sigma^2_{p,t-1} - 2t^{-1}\dot{p}_{t-1}(p_t - \dot{p}_{t-1}) + t^{-1}(p_t^2 - \sigma^2_{p,t-1} - \dot{p}^2_{t-1}) - t^{-2}(p_t - \dot{p}_{t-1})^2 \\
&= \sigma^2_{p,t-1} - 2t^{-1}\dot{p}_{t-1}(p_t - \dot{p}_{t-1}) + t^{-1}(p_t^2 - \sigma^2_{p,t-1} - \dot{p}^2_{t-1}) + t^{-2}(2p_t\dot{p}_{t-1} - p_t^2 - \dot{p}^2_{t-1})
\end{aligned}
$$

The second-order term $t^{-2}(2p_t\dot{p}_{t-1} - p_t^2 - \dot{p}^2_{t-1})$ shrinks to 0 extremely quickly when $t \to \infty$ and therefore $\sigma^2_{p,t}$ can approximately written as

$$
\begin{aligned}
\sigma^2_{p,t} &\approx \sigma^2_{p,t-1} + t^{-1}(2\dot{p}^2_{t-1} - 2\dot{p}_{t-1}p_t) + t^{-1}(p_t^2 - \sigma^2_{p,t-1} - \dot{p}^2_{t-1}) \\
&= \sigma^2_{p,t-1} + t^{-1}(p_t^2 - \sigma^2_{p,t-1} - \dot{p}^2_{t-1} - 2\dot{p}_{t-1}p_t + 2\dot{p}^2_{t-1}) \\
&= \sigma^2_{p,t-1} + t^{-1}(p_t^2 - 2\dot{p}_{t-1}p_t + \dot{p}^2_{t-1} - \sigma^2_{p,t-1}) \\
&= \sigma^2_{p,t-1} + t^{-1}((p_t - \dot{p}_{t-1})^2 - \sigma^2_{p,t-1})
\end{aligned}
$$

All RLS equations are listed as follows.

$$
\begin{aligned}
\theta_{1,t} &= \theta_{1,t-1} + t^{-1}S_t^{-1}X_t(p_t - \theta'_{1,t-1}X_t) \\
S_t &= S_{t-1} + t^{-1}(X_tX_t' - S_{t-1}) \\
\sigma^2_{p,t} &= \sigma^2_{p,t-1} + t^{-1}\left((p_t - \dot{p}_{t-1})^2 - \sigma^2_{p,t-1}\right)
\end{aligned}
$$

I express the RLS equations in the form of a stochastic recursive algorithm in equation (3.25) shown below.

$$
\phi_t = \phi_{t-1} + \gamma_t \mathcal{H}(t, \phi_{t-1}, D_t)
$$

Under decreasing-gain learning, $\gamma_t = t^{-1}$. I redefine D_t and ϕ_t.

$$
D_t = \begin{pmatrix} p_t \\ p_{t-1} \\ u_t \\ v_t \end{pmatrix} \qquad \phi_t = \begin{pmatrix} k_{1,t} \\ c_{1,t} \\ p_t \\ p_t^2 \\ \sigma^2_{p,t} \end{pmatrix}
$$

I redefine the T-map in equation (4.17) as $T(\theta_1; \sigma^2)$ where $\theta = (k_1, c_1)'$ due to the dependence of θ_1 on σ^2. Equation (4.18) can be written as

$$p_t = T(\theta_{1,t-1}; \sigma^2_{t-1})'(1, p_{t-1})' - \beta a \sigma^2_{t-1} v_t$$

I express the RLS equations as the term $\mathcal{H}(t, \phi_{t-1}, D_t)$ in the form of a stochastic recursive algorithm in equation (3.25).

$$\begin{aligned}
\mathcal{H}_\theta &= S_t^{-1} X_t \left(X_t' \left(T(\theta_{1,t-1}; \sigma^2_{t-1}) - \theta_{1,t-1} \right) - \beta a \sigma^2_{t-1} v_t \right) \\
\mathcal{H}_S &= X_t X_t' - S_{t-1} \\
\mathcal{H}_{\sigma_p^2} &= (p_t - \dot{p}_{t-1})^2 - \sigma^2_{p,t-1}
\end{aligned}$$

Then apply Lemma 3.4.2, $h(\phi)$ becomes

$$\begin{aligned}
h_\theta &= S^{-1} M(\theta_1, \sigma_1^2)(T(\theta_1; \sigma^2) - \theta_1) \\
h_S &= M(\theta_1, \sigma^2) - S \\
h_{\sigma_p^2} &= \lim_{t \to \infty} E[(p_t - \dot{p}_{t-1})^2 - \sigma^2_{p,t-1}] \\
&= E[(-a\beta\sigma^2 v)^2 - \sigma_p^2] \\
&= a^2\beta^2\sigma_v^2(\sigma_p^2 + \sigma_u^2)^2 - \sigma_p^2 \\
&= a^2\beta^2\sigma_v^2(\sigma_p^2)^2 + 2a^2\beta^2\sigma_v^2\sigma_u^2\sigma_p^2 + a^2\beta^2\sigma_v^2(\sigma_u^2)^2 - \sigma_p^2
\end{aligned}$$

Similar to Section 3.4.3, $h_S = M(\theta_1, \sigma^2) - S$ and $S \to M$ where $M = EX(\theta_1, \sigma^2)X(\theta_1, \sigma^2)'$ since S_t is an estimate of EX_tX_t' and $h_{\theta_1} = T(\theta_1; \sigma^2) - \theta_1$. $h(\phi)$ is simplified to

$$\begin{aligned}
h_{\theta_1} &= T(\theta_1; \sigma^2) - \theta_1 \\
h_S &= M(\theta_1, \sigma^2) - S \\
h_{\sigma_p^2} &= a^2\beta^2\sigma_v^2(\sigma_p^2)^2 + 2a^2\beta^2\sigma_v^2\sigma_u^2\sigma_p^2 + a^2\beta^2\sigma_v^2(\sigma_u^2)^2 - \sigma_p^2
\end{aligned}$$

The Jacobian matrix of $h(\phi)$ evaluated at the equilibrium is

$$\begin{pmatrix}
\beta n_1(1 + c_1) - 1 & \beta n_1 k_1 & 0 & 0 & -\beta a s_0 \\
0 & 2\beta n_1 c_1 - 1 & 0 & 0 & 0 \\
\dfrac{\partial M(1,2)}{\partial k_1} & \dfrac{\partial M(1,2)}{\partial c_1} & -1 & 0 & \dfrac{\partial M(1,2)}{\partial \sigma_p^2} \\
\dfrac{\partial M(2,2)}{\partial k_1} & \dfrac{\partial M(2,2)}{\partial c_1} & 0 & -1 & \dfrac{\partial M(2,2)}{\partial \sigma_p^2} \\
0 & 0 & 0 & 0 & 2a^2\beta^2\sigma_v^2(\sigma_p^2 + \sigma_u^2) - 1
\end{pmatrix}$$

The eigenvalues of the matrix are on the diagonal of the matrix (confirmed in Mathematica):

$$
\begin{aligned}
&-1+\beta n_1(1+c_1) && (4.29)\\
&-1+2\beta n_1 c_1 && (4.30)\\
&-1+2a^2\beta^2\sigma_v^2(\sigma_p^2+\sigma_u^2) && (4.31)
\end{aligned}
$$

and repeated values of -1. First I check whether the first two eigenvalues are negative. For $\bar{c}_1 = \bar{c}_{1,+} = \dfrac{1+\sqrt{1-4\beta^2 n_1 n_2 c_2^2}}{2\beta n_1}$, the eigenvalues are

$$
\begin{aligned}
-1+\beta n_1\left[1+\frac{1+\sqrt{1-4\beta^2 n_1 n_2 c_2^2}}{2\beta n_1}\right] &= -1+\beta n_1+\frac{1+\sqrt{1-4\beta^2 n_1 n_2 c_2^2}}{2} && (4.32)\\
-1+2\beta n_1\frac{1+\sqrt{1-4\beta^2 n_1 n_2 c_2^2}}{2\beta n_1} &= \sqrt{1-4\beta^2 n_1 n_2 c_2^2} && (4.33)
\end{aligned}
$$

For $\bar{c}_1 = \bar{c}_{1,-} = \dfrac{1-\sqrt{1-4\beta^2 n_1 n_2 c_2^2}}{2\beta n_1}$, the eigenvalues are

$$
\begin{aligned}
-1+\beta n_1\left[1+\frac{1-\sqrt{1-4\beta^2 n_1 n_2 c_2^2}}{2\beta n_1}\right] &= -1+\beta n_1+\frac{1-\sqrt{1-4\beta^2 n_1 n_2 c_2^2}}{2} && (4.34)\\
-1+2\beta n_1\frac{1-\sqrt{1-4\beta^2 n_1 n_2 c_2^2}}{2\beta n_1} &= -\sqrt{1-4\beta^2 n_1 n_2 c_2^2} && (4.35)
\end{aligned}
$$

The eigenvalues in equation (4.32) and (4.33) govern the stability of any equilibrium corresponding to $\bar{c}_{1,+}$. Similarly, the eigenvalues in equation (4.34) and (4.35) govern the stability of any equilibrium corresponding to $\bar{c}_{1,-}$. All the eigenvalues shown above depend on the discriminant Δ_{Homo}. Section 4.1.4 shows Δ_{Homo} is always greater than 0. The eigenvalue in equation (4.33) is thus always positive and thus $\bar{c}_{1,+}$ is unstable even if the eigenvalue in equation (4.32) is negative.

For $\bar{c}_{1,-}$, the eigenvalue in equation (4.35) is always negative since $\Delta_{Homo} > 0$. Mathematica solves the eigenvalue in equation (4.34). It shows as long as the conditions in equation (4.1), (4.2) and (4.6) are satisfied, the eigenvalue is always negative. Therefore, there are two possible stable equilibria combining $\bar{c}_{1,-}$ with $\bar{\sigma}^2$ which has two roots: σ_+^2 and σ_-^2. In order to check the stability of these two equilibria, I substitute each into the eigenvalue in equation (4.31). For σ_+^2,

$$\begin{aligned}
&-1+2a^2\beta^2\sigma_v^2\sigma_+^2 \\
=&-1+2a^2\beta^2\sigma_v^2\frac{(1-c_1^2)\left[1+\sqrt{1-\frac{4a^2\beta^2\sigma_v^2\sigma_u^2}{1-c_1^2}}\right]}{2a^2\beta^2\sigma_v^2} \\
=&-1+(1-c_1^2)\left[1+\sqrt{1-\frac{4a^2\beta^2\sigma_v^2\sigma_u^2}{1-c_1^2}}\right]
\end{aligned}$$

For $\bar{\sigma}_-^2$,

$$\begin{aligned}
&-1+2a^2\beta^2\sigma_v^2\sigma_-^2 \\
=&-1+2a^2\beta^2\sigma_v^2\frac{(1-c_1^2)\left[1-\sqrt{1-\frac{4a^2\beta^2\sigma_v^2\sigma_u^2}{1-c_1^2}}\right]}{2a^2\beta^2\sigma_v^2} \\
=&-1+(1-c_1^2)\left[1-\sqrt{1-\frac{4a^2\beta^2\sigma_v^2\sigma_u^2}{1-c_1^2}}\right]
\end{aligned}$$

Hence the condition for eigenvalues for σ_+^2 and σ_-^2 can be simplified to

$$\text{Eigenvalue for } \sigma_+^2: \qquad (1-c_1^2)\left[1+\sqrt{1-\frac{4a^2\beta^2\sigma_v^2\sigma_u^2}{1-c_1^2}}\right]<1 \tag{4.36}$$

$$\text{Eigenvalue for } \sigma_-^2: \qquad (1-c_1^2)\left[1-\sqrt{1-\frac{4a^2\beta^2\sigma_v^2\sigma_u^2}{1-c_1^2}}\right]<1 \tag{4.37}$$

Both inequality can be solved by Mathematica. For equation (4.36), it solves the following conditions

$$0<\beta, a<1$$

$$0<\sigma_v$$

$$0<\sigma_u<\frac{1}{2a\beta\sigma_v} \tag{4.38}$$

$$\sqrt{1+\frac{1}{4a^2\beta^2\sigma_u^2\sigma_v^2-2}}<c_1\leq\sqrt{1-4a^2\beta^2\sigma_u^2\sigma_v^2} \tag{4.39}$$

The first two conditions are included in the conditions in equation (4.1) and (4.2). Equation (4.38) depends on σ_v and shows if the chosen value of σ_v is high, the range of values of σ_u that can be chosen from is narrow. For equation (4.39), it can be rearranged to

$$\sqrt{\frac{4a^2\beta^2\sigma_u^2\sigma_v^2-1}{4a^2\beta^2\sigma_u^2\sigma_v^2-2}} < c_1 \leq \sqrt{1-4a^2\beta^2\sigma_u^2\sigma_v^2}$$
$$\sqrt{\frac{1-4a^2\beta^2\sigma_u^2\sigma_v^2}{2-4a^2\beta^2\sigma_u^2\sigma_v^2}} < c_1 \leq \sqrt{1-4a^2\beta^2\sigma_u^2\sigma_v^2}$$

If $1-4a^2\beta^2\sigma_u^2\sigma_v^2 < 0$ then it has complex solutions and is not considered. For $1-4a^2\beta^2\sigma_u^2\sigma_v^2 > 0$, c_1 has a very narrow range to satisfy the condition in (4.39) and so the condition in equation (4.36). Hence there is only a limited set of values of c_1 that leads $\bar{\sigma}_+^2$ to be stable. For $\bar{\sigma}_-^2$, the inequality in equation (4.37) is satisfied if the following conditions solved by Mathematica are satisfied.

$$\begin{aligned} &0 < \beta, a < 1 \\ &0 < \sigma_v \\ &0 < \sigma_u < \frac{1}{2a\beta\sigma_v} \\ &0 < c_1 \leq \sqrt{1-4a^2\beta^2\sigma_u^2\sigma_v^2} \end{aligned} \tag{4.40}$$

The only condition different from the conditions for the eigenvalue in equation (4.36) is the condition in equation (4.40). For $1-4a^2\beta^2\sigma_u^2\sigma_v^2 > 0$, the greater $4a^2\beta^2\sigma_u^2\sigma_v^2$ is, the narrower the range for c_1 is. However, equation (4.40) allows a wider range for c_1 than equation (4.39). Therefore, the condition for eigenvalue in equation (4.37) is more likely to be satisfied than that in equation (4.36) and thus $\bar{\sigma}_-^2$ is more likely to be stable than $\bar{\sigma}_+^2$. Overall, there are four equilibria solved but only the equilibrium corresponding to $\bar{c}_{1,-}$ and $\bar{\sigma}_{1,-}^2$ in equation (4.28) is likely to be stable.

4.1.6 Random Walk Beliefs

This section examines the PLM under random walk beliefs in the Homogeneous Variance Case. Since T-map cannot capture this particular equilibrium, I refer to this particular equilibrium as random walk equilibrium and the stationary PLM equilibrium in Section 4.1.4 as T-map equilibrium. Section 3.4.5 illustrates major properties of the random walk equilibrium in the Basic Model. Stock price at the random walk equilibrium has the identical form to that at the stable T-map equilibrium. The random walk beliefs lead to a substantial amount of market volatility and thus lower the stock price at the random walk equilibrium comparing to the stable T-map equilibrium. I follow the notations defined in Section 3.4.5 (i.e. $\tilde{p}$, $\tilde{\sigma}_p^2$ and $\tilde{\sigma}^2$ denote the fixed point of p_t, $\sigma_{p,t}^2$ and σ_t^2 at random walk equilibrium respectively). If learning agents perceive that stock prices follow a random walk, the PLM in equation (4.4) yields

$$\begin{aligned}
p_{1,t} &= p_{t-1} + \varepsilon_{1,t} \\
&= Lp_t + \varepsilon_{1,t}
\end{aligned}$$

Assuming σ^2_{t-1} is at its fixed point $\tilde{\sigma}^2$, the ALM in equation (4.9) under random walk beliefs can be expressed as

$$\begin{aligned}
p_t &= \beta(d_0 - a\tilde{\sigma}^2 s_0 + n_2k_2(1+c_2)) + \beta(n_1 + n_2c_2^2)Lp_t - \beta a\tilde{\sigma}^2 v_t \\
(1-\beta(n_1+n_2c_2^2)L)p_t &= \beta(d_0 - a\tilde{\sigma}^2 s_0 + n_2k_2(1+c_2)) - \beta a\tilde{\sigma}^2 v_t \\
p_t &= \frac{\beta(d_0 - a\tilde{\sigma}^2 s_0 + n_2k_2(1+c_2))}{1-\beta(n_1+n_2c_2^2)} + f(L)v_t
\end{aligned}$$

where

$$f(L) = \frac{-\beta a\tilde{\sigma}^2}{1-\beta(n_1+n_2c_2^2)L}$$

Hence the ALM under random walk beliefs in the Two-Beliefs Model has a similar structural form to that in the Basic Model in Section 3.4.5. Similarly, express $f(L)$ into a geometric series.

$$f(L) = \frac{-\beta a\tilde{\sigma}^2}{1-\beta(n_1+n_2c_2^2)L} = -\beta a\tilde{\sigma}^2 \sum_{i=0}^{\infty} (\beta(n_1+n_2c_2^2)L)^i$$

The ALM can be expressed into a similar form as in equation (3.41) in the Basic Model.

$$p_t = \frac{\beta(d_0 - a\tilde{\sigma}^2 s_0 + n_2k_2(1+c_2))}{1-\beta(n_1+n_2c_2^2)} - \beta a\tilde{\sigma}^2 \sum_{i=0}^{\infty} (\beta(n_1+n_2c_2^2)L)^i v_t \tag{4.41}$$

Therefore, the ALM under random walk beliefs involves infinite lagged v_t. It also introduces serial correlation into the model. I then show in the Homogeneous Variance Case, the stock price at the random walk equilibrium (i.e. $\tilde{p}$) has the identical form to that at the stable T-map equilibrium (i.e. $\bar{p}$). Derive $\tilde{p}$ by taking unconditional expectation on both sides of the ALM in equation (4.41).

$$\begin{aligned}
E(p_t) &= E\left(\frac{\beta(d_0 - a\tilde{\sigma}^2 s_0 + n_2k_2(1+c_2))}{1-\beta(n_1+n_2c_2^2)} - \beta a\tilde{\sigma}^2 \sum_{i=0}^{\infty} (\beta(n_1+n_2c_2^2)L)^i v_t\right) \\
\tilde{p} &= \frac{\beta(d_0 - a\tilde{\sigma}^2 s_0 + n_2k_2(1+c_2))}{1-\beta(n_1+n_2c_2^2)}
\end{aligned}$$

Derive $\bar{p}$ by substituting $\bar{c}_{1,-}$, $\bar{k}_1$ and $\bar{\sigma}^2_-$ into $\bar{p}$ in equation (4.28).

$$\bar{p} = \frac{\beta(d_0 - a\bar{\sigma}^2 s_0 + n_2 k_2(1+c_2))}{1-\beta(n_1 + n_2 c_2^2)}$$

Hence $\tilde{p}$ and $\bar{p}$ have the identical form. They are only equal if $\tilde{\sigma}^2 = \bar{\sigma}^2$. However, random walk beliefs lead to a substantial amount of excess volatility. It can be shown by comparing the variance of price at the random walk equilibrium and at the stable T-map equilibrium. Then I solve for $\tilde{\sigma}_p^2$ by taking unconditional variance on both side of the ALM in equation (4.41).

$$Var(p_t) = Var\left(\frac{\beta(d_0 - a\tilde{\sigma}^2 s_0 + n_2 k_2(1+c_2))}{1-\beta(n_1 + n_2 c_2^2)} - \beta a \tilde{\sigma}^2 \sum_{i=0}^{\infty} (\beta(n_1 + n_2 c_2^2)L)^i v_t\right)$$
$$\tilde{\sigma}_p^2 = \frac{\beta^2 a^2 (\tilde{\sigma}^2)^2 \sigma_v^2}{(1-\beta(n_1+n_2c_2^2))^2}$$

The issue becomes whether the unconditional variance of price (i.e. $\tilde{\sigma}_p^2$) at the random walk equilibrium is higher than that at the stable T-map equilibrium (i.e. $\bar{\sigma}_p^2$) in equation (4.20). It can be expressed as the following inequality.

$$\tilde{\sigma}_p^2 > \bar{\sigma}_p^2$$
$$\frac{a^2\beta^2(\tilde{\sigma}^2)^2\sigma_v^2}{(1-\beta(n_1+n_2c_2^2))^2} > \frac{a^2\beta^2(\bar{\sigma}^2)^2\sigma_v^2}{1-\beta^2(n_1\bar{c}_1^2 + n_2c_2^2)^2}$$

If $\bar{\sigma}^2 = \tilde{\sigma}^2$, the inequality can be simplified to

$$\frac{1}{(1-\beta(n_1+n_2c_2^2))^2} > \frac{1}{1-\beta^2(n_1\bar{c}_1^2 + n_2c_2^2)^2} \tag{4.42}$$

As long as the conditions in equation (4.1), (4.2) and (4.6) are satisfied, the inequality in equation (4.42) is always satisfied. The variance of price at the random walk equilibrium is thus always higher than that at the stable T-map equilibrium, that is $\tilde{\sigma}_p^2 > \bar{\sigma}_p^2$. Moreover, variance of excess returns $\sigma^2 = \sigma_p^2 + \sigma_u^2$, as $\tilde{\sigma}_p^2 > \bar{\sigma}_p^2$, at equilibrium it holds that $\tilde{\sigma}^2 > \bar{\sigma}^2$. Therefore, the stock price at the random walk equilibrium is lower than that at the stable T-map equilibrium. It shows the random walk equilibrium in the Homogeneous Variance Case follows major properties of the random walk equilibrium in the Basic Model.

4.2 Two-Beliefs Model: Learning vs Non-learning Heterogeneous Variance Case

This section relaxes the assumption in the Homogeneous Variance Case in Section 4.1 that learning and non-learning agents have identical estimate on variance of excess returns. In this case, each group has their individual estimate on the variance of excess returns. The analysis of the model follows a similar approach as in the Homogeneous Variance Case.

4.2.1 Aggregate Demand

Following the derivation of aggregate demand in Section 4.1.1, the individual demand $z_{h,dt}$ with heterogeneous variance of excess returns (i.e. $Var_{h,t}(p_{t+1} + d_{t+1} - Rp_t) \equiv \sigma^2_{h,t}$) becomes

$$z_{h,dt} = \hat{E}_{h,t}(p_{t+1} + d_{t+1} - Rp_t)/a\sigma^2_{h,t}$$

The aggregate demand z_{dt} yields

$$\begin{aligned} z_{dt} &= \sum_{h=1}^{2} n_h z_{h,dt} \\ &= \sum_{h=1}^{2} n_h \left\{ \hat{E}_{h,t}(p_{t+1} + d_{t+1} - Rp_t)/a\sigma^2_{h,t} \right\} \end{aligned}$$

At market clearing conditions $z_{st} = z_{dt}$, it yields

$$\begin{aligned} \frac{n_1(\hat{E}_{1,t}(p_{t+1} + d_{t+1}) - Rp_t)}{a\sigma^2_{1,t}} + \frac{n_2(\hat{E}_{2,t}(p_{t+1} + d_{t+1}) - Rp_t)}{a\sigma^2_{2,t}} &= z_{st} \\ \frac{\sigma^2_{2,t} n_1(\hat{E}_{1,t}(p_{t+1} + d_{t+1}) - Rp_t) + \sigma^2_{1,t} n_2(\hat{E}_{2,t}(p_{t+1} + d_{t+1}) - Rp_t)}{a\sigma^2_{1,t}\sigma^2_{2,t}} &= z_{st} \\ \sigma^2_{2,t} n_1 \hat{E}_{1,t}(p_{t+1} + d_{t+1}) + \sigma^2_{1,t} n_2 \hat{E}_{2,t}(p_{t+1} + d_{t+1}) - Rp_t(\sigma^2_{2,t} n_{1,t} + \sigma^2_{1,t} n_{2,t}) &= a\sigma^2_{1,t}\sigma^2_{2,t} z_{st} \\ \frac{\sigma^2_{2,t} n_1 \hat{E}_{1,t}(p_{t+1} + d_{t+1}) + \sigma^2_{1,t} n_2 \hat{E}_{2,t}(p_{t+1} + d_{t+1})}{R(\sigma^2_{2,t} n_1 + \sigma^2_{1,t} n_2)} - \frac{a\sigma^2_{1,t}\sigma^2_{2,t} z_{st}}{R(\sigma^2_{2,t} n_1 + \sigma^2_{1} n_{2,t})} &= p_t \end{aligned}$$

In the environment introduced in Section 4.1, non-learning agents do not update parameters in their PLM. It is thus more natural to assume non-learning agents also do not update their estimate on the variance of excess returns (i.e. $\sigma^2_{2,t} = \sigma^2_2$). Given $\beta = R^{-1}$, the true stochastic process of the risky asset price becomes

$$p_t = \frac{\beta\sigma^2_2 n_1 \hat{E}_{1,t}(p_{t+1} + d_{t+1}) + \beta\sigma^2_{1,t} n_2 \hat{E}_{2,t}(p_{t+1} + d_{t+1})}{\sigma^2_2 n_1 + \sigma^2_{1,t} n_2} - \frac{\beta a\sigma^2_{1,t}\sigma^2_2 z_{st}}{\sigma^2_2 n_1 + \sigma^2_{1,t} n_2} \tag{4.43}$$

The Two-Beliefs Model under the Heterogeneous Variance Case follows the conditions in equation (4.1), (4.2) and (4.6) in the Homogeneous Variance Case and adds two extra conditions on $\sigma^2_{1,t}$ and σ^2_2. All the conditions are listed as follows.

$$0 \le p_t, \sigma^2_{1,t}, \sigma^2_u, \sigma^2_v, \sigma^2_2 \tag{4.44}$$

$$0 < \beta, a, n_1 < 1 \tag{4.45}$$

$$-1 < c_2 < 1 \tag{4.46}$$

The model also follows Assumption 3.4.1 that no contemporaneous variables and information are observable to all agents. Following the notation $\hat{E}_{h,t}(\cdot|\Omega_{t-1})$ introduced in Section 4.1.1, equation (4.43) becomes

$$p_t = \frac{\beta\sigma^2_2 n_1 \hat{E}_{1,t}(p_{t+1} + d_{t+1}|\Omega_{t-1}) + \beta\sigma^2_{1,t-1} n_2 \hat{E}_{2,t}(p_{t+1} + d_{t+1}|\Omega_{t-1})}{\sigma^2_2 n_1 + \sigma^2_{1,t-1} n_2} - \frac{\beta a \sigma^2_{1,t-1}\sigma^2_2 z_{st}}{\sigma^2_2 n_1 + \sigma^2_{1,t-1} n_2}$$

Given dividend is independent to price, it yields

$$p_t = \frac{\beta\sigma^2_2 n_1 (d_0 + \hat{E}_{1,t}(p_{t+1}|\Omega_{t-1})) + \beta\sigma^2_{1,t-1} n_2 (d_0 + \hat{E}_{2,t}(p_{t+1}|\Omega_{t-1}))}{\sigma^2_2 n_1 + \sigma^2_{1,t-1} n_2} - \frac{\beta a \sigma^2_{1,t-1}\sigma^2_2 z_{st}}{\sigma^2_2 n_1 + \sigma^2_{1,t-1} n_2} \tag{4.47}$$

The price process in this case is more complicated than that in the Homogeneous Variance Case. In the Heterogeneous Variance Case, the variance of excess returns $\sigma^2_{1,t}$ and σ^2_2 enters the ALM. They affects not only the share supply noise v_t weighed in the ALM as in the Basic Model and the Homogeneous Variance Case but also the fractions of expectations weighed in the ALM. I refer the terms $\sigma^2_2 n_1/(\sigma^2_2 n_1 + \sigma^2_{1,t-1} n_2)$ and $\sigma^2_{1,t-1} n_2/(\sigma^2_2 n_1 + \sigma^2_{1,t-1} n_2)$ as the effective fractions. The effective fractions determine the share of each group's expectations on future price contributes to stock price.

Intuitively, when σ^2_2 is relatively higher than σ^2_1, it implies that non-learning agents perceive the stock is riskier to invest than learning agents believe. Since agents are risk averse and follow mean-variance preference, non-learning agents have relatively low demand in the risky asset so learning agents dominate the market and have a larger influence on the price process in equation (4.47). Alternatively, when $\sigma^2_{1,t}$ is relatively higher than σ^2_2, non-learning agents dominate the market.

4.2.2 Perceived and Actual Law of Motion

The PLM of learning and non-learning agents are identical to equation (4.4) and (4.5) derived in the Homogeneous Variance Case. Learning and non-learning agents form expectations on future price same as before.

$$\hat{E}_{1,t}(p_{1,t+1}|\Omega_{t-1}) = k_{1,t-1}(1 + c_{1,t-1}) + c_{1,t-1}^2 p_{t-1} \tag{4.48}$$
$$\hat{E}_{2,t}(p_{2,t+1}|\Omega_{t-1}) = k_2(1 + c_2) + c_2^2 p_{t-1} \tag{4.49}$$

Since there are separated estimates on variance of excess returns $\sigma_{1,t}^2$ and σ_2^2, the forecast errors of learning agents can be derived from their PLM. Therefore, the Heterogeneous Variance Case follows the approach in the Basic Model analysed in Section 3.4.1. Learning agents estimate variance of excess returns by the forecast error between realized p_t and their PLM as well as the dividend noise u_t.

$$\begin{aligned}
\sigma_{1,t-1}^2 &= Var_t((p_{1,t} + d_t)|\Omega_{t-1}) \\
&= E_t[(p_t - \hat{E}_{1,t}(p_{1,t}|\Omega_{t-1}) + d_t - \hat{E}_t(d_t|\Omega_{t-1}))^2] \\
&= E_t[(p_t - \hat{E}_{1,t}(p_{1,t}|\Omega_{t-1}) + u_t)^2]
\end{aligned} \tag{4.50}$$

I then derive the ALM. From equation (4.47), (4.48) and (4.49), the ALM becomes

$$\begin{aligned}
p_t &= \frac{\beta\sigma_2^2 n_1(d_0 + \hat{E}_{1,t}(p_{t+1}|\Omega_{t-1})) + \beta\sigma_{1,t-1}^2 n_2(d_0 + \hat{E}_{2,t}(p_{t+1}|\Omega_{t-1}))}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} - \frac{\beta a \sigma_{1,t-1}^2 \sigma_2^2 z_{st}}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} \\
&= \frac{\beta\sigma_2^2 n_1 \left(d_0 + k_{1,t-1}(1 + c_{1,t-1}) + c_{1,t-1}^2 p_{t-1}\right) + \beta\sigma_{1,t-1}^2 n_2 \left(d_0 + k_2(1 + c_2) + c_2^2 p_{t-1}\right)}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} \\
&\quad - \frac{\beta a \sigma_{1,t-1}^2 \sigma_2^2 z_{st}}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} \\
&= \frac{\beta\left[\sigma_2^2 n_1(d_0 + k_{1,t-1}(1 + c_{1,t-1})) + \sigma_{1,t-1}^2 n_2(d_0 + k_2(1 + c_2)) - a\sigma_{1,t-1}^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} \\
&\quad + \frac{\beta\left(\sigma_2^2 n_1 c_{1,t-1}^2 + \sigma_{1,t-1}^2 n_2 c_2^2\right) p_{t-1}}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} - \frac{\beta a \sigma_{1,t-1}^2 \sigma_2^2 v_t}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2}
\end{aligned} \tag{4.51}$$

The ALM here has a much more complicated structural form than that in the Homogeneous Variance Case. The variance of excess returns $\sigma_{1,t-1}^2$ and σ_2^2 enter every term in the ALM. Define

$$\begin{aligned}
w_{1,t-1} &= \frac{\sigma_2^2 n_1}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} \\
w_{2,t-1} &= \frac{\sigma_{1,t-1}^2 n_2}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} \\
\psi_{t-1} &= \frac{\sigma_{1,t-1}^2 \sigma_2^2}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2}
\end{aligned}$$

Weights $w_{1,t-1}$ and $w_{2,t-1}$ can be viewed as the effective fractions. Then the ALM can be expressed in a form similar to that in the Homogeneous Variance Case.

$$p_t = \beta \left[w_{1,t-1}(d_0 + k_{1,t-1}(1 + c_{1,t-1})) + w_{2,t-1}(d_0 + k_2(1 + c_2)) - a\psi_{t-1}s_0\right] + \beta \left(w_{1,t-1}c_{1,t-1}^2 + w_{2,t-1}c_2^2\right) p_{t-1} - a\psi_{t-1}v_t \tag{4.52}$$

4.2.3 Equilibrium

Similar to the Homogeneous Variance Case, T-map maps the PLM to the ALM. Let $\bar{p}$, $\bar{k}_1$, $\bar{c}_1$ and $\bar{\sigma}_1^2$ denote the fixed point for p_t, $k_{1,t}$, $c_{1,t}$ and $\sigma_{1,t}^2$. The ALM in equation (4.51) is expressed in the form of a T-map. Noted that this excludes σ^2 which I derive later. T-map is defined as

$$T\begin{pmatrix} k_1 \\ c_1 \end{pmatrix} = \begin{Bmatrix} \dfrac{\beta\left[\sigma_2^2 n_1\left(d_0 + k_1(1+c_1)\right) + \sigma_1^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\sigma_1^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \sigma_1^2 n_2} \\ \dfrac{\beta\left(\sigma_2^2 n_1 c_1^2 + \sigma_1^2 n_2 c_2^2\right)}{\sigma_2^2 n_1 + \sigma_1^2 n_2} \end{Bmatrix} \tag{4.53}$$

Hence p_t can be written as

$$p_t = T\begin{pmatrix} k_{1,t-1} \\ c_{1,t-1} \end{pmatrix}' (1, p_{t-1})' - \frac{\beta a \sigma_{1,t-1}^2 \sigma_2^2 v_t}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} \tag{4.54}$$

Solving for fixed point of the T-map,

$$T\begin{pmatrix} \bar{k}_1 \\ \bar{c}_1 \end{pmatrix} = \begin{pmatrix} \bar{k}_1 \\ \bar{c}_1 \end{pmatrix}$$

Then,

$$\frac{\beta\left(\sigma_2^2 n_1 \bar{c}_1^2 + \bar{\sigma}_1^2 n_2 c_2^2\right)}{\sigma_2^2 n_1 + \bar{\sigma}^2 n_2} = \bar{c}_1$$

$$\beta\left(\sigma_2^2 n_1 \bar{c}_1^2 + \bar{\sigma}_1^2 n_2 c_2^2\right) = \bar{c}_1(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)$$

$$\beta\sigma_2^2 n_1 \bar{c}_1^2 - \bar{c}_1(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2) + \beta\bar{\sigma}_1^2 n_2 c_2^2 = 0$$

$$\bar{c}_1 = \frac{(\bar{\sigma}_1^2 n_2 + \sigma_2^2 n_1)\left(1 \pm \sqrt{1 - \dfrac{4\beta^2\bar{\sigma}_1^2\sigma_2^2 n_1 n_2 c_2^2}{\left(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2\right)^2}}\right)}{2\beta\sigma_2^2 n_1}$$

It can be expressed in an alternative form.

$$\bar{c}_1 = \frac{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2) \pm \sqrt{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)^2 - 4\beta^2\bar{\sigma}_1^2\sigma_2^2 n_1 n_2 c_2^2}}{2\beta\sigma_2^2 n_1}$$

In the Homogeneous Variance Case, $\bar{p}$, $\bar{k}_1$ and $\bar{\sigma}^2$ all depend on $\bar{c}_1$ which only involves exogenous parameters. However, $\bar{c}_1$ in this case involves not only exogenous parameters but also $\bar{\sigma}_1^2$. Hence $\bar{c}_1$ depends on $\bar{\sigma}_1^2$ instead. Let the discriminant in $\bar{c}_1$ as Δ_{Hete}

$$\Delta_{Het} = 1 - \frac{4\beta^2\bar{\sigma}_1^2\sigma_2^2 n_1 n_2 c_2^2}{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)^2}$$

Discriminant $\Delta_{Hete} > 0$ is always satisfied given the conditions in equation (4.44) - (4.46) and it imposes a condition on c_2.

$$\begin{aligned}(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)^2 - 4\beta^2\bar{\sigma}_1^2\sigma_2^2 n_1 n_2 c_2^2 > 0 \\ c_2^2 &< \frac{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)^2}{4\beta^2\bar{\sigma}_1^2\sigma_2^2 n_1 n_2} \\ |c_2| &< \frac{\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2}{2\beta\bar{\sigma}_1\sigma_2\sqrt{n_1 n_2}} \qquad (4.55)\end{aligned}$$

Similar to equation (4.19), the condition on c_2 in equation (4.55) only matters if it imposes a narrower range on c_2 given the stationary condition on c_2 (i.e. $-1 < c_2 < 1$). It involves $\bar{\sigma}^2$. Later analysis shows $\bar{\sigma}^2$ can only be solved numerically and thus this condition can also only be solved numerically. Then I solve for $\bar{k}_1$

$$\bar{k}_1 = \frac{\beta\left[\sigma_2^2 n_1\left(d_0 + \bar{k}_1(1+\bar{c}_1)\right) + \bar{\sigma}_1^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\bar{\sigma}_1^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2}$$

$$\begin{aligned}\bar{k}_1 \big[(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2) \\ -\beta\sigma_2^2 n_1(1+\bar{c}_1)\big] &= \beta\sigma_2^2 n_1 d_0 + \beta\bar{\sigma}_1^2 n_2\left(d_0 + k_2(1+c_2)\right) - \beta a\bar{\sigma}_1^2\sigma_2^2 s_0 \\ \bar{k}_1 &= \frac{\beta\left[\sigma_2^2 n_1 d_0 + \bar{\sigma}_1^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\bar{\sigma}_1^2\sigma_2^2 s_0\right]}{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2) - \beta\sigma_2^2 n_1(1+\bar{c}_1)}\end{aligned}$$

I then solve for $\bar{p}$. Taking unconditional expectation on both sides of the ALM in equation (4.51),

$$\begin{aligned}E(p_t) = E&\left[\frac{\beta\left[\sigma_2^2 n_1\left(d_0 + k_{1,t-1}(1+c_{1,t-1})\right) + \sigma_{1,t-1}^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\sigma_{1,t-1}^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2}\right. \\ &\left. + \frac{\beta\left(\sigma_2^2 n_1 c_{1,t-1}^2 + \sigma_{1,t-1}^2 n_2 c_2^2\right) p_{t-1}}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2}\right] \\ = &\frac{\beta\left[\sigma_2^2 n_1\left(d_0 + k_{1,t-1}(1+c_{1,t-1})\right) + \sigma_{1,t-1}^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\sigma_{1,t-1}^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} \\ &+ \frac{\beta\left(\sigma_2^2 n_1 c_{1,t-1}^2 + \sigma_{1,t-1}^2 n_2 c_2^2\right) E(p_{t-1})}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2}\end{aligned}$$

As before, define the fixed point of th price as $E(p_t) = E(p_{t-1}) = \bar{p}$. At equilibrium, $k_{1,t-1}$, $c_{1,t-1}$ and $\sigma^2_{1,t-1}$ converge to $\bar{k}_1$, $\bar{c}_1$ and $\bar{\sigma}^2_1$ respectively. Then,

$$
\begin{aligned}
(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)\bar{p} &= \beta\left[\sigma_2^2 n_1\left(d_0 + \bar{k}_1(1+\bar{c}_1)\right) + \bar{\sigma}_1^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\bar{\sigma}_1^2\sigma_2^2 s_0\right] \\
&\quad + \beta\left(\sigma_2^2 n_1 \bar{c}_1^2 + \bar{\sigma}_1^2 n_2 c_2^2\right)\bar{p} \\
\bar{p}\left[\sigma_2^2 n_1(1-\beta\bar{c}_1^2)\right. \\
\left.+\bar{\sigma}_1^2 n_2(1-\beta c_2^2)\right] &= \beta\left[\sigma_2^2 n_1\left(d_0 + \bar{k}_1(1+\bar{c}_1)\right) + \bar{\sigma}_1^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\bar{\sigma}_1^2\sigma_2^2 s_0\right] \\
\bar{p} &= \frac{\beta\left[\sigma_2^2 n_1\left(d_0 + \bar{k}_1(1+\bar{c}_1)\right) + \bar{\sigma}_1^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\bar{\sigma}_1^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1(1-\beta\bar{c}_1^2) + \bar{\sigma}_1^2 n_2(1-\beta c_2^2)}
\end{aligned}
$$

Similar to the Homogeneous Variance Case, $\bar{k}_1$ and $\bar{p}$ depend on $\bar{\sigma}_1^2$ and $\bar{c}_1$. I then solve for $\bar{\sigma}_1^2$. At equilibrium, the PLM is equivalent to the ALM and the parameters in the PLM $\bar{k}_1$ and $\bar{c}_1$ are equal to the T-map $T(\bar{k}_1, \bar{c}_1)$ in the ALM. The variance of excess returns in equation (4.50) becomes

$$
\begin{aligned}
\bar{\sigma}_1^2 &= E\left[\left(-\frac{\beta a\bar{\sigma}_1^2\sigma_2^2 v}{\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2} + u\right)^2\right] \\
&= \frac{\beta^2 a^2(\bar{\sigma}_1^2)^2(\sigma_2^2)^2\sigma_v^2}{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)^2} + \sigma_u^2
\end{aligned}
$$

It can be simplified to

$$
(\bar{\sigma}_1^2)^3 n_2^2 + (\bar{\sigma}_1^2)^2(2n_2\sigma_2^2 n_1 - \beta a^2(\sigma_2^2)^2\sigma_v^2 - \sigma_u^2 n_2^2) + (\bar{\sigma}_1^2)((\sigma_2^2)^2 n_1^2 - 2n_2\sigma^2 n_1\sigma_u^2) = 0
$$

Different from the Basic Model and the Homogeneous Variance Case, $\bar{\sigma}_1^2$ is no longer a quadratic equation. There are three solutions for σ_1^2. However, due to the complicated functional form, $\bar{\sigma}_1^2$ can only be solved numerically. There is also a high chance that some of the solutions are complex and I only consider real solutions. Also the real roots of $\bar{\sigma}_1^2$ have to be non-negative. I denote each root of $\bar{\sigma}_1^2$ as $\bar{\sigma}^2_{1,i\in(1,2,3)}$. All possible equilibria $\bar{p}$, $\bar{k}_1$, $\bar{c}_1$ and $\bar{\sigma}_1^2$ are summarized as follows.

$$\bar{p} = \frac{\beta\left[\sigma_2^2 n_1\left(d_0 + \bar{k}_1(1+\bar{c}_1)\right) + \bar{\sigma}_1^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\bar{\sigma}_1^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1(1-\beta\bar{c}_1^2) + \bar{\sigma}_1^2 n_2(1-\beta c_2^2)} \tag{4.56}$$

$$\bar{k}_1 = \frac{\beta\left[\sigma_2^2 n_1 d_0 + \bar{\sigma}_1^2 n_2\left(d_0 + k_2(1+c_2)\right) - a\bar{\sigma}_1^2\sigma_2^2 s_0\right]}{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2) - \beta\sigma_2^2 n_1(1+\bar{c}_1)} \tag{4.57}$$

$$\bar{c}_1 = \begin{cases} \dfrac{(\bar{\sigma}_1^2 n_2 + \sigma_2^2 n_1)\left(1 + \sqrt{1 - \dfrac{4\beta^2\bar{\sigma}_1^2\sigma_2^2 n_1 n_2 c_2^2}{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)^2}}\right)}{2\beta\sigma_2^2 n_1} \rightarrow \bar{c}_{1,+} \\ \dfrac{(\bar{\sigma}_1^2 n_2 + \sigma_2^2 n_1)\left(1 - \sqrt{1 - \dfrac{4\beta^2\bar{\sigma}_1^2\sigma_2^2 n_1 n_2 c_2^2}{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)^2}}\right)}{2\beta\sigma_2^2 n_1} \rightarrow \bar{c}_{1,-} \end{cases} \tag{4.58}$$

$$\bar{\sigma}_1^2 = \begin{cases} \bar{\sigma}_{1,1}^2 \\ \bar{\sigma}_{1,2}^2 \\ \bar{\sigma}_{1,3}^2 \end{cases} \tag{4.59}$$

Combining two solutions for $\bar{c}_1$ with three solutions for $\bar{\sigma}_1^2$, there are six equilibria in the Heterogeneous Variance Case. Since $\bar{p}$ and $\bar{k}_1$ are functions of $\bar{c}_1$ and $\bar{\sigma}_1^2$. Let $[\bar{p}(\bar{c}_1, \bar{\sigma}_1^2), \bar{k}_1(\bar{c}_1, \bar{\sigma}_1^2), \bar{c}_1, \bar{\sigma}_1^2]$ denote an equilibrium. All six equilibria are listed as follows.

$$\begin{gathered}
\left[\bar{p}(\bar{c}_{1,+}, \bar{\sigma}_{1,1}^2), \bar{k}_1(\bar{c}_{1,+}, \bar{\sigma}_{1,1}^2), \bar{c}_{1,+}, \bar{\sigma}_{1,1}^2\right] \\
\left[\bar{p}(\bar{c}_{1,+}, \bar{\sigma}_{1,2}^2), \bar{k}_1(\bar{c}_{1,+}, \bar{\sigma}_{1,2}^2), \bar{c}_{1,+}, \bar{\sigma}_{1,2}^2\right] \\
\left[\bar{p}(\bar{c}_{1,+}, \bar{\sigma}_{1,3}^2), \bar{k}_1(\bar{c}_{1,+}, \bar{\sigma}_{1,3}^2), \bar{c}_{1,+}, \bar{\sigma}_{1,3}^2\right] \\
\left[\bar{p}(\bar{c}_{1,-}, \bar{\sigma}_{1,1}^2), \bar{k}_1(\bar{c}_{1,-}, \bar{\sigma}_{1,1}^2), \bar{c}_{1,-}, \bar{\sigma}_{1,1}^2\right] \\
\left[\bar{p}(\bar{c}_{1,-}, \bar{\sigma}_{1,2}^2), \bar{k}_1(\bar{c}_{1,-}, \bar{\sigma}_{1,2}^2), \bar{c}_{1,-}, \bar{\sigma}_{1,2}^2\right] \\
\left[\bar{p}(\bar{c}_{1,-}, \bar{\sigma}_{1,3}^2), \bar{k}_1(\bar{c}_{1,-}, \bar{\sigma}_{1,3}^2), \bar{c}_{1,-}, \bar{\sigma}_{1,3}^2\right]
\end{gathered}$$

4.2.4 Stability Analysis

First recall the form of a stochastic recursive algorithm in equation (3.25).

$$\phi_t = \phi_{t-1} + \gamma_t \mathcal{H}(t, \phi_{t-1}, D_t)$$

I redefine D_t and ϕ_t.

$$D_t = \begin{pmatrix} p_t \\ p_{t-1} \\ u_t \\ v_t \end{pmatrix} \qquad \phi_t = \begin{pmatrix} k_{1,t} \\ c_{1,t} \\ p_t \\ p_t^2 \\ \sigma_{1,t}^2 \end{pmatrix}$$

In this case, the RLS equations are very similar to that in the Basic Model and listed as follows.

$$\theta_{1,t} = \theta_{1,t-1} + \gamma_{1,t} S_t^{-1} X_t (p_t - \theta_{1,t-1}' X_t) \tag{4.60}$$

$$S_t = S_{t-1} + \gamma_{1,t}(X_t X_t' - S_{t-1}) \tag{4.61}$$

$$\sigma_{1,t}^2 = \sigma_{1,t-1}^2 + \gamma_{2,t}\left((p_t - \theta_{1,t-1}' X_t + u_t)^2 - \sigma_{1,t-1}^2\right) \tag{4.62}$$

I redefine the T-map in equation (4.53) as $T(\theta_1; \sigma_1^2)$ where $\theta = (k_1, c_1)'$ due to the dependence of θ_1 on σ_1^2. Equation (4.54) can be written as

$$p_t = T(\theta_{1,t-1}; \sigma_{1,t-1}^2)'(1, p_{t-1})' - \frac{\beta a \sigma_{1,t-1}^2 \sigma_2^2 v_t}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2}$$

I express the RLS equations as the term $\mathcal{H}(t, \phi_{t-1}, D_t)$ in the form of a stochastic recursive algorithm in equation (3.25).

$$\mathcal{H}_{\theta_1} = S_t^{-1} X_t \left(X_t' \left(T(\theta_{1,t-1}; \sigma_{1,t-1}^2) - \theta_{1,t-1}\right) - \frac{\beta a \sigma_{1,t-1}^2 \sigma_2^2 v_t}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} \right) \tag{4.63}$$

$$\mathcal{H}_S = X_t X_t' - S_{t-1} \tag{4.64}$$

$$\mathcal{H}_{\sigma_1^2} = (z_t z_t' - \sigma_{1,t-1}^2) \tag{4.65}$$

where

$$z_t z_t' = (T(\theta_{1,t-1}; \sigma_{1,t-1}^2) - \theta_{1,t-1})' X_t X_t' (T(\theta_{1,t-1}; \sigma_{1,t-1}^2) - \theta_{1,t-1}) + \frac{\beta^2 a^2 (\sigma_{1,t-1}^2)^2 (\sigma_2^2)^2 v_t^2}{(\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2)^2} + u_t^2$$

Then, apply Lemma 3.4.2, $h(\phi)$ becomes

$$h_\theta = S^{-1} M(\theta_1, \sigma_1^2)(T(\theta_1; \sigma_1^2) - \theta_1)$$

$$h_S = M(\theta_1, \sigma_1^2) - S$$

$$h_{\sigma_1^2} = \left(T(\theta_1; \sigma_1^2) - \theta_1\right)' M(\theta_1, \sigma_1^2) \left(T(\theta_1; \sigma_1^2) - \theta_1\right) + \frac{\beta^2 a^2 (\sigma_1^2)^2 (\sigma_2^2)^2 \sigma_v^2}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2} + \sigma_u^2 - \sigma_1^2$$

Section 3.4.4 shows h_S gives $S \to M$ where M is $EX(\theta_1, \sigma_1^2)X(\theta_1, \sigma_1^2)'$ and hence h_{θ_1} becomes $T(\theta_1; \sigma_1^2) - \theta_1$. The stability of the entire system is determined by the stability of a smaller dimension system. $h(\phi)$ is simplified to

$$\begin{aligned}
h_{\theta_1} &= T(\theta_1; \sigma_1^2) - \theta_1 \\
h_S &= M(\theta_1, \sigma^2) - S \\
h_{\sigma_1^2} &= \left(T(\theta_1; \sigma_1^2) - \theta_1\right)' M(\theta_1, \sigma_1^2) \left(T(\theta_1; \sigma_1^2) - \theta_1\right) + \frac{\beta^2 a^2 (\sigma_1^2)^2 (\sigma_2^2)^2 \sigma_v^2}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2} + \sigma_u^2 - \sigma_1^2
\end{aligned}$$

The Jacobian matrix of $h(\phi)$ evaluated at the equilibrium is

$$\begin{pmatrix}
\frac{\beta\sigma_2^2 n_1(1+c_1)}{\sigma_2^2 n_1 + \sigma_1^2 n_2} - 1 & \frac{\beta\sigma_2^2 n_1 k_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2} & 0 & 0 & -\frac{\beta a n_1 \sigma_2^4 s_0}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2} \\
0 & \frac{2\beta\sigma_2^2 n_1 c_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2} - 1 & 0 & 0 & \frac{\beta n_1 n_2 \sigma_2^2 (c_2^2 - c_1^2)}{\sigma_2^2 n_1 + \sigma_1^2 n_2} \\
\frac{\partial M(1,2)}{\partial k_1} & \frac{\partial M(1,2)}{\partial c_1} & -1 & 0 & \frac{\partial M(1,2)}{\partial \sigma_1^2} \\
\frac{\partial M(2,2)}{\partial k_1} & \frac{\partial M(2,2)}{\partial c_1} & 0 & -1 & \frac{\partial M(2,2)}{\partial \sigma_1^2} \\
0 & 0 & 0 & 0 & \frac{2a^2\beta^2 n_1 \sigma_1^2 \sigma_2^6 \sigma_v^2}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^3} - 1
\end{pmatrix}$$

The eigenvalues of the matrix are on the diagonal of the matrix (confirmed in Mathematica):

$$-1 + \frac{\beta\sigma_2^2 n_1(1+c_1)}{\sigma_2^2 n_1 + \sigma_1^2 n_2} \tag{4.66}$$

$$-1 + \frac{2\beta\sigma_2^2 n_1 c_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2} \tag{4.67}$$

$$-1 + \frac{2a^2\beta^2 n_1 \sigma_1^2 \sigma_2^6 \sigma_v^2}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^3} \tag{4.68}$$

and repeated values of -1. Section 4.2.3 solves six equilibria. First I check whether the first two eigenvalues are negative. For $\bar{c}_1 = \bar{c}_{1,+}$, the eigenvalues are

$$\begin{aligned} & -1 + \frac{\beta\sigma_2^2 n_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2}\left[1 + \frac{(\sigma_2^2 n_1 + \sigma_1^2 n_2) + \sqrt{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2 - 4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}}{2\beta\sigma_2^2 n_1}\right] \\ =& -1 + \frac{\beta\sigma_2^2 n_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2} + \frac{1}{2} + \frac{\sqrt{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2 - 4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}}{2(\sigma_2^2 n_1 + \sigma_1^2 n_2)} \\ =& -\frac{1}{2} + \frac{\beta\sigma_2^2 n_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2} + \sqrt{1 - \frac{4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2}} \end{aligned} \tag{4.69}$$

$$\begin{aligned} & -1 + \frac{2\beta\sigma_2^2 n_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2}\left[\frac{(\sigma_2^2 n_1 + \sigma_1^2 n_2) + \sqrt{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2 - 4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}}{2\beta\sigma_2^2 n_1}\right] \\ =& -1 + \frac{(\sigma_2^2 n_1 + \sigma_1^2 n_2) + \sqrt{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2 - 4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}}{\sigma_2^2 n_1 + \sigma_1^2 n_2} \\ =& \sqrt{1 - \frac{4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2}} \end{aligned} \tag{4.70}$$

For $\bar{c}_1 = \bar{c}_{1,-}$, the eigenvalues are

$$\begin{aligned} & -1 + \frac{\beta\sigma_2^2 n_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2}\left[1 + \frac{(\sigma_2^2 n_1 + \sigma_1^2 n_2) - \sqrt{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2 - 4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}}{2\beta\sigma_2^2 n_1}\right] \\ =& -\frac{1}{2} + \frac{\beta\sigma_2^2 n_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2} - \sqrt{1 - \frac{4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2}} \end{aligned} \tag{4.71}$$

$$\begin{aligned} & -1 + \frac{2\beta\sigma_2^2 n_1}{\sigma_2^2 n_1 + \sigma_1^2 n_2}\left[\frac{(\sigma_2^2 n_1 + \sigma_1^2 n_2) - \sqrt{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2 - 4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}}{2\beta\sigma_2^2 n_1}\right] \\ =& -\sqrt{1 - \frac{4\beta^2\sigma_1^2\sigma_2^2 n_1 n_2 c_2^2}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2}} \end{aligned} \tag{4.72}$$

The eigenvalues in equation (4.69) and (4.70) govern the stability of any equilibrium corresponding to $\bar{c}_{1,+}$. Similarly, the eigenvalues in equation (4.71) and (4.72) govern the stability of any equilibrium corresponding to $\bar{c}_{1,-}$. All the eigenvalues shown above depend on the discriminant Δ_{Hete}. Section 4.2.3 shows Δ_{Hete} is always greater than 0. The eigenvalue in equation (4.70) is thus always positive and hence $\bar{c}_{1,+}$ is unstable even if the eigenvalue in equation (4.69) is negative.

For $\bar{c}_{1,-}$, the eigenvalue in equation (4.72) is always negative since $\Delta_{Hete} > 0$. Mathematica solves the eigenvalue in equation (4.71). It shows as long as the conditions in equation (4.44) - (4.46) are satisfied, the eigenvalue is always negative. Hence $\bar{c}_{1,-}$ is stable. Overall, three equilibria corresponding $\bar{c}_{1,+}$ are unstable. The stability of the remaining three equilibria corresponding to $\bar{c}_{1,-}$ depends on $\bar{\sigma}_1^2$ and can only be solved numerically.

4.2.5 Random Walk Beliefs

This section examines the PLM under the random walk beliefs in the Heterogeneous Variance Case. Since T-map cannot capture this particular equilibrium, I refer to this particular equilibrium as random walk equilibrium and the stationary PLM equilibrium in Section 4.2.3 as T-map equilibrium. I first solve for random walk equilibrium of price $\tilde{p}$ and the variance of price $\tilde{\sigma}_p^2$. If learning agents hold random walk beliefs, the PLM in equation (4.4) yields

$$p_{1,t} = Lp_t + \varepsilon_{1,t}$$

Assuming $\sigma_{1,t-1}^2$ is at its fixed point $\tilde{\sigma}_1^2$, the ALM in equation (4.51) under random walk beliefs can be expressed as

$$\begin{aligned}
p_t &= \frac{\beta\left[\sigma_2^2 n_1 d_0 + \tilde{\sigma}_1^2 n_2 (d_0 + k_2(1+c_2)) - a\tilde{\sigma}_1^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2} \\
&\quad + \frac{\beta\left(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2\right) L p_t}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2} - \frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2 v_t}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2} \\
\left(1 - \frac{\beta\left(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2\right) L}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2}\right) p_t &= \frac{\beta\left[\sigma_2^2 n_1 d_0 + \tilde{\sigma}_1^2 n_2 (d_0 + k_2(1+c_2)) - a\tilde{\sigma}_1^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2} - \frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2 v_t}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2} \\
p_t &= \frac{\beta\left[\sigma_2^2 n_1 d_0 + \tilde{\sigma}_1^2 n_2 (d_0 + k_2(1+c_2)) - a\tilde{\sigma}_1^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2} \\
&\quad \left(1 - \frac{\beta\left(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2\right)}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2}\right)^{-1} + f(L) v_t \\
p_t &= \frac{\beta\left[\sigma_2^2 n_1 d_0 + \tilde{\sigma}_1^2 n_2 (d_0 + k_2(1+c_2)) - a\tilde{\sigma}_1^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 (1-\beta) + \tilde{\sigma}_1^2 n_2 (1 - \beta c_2^2)} + f(L) v_t
\end{aligned}$$

where

$$\begin{aligned}
f(L) &= -\frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2}\left(1 - \frac{\beta\left(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2\right) L}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2}\right)^{-1} \\
&= -\frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2}{(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2) - \beta(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2) L}
\end{aligned}$$

Express $f(L)$ into a geometric series.

$$\begin{aligned} f(L) &= -\frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2}{(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2) - \beta(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2)L} \\ &= -\frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2}{(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2)} \sum_{i=0}^{\infty} \left(\frac{\beta\left(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2\right) L}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2} \right)^i v_t \end{aligned}$$

The ALM can be expressed into an alternative form.

$$\begin{aligned} p_t = & \frac{\beta\left[\sigma_2^2 n_1 d_0 + \tilde{\sigma}_1^2 n_2 (d_0 + k_2(1+c_2)) - a \tilde{\sigma}_1^2 \sigma_2^2 s_0\right]}{\sigma_2^2 n_1 (1-\beta) + \tilde{\sigma}_1^2 n_2 (1-\beta c_2^2)} - \\ & \frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2}{(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2)} \sum_{i=0}^{\infty} \left(\frac{\beta\left(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2\right) L}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2} \right)^i v_t \end{aligned} \tag{4.73}$$

Therefore, the ALM in the Heterogeneous Variance Case also involves infinite lagged values v_t similar to the Basic Model and the Homogeneous Variance Case. I then show the stock price at the random walk equilibrium (i.e. $\tilde{p}$) has the identical form to that at the stable T-map equilibrium (i.e. $\bar{p}$). Derive $\tilde{p}$ by taking the unconditional expectation on both sides of the ALM in equation (4.73).

$$\begin{aligned} E(p_t) = E & \left(\frac{\beta\left[\sigma_2^2 n_1 d_0 + \tilde{\sigma}_1^2 n_2 (d_0 + k_2(1+c_2)) - a \tilde{\sigma}_1^2 \sigma_2^2 s_0\right]}{\sigma_2^2 n_1 (1-\beta) + \tilde{\sigma}_1^2 n_2 (1-\beta c_2^2)} \right. \\ & \left. - \frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2}{(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2)} \sum_{i=0}^{\infty} \left(\frac{\beta\left(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2\right) L}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2} \right)^i v_t \right) \\ \tilde{p} = & \frac{\beta\left[\sigma_2^2 n_1 d_0 + \tilde{\sigma}_1^2 n_2 (d_0 + k_2(1+c_2)) - a \tilde{\sigma}_1^2 \sigma_2^2 s_0\right]}{\sigma_2^2 n_1 (1-\beta) + \tilde{\sigma}_1^2 n_2 (1-\beta c_2^2)} \end{aligned}$$

Derive $\bar{p}$ by substituting $\bar{c}_{1,-}$ and $\bar{k}_1$ into $\bar{p}$ in equation (4.56). Mathematica solves for $\bar{p}$.

$$\bar{p} = \frac{\beta\left[\sigma_2^2 n_1 d_0 + \bar{\sigma}_1^2 n_2 (d_0 + k_2(1+c_2)) - a \bar{\sigma}_1^2 \sigma_2^2 s_0\right]}{\sigma_2^2 n_1 (1-\beta) + \bar{\sigma}_1^2 n_2 (1-\beta c_2^2)}$$

Hence $\tilde{p}$ and $\bar{p}$ have the identical form. They are only equal if $\tilde{\sigma}_p^2 = \bar{\sigma}_p^2$. However, random walk beliefs lead to a substantial amount of excess volatility. It can be shown by comparing the variance of price at the random walk equilibrium and at the stable T-map equilibrium. Then I solve for $\tilde{\sigma}_p^2$ by taking unconditional variance on both sides of the ALM in equation (4.73),

$$
\begin{aligned}
Var(p_t) &= Var\left(\frac{\beta\left[\sigma_2^2 n_1 d_0 + \tilde{\sigma}_1^2 n_2(d_0 + k_2(1+c_2)) - a\tilde{\sigma}_1^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2}\right. \\
&\quad \left. - \frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2}{(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2)} \sum_{i=0}^{\infty} \left(\frac{\beta\left(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2\right) L}{\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2}\right)^i v_t\right) \\
&= Var\left(-\frac{\beta a \tilde{\sigma}_1^2 \sigma_2^2 v_t}{(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2) - \beta(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2) L}\right) \\
\tilde{\sigma}_p^2 &= \frac{\beta^2 a^2 (\tilde{\sigma}_1^2)^2 (\sigma_2^2)^2 \sigma_v^2}{\left((\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2) - \beta(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2)\right)^2}
\end{aligned}
$$

Similar to the Homogeneous Variance Case, the issue is whether $\tilde{\sigma}_p^2$ is higher than $\bar{\sigma}_p^2$ which is

$$
\bar{\sigma}_p^2 = \frac{\beta^2 a^2 (\bar{\sigma}_1^2)^2 (\sigma_2^2)^2 \sigma_v^2}{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)^2}
$$

Therefore the inequality is as follows.

$$
\begin{aligned}
\tilde{\sigma}_p^2 &> \bar{\sigma}_p^2 \\
\frac{\beta^2 a^2 (\tilde{\sigma}_1^2)^2 (\sigma_2^2)^2 \sigma_v^2}{\left((\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2) - \beta(\sigma_2^2 n_1 + \tilde{\sigma}_1^2 n_2 c_2^2)\right)^2} &> \frac{\beta^2 a^2 (\bar{\sigma}_1^2)^2 (\sigma_2^2)^2 \sigma_v^2}{(\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2)^2}
\end{aligned}
$$

If $\bar{\sigma}_1^2 = \tilde{\sigma}_1^2$, the inequality can be simplified to

$$
\frac{1}{\left((\sigma_2^2 n_1 + \sigma_1^2 n_2) - \beta(\sigma_2^2 n_1 + \sigma_1^2 n_2 c_2^2)\right)^2} > \frac{1}{(\sigma_2^2 n_1 + \sigma_1^2 n_2)^2} \qquad (4.74)
$$

As long as the conditions in equation (4.44) - (4.46) are satisfied, the inequality in equation (4.74) is always satisfied. The variance of price at the random walk equilibrium is thus always higher than that at the stable T-map equilibrium, that is $\tilde{\sigma}_p^2 > \bar{\sigma}_p^2$. Moreover, variance of excess returns $\sigma^2 = \sigma_p^2 + \sigma_u^2$, as $\tilde{\sigma}_p^2 > \bar{\sigma}_p^2$, at equilibrium it holds that $\tilde{\sigma}^2 > \bar{\sigma}^2$. As a result, the stock price at the random walk equilibrium is actually lower than that at the stable T-map equilibrium. As a result, the random walk equilibrium in the Heterogeneous Variance Case of the Two-Beliefs Model also follows all major properties of the random walk equilibrium in the Basic Model.

Chapter 5
Results

Chapter 4 introduces the Two-Beliefs Model. It follows some features of the Basic Model and investigates dynamics of a stock market where learning and non-learning agents exist. Both groups of agents are concerned about risk and returns. Learning agents recursively update their forecast of future price and variance of excess returns whereas non-learning agents always have static forecasts. In the Heterogeneous Variance Case, both groups of agents identically estimate the variance of excess returns by sample variance of price based on historical prices. In the Heterogeneous Variance Case, they have separated estimates.

The Two-Beliefs Model combines heterogeneous beliefs with adaptive learning. Learning agents can be considered as rational traders who adjust their forecasts when new information is available. Non-learning agents are irrational traders or those who do not have knowledge or resources to adjust their forecasts. Non-learning agents create additional disturbance to stock price. Although learning agents do not know about the existence of the other group directly, they observe and react on changes in prices caused in the past by the other group.

Section 4.1 derives four equilibria in the Homogeneous Variance Case. The stability analysis shows there is only one stable equilibrium. Section 4.2 solves six equilibria in the Heterogeneous Variance Case. Stability analysis shows that three of those six equilibria are unstable whereas the stability conditions of the remaining three equilibria can only be solved numerically. In both cases, when learning agents hold random walk beliefs in their PLM, it may also lead to an equilibrium in the ALM that T-map cannot capture.

This chapter first discusses the choice of parameter values. It is followed by numerical simulations to validate the analytical findings in Chapter 4 and bifurcation diagrams which demonstrate how the state variables respond to changes in certain exogenous parameters in the Two-Beliefs Model under the Heterogeneous and Homogeneous Variance Case.

5.1 Choice of Parameter Values

Since the Two-Beliefs Model follows a similar structure of the Basic Model, I start numerical simulations using parameter values chosen by BE. The entire model has two important exogenous shocks: the dividend shock and the share supply shock. Section 3.2 demonstrates that both shocks follow white noise processes with respective variances σ_u^2 and σ_v^2.

$$d_t \sim WN(d_0, \sigma_u^2)$$
$$z_{st} \sim WN(s_0, \sigma_v^2)$$

Variances of shocks σ_u^2 and σ_v^2 in part determine the volatility of the stock market since the share supply shock v_t enters the ALM whereas the dividend shock u_t enters the RLS equation for the variance of excess returns. At equilibrium, the variance of excess returns $\bar{\sigma}^2$ in the Homogeneous Variance Case and $\bar{\sigma}_1^2$ in the Heterogeneous Variance Case depend on σ_u^2 and σ_v^2. The larger the values of σ_u^2 and σ_v^2 are, the higher the market volatility is. Different values of σ_u^2 and σ_v^2 may lead to different simulation results. I consider two circumstances: 1) Large noise where $\sigma_u = 0.9$ and $\sigma_v = 0.5$ and 2) Small noise where $\sigma_u = 0.09$ and $\sigma_v = 0.05$.

	β	a	d_0	s_0	σ_u	σ_v	n_1	k_2	c_2	σ_2
(a) Large Noise (LN)	0.95	0.75	1.5	1	0.9	0.5	0.5	0.5	0.8	0.25
(b) Small Noise (SN)	0.95	0.75	1.5	1	0.09	0.05	0.5	0.5	0.8	0.25

Table 5.1: Parameter values

Table 5.1 lists all parameter values for numerical simulations under Large and Small Noise. The only difference between Large Noise and Small Noise are σ_v and σ_u. Discount factor $\beta = 0.95$ is equivalent to the interest rate $r = 5.26\%$. Risk aversion factor $a = 0.75$ indicates that all agents are risk averse. The Two-Beliefs Model introduces the fractions of learning and non-learning agents in the market n_1 and n_2 as well as the parameters in PLM of non-learning agents k_2 and c_2. I choose $n_1 = n_2 = 0.5$, $k_2 = 0.5$ and $c_2 = 0.8$. When $n_1 = n_2 = 0.5$, half of agents in the market learn whereas the remaining half does not. When considering $k_2 = 0.5$ and $c_2 = 0.8$, the PLM of non-learning agents in equation (4.5) becomes

$$p_{2,t} = 0.5 + 0.8p_{t-1} + \varepsilon_{2,t}$$

It indicates that non-learning agents are optimistic about future price and believe stock price depends on its lags. In the Heterogeneous Variance Case, there is an extra exogenous parameter σ_2^2 which is the non-learning agents' estimate on variance of excess returns. I choose $\sigma_2 = 0.25$. I use these two sets of parameter values in Table 5.1 for numerical simulations in Section 5.2 for the Homogeneous Variance Case and Section 5.4 for the Heterogeneous Variance Case. The sensitivity of results to parameter values is analysed with bifurcation in Section 5.3 and 5.5.

5.2 Numerical Simulation: Homogeneous Variance Case

The main purpose of numerical simulations shown in this section is to investigate the dynamic behaviour of the system and verify the analytical findings from Section 4.1. A long period of simulation demonstrates the time path and the dynamics of state variables. It also gives the converged values or the end points of the state variables which may not exactly be to the theoretical equilibrium, but are close to these values.

Reminded that I refer to T-map equilibrium as the equilibrium for which the PLM is stationary as opposed to random walk equilibrium for which the PLM is random walk and thus non-stationary. Table 5.2 and first twelve rows of Table 5.3 list the numerical solutions for all theoretical equilibria and the corresponding eigenvalues in the Basic Model and in the Two-Beliefs Model under the Homogeneous Variance Case respectively. The considered parameter values are listed in Table 5.1. Each row indicates one equilibrium. The numerical equilibria in Table 5.2 confirm the analytical result in Chapter 3 that there is only one stable T-map equilibrium (i.e. Eq 4 for Large Noise and Eq 10 for Small Noise). Under Large Noise, all random walk equilibria are complex and hence are not attainable. Under Small Noise, there are two real random walk equilibria.

In the Homogeneous Variance Case, Eq 13 to 16 refer to the T-map equilibria under Large Noise. An equilibrium is stable if all the eigenvalues have negative real parts. Eq 13 and 15 are not attainable since $\bar{\sigma}^2$ in Eq 13 and $\bar{p}$ in Eq 15 are negative. Not all the eigenvalues of Eq 13 to 15 are negative and therefore they are unstable. All eigenvalues of Eq 16 are negative. Hence only Eq 16 is stable for the considered values of parameters under Large Noise. Eq 19 to 22 refer to the T-map equilibria under Small Noise. All eigenvalues of Eq 22 are negative whereas the other three equilibria Eq 19 to 21 are not. Hence only Eq 22 is stable for the considered values of parameters under Small Noise. The numerical solutions in Table 5.3 validate the derivation of equilibrium and its stability analysis in Section 4.1.4 and 4.1.5 which show any equilibrium corresponding to $\bar{c}_{1,+}$ is unstable. Eq 13 and 14 under Large Noise as well as Eq 19 and 20 under Small Noise corresponding to $\bar{c}_{1,+}$ are unstable. It further shows that combination of $\bar{c}_{1,-}$ and $\bar{\sigma}^2_{1,-}$ is more likely to be stable than $\bar{c}_{1,-}$ and $\bar{\sigma}^2_{1,+}$. Here it shows only Eq 16 and 22 corresponding to $\bar{c}_{1,-}$ and $\bar{\sigma}^2_{1,-}$ are stable whereas Eq 15 and 21 corresponding to $\bar{c}_{1,-}$ and $\bar{\sigma}^2_{1,+}$ are unstable.

			$\bar{p}$	k	$\bar{c}$	$\bar{\sigma}^2$	Eigenvalues		
			eq.(3.21)	eq.(3.22)	eq.(3.23)	eq.(3.24)	eq.(3.35)	eq.(3.36)	eq.(3.37)
Large Noise	T-map equilibrium	Eq 1	-70.7186	-70.7186	1.0526	6.9627	0.9500	1.0000	0.7673
		Eq 2	15.4379	15.4379	1.0526	0.9166	0.9500	1.0000	-0.7673
		Eq 3	-70.7186	-6.6569	0.0000	6.9627	-0.0500	-1.0000	0.4899
		Eq 4	**15.4379**	**15.4379**	**0.0000**	**0.9166**	**−0.0500**	**−1.0000**	**−0.7673**
	Random walk equilibrium	Eq 5	Complex solutions						
		Eq 6							
Small Noise	T-map equilibrium	Eq 7	-11199.4547	-11199.4547	1.0526	787.9266	0.9500	1.0000	1.0000
		Eq 8	28.3846	28.3846	1.0526	0.0081	0.9500	1.0000	-1.0000
		Eq 9	-11199.4547	-11199.4547	0.0000	787.9266	-0.0500	-1.0000	1.0000
		Eq 10	**28.3846**	**28.3846**	**0.0000**	**0.0081**	**−0.0500**	**−1.0000**	**−1.0000**
	Random walk equilibrium	Eq 11	28.3841	0.0000	1.0000	0.0081			
		Eq 12	0.5457	0.0000	1.0000	1.9617			

Table 5.2: Equilibria and eigenvalues in the Basic Model

Notes: Bold font indicates stable equilibria.

			$\bar{p}$	k_1	$\bar{c}_1$	$\bar{\sigma}^2$	Eigenvalues		
			eq.(4.21)	eq.(4.22)	eq.(4.23)	eq.(4.24)	eq.(4.29)	eq.(4.30)	eq.(4.31)
Large Noise	T-map equilibrium	Eq 13	62.0935	−45.7481	1.7368	−16.6599	0.3000	0.6499	−5.2287
		Eq 14	5.8920	−4.3410	1.7368	0.7724	0.3000	0.6499	−0.8039
		Eq 15	−10.5415	−6.6569	0.3685	5.8697	−0.3500	−0.6499	0.4899
		Eq 16	**5.3529**	**3.3803**	**0.3685**	**0.9397**	**−0.3500**	**−0.6499**	**−0.7615**
	Random walk equilibrium	Eq 17	Complex solutions						
		Eq 18							
Small Noise	T-map equilibrium	Eq 19	5130.4895	−3779.9452	1.7368	−1588.7518	0.3000	0.6499	−5.0327
		Eq 20	8.3562	−6.1566	1.7368	0.0081	0.3000	0.6499	−1.0000
		Eq 21	−2186.9237	−1381.0379	0.3685	680.9300	−0.3500	−0.6499	0.7284
		Eq 22	**8.3562**	**5.2769**	**0.3685**	**0.0081**	**−0.3500**	**−0.6499**	**−1.0000**
	Random walk equilibrium	Eq 23	8.3562	0.0000	1.0000	0.0081			
		Eq 24	−115.6617	0.0000	1.0000	38.4574			
End points	Figure 5.1	(a) LN	5.3685	3.4239	0.3685	0.9338			
		(b) SN	8.3563	5.2769	0.3685	0.0081			
	Figure 5.2	(a) LN	5.2832	3.3432	0.3750	0.9403			
		(b) SN	8.3237	0.5629	0.9320	0.0144			

Table 5.3: Equilibria, eigenvalues and end points in the Homogeneous Variance Case

Notes: 1) Bold font indicates stable equilibria; 2) The end points listed are the average of the last 100 points from the corresponding simulation.

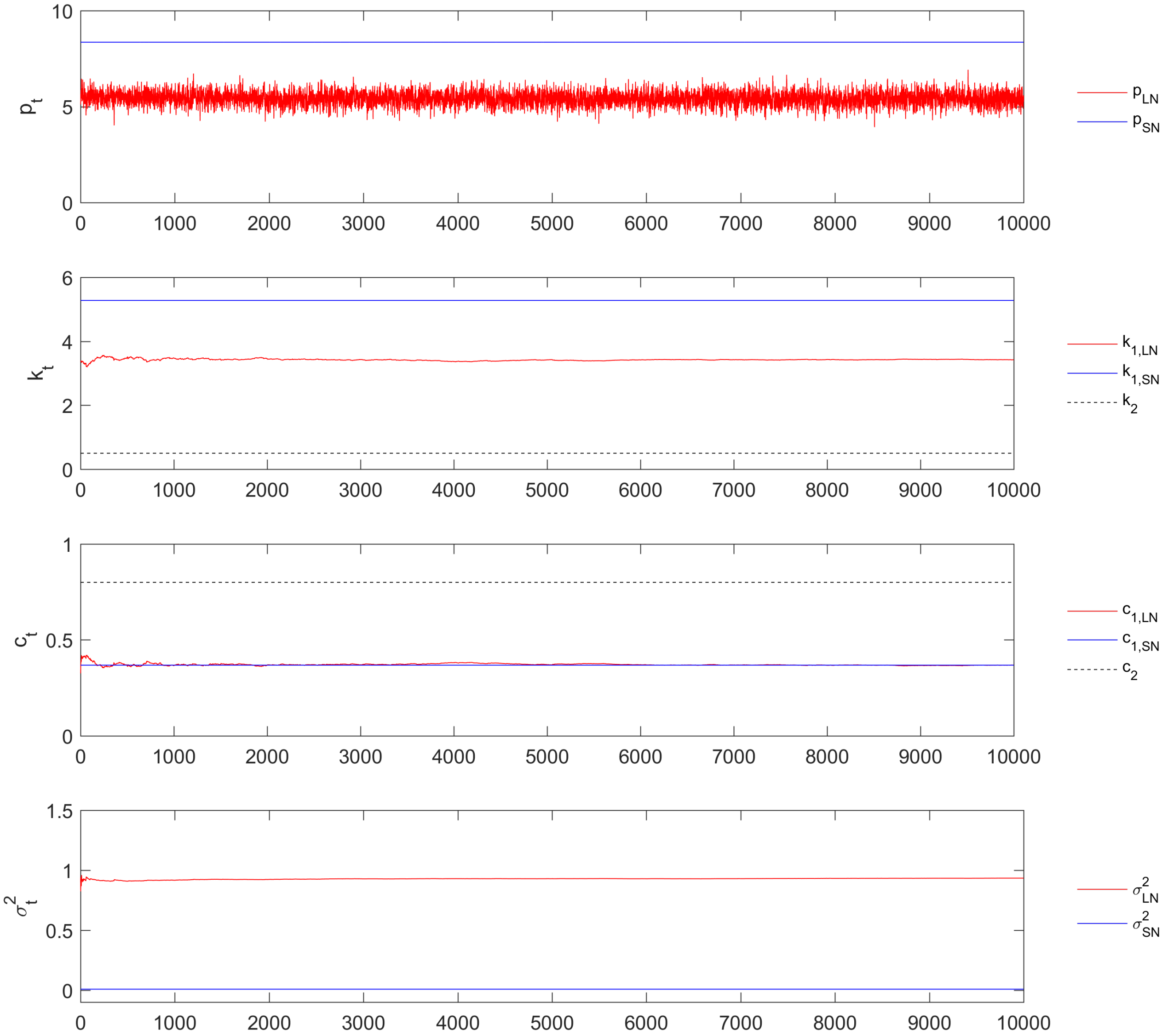

Figure 5.1: Numerical simulation (baseline) in the Homogeneous Variance Case

Figure 5.1 shows a simulation of length 10,000 with Large and Small Noise. The initial values are near Eq 16 for Large Noise and near Eq 22 for Small Noise in Table 5.3. The top panel plots the stock price p_t. The two middle panels plot the estimated time-varying parameters in the PLM for learning agents $k_{1,t}, c_{1,t}$ and the fixed parameters for non-learning agents k_2, c_2. The bottom panel plots the variance of excess returns σ_t^2. It shows that stock price realized from the ALM is stationary under both noises. As I would expect, all state variables p_t, $k_{1,t}$, $c_{1,t}$ and σ_t^2 are more stable under Small Noise than under Large Noise. Since $\bar{c}_1$ in equation (4.23) does not involve σ_v^2 and σ_u^2, Figure 5.1 shows $c_{1,t}$ converges identically despite the different size of noises given. The variance of excess returns σ_t^2 under Small Noise is much lower than that under Large Noise. Therefore, the stock price under Small Noise is higher than that under Large Noise. It justifies that stock price declines with variance. This resembles real markets where stock prices typically decline after turbulent periods. Table 5.3 shows that the end points of Figure 5.1 (a) LN and (b) SN are very close to Eq 16 and Eq 22 respectively. Therefore, the simulation confirms the model converges to the theoretical stable T-map equilibrium for both noises.

5.2.1 Stability

Another issue is whether the model reverts back to the stable T-map equilibrium if there is a small deviation from it. I choose some initial values away from the stable T-map equilibrium and run a long period of simulation. The analysis in Section 4.1.5 only considers local stability of an equilibrium which is bounded by the region of attraction (ROA). If the chosen initial values are out of the region, the model may not necessarily converge to the theoretical stable T-map equilibrium even though that equilibrium is theoretically stable.

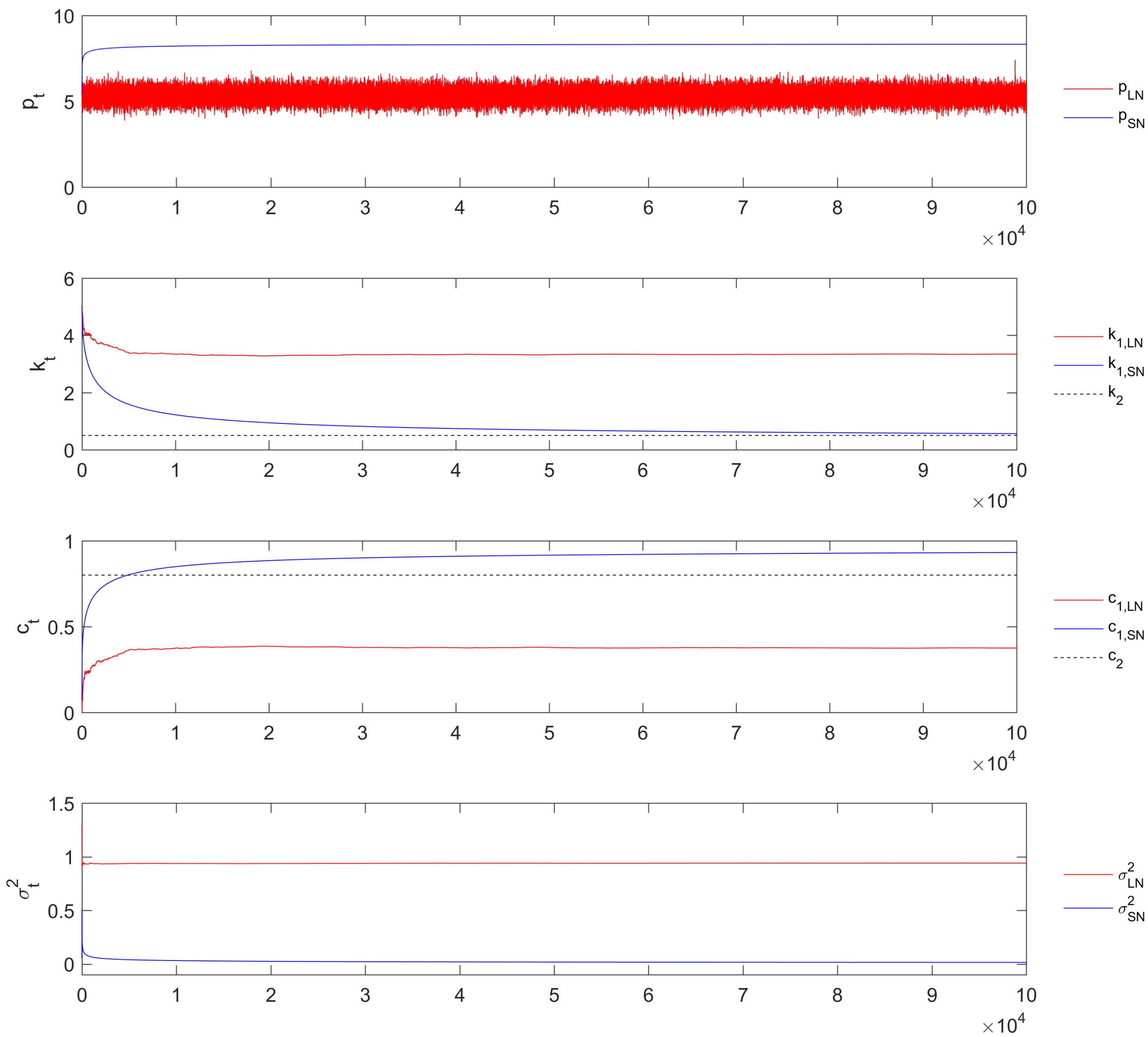

Figure 5.2: Numerical simulation (stability) in the Homogeneous Variance Case

Figure 5.2 shows a simulation of length 100,000 with Large and Small Noise. The initial values are $p = 5$, $k_1 = 5$, $c_1 = 0$ and $\sigma^2 = 0.5$. I run a longer period of iterations here since when the initial values are non-equilibrium values, the model requires more time to converge. Comparing to Figure 5.2, all state variables require more time to converge and eventually become stable. More importantly, different noises lead to different results. Table 5.3 shows the end points of Figure 5.2 (a) LN are close to the stable T-map equilibrium Eq 16. However, under Small Noise, it does not converge to the stable T-map equilibrium. The end points of Figure 5.2 (b) SN are closer to the random walk equilibrium than the stable T-map equilibrium. Intuitively, learning agents starting from some non-equilibrium beliefs eventually learn the true stochastic process of the stock market when market volatility is high. In the case of Small Noise, when random walk beliefs are attainable, learning agents converge to the random walk beliefs.

If some sequence of shocks leads learning agents to hold random walk beliefs, these beliefs due to its self-fulling nature would persist for a substantial amount of time when market volatility is low. In addition, learning agents cannot distinguish the random walk equilibrium from the stable T-map equilibrium since stock prices are close at both equilibria. Even though the ALM is not a random walk, under random walk PLM, the price becomes highly persistent. High persistence of the price is a realistic behaviour of real stock markets.

Figure 5.1 and 5.2 show in the Two-Beliefs Model where learning and non-learning agents exist, c_1 in the PLM of learning agents converges to some positive value, but not to the beliefs of non-learning agents c_2. Comparing to the Basic Model which only considers homogeneous learning agents, c in the PLM converges to 0. At the stable T-map equilibrium, the PLM of learning agents and the ALM in the Homogeneous Variance Case are

$$\text{PLM:} \qquad p_{1,t} = \bar{k}_1 + \bar{c}_1 p_{t-1} + \varepsilon_{1,t} \tag{5.1}$$

$$\text{ALM:} \qquad p_t = \bar{k}_1 + \bar{c}_1 p_{t-1} - \beta a \bar{\sigma}^2 v_t \tag{5.2}$$

While the corresponding PLM and ALM in the Basic Model are

$$\text{PLM:} \qquad p_t = \bar{k} + \varepsilon_t \tag{5.3}$$

$$\text{ALM:} \qquad p_t = \bar{k} - \beta a \bar{\sigma}^2 v_t \tag{5.4}$$

Hence the ALM in the Two-Beliefs model is an AR(1) process with some white noise v_t. Therefore, the REE in the Basic Model which has the structural form of a constant with some white noise v_t in equation (5.4) is not attainable in the Two-Beliefs Model. However, it does not imply the REE is never attainable. For example, when $n_1 = 1$ where all agents are learning, the Two-Beliefs Model reduces to the Basic Model and the REE is thus attainable. Moreover, Table 5.2 and 5.3 show stock price at the stable T-map equilibrium in the Homogeneous Variance Case is lower than that in the Basic Model for Large and Small Noise. It illustrates that the existence of non-learning agents creates additional disturbance in the market and decreases the average stock price.

5.2.2 Profit of Learning and Non-learning Agents

So far I select the shares of learning and non-learning types exogenously. One way to justify the existence or prevalence of one type over another is through their profits. This section examines profit of learning and non-learning agents. I denote $\pi_{h,t}$ as the profit of group h at time t and Π_h as the average profit of group h. $\pi_{h,t}$ is defined as

$$\begin{aligned}\pi_{h,t} &= (p_{t+1} + d_{t+1} - Rp_t)z_{h,dt} \\ &= (p_{t+1} + d_{t+1} - Rp_t)\left[\hat{E}_{h,t}(p_{t+1} + d_{t+1} - Rp_t)/a\sigma^2_{t-1}\right] \\ &= (p_{t+1} + d_{t+1} - Rp_t)\left[(\hat{E}_{h,t}(p_{t+1}) + d_0 - Rp_t)/a\sigma^2_{t-1}\right] \end{aligned} \tag{5.5}$$

Profit at time t is realized when p_{t+1} is realized at time $t+1$. When simulating T periods, only $T-1$ periods of profits are realized. Π_h is therefore defined as

$$\Pi_h = \frac{1}{T-1}\sum_{i=1}^{T-1} \pi_{h,t} \tag{5.6}$$

Profit of each group depends on the excess returns $(p_{t+1} + d_{t+1} - Rp_t)$ and their demand $z_{h,dt}$. At equilibrium, the excess returns can be simplified to $d_0 - r\bar{p}$. When stock price increases, the excess returns decrease and vice versa. Since the excess returns are realised from the market and identical for both groups, the profit difference between the two groups entirely depends on the difference in demand $z_{h,dt}$ driven by the difference in expectations on future price $\hat{E}_{h,t}(p_{t+1})$. Positive demand implies the expectations on future price $\hat{E}_{h,t}(p_{t+1})$ is higher than the cost of investing in the risky asset Rp_t. Therefore agents receive future dividend d_{t+1} and price p_{t+1} but pay Rp_t. Alternatively, negative demand implies short selling and agents pay future dividend and price but receive Rp_t. They earn profits only if stock price decline next period.

Table 5.4 shows average profit of each group from a simulation of length 10,000.

Learning agents earn significantly higher profit than non-learning agents. Actively updating forecasts requires time, knowledge and resources. This induces costs to the learning agents. Since the model does not impose a cost on learning, the result that learning group perform better is expected.

Large Noise	Π_1	2.2756
	Π_2	0.4020
Small Noise	Π_1	184.9637
	Π_2	-182.8434

Table 5.4: Average profits in the Homogeneous Variance Case

Under Large Noise, the expectations on future price of learning agents are higher than that of non-learning agents but both are positive. The difference in expectations leads to difference in demand. Therefore, the profit gap exists. Given that profits are positive, both groups may coexist. Under Small Noise, when the variance is lower, the stock price is higher and therefore the excess returns decrease. However, the higher stock price also leads to a larger difference in demand between the two groups. For non-learning agents, the cost of investing in the risky asset is much larger than their expectations. Therefore, their demand switches from barely positive under Large Noise to negative whereas learning agents' positive demand increases significantly. It thus creates a larger profit gap between the two groups under Small Noise. In this case, the existence of non-learning agents in the market is unlikely.

5.3 Bifurcation: Homogeneous Variance Case

This section discusses how the state variables and profit of each group respond to changes in the selected exogenous parameters using bifurcation diagrams. Among all the exogenous parameters, the discount factor β, the belief of non-learning agents c_2 and the market share of learning agents n_1 are the most intriguing to look into.

5.3.1 Method

The method I used to generate bifurcation diagrams is as follows:

1. Starts with $n = 1, ..., N$ evenly spaced points of an exogenous parameter H (i.e. β, c_2 or n_1) with the range in Table 5.5 while all other exogenous parameters are fixed.

2. Arbitrarily choose a value for p_0

3. Create a relatively close neighbourhood around p_0 (i.e. $\pm 10\%$) with $k = 1, ..., K$ evenly spaced points in this neighbourhood and each of these points is considered as an initial value for p_t

4. Initial values of other state variables $k_{1,t}$, $c_{1,t}$ and σ_t^2 are slightly away from their theoretical stable T-map equilibrium values (i.e. $+1\%$).

5. For each k, all state variables p_t, $k_{1,t}$, $c_{1,t}$ and σ_t^2 converge after T iterations

6. The last 50 points of the simulation are plotted on the bifurcation diagram for each state variable.

7. After repeating for all K, the end point of p_t converged from p_0 becomes the new p_0 for $n + 1$.

8. Repeat Step 3 to 7 for all N

H	Lower bound	Upper Bound
β	0.80	0.99
c_2	-0.95	0.95
n_1	0.05	0.95

Table 5.5: Range of β, c_2 and n_1 in bifurcation

Bifurcation diagrams capture convergence of state variables when changing values of a single parameter holding others fixed. There are different methods to generate bifurcation diagrams. I adopt multiple methods with different settings and the results are very consistent regardless of which method I used. Therefore, I decide to show bifurcation diagrams generated from the above method since by allowing initial values to be a changing neighbourhood around price instead of a static value, it takes the impact of different initial values on simulations into account. In addition, Step 4 allows the model does not start exactly at the theoretical stable T-map equilibrium values otherwise it eliminates any possibility that the model does not converge to the stable equilibrium. The considered parameter values for the bifurcation diagrams are (b) Small Noise in Table 5.1. When the dividend and share supply noise are small, the convergence is theoretically clearer than using Large Noise. For the Homogeneous Variance Case, $p_0 = 5$, $T = 15,050$, $K = 21$ and $N = 500$.

5.3.2 Discount Factor

Panel (a) to (c) in Figure 5.3 show p, k_1 and c_1 against discount factor β. When β increases, stock price increases. This relationship can be interpreted as follows. In the model, there are only risky and risk free asset. When interest rate is high (i.e β is low), the perceived risk to invest in the risky asset is high as the outside option that agents can invest in the risk free asset yields a relatively high return. Therefore, the demand for risky asset and its price are low and vice versa. This explanation also applies to k_1 and c_1 which also increase with β. Learning agents observe the increasing trend of price and adjust their forecast accordingly over time. Their expectations on future price formed from the PLM are higher. Non-learning agents also have higher expectations since their PLM is an AR(1) process. These expectations enter the ALM and thus push the stock price up.

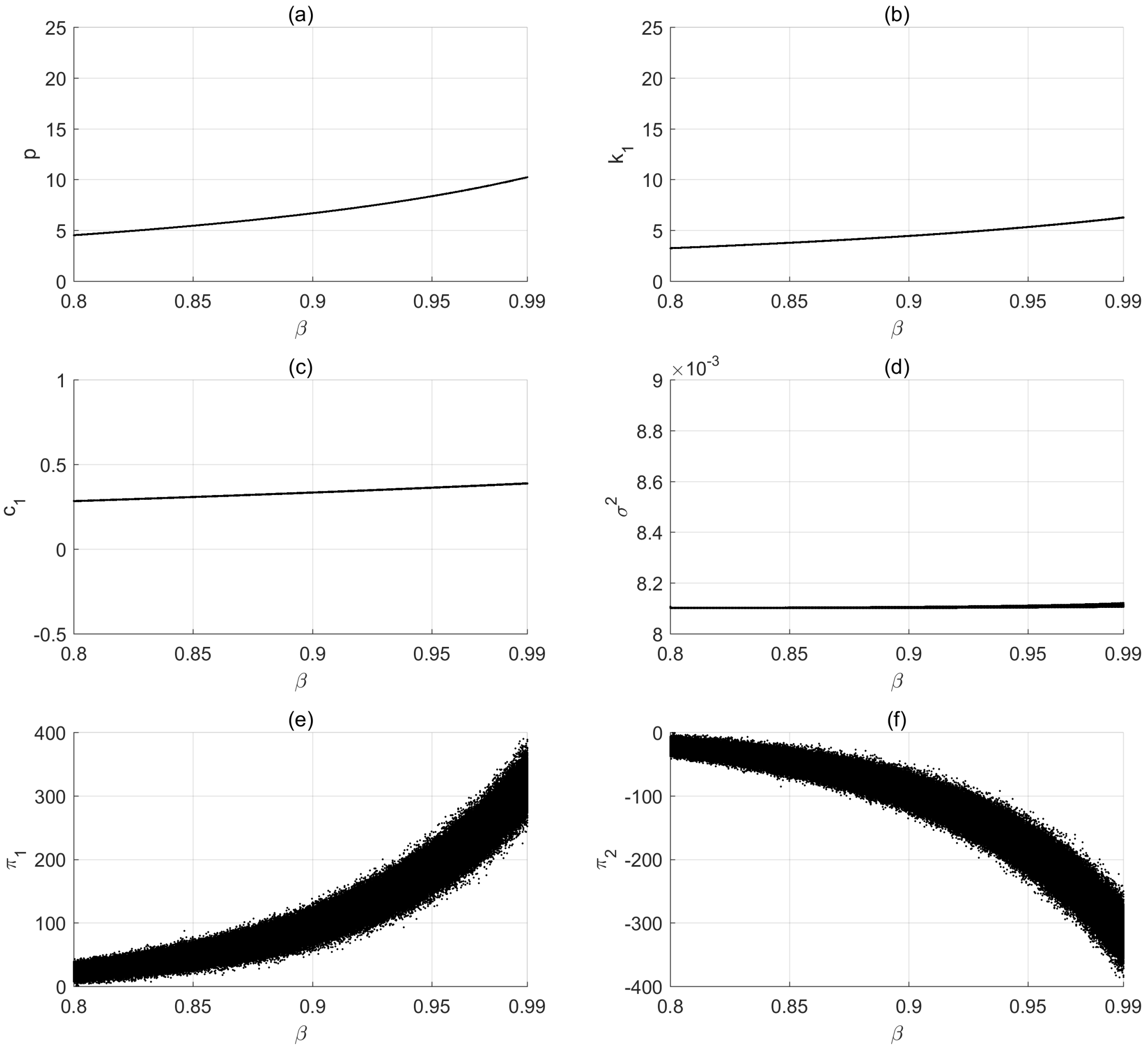

Figure 5.3: Bifurcation diagrams (β) in the Homogeneous Variance Case

Panel (d) shows σ^2 barely responds to β. The market volatility is fairly stable except when β gets closer to 1, the variance slightly increases. This increase is likely due to numerical accuracy and finite number of iterations rather than any explicit relationship between σ^2 and β. Under Small Noise, the impact of variance on stock price is minimal and therefore the increasing trend of stock price against β is fairly consistent. As a result, the discount factor β has an increasing relationship with state variables p, k_1 and c_1 except σ^2 in which β does not have much influence.

Panel (e) and (f) show profit of learning and non-learning agents calculated from equation (5.5). Learning agents always earn higher profit than non-learning agents for all β. It illustrates the advantage of learning. Comparing to learning agents, non-learning agents always have lower expectations on future price which translate to negative demand and thus they short sell. The profit difference between the two groups rapidly increases as β increases. When interest rate decreases, more agents invest in the risky asset since investing in the risk free asset becomes less attractive. With more learning agents trading in the market and stock price thus increase, learning agents benefit from it whereas non-learning agents are punished more heavily from not willing to learn.

5.3.3 Belief of Non-Learning Agents

Panel (c) of Figure 5.4 shows c_1 has a very symmetric relationship against c_2. Recall $\bar{c}_1$ in equation (4.23).

$$\bar{c}_1 = \frac{1 \pm \sqrt{1 - 4\beta^2 n_1 n_2 c_2^2}}{2\beta n_1}$$

Section 4.1.4 shows Δ_{Homo} is always greater than 0 and there are two real fixed points of c_1. Since any equilibrium corresponding to $\bar{c}_{1,+}$ is unstable, the model should theoretically always converge to $\bar{c}_{1,-}$. The sign of c_2 does not affect $\bar{c}_1$ since c_2 is squared in $\bar{c}_1$. It thus creates the symmetric behaviour of c_1 against c_2 and also shows $\bar{c}_{1,-}$ is minimized and equal to 0 when $c_2 = 0$. Since the beliefs of learning and non-learning agents c_1 and c_2 enter the ALM, the symmetric property creates a ripple effect to other state variables. Therefore, all bifurcation diagrams for c_2 also have a symmetric shape.

Panel (d) plots σ^2 against c_2. The variance of excess returns increases dramatically when c_2 gets close to its end values. For other values of c_2, σ^2 are fairly stable. Although equation (4.24) shows $\bar{\sigma}^2$ does not directly involve c_2, it involves $\bar{c}_1$ which has a symmetric relationship with c_2. Hence σ^2 also follows a symmetric behaviour against c_2. The impact of σ^2 on p is minimal since variance is small under Small Noise. The higher values of σ^2 on both ends thus do not decrease stock price.

Panel (a) and (b) shows p and k_1 also follow the symmetric behaviour of c_1 and σ^2 since $\bar{p}$ and $\bar{k}_1$ depend on $\bar{c}_1$ and $\bar{\sigma}^2$. However, they are not as symmetric as c_1 against c_2 since stock price responds more extensively to positive c_2 than negative c_2. For negative c_2, c_1 converges to some positive values and it implies that the learning group believes opposite to what the non-learning group believes. Therefore, two opposite beliefs force stock price to decrease. Alternatively, for positive c_2, c_1 are also positive. Hence the two beliefs are coherent and push the stock price even higher.

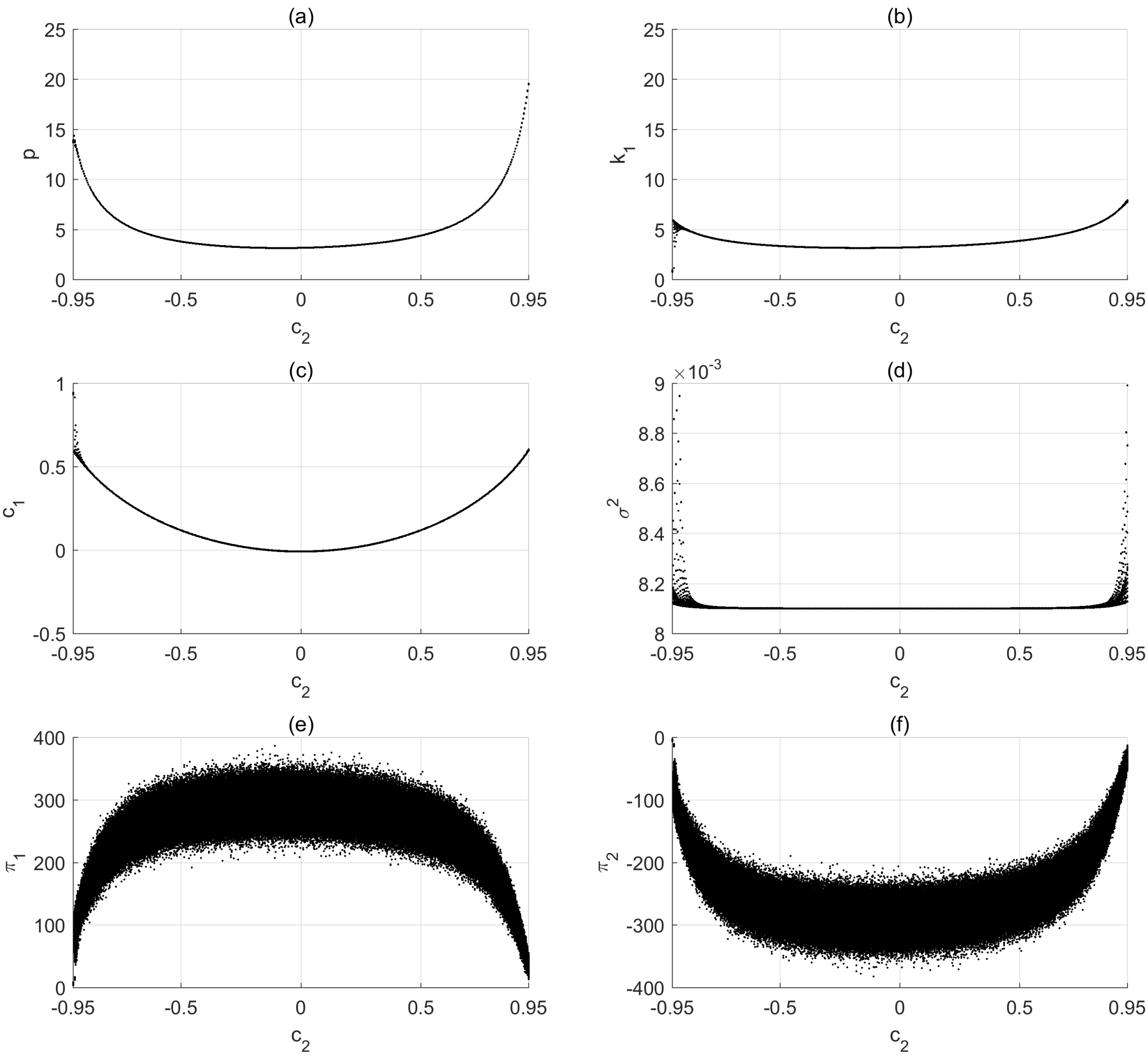

Figure 5.4: Bifurcation diagrams (c_2) in the Homogeneous Variance Case

Panel (e) and (f) show learning agents earn higher profit for all c_2. More interestingly, the profit gap increases when $|c_2|$ decreases and reaches its largest level when $c_2 = c_1 = 0$. At that point, the PLM of learning and non-learning agents becomes

$$p_{1,t} = k_{1,t-1} + \varepsilon_{1,t}$$
$$p_{2,t} = k_2 + \varepsilon_{2,t}$$

The only difference between the two PLM is $k_{1,t}$ and k_2. Panel (b) shows k_1 at $c_2 = 0$ is around 3 whereas k_2 is fixed at 0.5. As a result, the difference in expectations on future price between learning agents and non-learning agents is largest. It translates to the largest difference in demand among all c_2 as well. In addition, when $c_2 = 0$, stock price is lowest and the excess returns are thus highest. Combining both effects, the profit gap is thus largest when $c_2 = 0$.

5.3.4 Market Share of Learning Agents

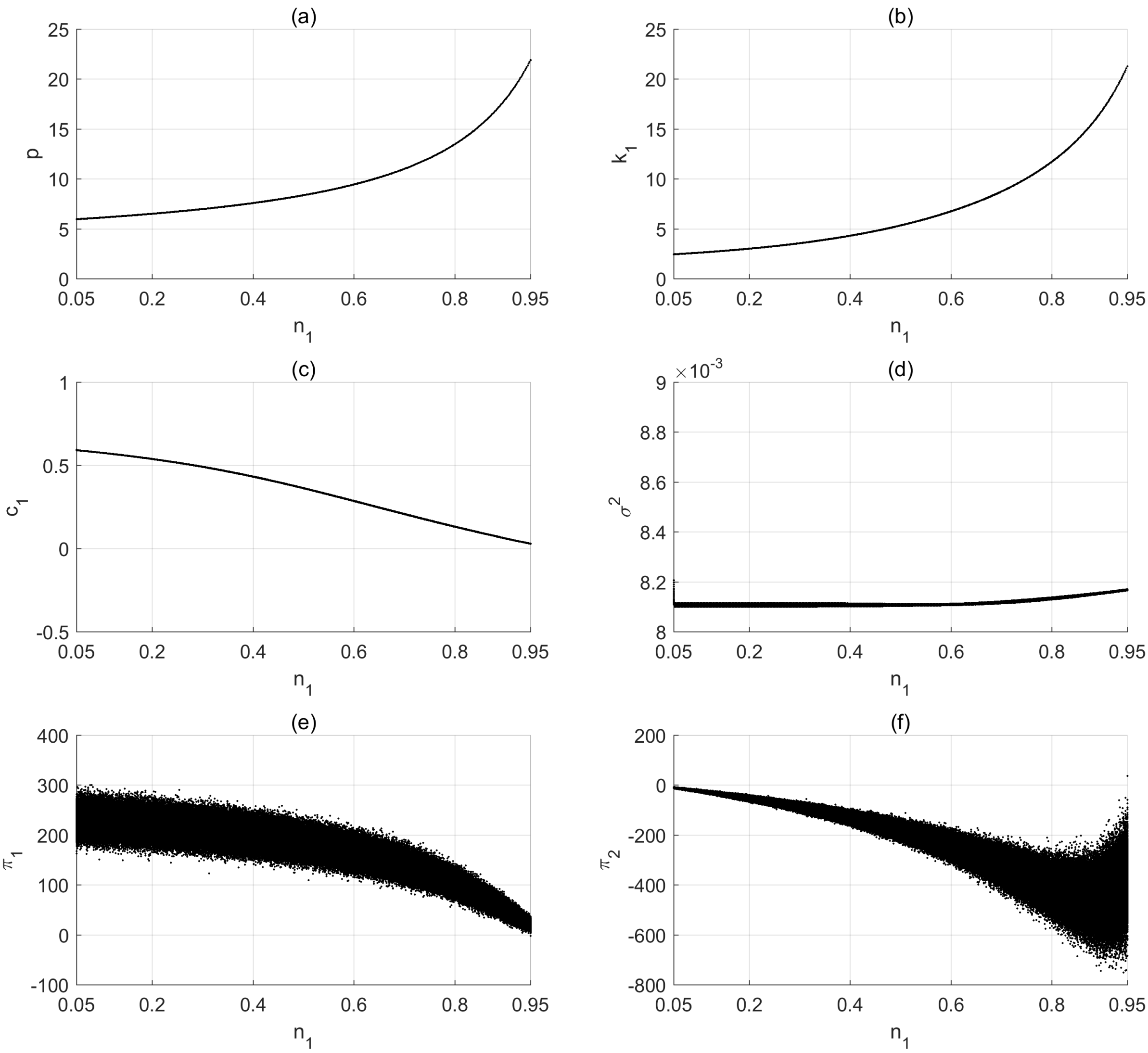

Figure 5.5: Bifurcation diagrams (n_1) in the Homogeneous Variance Case

Another important parameter in the model is the market share of learning agents n_1. When $n_1 = 1$, the Two-Beliefs model reduces to the Basic Model. Panel (c) of Figure 5.5 shows when learning agents dominate the market (i.e. n_1 gets close to 1), c_1 decreases gradually to 0. This result justifies the stable T-map equilibrium in the Basic Model (i.e. Eq 10 in Table 5.2 where $\bar{p} = 28.3845, \bar{k} = 28.3845, \bar{c} = 0, \bar{\sigma}^2 = 0.0081$ given the parameter values in Table 5.1 (b) Small Noise). When non-learning agents dominate the market (i.e. n_1 gets closer to 0), c_1 slowly increases.

Stock price p in Panel (a) increases with n_1 since when more learning agents trade in the market, the negative impact of non-learning agents on stock price decreases. It again justifies the stable T-map equilibrium in the Basic Model where $\bar{p}$ is significantly higher than $\bar{p}$ in the Homogeneous Variance Case. Panel (d) of Figure 5.5 illustrates the variance of excess returns σ^2 barely responds to n_1. When n_1 is roughly above 0.6, the variance increases slightly. The increase is more likely due to numerical accuracy and finite number of iterations rather than any explicit relationship between σ^2 and n_1.

Similar to β and c_2, Panel (e) and (f) show learning agents earn higher profit for all n_1. They always have positive demand. When more learning agents enter the market, the profit advantage for learning is diversified by the new comers and slowly shrinks to zero. Non-learning agents who face negative demand for all n_1 always suffer loss. Moreover, when n_1 increases, the demand of learning agents decreases slowly whereas the demand of non-learning agents decreases at an accelerating rate. Therefore, the profit gap increases with n_1. It demonstrates that the non-learning group is more severely punished from not learning when more agents are willing to learn in the market.

5.4 Numerical Simulation: Heterogeneous Variance Case

This section shows numerical simulations in the Heterogeneous Variance Case where learning and non-learning agents have different estimates on variance of excess returns in order to validate the analytical findings in Section 4.2.

Table 5.6 lists the numerical solutions for all theoretical equilibria and the corresponding eigenvalues given the parameter values with Large Noise and Small Noise in Table 5.1. Eq 25 to 30 refer to the T-map equilibria under Large Noise. Four of them are complex solutions and only Eq 29 and 30 are real solutions. All eigenvalues of Eq 30 are negative whereas two eigenvalues of Eq 29 are positive. Therefore, Eq 30 corresponding to $\bar{c}_{1,-}$ is stable but Eq 29 corresponding to $\bar{c}_{1,+}$ is unstable. Similarly, for the T-map equilibria under Small Noise, Eq 34 to 37 are complex solutions. The eigenvalues show only Eq 39 corresponding to $\bar{c}_{1,-}$ is stable whereas Eq 38 corresponding to $\bar{c}_{1,+}$ is unstable. It is consistent with the derivation of equilibrium in Section 4.2.4 which shows any equilibrium corresponding to $\bar{c}_{1,+}$ is unstable. Since two of the three solutions for σ_1^2 are complex, it leads to the complex equilibria under Large and Small Noise.

Figure 5.6 shows a simulation of length 10,000 with Large and Small Noise. The initial values are near Eq 30 for Large Noise and near Eq 39 for Small Noise in Table 5.6. The top panel shows stock price is stationary for both noises. The three middle panels show all other state variables $k_{1,t}$, $c_{1,t}$ and $\sigma^2_{1,t}$ are fairly stable after 2,000 iterations. Their end points listed in Table 5.6 (a) LN are very close to Eq 30 and (b) SN are very close to Eq 39. Therefore, the numerical simulation confirms the model converges to the theoretical stable T-map equilibrium for both noises.

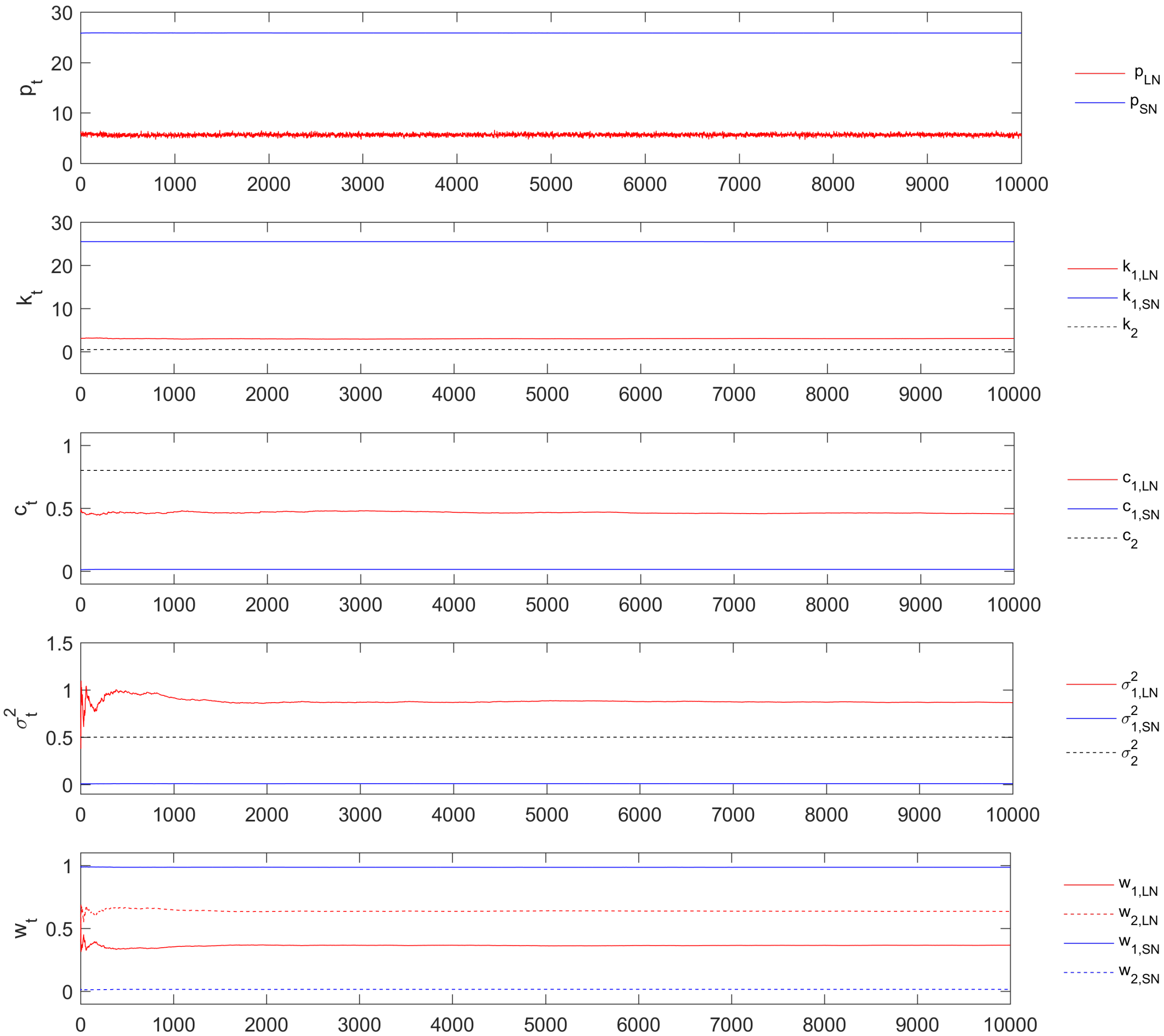

Figure 5.6: Numerical simulation (baseline) in the Heterogeneous Variance Case

The last panel in Figure 5.6 shows the effective fraction $w_{h,t}$ in the ALM defined in equation (4.52). Under Large Noise, the effective fraction of learning agents is roughly 0.4. Since learning agents have higher estimate on variance than non-learning agents (i.e. $\sigma_2^2 = 0.5$), they are unwilling to purchase risky asset. Therefore, non-learning agents dominate the stock market. Under Small Noise, $\sigma^2_{1,t}$ is significantly lower than σ_2^2. The learning agents dominate the stock market instead and $w_{1,t}$ is close to 1.

			$\bar{p}$	k_1	$\bar{c}_1$	$\bar{\sigma}_1^2$	Eigenvalues		
			eq.(4.56)	eq.(4.57)	eq.(4.58)	eq.(4.59)	eq.(4.66)	eq.(4.67)	eq.(4.68)
Large Noise	T-map equilibrium	Eq 25	Complex solutions						
		Eq 26							
		Eq 27							
		Eq 28							
		Eq 29	5.6888	-8.0046	2.4071	0.8608	0.1893	0.6804	-0.9566
		Eq 30	**5.6888**	**3.0848**	**0.4577**	**0.8608**	$\mathbf{-0.4912}$	$\mathbf{-0.6804}$	$\mathbf{-0.9566}$
	Random walk equilibrium	Eq 31	Complex solutions						
		Eq 32							
		Eq 33	4.9535	0.0000	1.0000	1.5728			
Small Noise	T-map equilibrium	Eq 34	Complex solutions						
		Eq 35							
		Eq 36							
		Eq 37							
		Eq 38	25.7387	-1.5418	1.0599	0.0081	0.9257	0.9817	-0.9999
		Eq 39	**25.7387**	**25.4869**	**0.0098**	**0.0081**	$\mathbf{-0.0560}$	$\mathbf{-0.9817}$	$\mathbf{-0.9999}$
	Random walk equilibrium	Eq 40	Complex solutions						
		Eq 41							
		Eq 42	25.7064	0.0000	1.0000	0.0082			
End points	Figure 5.6	(a) LN	5.6200	3.0739	0.4559	0.8662			
		(b) SN	25.8455	25.4724	0.0146	0.0080			
	Figure 5.7	(a) LN	5.6222	3.0851	0.4590	0.8639			
		(b) SN	25.6134	0.0464	0.9982	0.0085			
	Figure 5.8	(a) LN	5.6479	2.7446	0.5181	0.8569			
		(b) SN	25.5642	0.0026	0.9999	0.0087			
	Figure 5.9	(a) SN	25.2279	0.0045	0.9998	0.0098			
		(b) SN	25.7762	25.4431	0.0130	0.0081			

Table 5.6: Equilibria, eigenvalues and end points in the Heterogeneous Variance Case

Notes: 1) Bold font indicates stable equilibria; 2) The end points listed are the average of the last 100 points from the corresponding simulation.

5.4.1 Stability

Similar to Section 5.2.1, this section examines whether the model reverts back to the stable T-map equilibrium when starting from some non-equilibrium values. Figure 5.7 shows a simulation of length 100,000. The initial values are $p = 5$, $k_1 = 5$, $c_1 = 0$ and $\sigma_1^2 = 0.5$. The top panel shows the stock price is still stationary for both noises. For Large Noise, state variables $k_{1,t}$, $c_{1,t}$ and $\sigma_{1,t}^2$ in the three middle panels quickly converge and become fairly stable. The end points in Table 5.6 (a) LN are fairly close to Eq 30. For Small Noise, it does not converge to the stable T-map equilibrium Eq 39. Instead, the state variables converge to the random walk equilibrium since the end points in Table 5.6 (b) SN are close to Eq 42. This result that the system responds differently for different noises is identical to the Homogeneous Variance Case. Eventually, learning agents hold random walk beliefs when market volatility is low whereas they learn the true stochastic process when market volatility is high.

Figure 5.6 and Figure 5.7 show c_1 in the Heterogeneous Variance Case converges to some positive value. At the stable T-map equilibrium, the PLM of learning agents and the ALM are

$$\begin{aligned} \text{PLM:} \qquad & p_{1,t} = \bar{k}_1 + \bar{c}_1 p_{t-1} + \varepsilon_{1,t} \\ \text{ALM:} \qquad & p_t \;\; = \bar{k}_1 + \bar{c}_1 p_{t-1} - \frac{\beta a \bar{\sigma}_1^2 \sigma_2^2 v_t}{\sigma_2^2 n_1 + \bar{\sigma}_1^2 n_2} \end{aligned}$$

The ALM in the Heterogeneous Variance Case is an AR(1) process with some white noise v_t similar to that in the Homogeneous Variance Case in equation (5.2). The structural form of ALM in the Heterogeneous Variance Case is different from that in the Basic Model which is a constant with some white noise v_t. Therefore, the REE is also not attainable for the considered parameter values in the Heterogeneous Variance Case. In addition, the existence of non-learning agents in this case also creates additional disturbance to the market and leads average stock price to be lower than that in the Basic Model.

More interestingly, all the stable T-map and random walk equilibria in the Heterogeneous Variance Case yield higher stock price than that in the Homogeneous Variance Case since learning agents estimate variance by measuring the forecast error between the observed price and their estimated variance from last period. They actually underestimate the variance as they do not take the non-learning group's fixed estimate on variance into account explicitly. The underestimated variance leads to higher demand and thus pushes stock price up in the Heterogeneous Variance Case comparing to the Homogeneous Variance Case. It shows that when learning and non-learning agents have different risk estimates, the stock market is less affected by the disturbance created by non-learning agents.

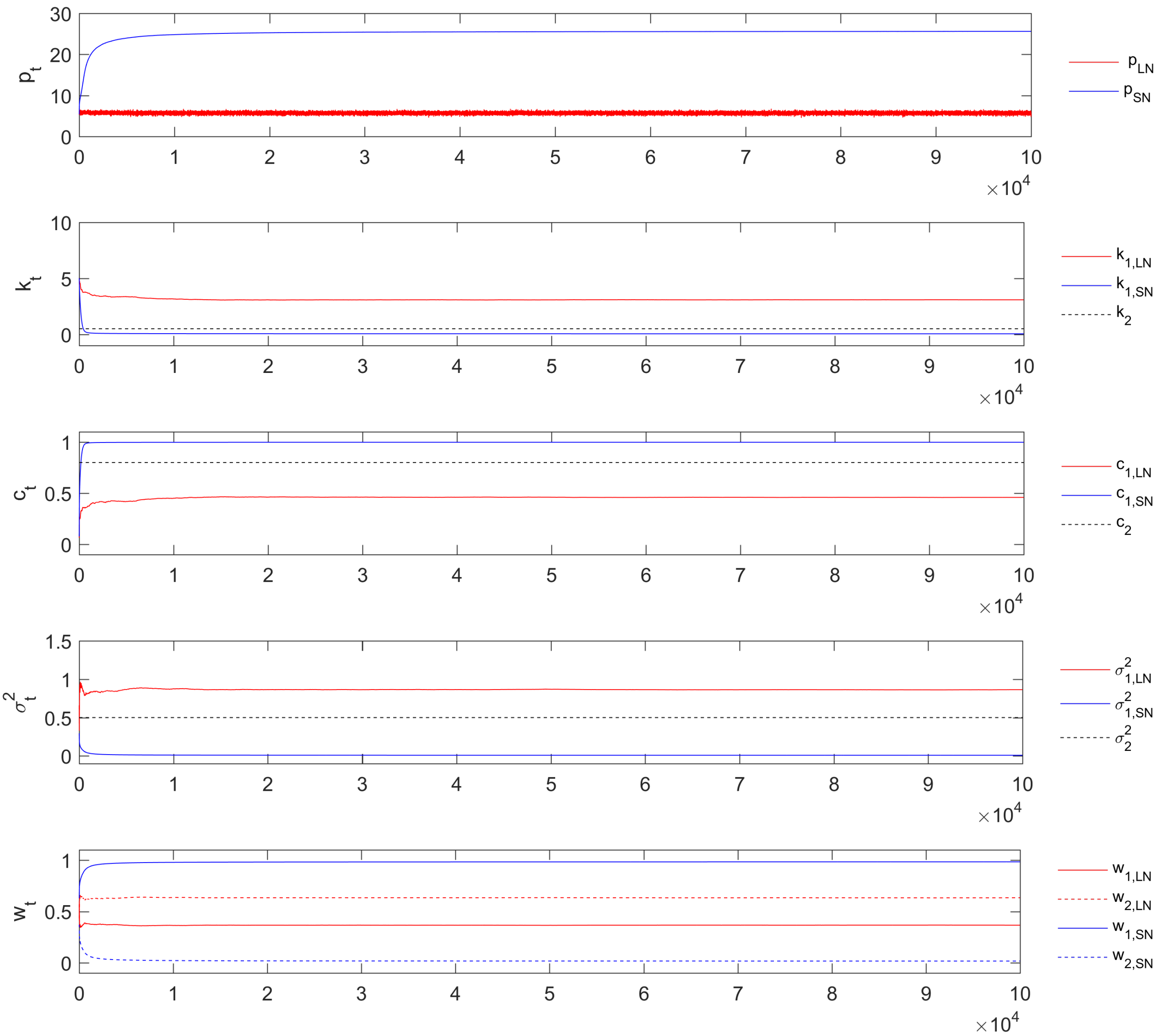

Figure 5.7: Numerical simulation (stability) in the Heterogeneous Variance Case

5.4.2 Profits of Learning and Non-learning Agents

Profit $\pi_{h,t}$ is calculated similar to the Homogeneous Variance Case. Since learning and non-learning agents have different estimates on variance, equation (5.5) becomes

$$\pi_{1,t} = (p_{t+1} + d_{t+1} - Rp_t) \left[(\hat{E}_{h,t}(p_{t+1}) + d_0 - Rp_t)/a\sigma^2_{1,t-1} \right] \tag{5.7}$$

$$\pi_{1,t} = (p_{t+1} + d_{t+1} - Rp_t) \left[(\hat{E}_{h,t}(p_{t+1}) + d_0 - Rp_t)/a\sigma^2_2 \right] \tag{5.8}$$

The average profit of each group Π_h follows equation (5.6). Table 5.7 shows the average profit of each group from a simulation of length 10,000. Similar to the

Homogeneous Variance Case, learning agents earn significantly higher profit than non-learning agents. Under Large Noise, although the estimate on variance of excess returns by learning agents is higher than that by non-learning agents, learning agents, who have much higher expectation on future price, still earn higher profit than non-learning agents. Under Small Noise, the difference in expectations are even larger and the demand of non-learning agents switches from positive to negative. In addition, the estimate on variance by learning agents is lower than that by non-learning agents. As a result, the profit gap is even larger.

In addition, the profit gap under Small Noise is significantly smaller comparing to the Homogeneous Variance Case. In the Heterogeneous Variance Case, learning agents always underestimate the variance of excess returns and pushes stock price up. The excess returns decline with stock price and therefore the profit gap is smaller. However, for Large Noise, the profit gap remains similar to the Homogeneous Variance Case. Any underestimation of variance is amplified under Small Noise whereas it becomes minimal under Large Noise since variance under Large Noise are much larger than that under Small Noise. As a result, profit gap in the Homogeneous and Heterogeneous Variance Case are much closer under Large Noise than under Small Noise.

Large Noise	Π_1	2.2730
	Π_2	0.2834
Small Noise	Π_1	3.4514
	Π_2	−3.1630

Table 5.7: Average profits in the Heterogeneous Variance Case

5.4.3 Random Walk Equilibrium

This section checks for the existence of random walk equilibrium by running a long period of simulation where initial values are near random walk (i.e. $p = 20$, $c_1 = 1$ $k_1 = 0$, $\sigma^2 = 0.5$). Table 5.6 lists the theoretical random walk equilibria. Under Large Noise, two of the three random walk equilibria are complex and there is only one real random walk equilibrium Eq 33. This is different to the Homogeneous Variance Case where they are no random walk equilibrium under Large Noise. Under Small Noise, only Eq 42 may be attainable. Moreover, both theoretical random walk equilibria yield higher variance but lower stock price than the stable T-map equilibria. It justifies the finding in Section 4.2.5 that random walk beliefs increase market volatility and decrease average stock price.

Figure 5.8 shows Large and Small Noise give different simulation results when learning agents start from random walk beliefs. Under Small Noise, c_1 starting from 1 first drops to around 0.7 and eventually converges back to random walk after 100,000 iterations. Stock price then recovers quickly and becomes stable after 10,000 iterations. However, under Large Noise, c_1 also starts from 1 and drops to 0.7 but it continually decreases to 0.5181. The end points of Figure 5.8 (a) LN in Table 5.6 are fairly close to Eq 30 which is the stable T-map equilibrium. Under Small Noise, the end points of (b) SN are close to Eq 42. Therefore the model converges to the stable T-map equilibrium under Large Noise and random walk equilibrium under Small Noise, which I also observe in the Homogeneous Variance Case.

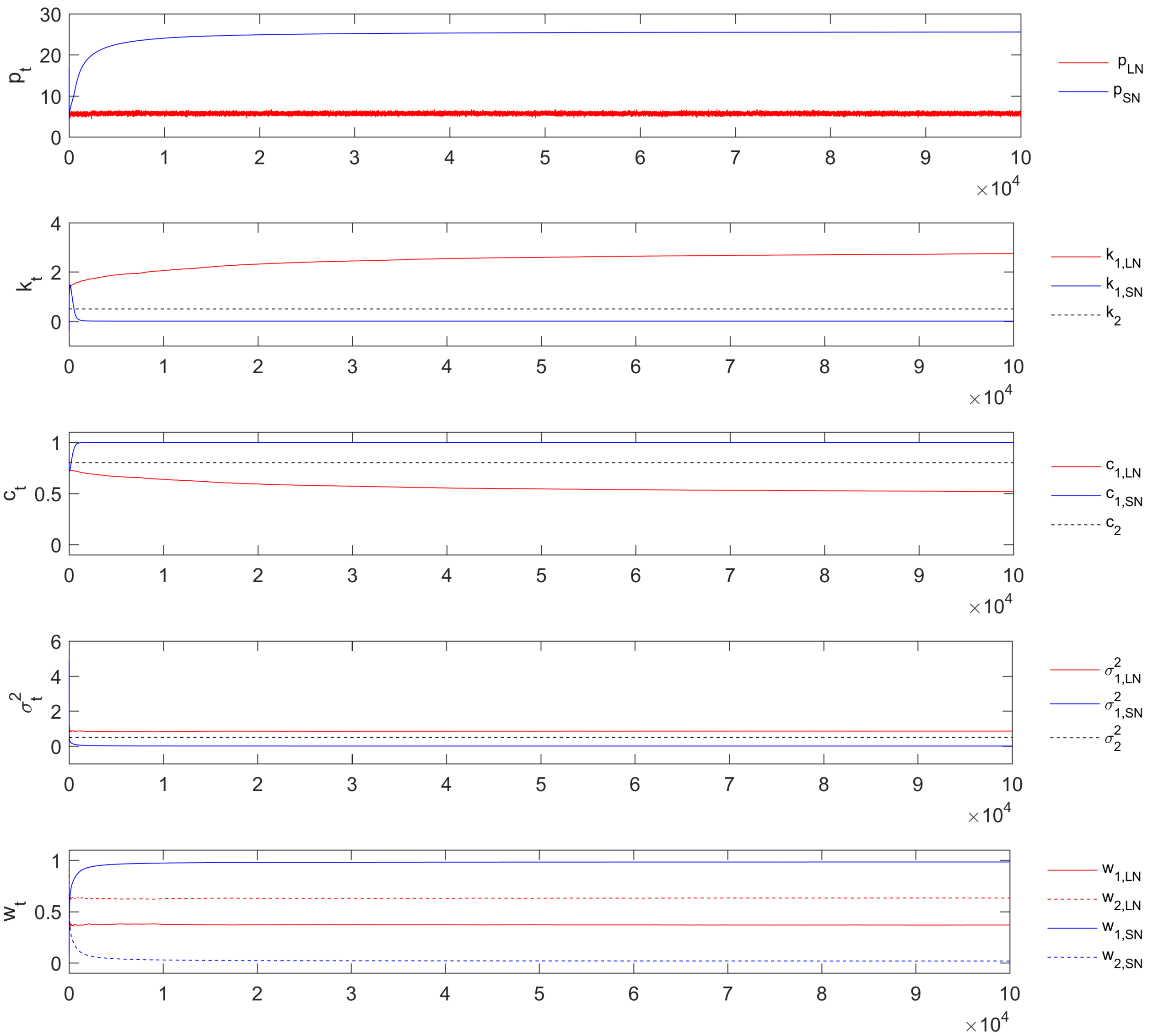

Figure 5.8: Numerical simulation (random walk) in the Heterogeneous Variance Case

The remaining difference between the end points and the theoretical equilibrium can be explained by the slow convergence. The end points of k_1 and c_1 at 10,000 and 50,000 iterations in Table 5.8 further shows the model is slowly converging to the stable T-map equilibrium under Large Noise and to random walk equilibrium for Small Noise. Similar to the Homogeneous Variance Case, it shows high market volatility allows learning agents to escape from the random walk beliefs whereas low market volatility does not.

	Large Noise		Small Noise	
Iterations	10,000	50,000	10,000	50,000
k_1	2.1166	2.5670	0.0029	0.0026
c_1	0.6204	0.5447	0.9999	0.9999

Table 5.8: Values for k_1 and c_1 in Figure 5.8

Another finding related to random walk equilibrium is under Small Noise, the model starting near the theoretical stable T-map equilibrium (i.e. Eq 39 in Table 5.6) may converge to the random walk equilibrium instead. Neither the Basic Model or the Homogeneous Variance Case illustrates this result since both models converge to the stable T-map equilibrium when starting near the stable T-map equilibrium.

Figure 5.9 compares this exceptional case to the normal case. In the exceptional case, k_1 and c_1 quickly converge to random walk and become fairly stable. The top panel shows the stock price suddenly drops from its initial value and slowly increases over time. After 500,000 iterations, the stock price eventually reverts back near its original position. The end points of the exceptional case which is (a) SN in Table 5.6 are close to the theoretical random walk equilibrium. In the normal case, all the state variables consistently stay at the stable T-map equilibrium. The end points of normal case which is (b) SN in Table 5.6 are close to the theoretical stable T-map equilibrium.

One possible explanation of the drop in stock price in the exceptional case is the abrupt increase in σ_1^2. At the beginning of the simulation, variance increases rapidly to above 30 in a very short period of time. It thus decreases stock price and pushes c_1 and k_1 away from its stable T-map equilibrium. Since the two equilibria have nearly identical stock price and variance, learning agents are incapable of distinguishing the two equilibria by only observing stock prices. In some occasions, they result in holding random walk beliefs. The non-stationary PLM of learning agents still leads to an equilibrium and stock price slowly recovers. Moreover, this exceptional case suggests the region of attraction (ROA) of the random walk equilibrium may be larger than that of the stable T-map equilibrium. The sudden drop in price forces the state variables away from the ROA of the stable T-map equilibrium and enters the ROA of the random walk equilibrium instead. Hence it never reverts back to the stable T-map equilibrium but converges to the random walk equilibrium.

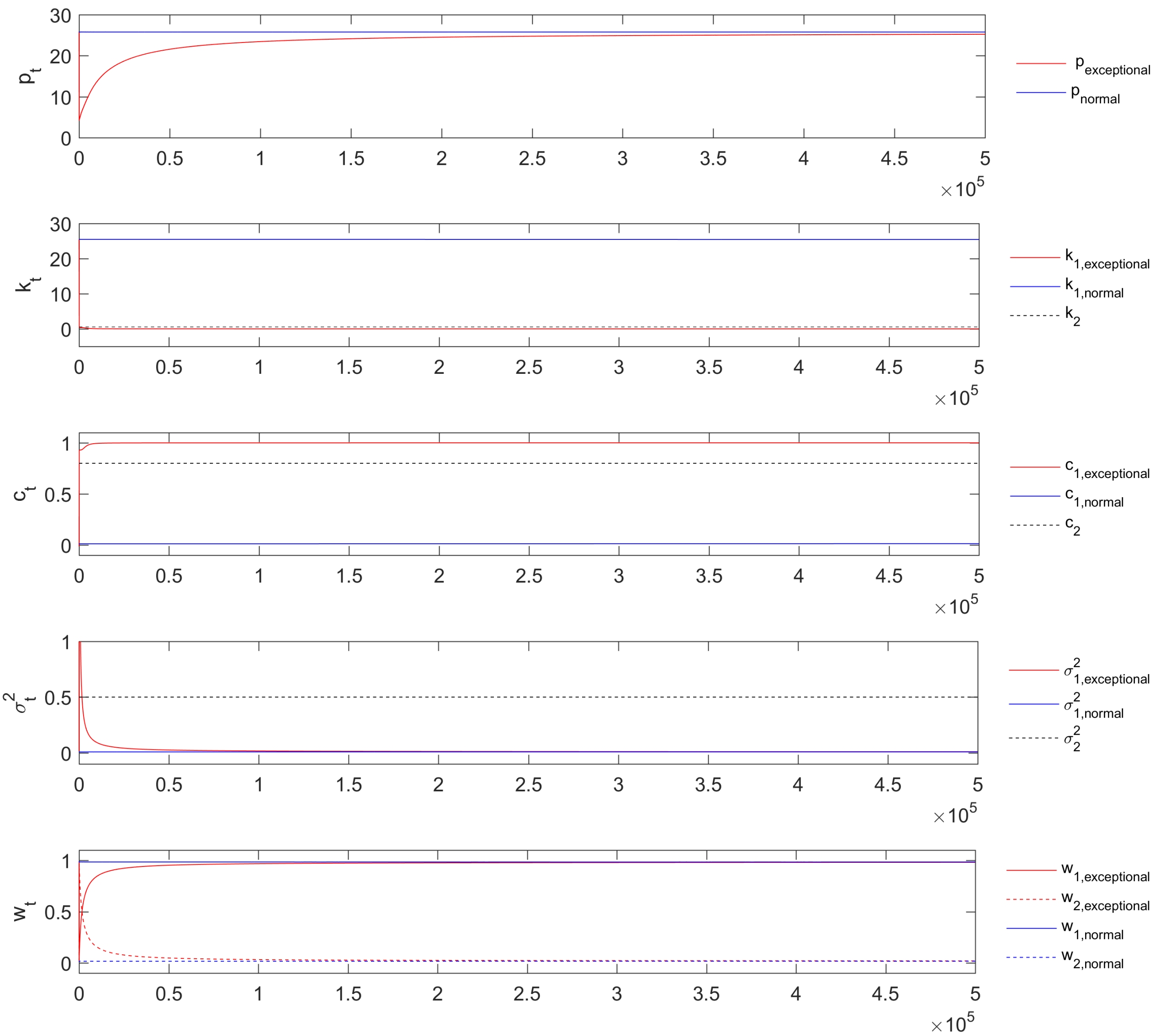

Figure 5.9: Numerical simulation (exceptional case) in the Heterogeneous Variance Case

5.5 Bifurcation: Heterogeneous Variance Case

This section discusses how the state variables and profit of each group respond to changes in the selected exogenous parameters in the Heterogeneous Variance Case. Bifurcation diagrams are generated from the same method described in Section 5.3.1. For the Heterogeneous Variance Case, $p_0 = 5$, $T = 150,050$, $K = 21$ and $N = 500$. Since the exceptional case demonstrated in Section 5.4.3 may arise in some occasions, the model requires more time to converge and I therefore run a longer period of simulation than the Homogeneous Variance Case where $T = 15,050$.

5.5.1 Discount Factor

Panel (c) of Figure 5.10 shows c_1 converges to two equilibria. When β is low, c_1 converges to the theoretical stable T-map equilibrium since majority of c_1 is close to 0. It starts to converge to random walk as β increases. When β is around 0.97, c_1 always converges to 1. The random walk in c_1 causes stock price in Panel (a) to bifurcate into two trajectories. The upper trajectory shows stock price increases with β. When β increases, the perceived risk of investing in the risky asset decreases and the outside option yields higher returns. The demand for risky asset is higher and thus the price increases. However, the lower trajectory shows stock price decreases with β when it gets closer to 0.99.

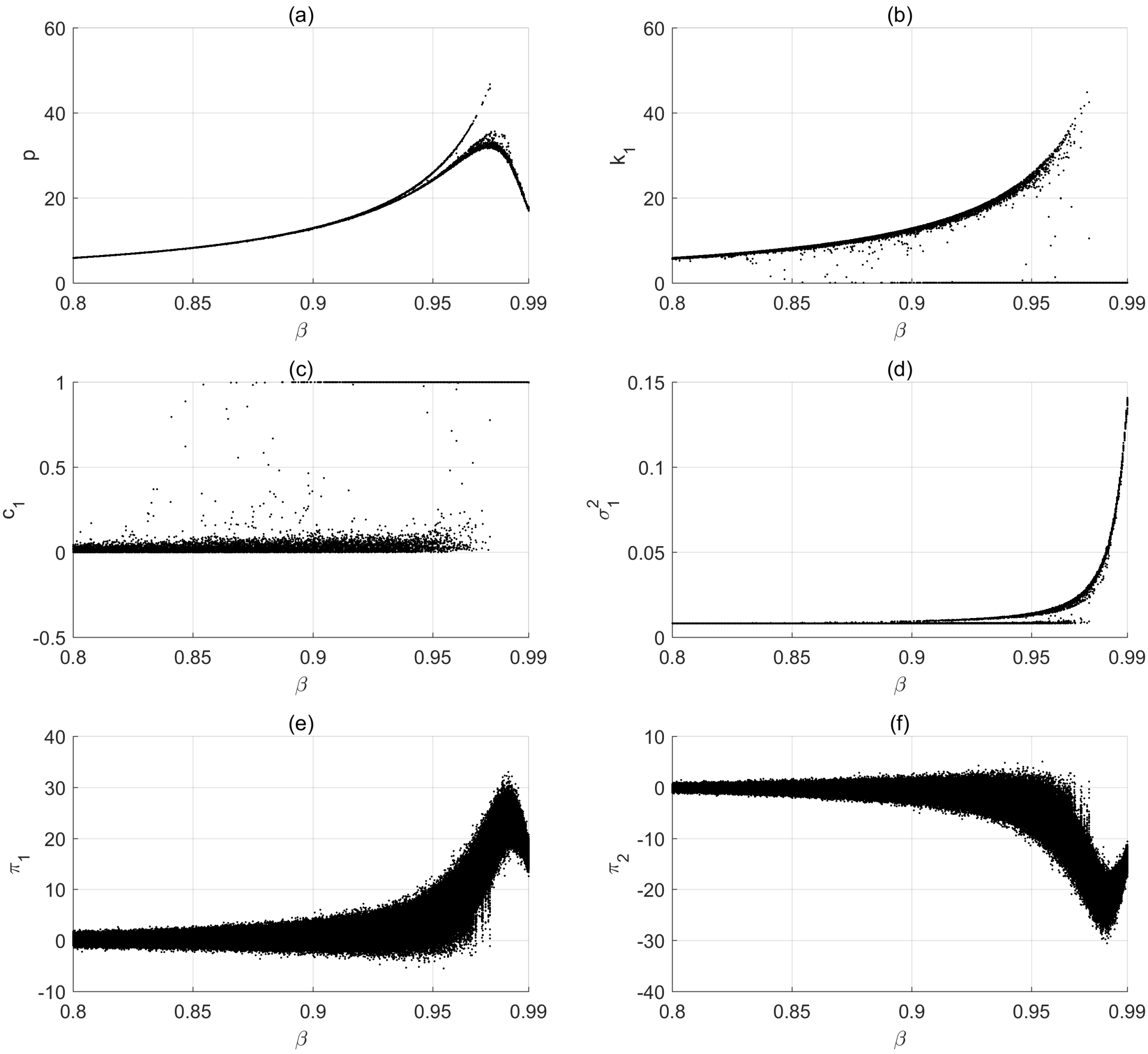

Figure 5.10: Bifurcation diagrams (β) in the Heterogeneous Variance Case

Before explaining this non-trivial behaviour of stock price (i.e. the lower trajectory), I examine the theoretical stable T-map and random walk equilibria for all β in this range. Stock price of both equilibria increases with β and the values are very close to each other for all β. The theoretical stable T-map equilibrium values of σ_1^2 barely respond to β whereas the theoretical random walk equilibrium values of σ_1^2 increase slightly with β but to a much lesser extent than Panel (d) shows. The higher simulated values of σ^2 on Panel (d) are likely due to slow convergence in the exceptional case described in Section 5.4.3. Since one of the eigenvalues of the stable T-map equilibrium gets closer to 0 when β increases, the equilibrium is on the border of stability and potential instability around it may arise. Therefore, the status of the stable T-map equilibrium may switch from stable to unstable when noises in the model shift the eigenvalues from barely negative to non-negative and vice versa. In some occasions, an abrupt increase in σ^2 leads stock price to drop significantly. Since learning agents cannot distinguish the two equilibria by only observing stock prices, random walk equilibrium occurs. The sudden but significant drop in price takes extremely long period of time to recover due to slow convergence. As a result, even after 150,050 iterations, p and σ_1^2 do not recover and the end points plotted on the diagrams of p and σ^2 are still relatively far from the theoretical random walk equilibrium. Therefore, the upper trajectory only captures the incomplete convergence path of the random walk equilibrium against β unlike the lower trajectory which captures the full convergence of the stable T-map equilibrium.

Panel (e) and (f) show learning agents always earn higher profit than non-learning agents similar to the Homogeneous Variance Case. The profit of learning agents increases when β is around 0.95 to 0.98. It is followed by a sudden drop at the end of β. Since profit strongly depends on stock price, the drop in profit basically follows the behaviour of stock price. The profit gap increases with β since learning agents have higher expectations on future price than non-learning agent when β increases.

5.5.2 Belief of Non-Learning Agents

I first examine the theoretical stable T-map equilibrium and random walk equilibrium for all c_2 in this range. Stock price of both equilibria has symmetric behaviour against c_2 and their values are very close. The stable T-map equilibrium values of σ_1^2 barely responds to c_2. The random walk equilibrium values of σ_1^2 have a symmetric behaviour against c_2 and are higher than the stable T-map equilibrium but lower than Panel (d) shows. It can again be explained by the exceptional case in Section 5.4.3. Similar to the case of β, after the sudden drop in price due to an abrupt increase in variance, p and σ_1^2 do not converge very close to the theoretical random walk equilibrium due to slow convergence. However, they recover much quicker than the case of β since the eigenvalues in this case are close to 0 for all c_2 but not as close as in the case of β. Potential instability around the theoretical stable T-map equilibrium still arises but less frequently comparing to the case of β and thus the convergence of state variables is likely to be faster.

Figure 5.11 shows state variables follow the symmetric behaviour against c_2 demonstrated in the Homogeneous Variance Case. Panel (c) shows c_1 always converges to 1 for all c_2. In some occasions, it also converges close to 0. It happens when c_2 gets close to its end values but less frequently when $-0.5 < c_2 < 0.5$. The random walk in c_1 causes stock price in Panel (a) to bifurcate into two trajectories. The upper trajectory represents the stable T-map equilibrium whereas the lower trajectory represents the random walk equilibrium. Since the instability issue is less severe for c_2, the two trajectories are very close to each other as the two theoretical equilibria suggest.

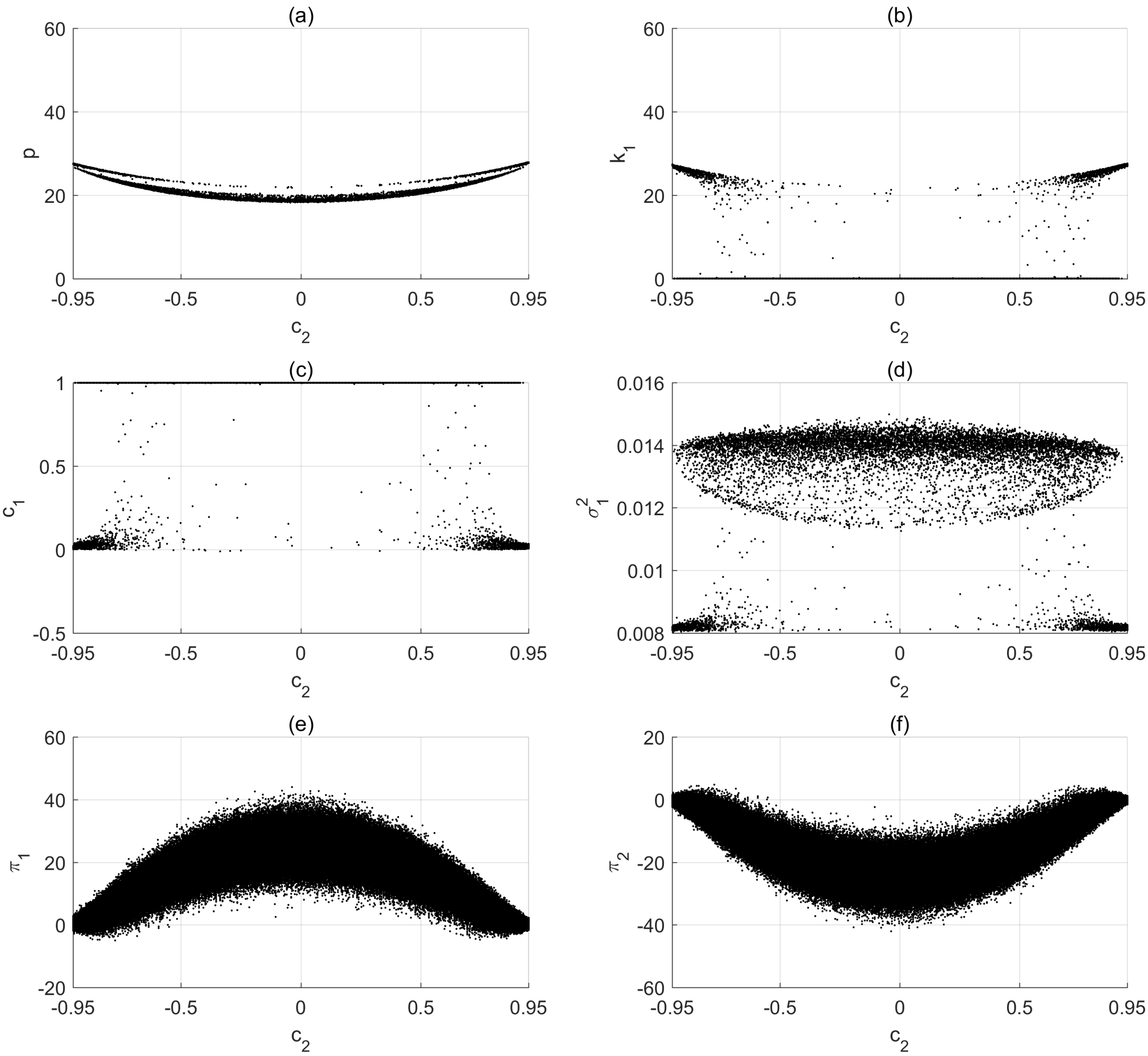

Figure 5.11: Bifurcation diagrams (c_2) in the Heterogeneous Variance Case

Panel (e) and (f) show learning agents earn higher profit in both equilibrium for all c_2 since they have higher expectations on future price and thus positive demand. Non-learning agents have relatively lower expectations than learning agents. Although their expectations are still positive, these expectations are lower than the cost of investing in the risky asset. Therefore, non-learning agents have negative demand and short sell. Given the excess returns are always positive for all c_2, the non-learning group always suffers loss. Similar to the Homogeneous Variance Case, the profit gap between the two groups is largest when $c_2 = 0$. When $c_2 = 0$, c_1 always converges to 1. At $c_2 = 0$, the PLM of non-learning agents yields

$$\begin{aligned} p_{1,t} &= k_{1,t-1} + c_{1,t-1}p_{1,t-1} + \varepsilon_{1,t} \\ p_{2,t} &= k_2 + \varepsilon_{2,t} \end{aligned}$$

where $k_{1,t-1}$ is close to 0 and $c_{1,t-1}$ is very close to 1. The expectations of non-learning agents reach its lowest level when $c_2 = 0$ (i.e. $\hat{E}_t(p_{t+1}|\Omega_{t-1}) = k_2$) whereas the expectations of learning agents reach its highest level (i.e. $\hat{E}_{1,t}(p_{t+1}|\Omega_{t-1}) = p_{t-1}$). The largest difference in expectations translates to the largest difference in demand. Since stock price is lowest when $c_2 = 0$, the excess returns are highest among all c_2. Combining both effects, the profit gap is largest when $c_2 = 0$.

5.5.3 Market Share of Learning Agents

I first examine the theoretical stable T-map equilibrium and random walk equilibrium. Stock price of both equilibria increases with n_1. Similar to the case of β and c_2, the values of the two equilibria are very close. The stable T-map equilibrium values of σ_1^2 barely respond to n_1. The random walk equilibrium values of σ_1^2 decrease with n_1. The values are also higher than the stable T-map equilibrium values but are lower than Panel (d) shows. Similar to the case of c_2, it can again be explained by the exceptional case in Section 5.4.3. The eigenvalues are close to 0 for all n_1 but not as close as in the case of β. Similar to c_2, the instability and the slow convergence may still arise but less severe than the case of β.

Panel (c) of Figure 5.12 shows when most agents are willing to learn (i.e. n_1 is close to 1), no random walk equilibrium occurs and the model always converges to the stable T-map equilibrium. This result is consistent with that in the Basic Model since BE show the beliefs of learning agents c is 0 at REE. They also shows no random walk equilibrium occurs under decreasing-gain learning. However, if non-learning agents dominate the market (i.e. n_1 is close to 0), only random walk equilibrium occurs. At the random walk equilibrium, learning agents are no longer considered as rationally learning agents in the market. Instead, there are two types of agents who have "fixed" parameters in their PLM. One has $c_1 = 1$ and the other has $c_2 = 0.8$. The beliefs of learning agents start shifting from random walk equilibrium to the stable T-map equilibrium when n_1 is around 0.3. Panel (a) shows that stock price bifurcates into two trajectories. Similar to c_2, the upper trajectory represents the stable T-map equilibrium whereas the lower trajectory represents the random walk equilibrium. The two trajectories are very close as the theoretical equilibria suggest. Since stock price increases with n_1, it identifies the major finding of the Two-Beliefs Model where non-learning agents create additional disturbance to the market and decrease the average stock price.

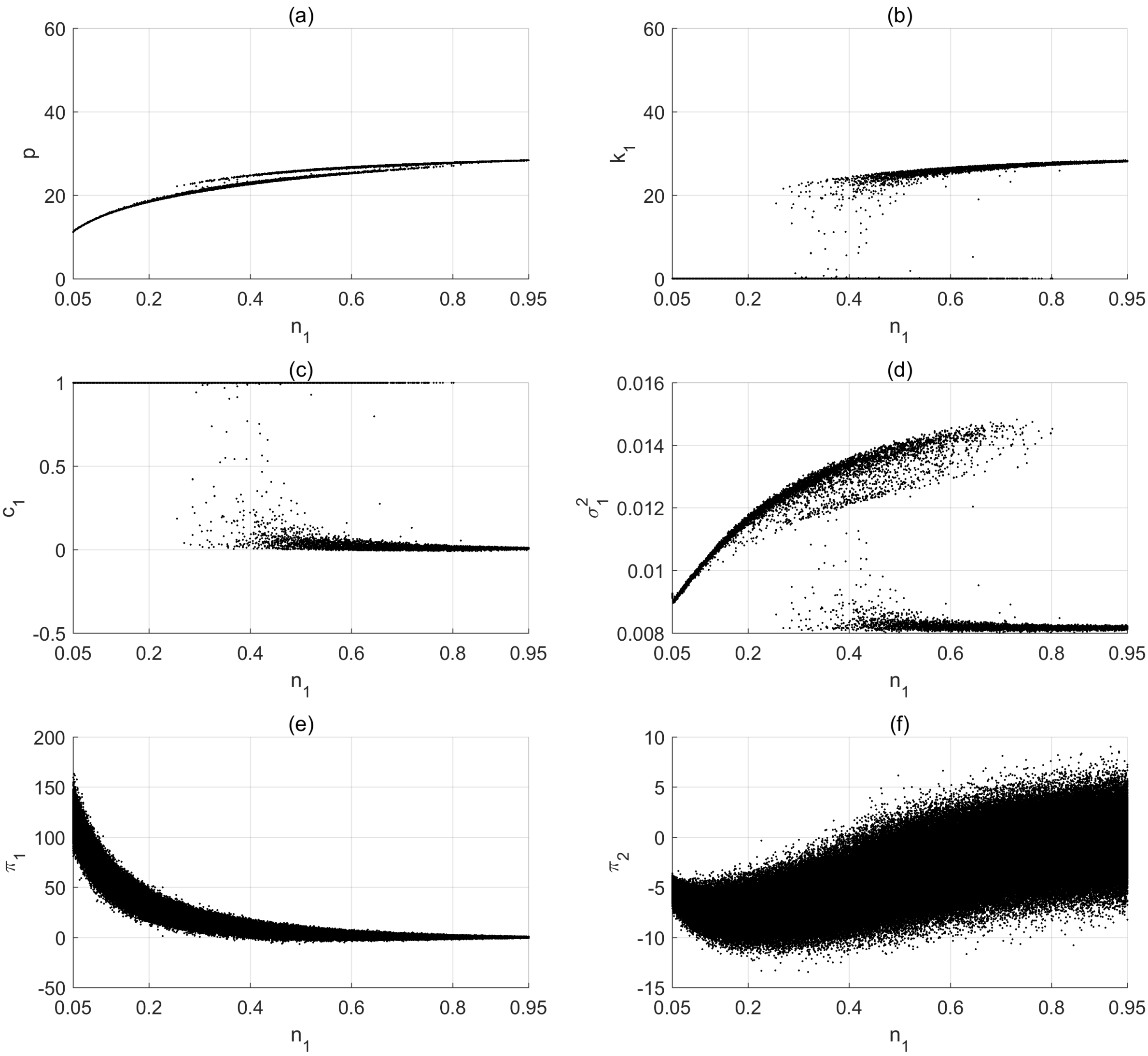

Figure 5.12: Bifurcation diagrams (n_1) in the Heterogeneous Variance Case

Different from β and c_2, Panel (e) and (f) show learning agents do not always earn higher profit than non-learning agents. When non-learning agents dominate the market (i.e. $n_1 = 0.05$), learning agents earn highest profit where they hold random walk beliefs. The profit advantage of learning are diversified by the new comers when all investors are willing to learn. More interestingly, the profit gap does not grow when more learning agents in the market. Instead, the profit of non-learning agents seems to increase with n_1. Eventually, when learning agents dominate the market, the non-learning group earns higher profit.

5.6 Constant-Gain Learning

All the results shown so far are under decreasing-gain learning where $\gamma_t = 1/t$. BE illustrates constant-gain learning where γ_t is a small and fixed constant allows learning agents to be more reactive to structural change and put a higher weight on the recent observations. Under constant-gain learning, the state variables converge to a stationary process around the equilibrium instead of a static equilibrium. This section briefly provides insight into how the state variables in the Two-Beliefs Model respond under constant-gain learning. However, I do not investigate this case in details and leave it for further research.

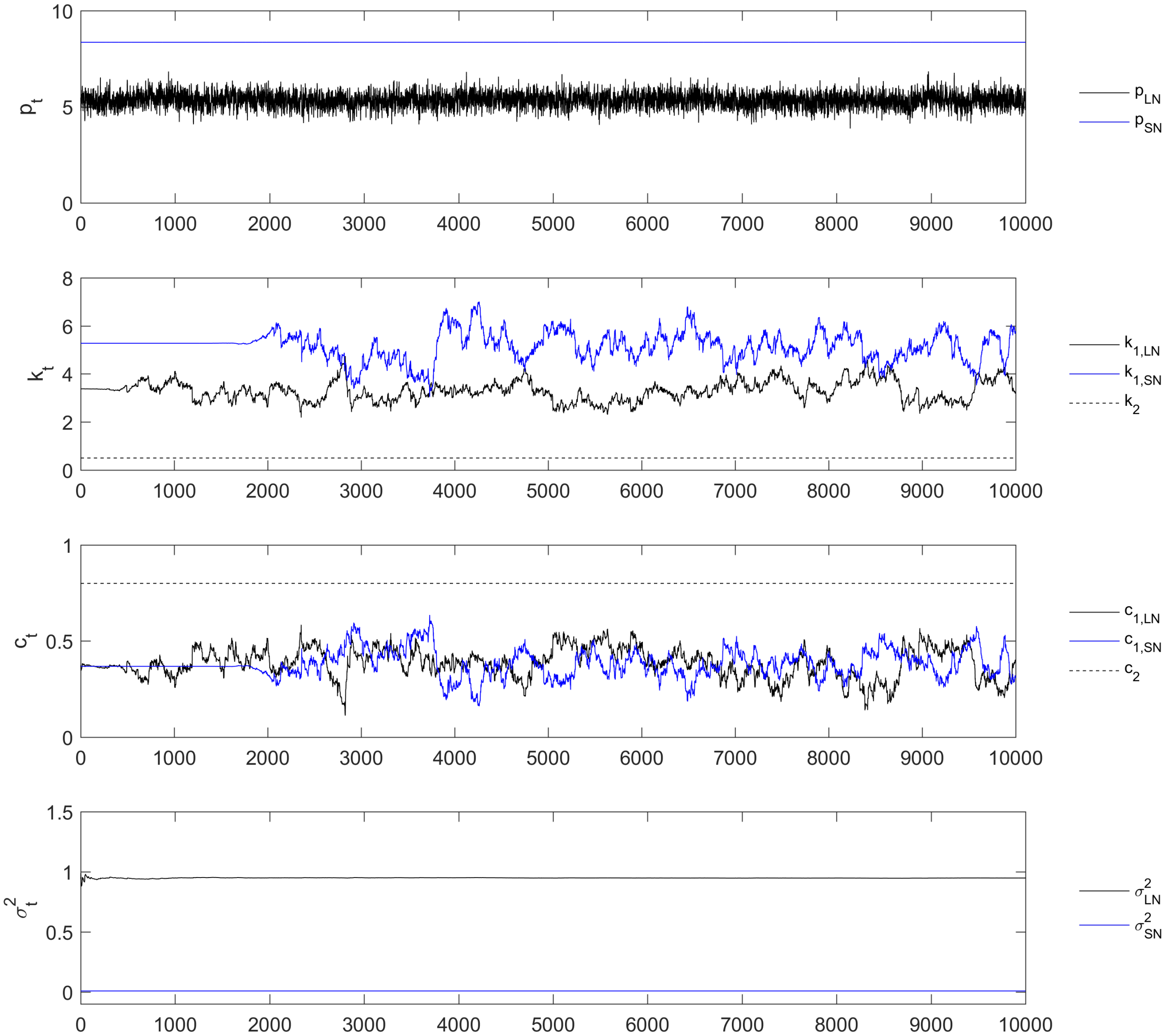

Figure 5.13: Numerical simulation (constant-gain learning) in the Homogeneous Variance Case

Figure 5.13 shows a simulation of length 10,000 in the Homogeneous Variance Case. The initial values are near the theoretical stable T-map equilibrium (i.e. Eq 16 for

Large Noise and Eq 22 for Small Noise in Table 5.3).The gain parameter $\gamma_{1,t}$ is set to be 0.01 in equation (4.12) and (4.13). The simulation is corresponding to that in Figure 5.1 under decreasing-gain learning. The only difference between the two figures is the time-varying estimates of learning agents, k_1 and c_1. Both converges to stationary process around the theoretical stable T-map equilibrium. Under Large Noise, k_1 is higher than that under Small Noise whereas c_1 converges to similar value regardless of the size of noise given. This result is consistent with that in Figure 5.1.

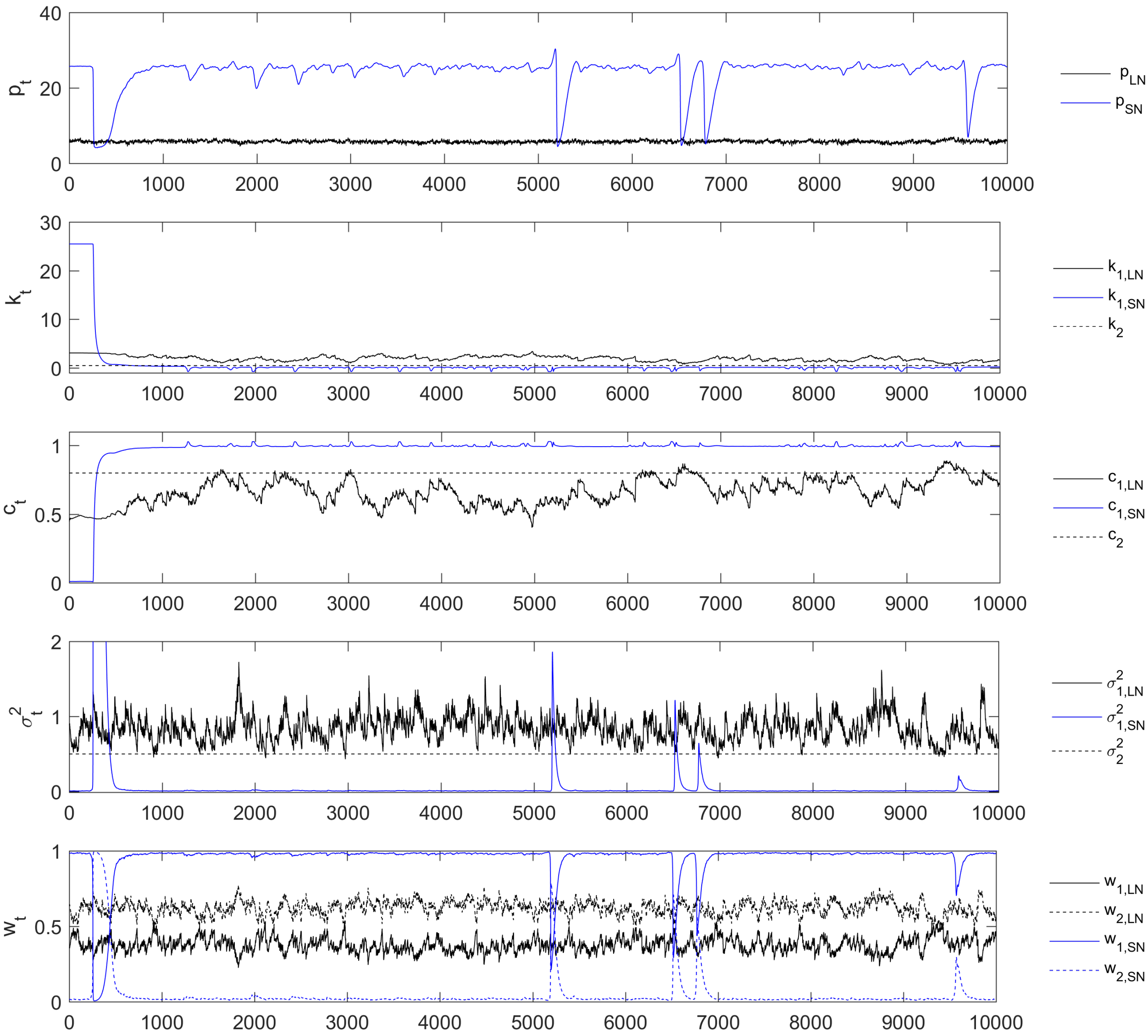

Figure 5.14: Numerical simulation (constant-gain learning) in the Heterogeneous Variance Case

Figure 5.14 shows a simulation of length 10,000 in the Heterogeneous Variance Case. The initial values are near the theoretical stable T-map equilibrium (i.e. Eq 30 for Large Noise and Eq 39 for Small Noise in Table 5.6). The gain parameters follow the setting in the Basic Model where $\gamma_{1,t}$ is set to be 0.01 in equation (4.60) and (4.61) and $\gamma_{2,t}$ is set to be 0.04 in equation (4.62). Under Small Noise, around period 250 to 350, there is a huge jump in σ_1^2 from around 0.05 to over 82. It causes stock price drops significantly. Learning agents respond to the endogenous shock and hold random walk beliefs as c_1 quickly converges close to 1. The stock price then recovers to its original position. After this initial crash in the market, learning agents continue to hold random walk beliefs.

Moreover, the top panel demonstrates multiple crashes in the stock market under constant-gain learning. Each crash seems to be driven by a sudden increase in variance. After each crash, the price recovers back to its original value while learning agents keep holding random walk beliefs in the entire simulation. The effective fractions in the bottom panel switch positions shortly when there is a crash. When price recovers to its original position, the effective fractions revert back to its pre-crash position. It shows occasional endogenous shocks are the major cause of recurrent crashes in the model. Given these features, the model with constant-gain learning has greater potential to reflect the price behaviour in actual stock markets.

Chapter 6
Conclusion

This is predominantly a theoretical work where I investigate the relationship between expectations and risky asset prices. The Basic Model of BE is extended to include heterogeneous beliefs under adaptive learning. The Two-Beliefs Model introduces two groups of agents: learning agents and non-learning agents. I consider two cases. In the Homogeneous Variance Case, both groups have identical estimate on variance of excess returns. In the Heterogeneous Variance Case, they have different estimates.

In the Basic Model, BE solve four equilibria and identify a unique stable equilibrium which corresponds to the rational expectations equilibrium (REE). They also discuss the PLM under random walk beliefs may lead to an equilibrium which T-map cannot capture. As I extend the model, in the Homogeneous Variance Case, I find four equilibria. Similar to the Basic Model, there is only one stable equilibrium although it does not necessarily correspond to the REE. In the Heterogeneous Variance Case, I find six equilibria. There is only one possibly stable equilibrium whereas the others are either non-attainable or complex solutions. In addition, when learning agents perceive stock price follow a random walk, it may also lead to an equilibrium in the model for both cases. However, the stability of the random walk equilibria remains unsolved.

The results of numerical simulations validate my analytical finding and suggest that non-learning agents create additional disturbance in the stock market and decrease average stock price. Learning agents need to adjust their forecasts and demands in light of the existence of non-learning agents. Random walk equilibrium occurs even under decreasing-gain learning. It happens more frequently when two groups have different risk estimates. Due to its self-fulfilling nature, random walk beliefs last for a substantial amount of time. Since stock price at the random walk equilibrium and the stable T-map equilibrium are nearly identical, learning agents who only observe prices realised from the market are not capable of recognising the difference between the two equilibria and fall into the random walk equilibrium.

Moreover, profits can be considered as motivation for learning since learning agents always earn higher profits than non-learning agents except when learning agents dominate the stock market and diversify the profit from learning extensively. In addition, bifurcation diagrams show the dynamics of the Two-Beliefs Model when changing three important exogenous parameters in the model, including the discount factor, the belief of non-learning agents and the market share of learning agents. In the Heterogeneous Variance Case, stock price bifurcates into two trajectories since the belief of learning agents converges to either the random walk equilibrium or the stable T-map equilibrium. Instability around the theoretical equilibrium may arise when changing values of certain parameters and further creates additional volatility in the market.

Although the Two-Beliefs Model under constant-gain learning is not investigated in detail, I observe that heterogeneous beliefs under adaptive learning may generate recurrent bubbles and crashes. Occasional endogenous shocks drive learning agents from their stable equilibrium toward random walk beliefs. Recurrent crashes lead the effective market share of learning and non-learning agents to switch positions temporarily. My analysis assumes the non-learning group has fixed estimates and therefore the arbitrary chosen values of the exogenous parameters in simulation have strong influence on the results. Further research, including the role of non-learning agents in generating recurrent bubbles and crashes as well as extending the Two-Beliefs Model to allow multiple groups of agents who forecast prices differently, would be useful. Furthermore, it would be interesting to investigate endogenous switching as in Brock and Hommes (1998). Another possible avenue for future research is to estimate the parameters of the developed theoretical model from the actual stock prices.

Appendix A
Pesudocode

Algorithm 1: Homogeneous Variance Case

input : β, k_2 c_2 d_0 s_0 a n_1 n_2 σ_u^2 σ_v^2 T

output: p_t $k_{1,t}$ $c_{1,t}$ σ_t^2 $\quad \forall t = 1, ..., T$

while $c_1 = NaN$ **do**

  Assign initial values for k_1 c_1 p_{t-1} R σ^2 $\dot{p}_{t-1}$;

  for $t = 1$ **to** T **do**

    $v_t \sim WN(0, \sigma_v^2)$

    Actual law of motion

      $z_{st} = s_0 + v_t;$

      $p_t = \beta[(d_0+n_1 k_{1,t-1}(1+c_{1,t-1})+n_2 k_2(1+c_2))+(n_1 c_{1,t-1}^2+n_2 c_2^2)]p_{t-1}-\beta a \sigma_{t-1}^2 z_{st};$

    Assign values

      $\theta_{t-1} \to [k_{1,t-1}; c_{1,t-1}];$

      $X \to [1; p_{t-1}];$

    Recursive equations

      $\theta_{1,t} = \theta_{1,t-1} + t^{-1} S_t^{-1} X_t (p_t - \theta_{1,t-1}' X_t);$

      $S_t = S_{t-1} + t^{-1}(X_t X_t' - S_{t-1});$

      $\dot{p}_t = \dot{p}_{t-1} + t^{-1}(p_t - \dot{p}_{t-1});$

      $\sigma_{p,t}^2 = \sigma_{p,t-1}^2 + \dot{p}_{t-1}^2 - \dot{p}_t^2 + t^{-1}(p_t^2 - \sigma_{p,t-1}^2 - \dot{p}_{t-1}^2)$;

      $\sigma_t^2 = \sigma_{p,t}^2 + \sigma_u^2;$

    Updating the state variables

      $k_{1,t} = \theta_t(1);$

      $c_{1,t} = \theta_t(2);$

      $k_{1,t-1} = k_{1,t};$

      $c_{1,t-1} = c_{1,t};$

      $p_{t-1} = p_t;$

      $\sigma_{t-1}^2 = \sigma_t^2;$

  end

end

Algorithm 2: Heterogeneous Variance Case

input : β, k_2 c_2 d_0 s_0 a n_1 n_2 σ_2^2 σ_u^2 σ_v^2 T

output: p_t $k_{1,t}$ $c_{1,t}$ $\sigma_{1,t}^2$ $\quad \forall t = 1, ..., T$

while $c_1 = NaN$ **do**

  Assign initial values for k_1 c_1 p_{t-1} R σ_1^2 ;

  for $t = 1$ **to** T **do**

    $u_t \sim WN(0, \sigma_u^2)$

    $v_t \sim WN(0, \sigma_v^2)$

    Actual law of motion

      $z_{st} = s_0 + v_t;$

      $$p_t = \frac{\beta\left[\sigma_2^2 n_1(d_0 + k_{1,t-1}(1 + c_{1,t-1})) + \sigma_{1,t-1}^2 n_2(d_0 + k_2(1 + c_2)) - a\sigma_{1,t-1}^2\sigma_2^2 s_0\right]}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} + \frac{\beta\left(\sigma_2^2 n_1 c_{1,t-1}^2 + \sigma_{1,t-1}^2 n_2 c_2^2\right) p_{t-1}}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2} - \frac{\beta a \sigma_{1,t-1}^2 \sigma_2^2 v_t}{\sigma_2^2 n_1 + \sigma_{1,t-1}^2 n_2};$$

    Assign values

      $\theta_{1,t-1} \to [k_{1,t-1}; c_{1,t-1}];$

      $X_t \to [1; p_{t-1}];$

    Recursive equations

      $\theta_{1,t} = \theta_{1,t-1} + t^{-1} S_t^{-1} X_t (p_t - \theta_{1,t-1}' X_t);$

      $S_t = S_{t-1} + t^{-1}(X_t X_t' - S_{t-1});$

      $\sigma_{1,t}^2 = \sigma_{1,t-1}^2 + t^{-1}((p_t - \theta_{1,t-1}' X_{t-1} + u_t)^2 - \sigma_{1,t-1}^2);$

    Updating the state variables

      $k_{1,t} = \theta_{1,t}(1);$

      $c_{1,t} = \theta_{1,t}(2);$

      $k_{1,t-1} = k_{1,t};$

      $c_{1,t-1} = c_{1,t};$

      $p_{t-1} = p_t;$

      $\sigma_{1,t-1}^2 = \sigma_{1,t}^2;$

  end

end

Appendix B
Proof of Recursive Least Squares Equations

Ljung and Söderström (1983) provide a detailed proof of recursive least squares equations. Here I show the derivation of equations (3.15) and (3.16). First the standard least squares equation gives

$$\theta_T = \left(\sum_{i=1}^{T} X_i X_i'\right)^{-1} \left(\sum_{i=1}^{T} X_i p_i\right)$$

where T is the total period of time. The estimate of θ_t given information available at time t is

$$\theta_t = \left(\sum_{i=1}^{t} X_i X_i'\right)^{-1} \left(\sum_{i=1}^{t} X_i p_i\right) \tag{B.1}$$

It can be rewritten to a recursive fashion. First denote

$$\begin{aligned}
\bar{S}_t &= \sum_{i=1}^{t} X_i X_i' \\
&= \sum_{i=1}^{t-1} X_i X_i' + X_t X_t' \\
\bar{S}_{t-1} &= \sum_{i=1}^{t-1} X_i X_i' \\
&= \bar{S}_t - X_t X_t'
\end{aligned} \tag{B.2}$$

Then from equation (B.1),

$$\theta_t = \bar{S}_t^{-1} \left(\sum_{i=1}^{t} X_i p_i\right) \tag{B.3}$$

$$\sum_{i=1}^{t} X_i p_i = \bar{S}_t \theta_t$$

Similarly,

$$\sum_{i=1}^{t-1} X_i p_i = \bar{S}_{t-1} \theta_{t-1} \tag{B.4}$$

From equation (B.3) and (B.4),

$$\begin{aligned}
\theta_t &= \bar{S}_t^{-1}\left(\sum_{i=1}^{t} X_i p_i\right) \\
&= \bar{S}_t^{-1}\left(\sum_{i=1}^{t-1} X_i p_i + X_t p_t\right) \\
&= \bar{S}_t^{-1}\left(\bar{S}_{t-1}\theta_{t-1} + X_{t-1} p_t\right) \\
&= \bar{S}_t^{-1}\left[\left(\bar{S}_t - X_t X_t'\right)\theta_{t-1} + X_t p_t\right] \\
&= \bar{S}_t^{-1}\left[\bar{S}_t\theta_{t-1} + X_t\left(-X_t'\theta_{t-1} + p_t\right)\right] \\
&= \theta_{t-1} + \bar{S}_t^{-1} X_t\left(p_t - X_t'\theta_{t-1}\right) \\
&= \theta_{t-1} + \bar{S}_t^{-1} X_t\left(p_t - \theta_{t-1}' X_t\right)
\end{aligned}$$

From equation (B.2),

$$\bar{S}_t = \bar{S}_{t-1} + X_t X_t'$$

Define

$$\begin{aligned}
S_t &= t^{-1}\bar{S}_t \\
S_{t-1} &= (t-1)^{-1}\bar{S}_{t-1}
\end{aligned}$$

From equation (B.2), equation (3.16) is derived.

$$\begin{aligned}
S_t &= t^{-1}\left(\bar{S}_{t-1} + X_t X_t'\right) \\
&= (t-1)t^{-1} S_{t-1} + t^{-1} X_t X_t' \\
&= S_{t-1} + t^{-1}(X_t X_t' - S_{t-1})
\end{aligned}$$

Combining the two equations, I obtain the system of recursive equations with decreasing gain $\gamma_t = t^{-1}$.

$$\begin{aligned}
\theta_t &= \theta_{t-1} + t^{-1} S_t^{-1} X_t\left(p_t - \theta_{t-1}' X_t\right) \\
S_t &= S_{t-1} + t^{-1}(X_t X_t' - S_{t-1})
\end{aligned}$$

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