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Strategy and Fundraising in Sequential Majoritarian Elections

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Declaration

I hereby declare that the content of this thesis is my own work and that, to the best of my knowledge, it contains no material published or written by any other author or authors, except where acknowledged. This thesis has not been submitted for award of any other degree or diploma at the University of New South Wales or any other educational institution.

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Adam Solomon

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Contents

Abstract

We study dynamic sequential majoritarian campaigns between two players. The resource being expended is determined exogenously to the campaign and has no scrap value. We demonstrate, in a fairly general environment, that the unique subgame perfect equilibrium is symmetric. Using a novel technique, we show this equilibrium carries over to the stage and simultaneous versions of the campaign. We endogenize the budget choice prior to the campaign to analyze when symmetric equilibria still exist. We find that players with symmetric fundraising capacity are advantageous for a campaign organizer who cares about resource expenditure and participation. We discuss implications for publicly versus privately funded election campaigns.

Chapter 1
Introduction

The vast majority of competition involves repeated interaction between opponents. Whether it be political parties competing in multiple electorates, sports teams over multiple games or sets, or companies across multiple products, strategic interactions almost never occur just once. These games are not just repeated, but also possess a regular structure: the winning of a larger war requires victory in some number of similar or identical constituent battles. This regularity motivates our interest - real interactions tend to display some pattern - and injects some order into our analysis.

Success in any of these competitions depends on both the resources devoted to winning, and a dose of luck. Since the parties care both about winning each constituent interaction, and about winning a larger campaign, they face a difficult allocation problem. They must choose how to allocate their stock of resources across these multiple interactions so as to maximize their chances of winning, without knowing how their opponent will do so. In this paper, we find a solution to this allocation problem in the natural setting when players care about winning a majority of interactions.

The original formulation of the dynamic, majoritarian game has its foundations in military strategy. As posed by Borel (1921), two Colonels are to meet on multiple battlefields simultaneously and must allocate their finite armies across battlefields without knowing how their counterpart will do so. Each Colonel aims to send more troops to a majority of battlefields than their opponent. But this belies an abstract structure: a stock of generic resource must be allocated across repeated contests, each henceforth called a 'battle'. The resource increases (perhaps to certainty) the probability of winning whichever battle it is allocated to. The formulation is sufficiently general to apply to a wide variety of settings, including economic ones.

Political campaigns are often majoritarian fights over multiple battles. For instance, in the United States, to win the nomination of a political party to compete in the general election for President, a candidate must win the majority of delegates, which are awarded during a sequence of battles ('primaries') held in each state. In legislative democracies, to form a government, a party must win a majority of the individual elections that are held in each district. In such campaigns we might think of each candidate/party as facing an allocation decision with regards to their campaign funds. The more they spend in each district on advertising, events and so on, the greater their expected chances of winning that district should be.

Professional sports are another instance of this general structure. Tennis matches are played as best-of-three or five sets. Players must choose how to allocate their stock of energy amongst these battles, some of which might never be played. The NBA finals series is a best-of-seven campaign between the two finalist teams. In such contexts, one might think of the coaches as needing to allocate the feasible playing time of their players amongst the series of games, so as to maximize the chances of ultimate victory without any player 'burning out'.

Further, R&D races to develop a new technology are naturally modeled (e.g. Fudenberg, Gilbert, Stiglitz, and Tirole (1983)) as a sequence of battles to discover and patent intermediate technologies. Here, the time of employees and research funds, are spent by firms to try and win the campaign.

Abstractly, campaigns may feature battles fought in sequence or simultaneously. Further, the resource that each player expends might be either: (i) limitless but with a marginal cost of use at any battle; or (ii) it might be finite and useless outside of the campaign. Assuming (i) leads to either overly conservative or aggressive deployment of the resource in various ways. Contrastingly, assumption (ii) implies that a player might as well spend all of their resource and simply faces a dynamic allocation problem. Lastly, the probability of winning any particular battle, given resource allocations to that battle, might be either deterministic (an all-pay auction) or stochastic (governed by a contest success function). Given the exhaustive study of the former and the realism of the latter, we restrict our attention to the stochastic case.

We can categorize the examples above via these three characteristics. Patent races feature money and time being expended that certainly have scrap value, while in sports matches, especially in the final stages of a competition, competitors do not really need to 'leave anything in the tank', so to speak, and thus should optimally expend all of their available energy. Political elections might be interpreted either way: money for political advertising could be redirected into other activities, but given the magnitude of the prize being fought for, we might think of any surplus funds as worthless compared to the benefits of winning an important election. Alternatively, as in some jurisdictions with public funding, or regulations on the use of private donations, funds are provisioned explicitly for the current election cycle and thus should optimally all be spent. All of the above have some degree of randomness, in as much as exerting just a bit more effort than an opponent certainly does not guarantee a win, although it does help.

Most of these formulations have been studied extensively, as section 1.2 indicates. However, a significant gap in the literature remains with regards to campaigns played in sequence with no scrap value. In this paper we characterize the equilibrium of such a game. This allows us to pin down the effect on sequential campaigns of overall budgets being chosen exogenously versus endogenously and dynamically throughout the game. We find stark qualitative differences.

1.1 Literature review

This paper connects to two main literatures. The first literature on Colonel Blotto games canonically features stocks of resource with scrap value. Players must 'use it or lose it'. The original integer formulation was given by Blackett (1954) in which two military commanders distribute their armies simultaneously across m battles. The payoff function for the ith battle is $P_i(x, y)$, which depends only on resources committed to each generally heterogeneous battlefield. An alternative formulation was given by Gross and Wagner (1950) in which armies are unequal, continuously divisible and each particular battle is an all-pay auction worth $1/m$ to the winner. For asymmetric budgets where the larger budget is no larger than m times the smaller, it is well known that there are no pure strategy equilibria. Roberson (2006) characterizes the unique mixed strategy equilibria for the general game with equally weighted battles. The equilibrium exhibits a symmetry of sorts, with equilibrium strategies drawn from bounded uniform distributions.

In the second literature on sequential campaigns, an asymmetry in effort allocation across battles is typically observed. Often, losing players are discouraged from expending resources when those resources have scrap value (e.g. Klumpp and Polborn (2006)). This is because the cost of expending effort to try to resurrect a failing campaign is often too high to justify the prize on offer. This is most pronounced when each battle is an all-pay auction, but is still present when each battle is decided by a stochastic contest-success function. Bevia and Corchon (2013) find that a discouragement effect results from assuming that the outcome of prior battles affects the player's future battle technology. In particular, that the form of the contest-success function changes throughout the campaign to favour the leader. Duffy and Matros (2015) show that battles worth unequal amounts of points naturally lead to an asymmetry of effort allocation.

A full characterization of the equilibria for equally weighted battles and resources with scrap value is offered by Klumpp and Polborn (2006). Building on this, Konrad and Kovenock (2009) add prizes for each battle, and fully characterize the equilibria when each battle is decided by an all-pay auction. There the effort expendable in each battle is unlimited but with scrap value, thereby removing the direct dependence of tomorrow's budget on today's effort. They find that a discouragement effect is still present due to the high anticipated expenditure in states close to parity. Fu, Lu, and Pan (2015) study a game where each battle is fought by a different member of the same team. They generalize the contest function and introduce the possibility of prizes for winning each battle. The logic there is most relevant to this paper, since different players play in each battle, the temporal linkage between costs is broken, and the players play their particular battle independently of it's place in the overall campaign sequence. Nevertheless, certain substantive assumptions on the contest success functions (homogeneity of degree zero in the player's efforts) are made in that paper, which we do not make here.

This paper combines the sequential structure of the second literature with the budget constraints of the first. By combining these features, we are able to disentangle the source of some of the qualitative equilibrium characteristics observed in both. The contribution of this paper is to demonstrate the existence and uniqueness of a symmetric equilibrium with no discouragement effect in a sequential campaign with no scrap value resource. Thus, we find, à la Fu et al. (2015), behavior that is independent of the feedback mechanism and temporal structure of battles. That is, under rather general conditions, we see an equal treatment of each battle, as if they were fought simultaneously.

Indeed, in a working paper, Konrad (2016) studies best-of-three contests in which two players compete sequentially until a player wins two battles. The winner of each battle is determined by the basic Tullock contest success function $x/(x+y)$. With the assumption of no scrap value, Konrad verifies that a symmetric equilibrium without a discouragement effect is present. In this paper we extend the game in two ways: first, the contest is generalized to best of $2m-1$ and, second, the technology functions that translate resources expended into relative probabilities of winning are generalized to any non-negative, increasing and concave functions. The qualitative results are shown to be robust to vastly different budgets and technologies available to each player.

Of most direct relevance is the parallel work by Klumpp and Konrad (2018), who independently derive propositions 1 and 8. Their results cover a slightly larger class of functional forms for the contest-success function. On the other hand, in this paper, we focus on the intuition and implications of the main result, proposition 1. Via a novel technique, 'upward-induction', the result for the sequential game is extended to analogous existence and uniqueness results for the stage and simultaneous versions of this dynamic campaign.

This paper proceeds as follows. Section 2 describes the game. Section 3 provides intuition and characterizes the equilibria. Section 4 endogenizes the budget choice prior to the campaign. Section 5 offers some extensions, applications, avenues for future research, and concludes.

CHAPTER 2
Statement of the problem

Two players compete in a campaign that is decided by the best of $2m-1$ battles. That is, the campaign ends when one player wins m of those battles. Each player begins with a given (generally unequal) stock of resources that has no scrap value beyond the campaign in question. As such, an allocation of resources that leaves some of the budget unallocated is strictly dominated by exhausting all available resources.[1]

Given this assumption of no scrap value, the size of the prize being competed for is irrelevant, as all resources will be expended (in expectation). Therefore, players allocate resources across battles exactly to maximize their probability of winning the contest. To be more precise: two players, 1 and 2, have initial stocks of resource $b_1 > 0$ and $b_2 > 0$ respectively. In each battle, the probabilities of players 1 and 2 winning is respectively given by:

$$p_1(x,y) = \frac{f(x)}{f(x)+g(y)}, \text{ and } p_2(x,y) = 1 - p_1(x,y) = \frac{g(y)}{f(x)+g(y)}.$$

We require that f (respectively, g) is continuous, twice continuously differentiable, weakly increasing and weakly concave in x (y). We refer to the functions f and g as the 'battle technology' functions for players 1 and 2 respectively. These assumptions are sufficient for p_1 (p_2) to be continuous, twice continuously differentiable, increasing in x (y), decreasing in y (x) and concave in x (y). This nests, as special cases, many of the widely studied Tullock contest success functions.

It is helpful to visualize this first to m campaign as a grid. In figure 1, following Konrad and Kovenock (2009) and Konrad (2016), let the node (i,j) denote when the campaign is at the sub-battle prior to which player one requires i, and player two j, more battle wins to prevail in the overall campaign.

[1] To be clear, the stock of resources will generally not reach zero by the time the contest ends. This is because even when one player is very close to winning, resources should still be allocated, in precaution, to all the battles that might yet be played. So, more precisely, rational players must allocate all their resources in expectation of all the potential battles.

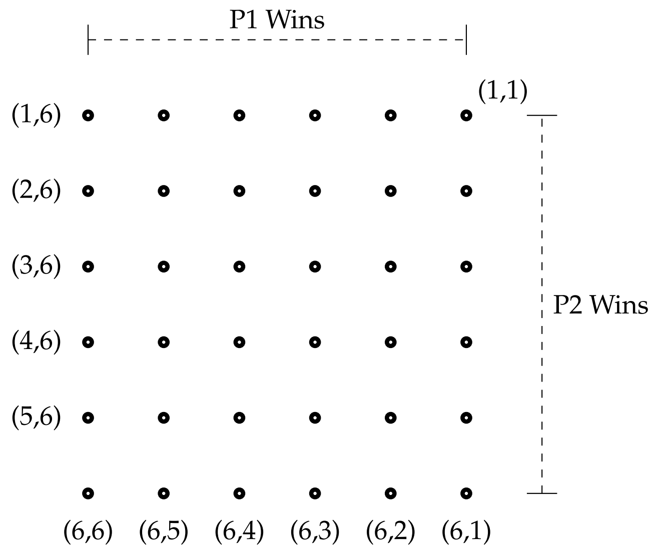

Figure 2.1: Visualization of the best-of-11 campaign

Each node represents the battle that is to take place there. From (i, j), when player one wins a battle the contest moves up one row in the grid to $(i - 1, j)$ and when player two wins a battle it moves one column to the right to $(i, j - 1)$.

At (i, j) we say that there are $n = i + j - 1 \leq m$ potential future battles toward which each player needs to allocate resources. In complete generality, a strategy for a player would specify, for each node and for each possible history that lead to that node, the proportion of that players remaining budget that gets spent at that node. The campaign ends when a node of the form $(i, 0)$ or $(0, j)$ is reached, where $i, j > 0$.

Even without the path dependence a strategy at (i, j) is an $i \times j$ matrix, subject to many path-dependent budget constraints, that specifies the portion of remaining budget to spend at each future battle.

CHAPTER 3
Equilibrium

Definition 1. *Given the contest is at the node (i, j), with $n = i + j - 1$ future battles to account for, the* ***symmetric strategy*** *means a player allocates $1/n$ of their current budget to each of the future (potential) battles.*

We note here that this departs from the usual definition of symmetry as it pertains to equilibrium strategies. Typically, a symmetric equilibrium means that players act identically within a given (potentially asymmetric) environment. Here, players act differently since they have different budgets, but each player plays symmetrically over each battle. It is this characteristic that we focus on: symmetric play across battles but not necessarily between players.
Given this definition, we can see that at the node (i, j), if both players are playing symmetrically, the probability that player 1 wins the campaign is given by

$$W_1(b_1, b_2) = \sum_{k=0}^{j-1} \left[\binom{i+k-1}{k} \left(\frac{f(b_1)}{f(b_1) + g(b_2)} \right)^i \left(\frac{g(b_2)}{f(b_1) + g(b_2)} \right)^k \right] \tag{3.1}$$

and similarly for player 2. We are now able to state the main result of this paper:

Proposition 1. *Each player playing the symmetric strategy is a subgame-perfect Nash equilibrium. Moreover, this is the unique subgame-perfect equilibrium.*

The remainder of the section will build the intuition behind this result. The formal proofs are relegated to the appendix.

3.1 NUMERICAL EXAMPLES

The temptation to play asymmetrically by over-exerting effort either today or in the future is intuitively most alluring for player 1 at a node of the form $(1, n)$, where player 1 needs to win only one of many battles to prevail in the campaign. The choice is between a concentration of resources at a small number of battles, or a symmetric strategy in the expectation that the chances of losing every single battle is low.

Let us now consider a campaign that arrives at $(1, 4)$. Suppose the battle technology function is the linear $f = g = id$ considered by Konrad (2016).

First, suppose each player has an equal budget $b_1 = b_2 = 4$ and player 2 is playing symmetrically. If player 1 plays symmetrically their chance of winning is $15/16 = 0.9375$. If they allocate all of their resource to today's battle, their chance of winning is $4/5$. If they abandon today's battle, and play symmetrically across the next three, their chance of winning is $316/343 \approx 0.92$.

When the budgets are asymmetrical: $b_1 = 4, b_2 = 40$, these probabilities respectively are 0.32 and 0.29. When the budgets are asymmetrically $b_1 = 40, b_2 = 4$, these probabilities respectively are 0.99 and 0.98. When we revert to equal budgets but change player 1's battle technology function to $f(x) = \log(x+1)$ with greater decreasing returns to scale, the probabilities become 0.88 and 0.62 respectively.[2]

Thus, it seems plausible that (robust to asymmetries in contest functions and in budget) the best response to an opponent playing symmetrically is to play symmetrically. Notice, though, that the assumption of concave technology functions is key. Without this assumption of decreasing returns to effort at each battle, it will certainly no longer be optimal in general to spread one's resources evenly.

3.2 First-order condition intuition

To simplify the exposition, consider the following diagram, which generically represents the paths to victory for player 1, accounting for the different possible outcomes of the next two battles.

The result in Proposition 1 - that the symmetric strategies form an equilibrium in the sub-campaign - implies two things. First, resources that are available when the sub-campaign begins will be split symmetrically. Second, in terms of first order conditions, this means that to an affine approximation near the symmetric strategy, reallocating a "small" amount of resource between future battles does not change the probability of winning the overall campaign.

Suppose we are at the node (i, j) and there are $n+1$ battles remaining. Now, for induction, suppose that both players are playing the equilibrium symmetric strategies from the next battle onwards. It remains to check that, at the current

[2]Note, this conclusion is sensitive to the assumption of concavity in the effort functions. Consider for example when $f(x) = e^x - 1 = g(x)$ with equal budgets. Then the probability of winning if player 1 plays symmetrically is $15/16 = 0.9375$, but if they divert all resources to the current battle this probability increases to approximately 0.97. Thus, with increasing marginal returns to effort, it is natural to want to concentrate effort at a minimum number of battles, which here means at just one.

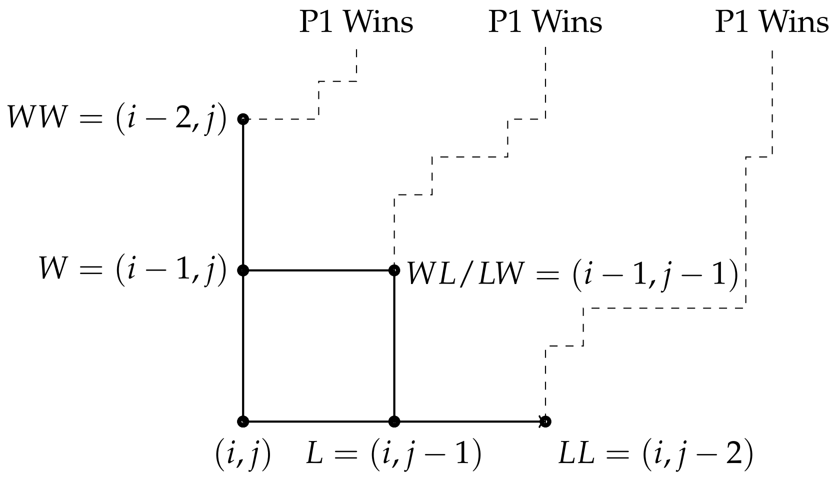

Figure 3.1: Generic paths to victory for player 1

battle, both players should mutually best respond to each other's symmetric strategy.

Thus, suppose player 2 is also playing the symmetric strategy $y = y' = b_2/(n+1)$ at the current battle. Define $x' = b_1/(n+1)$ similarly. Given player 2's effort choice is fixed, we can write $p(x) = f(x)/(f(x) + g(y'))$. By these suppositions, Player 1's choice is now to choose some δ such that at (i, j) they play $x' + n\delta$ of their budget, and then the remaining $x' - \delta$ at each future battle. For notational ease, label $\Pi(i-1, j)$ as $\pi(W)$, $\Pi(i-2, j)$ as $\pi(WW)$, $\Pi(i-1, j-1)$ as $\pi(WL)$ or LW and so on. Then, as a function of this choice of δ, player 1's chance of winning the campaign is

$$\begin{aligned}
\Pi(i, j, \delta) = &\, p(x' + \delta n)p(x' - \delta)\Pi(WW) \\
&+ p(x' + \delta n)(1 - p(x' - \delta))\Pi(WL) \\
&+ (1 - p(x' + \delta n))p(x' - \delta)\Pi(LW) \\
&+ (1 - p(x' + \delta n))(1 - p(x' - \delta))\Pi(LL).
\end{aligned}$$

Now, to examine the first-order behavior, suppose some small deviations are made so that player 1 is deploying $x' + \delta n + \epsilon$ units of resource today, and $x' - \delta - \epsilon/n$ from the next battle onwards. But, interpreting the equilibrium assumptions at $(i, j-1)$ and $(i-1, j)$ in the second way (that is, in terms of the first order conditions), we can amalgamate all of the $-\epsilon/n$ declines into one $-\epsilon$ decline at each of $(i, j-1)$ and $(i-1, j)$ with no change in the win probability to an affine approximation. So, after this ϵ increase at (i, j) and ϵ decrease at the next battle, the win probability for player 1 is now given by

$$\begin{aligned}\overline{\Pi}(i,j,\delta) &= p(x'+\delta n+\epsilon)p(x'-\delta-\epsilon)\Pi(WW) \\ &+ p(x'+\delta n+\epsilon)(1-p(x'-\delta-\epsilon))\Pi(WL) \\ &+ (1-p(x'+\delta n+\epsilon))p(x'-\delta-\epsilon)\Pi(LW) \\ &+ (1-p(x'+\delta n+\epsilon))(1-p(x'-\delta-\epsilon))\Pi(LL).\end{aligned}$$

Taking, by Taylor's Theorem, the best affine approximation, $v(x'+\delta n+\epsilon) = v(x'+\delta n)+\epsilon v'(x'+\delta n)$ and similarly for $v(x'-\delta-\epsilon)$, we can write this new win probability in terms of the original win probability:

$$\begin{aligned}\overline{\Pi}(i,j,\delta) &= \Pi(i,j,\delta) \\ &+ \epsilon\left[p'(x'+\delta n)\left(p(x'-\delta)(\Pi(WW)-\Pi(WL)) - (1-p(x'+\delta n))(\Pi(LW)-\Pi(LL))\right)\right] \\ &- \epsilon\left[p'(x'-\delta)\left(p(x'+\delta n)(\Pi(WW)-\Pi(WL)) - (1-p(x'-\delta))(\Pi(LW)-\Pi(LL))\right)\right] \\ &+ \mathcal{O}(\epsilon^2).\end{aligned}$$

Intuitively, this says that "overplaying" today makes it more likely you will win today, but less likely you will win tomorrow once you arrive at tomorrow's battle.

In general, it is hard to know which effect dominates. But, we can see that at the symmetric equilibrium, i.e. where $\delta = 0$, these effects exactly balance out. That is, the increased chance of getting to $(i-1, j)$ and decreased chance of winning once there balance out, and similarly at $(i, j-1)$. This demonstrates that when player 2 is playing symmetrically, player 1's best response function has a local turning point at the symmetric strategy.

3.3 Formal equilibrium analysis

We begin with the case where the campaign finds itself at a node of the form $(1, n)$ or $(n, 1)$. That is, where some player needs to win just one more battle to win the campaign, and the other must win $n \geq 2$. The case at $(1, 1)$ is immediate and vacuously symmetric. Lemmata 1 and 2 in the Appendix combine to give the following:

Proposition 2. *At all sub-campaigns starting at a node of the form $(1, n)$ or $(n, 1)$, the symmetric strategies form a subgame-perfect Nash equilibrium.*

We must now extend the induction to cover any node (i, j). Proposition 2 above becomes the one-dimensional inductive assumption. And so, we assume that i and j are no smaller than 2. Recall that the number of battles to budget for is labeled $n = i + j - 1 \geq 3$. Some combinatorics reveals the following:

Proposition 3. *Suppose at all subgames in the future the symmetric strategies form an equilibrium. Suppose further that player 2 is playing the symmetric strategy today, at (i, j). For player 1, the chance of winning the campaign at (i, j) is maximized by responding with the symmetric strategy.*

Now we can prove the main result of the paper.
Proof of Proposition 1.

Proof. It is clear that at $(1, 1)$ the players should spend all of their budget (which is trivially symmetric over the 1 battle remaining). Suppose we are at $(1, n)$ and player 2 is playing the symmetric strategy and, further, that it is an equilibrium to play the symmetric strategy at all nodes from $(1, n - 1)$ onwards. Proposition 2 establishes that, if the other player is playing symmetrically from the next battle onwards, each player should split their budget symmetrically at $(1, n)$ or $(n, 1)$ for any $n > 1$. By induction this shows that the symmetric strategies form a subgame-perfect Nash equilibrium at all battles starting at $(1, n)$ and $(n, 1)$ for all integers $n \geq 1$.

This forms the base case for our general induction. Suppose that the battle is at arbitrary (i, j) with $i, j \geq 2$ and the equilibrium at all sub-games in the future is to play the symmetric strategy. Proposition 3 demonstrates that if player 2 plays the symmetric strategy today the best best response for player 1 is to also play the symmetric strategy. But clearly players 1 and 2 are interchangeable in this context. Hence, the symmetric strategy is mutual unique best response. Thus, by induction the claim of existence is proven for a campaign of arbitrary length and starting point.

For, uniqueness, following Osborne and Rubinstein (1994) we say a finite player, finite stage extensive game satisfies the *no indifference condition* if, whenever one player is indifferent between two terminal histories, all other players are as well. Trivially, this constant sum two-player game satisfies the no indifference condition. By a proposition of Osborne and Rubinstein (1994), the subgame perfect equilibria are interchangeable. That is, whenever (s_1, s_2) and (s_1', s_2') are subgame-perfect equilibria then so is (s_1, s_2') and (s_1', s_2). We argued above that the symmetric strategies were unique best responses. Uniqueness of the equilibrium immediately follows, since if there were some other equilibrium, we would interchange one of those equilibrium strategies with the symmetric strategy implying that the symmetric strategy was not the unique best response, a contradiction. □

3.4 Discussion

This equilibrium is, in some dimensions, quite robust to large perturbations of the underlying parameters of the game, whilst in other dimensions it is a peculiar corner solution. The existence and uniqueness of the symmetric equilibrium carries through even when the players are wholly asymmetric in terms of their budgets and contest technologies. Moreover, the equilibrium strategies are not sensitive to the knowledge of the other player's technological or budgetary capacities. Thus, uncertainty and asymmetry (and uncertainty about the asymmetry) about the other player's capacities do not at all affect the equilibrium of the game. Section 3.5 explores this further.

Instead what drives the structure of the equilibrium is complete symmetry between battles. To be more precise, the equilibrium requires equivalence in the payoffs to the winner and loser across all terminal histories, and equivalence in the win function at any particular battle. Symmetric battles are only a necessary condition for this. We will see in Chapter 5 that any change, no matter how small, to this payoff equivalence across paths to victory or symmetry between battles will move us away from a symmetric equilibrium.

Finally, it is worth reinforcing the key qualitative difference between this symmetric equilibrium and the generally asymmetric equilibria observed in other configurations of these multi-stage dynamic campaigns. As Konrad (2012) points out, the outside value of the resource being expended discourages resource expenditure once one starts losing, and correspondingly encourages huge expenditure at even or close to even points in the campaign because of the sharp downside to losing. This feedback loop continues, further discouraging expenditure by the losing player, given they anticipate how bitterly the future battles will be fought in the best case that they win and restore the campaign to parity. Thus, part of the cause of the discouragement effect is the scrap value of the resource. However, a priori, it seems plausible that the fear of finding oneself in a losing state might motivate a similar overspending on the earlier games when the campaign is closer to parity. As such, we might expect a forward looking discouragement effect to arise. Indeed, the existing literature has been unable to disentangle these effects.

This paper, which disregards the assumption of scrap value, still has the potential for this forward looking discouragement effect. We do not find one, confirming that it is the scrap value of the resource, rather than the fear of low continuation values more generally, that drives the discouragement effect. Moreover, asymmetries in the players and their technologies are not sufficient to cause a discouragement effect.

3.5 Upward Induction to the Simultaneous and Stage Campaigns

The structure of this unique equilibrium is highly surprising. In particular, in equilibrium each player entirely ignores the sequentiality of the game in as much as the results of past battles do not inform future strategy. Additionally, the resource stock and technological strength of a player do not have any impact on how their opponent plays in equilibrium.

Formally, in the sequential game we have been considering, we can think of the state variables at battle (i, j) as consisting of $(i, j, n, b_1, b_2, f, g)$. Thus, a priori, we might guess that the equilibrium strategies, in particular the best response functions, are functions that involve all of these state variables. In equilibrium, however, this turns out not to be the case. The equilibrium strategy of player i depends only on n and b_i, and disregards the other state variables.

If we were to define a game identical to the one above except that the state variables were reduced, we would naturally expect the set of equilibria of this game with less information to coincide with the original game. This is intuitive: since players choose to ignore some information, when best responding, if we changed the game to hide that information unavailable from them, their behaviour should remain unchanged. Indeed, the fact that (b_{-i}, f, g) do not appear in the equilibrium immediately implies the first part of the following proposition. The fact that (i, j) do not appear in the equilibrium implies the second part. The result and terminology run parallel to Fu et al. (2015).

Proposition 4. *The existence of the symmetric equilibrium is:*

- *Technology and budget independent: full or partial uncertainty about the technology or budget of a player or their opponent does not affect equilibrium strategies or ex-ante expected payoffs;*
- *Temporal-structure independent: whether the campaign is sequential, simultaneous, or in between does not affect equilibrium strategies or ex-ante expected payoffs.*

First define the *stage campaign* corresponding to the sequential game above as a best of $2m - 1$ campaign in which the result of each intermediate battle is not revealed until all $2m - 1$ games have been played. We can think of this game as generalizing the sequential game, in that player i has state variables of (n, b_i, b_{-i}, f, g), a strict subset of those in the sequential game.

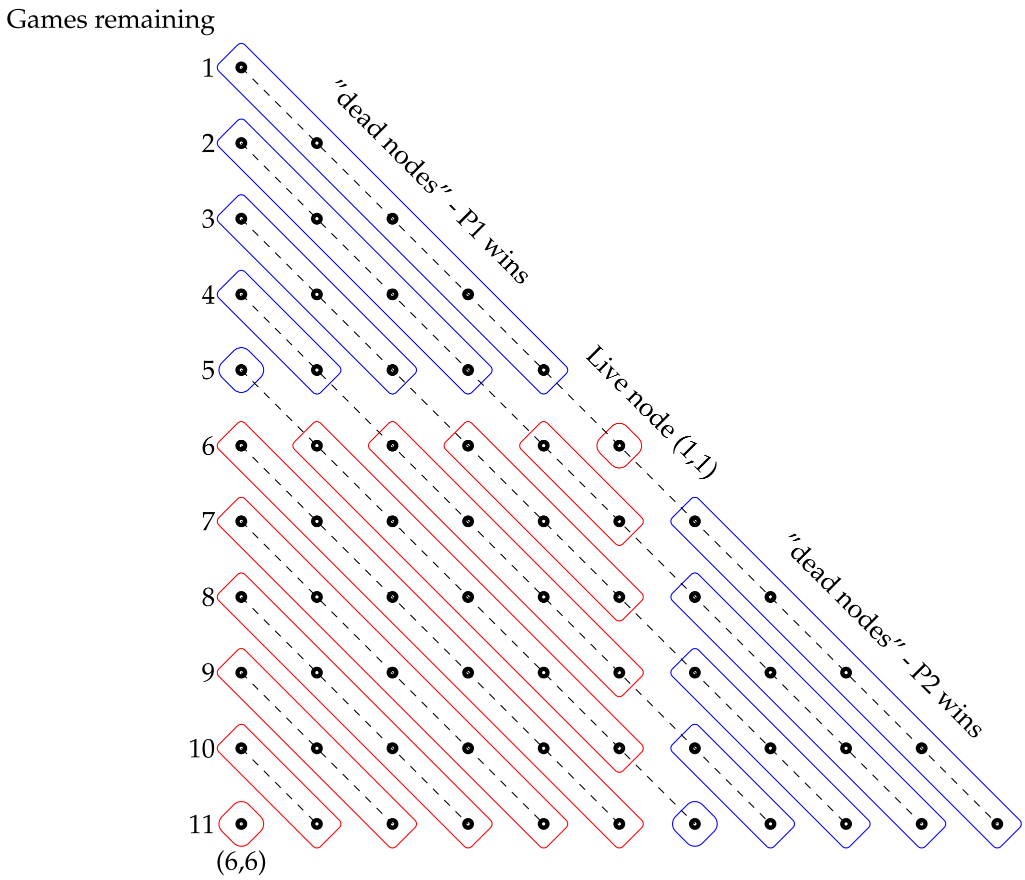

Figure 3.2: The 6 game stage campaign

Similarly, define the *simultaneous campaign*, as the best of $2m - 1$ campaign in which the results of the intermediate battles are not revealed until the end, and each player does not know their opponent's budget choice at each stage until the very end. This further generalizes the sequential game, since the state variables for player i are just (n, b_i, f, g). We then have the following as consequences of proposition 4 and some further argument in the appendix:

Corollary 1. *The symmetric strategies constitute the unique equilibrium of both the stage campaign and the simultaneous campaign.*

This existence and uniqueness result for the stage campaign is, in spirit, a generalization of the proof strategy of Klumpp and Konrad (2018). Most interesting is the result for simultaneous games, which generalises J. Clark and Konrad (2007), Friedman (1958) and Robson (2005). Since the $2m - 1$ games are ex ante symmetric in the simultaneous campaign, in one sense the existence of a symmetric equilibrium seems more plausible than in the sequential game. But in terms of computation, the search for an equilibrium of a simultaneous game is of much higher dimensionality than in a corresponding sequential game. Thus, the approach here, in which the equilibrium of the less general sequential game (with a 5 element state space) is lifted to the more general simultaneous game is novel. This general idea, in which structure is added to games until an equilibrium can be computed, and if we are lucky that equilibrium turns out to be independent of the structure just added and so can be *upwards-inducted* into the original general game, is potentially a very interesting avenue for future research.

CHAPTER 4
The fundraising of endogenous budgets

Until this point we have assumed that each player arrives at the beginning of the campaign with their budget exogenously given. Moreover, since the budget has no scrap value, the value of the prize that comes from winning the campaign is irrelevant, since each player simply plays to maximize their probability of winning given a budget with no scrap value.

However, given the closed form for the unique equilibrium described above, we can ask how players might choose their budgets prior to the campaign. To isolate the effects of fundraising capacity, we restrict our attention to symmetric battle technology functions $f = g$. We define the following two-stage game:

1. Two players raise their budgets b_i at costs $c(b_i)$ where $c(\cdot)$ is continuous, differentiable, convex and weakly increasing.[3]

2. The players compete in the first-to-m campaign described above for a prize normalized to 1.

Then a straightforward adaption of Lemma 4 gives the following:

Proposition 5. *The pure strategy subgame-perfect equilibrium to the two stage game, if it exists, features symmetric budget choice $b_1^* = b_2^*$ and symmetric play in the second stage.*

For example, if we put $c(b_i) = b_i$ and $f = g = id$ we recover the following form for the (potential) pure strategy Nash equilibrium:

$$b_1^* = b_2^* = \frac{m}{4^m} \binom{2m-1}{m}. \tag{4.1}$$

Note that when $m = 1$ we recover $b_1^* = b_2^* = 1/4$ which is a standard result for a Tullock contest. When $m = 2$ we recover $b_1^* = b_2^* = 3/8$ which confirms what Konrad (2016) calculated.

[3]This increasing cost of resource plausibly reflects the situation when a firm or organization must raise funds, or when individuals or teams must exert effort, or when political campaigns must elicit donations.

However, when m is sufficiently large these first order conditions will not produce an equilibrium. In particular, when $m \geq 13$ the solutions to the best response functions imply an effort of greater than unity, the prize value. This is clearly dominated by a deviation to a budget of zero, i.e. non-participation. Intuitively then, we have the following sufficient condition, valid for all symmetric $f = g$.

Proposition 6. *The symmetric budget choices $b_1^* = b_2^* = b^*$ from Proposition 5 form an equilibrium if and only if $c(b) \leq 1/2$. Otherwise there is no equilibrium in pure strategies.*

Thus we see that the game cannot be too expensive for the players to enter, lest they not participate at all. But, for any cost function that a pair of players has, there exists some m' such that, for all $m \geq m'$ there is no equilibrium. Or, put differently, all cost functions can be weakly ordered by the corresponding m'. This provides an ordering of cost functions corresponding to the level of efficiency with which the players can raise funds.

4.1 Asymmetric cost functions

Now, if we relax the assumption of a common cost function, we cannot in general determine the existence or uniqueness of the pure strategy equilibrium. However, we can study the asymptotic behavior, as m gets large, of these equilibria (b_1^*, b_2^*), should they exist. Such asymptotic behaviour of the equilibria exhibits qualitative differences between the cases of symmetric versus assymetric cost functions. Suppose, for tractability, that we have $c_i(b_i) = \gamma_i b_i$ for $i = 1, 2$, and $f = g = id$. We have the following:

Proposition 7. *If the cost functions are symmetric then the best response budget choices b_i^* tend to infinity for each $i = 1, 2$ as $m \to \infty$. Conversely, if the cost functions are asymmetric, then the best response budget choices b_i^* tend to zero for each $i = 1, 2$ as $m \to \infty$.*

Thus, the asymmetry has a qualitatively different effect on the dynamics of the equilibria: here an asymmetry in budgets is more greatly amplified in a longer match (where a 'law of large numbers' means any small budgetary advantage becomes more and more decisive).[4] Anticipating this, the player with the lower budget (the inefficient fundraiser) realizes a higher (but still lower than the other player's) budget is less likely to win the prize, and so drops their budget. In best response, so does the efficient fundraiser.

[4] This follows from proposition 8.

Thus, while for small m games with both symmetric and asymmetric costs will have unique pure strategy equilibria, for large m this will not be true in either case. In the former case, because a player will eventually choose to abandon the contest. Once that player chooses $b_i = 0$, the other player would like to use a budget of just tiny ϵ, which naturally leads into the dynamics of an all-pay auction. In the latter case, as best responses tend to zero, even a weaker player might eventually find it optimal to choose a budget just higher then their opponent for a decisive advantage. Such dynamics are also reminiscent of an all-pay auction, which we formalize in following subsection.

4.2 Asymptotic contest structure

We have seen that no matter whether the cost functions are symmetric or asymmetric, as the contest gets longer, i.e. as $m \to \infty$, pure strategy equilibria cease to exist, albeit for slightly different reasons. But in both cases, putting $f = g = id$, the dynamics give way to the following.

Proposition 8. *The campaign win function for player 1, considered as a sequence of functions in m, given by*

$$\{W_1(b_1, b_2)\}_m = \sum_{k=0}^{m-1} \left[\binom{m+k-1}{k} \left(\frac{b_1}{b_1 + b_2} \right)^m \left(\frac{b_2}{b_1 + b_2} \right)^k \right] \tag{4.2}$$

converges pointwise to the payoff function for an all pay auction:

$$W_1(b_1, b_2) = \begin{cases} 1 & \text{if } b_1 > b_2, \\ 1/2 & \text{if } b_1 = b_2, \\ 0 & \text{if } b_1 < b_2, \end{cases}$$

and similarly for player 2.

Now, we can discuss the equilibria of this asymptotic game. It is well known that there are no pure strategy equilibria for either symmetric or asymmetric cost structures.

When costs are symmetric, this is an all-pay auction with complete information. The equilibria were characterized by Baye, Kovenock, and de Vries (1996), and more generally in Siegel (2007). Note here that the effective 'valuation' of the campaign (i.e. the highest either player would plausibly bid) is given by the $v = b'$ where b' solves $c(b') = 1$. From this we can characterize the unique equilibrium of this all-pay auction. If G is the CDF of the distribution from which the mixed strategy is drawn, then we have that

$$G(b) = c(b) \text{ for all } b \in [0, v].$$

Conversely, suppose the cost functions are asymmetric, and say player 1, the efficient fundraiser, has $v_1 > v_2$ as defined above. Following the aforementioned paper, Siegel (2007), there is a unique equilibrium in mixed strategies, player 1 expects to receive utility of $1 - c_1(v_2) = c_1(v_1) - c_1(v_2)$. Further, player 2 expects to earn a profit of zero. Player 1 plays the atom-less mixed strategy $G_1(b) = c_2(b)$ for all $b \in [0, v_2]$ and player two plays the mixed strategy with atom at zero with weight $c_1(v_1) - c_1(v_2)$ and continuously over $[0, v_2]$ drawing from $G_2(b_2) = c_1(b_2)$.

Thus we have the following:

Corollary 2. *The sum of expected bids, and the probability of participation in the campaign, are higher under symmetric cost functions than asymmetric cost functions.*

That is, given a prize of the same size, more is expended in expectation in a more symmetric campaign. This has a natural interpretation, in that candidates of comparable ability will each think themselves competitive in both the fundraising and actual campaign. This stands in contrast to a campaign in which one player has an advantage, in which case both players compete less vigorously under the credible threat that the more effective campaigner could overwhelm their less effective opponent if needed.

CHAPTER 5
Discussion and conclusion

In this paper we have examined a class of campaign in which battles are fought sequentially and success in each particular battle is stochastic, but increasing in a player's effort level. Within the campaign, we have demonstrated that, robust to quite general asymmetries in the two players, the unique equilibrium is symmetric. That is, no matter the technological or budgetary differences between the two players (i.e. for general f, g) the best response to a symmetric strategy is a symmetric strategy. This is strong evidence for the fact that any potentially observable asymmetries in effort allocation (such as the discouragement effect) are due to asymmetries in the game structure, not in the players. In this section we discuss some applications and future research directions.

5.1 POLITICAL CAMPAIGNS

Perhaps the most important application is to elections fought over multiple districts, or states. Alternatively, one may think of each of the undecided voters as a battleground, some number of whom need to be convinced to vote for a candidate for that candidate to prevail. Whilst all voters in a district will generally vote at the same time, we can think of the times at which they declare (to themselves mentally, or perhaps to a pollster publicly) that they intend to vote for a particular candidate. Certainly, regular polling is conducted so that advertising money can be directed to the regions, populations and markets in which there are actually undecided voters left. Moreover, Kalla and Broockman (2017) show that campaign spending is indeed only useful well before election day, which compounds the evidence that modeling this game as a sequential competition (the sequence is of 'decision' dates) is appropriate.

However, one notes that the order of decision dates is now not exogenously given, but determined by the voter themselves. Nevertheless, the equilibrium we have found in this paper is notable for being history and order independent. Thus, treating each voter as equal, we can see that for any potential order of battles, a symmetric equilibrium exists. This is strong evidence that even for some random order of voters, the symmetric equilibrium exists. This assumes that each voter is 'equally' persuadable, which, in the language of this paper, means that the technologies f and g are the same for each battle. This is the first extension we highlight:

Battle dependent technologies One might expect that the efficiency with which a player's effort is translated into the probability of winning differs across battlefields. For example, in the case where some political districts 'lean right' or 'lean left', we would expect some minimal amount of effort exerted by a right-wing candidate to translate into a very different probability of winning than if effort were exerted in a left-leaning one. Or, in the case of sports campaigns, each battle may feature a different team having 'home-field' advantage. Both of those examples feature differences between battles that are exogenous. More generally, we might have 'momentum' effects, in which the result of previous battles affects the technologies at future battles. One might hope for some sort of split symmetric equilibrium in the exogenous case (ie treat all 'home' games symmetrically, and all 'away' games symmetrically). But the problem with endogenously determined technologies / contest success functions is far more difficult. A particular, very special case, analyzed by Bevia and Corchon (2013), demonstrates the existence of momentum effects, and thus of stark asymmetries in effort allocations.

In a similar vein, one might wish to incorporate 'safe' districts. If this simply means that s_1 districts will certainly accrue to player 1, and s_2 districts will certainly accrue to player 2, regardless of effort exerted players are really playing a campaign from $(m - s_1, m - s_2)$ instead of (m, m). Given this interpretation, the results of this paper are still applicable. If, however, no districts are truly safe, and the difference is just in the technology functions, then if these are idiosyncratic across battles we encounter the situation described in the preceding paragraph.

Unequally weighted battles In some applications - notably US elections where the electoral college, or primary campaigns, entail winning States - the candidates compete to win a majority of points awarded from unequally weighted battles. To see the difficulties involved, consider the following campaign.

Each candidate needs m points to win. Today's battle is worth $m - 1$ points, and all m (potential) battles thereafter are worth 1 point. From the analysis in this paper, assume each player is playing the symmetric strategies from tomorrow onwards, that is from $(1, m)$ or $(m, 1)$ onwards. Suppose $f = g = id$ and $b_1 = b_2 = 1$. The following table demonstrates, for m from 1 to 10, the unique solution to the first order conditions, as well as whether they constitute an equilibrium (i.e. if the second order conditions are satisfied).

Length of contest m	3	4	5	6	7	8
Pure strategy solutions to FOC	1/2	7/11	3/4	31/37	9/10	127/135
Equilibrium?	Yes	Yes	Yes	No	No	No

For example, when $m = 8$, supposing player 2 is playing 127/135 today and 1/135 at the future 8 battles, the chance of winning if player 1 mimics this is 1/2, but if player 1 deviates to playing ~ 0.188 today, the chance of winning rises to ~ 0.641. Indeed, for large enough m, the best response to such a strategy is actually to completely abandon the big game today. Thus the equilibrium will have to be in mixed strategies. The techniques in this paper are inadequate to address this question, a very interesting one for future research.

Implications for publicly funded elections The results of this paper reveal a stark asymmetry between campaigns fought by candidates who must raise money for themselves, and campaigns where each candidate is given campaign funds by the central government (determined exogenously, perhaps, by the results of a prior election). The results of section 4 demonstrate that in the former case, where $c_i = 0$ for both players, a symmetric equilibrium will result. Or, one might envisage a model where funds can be borrowed by each candidate at a constant (presumably) low rate, thus ensuring $c_1' = c_2'$. Conversely, if the players need to raise funds themselves, we would generically expect $c_1 \neq c_2$, and thus we would expect an asymmetric equilibrium of the form described.

This has two implications for the election designer. First, private funding leads, for small or large m, to a lesser expenditure of effort by the players owing to the asymmetry. Second, private funding, in long campaigns, features a non-zero chance of non-participation by the weaker fundraiser. From the point of view of a social planner who values electoral competition, both of these are sub-optimal.

5.2 Sports

Many sporting competitions fit into the structure of this model. For example, tennis matches are played as best of three or five sets, each of which is first to six games (with a tiebreak rule). This contest within a contest is within the scope of this analysis. Given a budget of energy, if players are playing the symmetric equilibrium within a set, then there clearly exists an equilibrium where they play symmetrically between sets. Thus, if indeed the resource being expended - energy (more exactly, energy on the day of the match) - has no scrap value, then this model would be appropriate. Certainly the outcome is stochastic in effort exerted, and in the case that the match being played is the final of some major tournament (where one does not need to play again in the imminent future) then the assumption of no scrap value seems reasonable.

However, one might argue that psychological effects from leading create an asymmetry in the technologies. As such, extensions identical to those suggested above might be preferable.

Intermediate prizes From a contest design point of view, if the organizer is aiming to maximize the effort expended by contestants, under general conditions it is optimal for all prize money to be allocated only to the ultimate winner of the campaign.[5]

Nevertheless, in many competitions prizes are awarded for prevailing in intermediate battles as well as in the overall campaign. In some setups, our analysis still applies. For example, suppose that there is a prize that accrues to the winner of each stage battle only if they turn out to be the overall campaign winner. Or, if the prize awarded for the overall campaign is much larger than the intermediate battle prizes. An example of this might be an election between two individuals, who both care lexicographically about winning the election. Conditional on winning, they care about the margin of victory in the sense that this is viewed as a larger 'governing mandate'. Or, in sports, think of a basketball team who cares first about winning the best of seven final series, and secondarily about doing so with the highest margin of victory. In this situation, it is immediate that the symmetric equilibrium still exists, since players are still just playing to maximize their chances of victory.

Conversely, if players care about winning the campaign, and conditional on doing so, wish to minimize the number of losses, symmetry breaks down, with players overplaying today as they now value paths to victory that feature fewer losses, as expected.

Generally, suppose that the winner of each battle wins an intermediate prize p regardless of the outcome of the overall campaign. Normalize the prize for winning the overall campaign to be 1. Take $b_1 = b_2 = 1$. Consider the following campaign beginning at $(1, 2)$ with $f = g = id$. Either player 1 wins the campaign

[5] See, for example, Fu and Lu (2009) and Fu and Lu (2012).

from $(1, 2)$ or from $(1, 1)$, in which case their payoff is $1 + p$. Player 2 can either win both battles for a payoff of $1 + 2p$, or win today's battle and lose the campaign from $(1, 1)$ for a payoff of p. The solution to the first order conditions are given by

$$x^* = \frac{1 + p}{2 + 3p}, y^* = \frac{1 + 2p}{2 + 3p}.$$

Notice player 1 is still just trying to maximize their probability of winning the campaign (and so, per our results above, would respond to a symmetric strategy with a symmetric strategy). However, player 2 has an additional incentive to overplay today, raise their chances of winning an intermediate stage prize, even if this detracts from their overall chance of winning the campaign (which is already small). Thus, y^* is increasing in p, and in best response, x^* is decreasing in p. Thus, even in this very simple case, for any $p > 0$, the symmetric equilibrium fails to exist.

In our original game, players did not care which path they took to prevail in the overall campaign, so long as they did eventually win. Here, the path to victory (or loss) matters in as much as it changes the winnings that can accrue from intermediate prizes. This demonstrates, conversely, the sensitivity of the existence of the symmetric equilibrium to such assumptions that ensure indifference between various paths to victory.

5.3 R & D RACES

The development of a new technology has been modelled (originally, see Harris and Vickers (1987)) as a sequence of battles to develop a sufficient number of intermediate technologies. Here the resource being expended might be the time of specialist employees. Given such specialization, the assumption of no scrap value can be sustained. Certainly there is uncertainty in the outcome of each battle, making a stochastic contest success function an appropriate modeling choice. Assuming each technology is of equal importance to the development of the ultimate innovation, the results above demonstrate that, in equilibrium, resources should be split symmetrically, even if one firm begins with a head-start in the campaign.

Multiple Players In many innovation races, $N > 2$ players compete in each round. Similarly, in many elections, many candidates are vying to win each intermediate district/state battle. Here, however, we must make a distinction between campaigns that are first to m points and those that are best of $2m - 1$

points. When there are two players, these are the same. But for more than two players, the former means that the campaign continues until someone accrues m points (this will take a maximum of $N(m-1)+1$ rounds), while the latter means the total number of rounds is fixed at $2m-1$ and the majority (or perhaps plurality) winner is then calculated.

In the latter case, our results are applicable. We stipulate that a player can only win the campaign if they win an absolute majority of the $2m-1$ battles. That is, from the point of view of each player, no one winning the overall campaign is equivalent to someone else winning the overall campaign. For example, we might say that a firm only gets a payoff if they develop a majority of new technologies and patent a new product. If some other firm does this, or if no firm manages the ultimate breakthrough, then (at least in the short term) the payoffs are equally zero. Or, consider an insurgent, outsider candidate in a multi-candidate primary election. If a candidate does not win an outright majority of battles, the winner might be chosen by some other process (such as a party convention in the case of the USA) that will always choose an establishment candidate. In this case, each player is actually competing against an amalgam of other players, and to win each player must win a majority of the battles before the amalgam player wins a majority. Once the amalgam player wins a majority (whether or not any particular other player has) the original player cannot win the campaign anymore and so might as well play symmetrically. Before that stage, this is then the standard two player situation we have studied in this paper. Thus the equilibrium results apply.

In the former case, or in the latter case with the winner determined by plurality, we conjecture that the symmetric equilibrium still exists. Heuristically, to win a first-to-m campaign against $N-1$ other players, a player must in particular win that first-to-m campaign against each other player. To maximize the probability of winning in each particular (shadow) campaign, the symmetric strategy is to be played. Thus, it seems plausible that this symmetric strategy also (in best response to everyone else playing symmetrically) maximizes the overall chance of winning a first-to-m campaign. The direct algebraic methods of this paper are insufficient to establish this, however. This would be a useful avenue for future research.

Asymmetric Win Payoffs The development of a new technology might be worth vastly different payoffs to different firms, owing to how the new technology might diversely improve current operations. However, since each player is just maximizing their probability of winning the campaign (regardless of winning payoff), this asymmetry does not directly affect the campaign equilibrium. However, differences occur when we allow for an endogenous budget choice before the campaign. Analysis virtually identical to section 4.1 demonstrates that should an equilibrium in pure strategies exist, the budget choices will be proportional to prizes: $b_1^*/b_2^* = p_1/p_2$ where p_1, p_2 are the payoffs to players one and two respectively from winning the overall campaign.

Similarly to section 4.1, these equilibrium payoffs are decreasing in cost length m. This is just the dual statement to section 4.1. When either the benefits to winning the campaign are different, or when costs to winning the campaign are different, the cost-benefit calculation differs. For the player who sees less to gain from winning the campaign, and who knows any asymmetry in budget choice will be compounded in a long campaign, they eventually (for large m) have less incentive to commit resources to this campaign. But, as in section 4.1, when these b_1^*, b_2^* get small enough, and m gets large enough, players will want to make a non-local deviation so as to almost guarantee victory by just overplaying relative to the opponent. Thus, the dynamics and asymptotics here are qualitatively the same as those analyzed for unequal cost functions in section 4.2.

Conclusions In this paper we have analyzed a campaign in which two players meet in a sequence of battles in which the winner of the campaign is determined by whomever wins the majority of the battles. We have demonstrated that, even for players with vastly different budgets and vastly different resource technologies, there exists an equilibrium in which players split their budget equally across all (potential) battles. Endogenizing the budget choice at some cost to be incurred before the campaign is to be played, we have demonstrated the natural conditions under which a symmetric equilibrium continues to exist in this two stage game, and identified the implications for budget choice and participation incentives when the symmetric equilibrium breaks down. Finally, some extensions to the campaign structure that lead to the symmetry to break down have been identified, and some directions for future research identified.This contributes to the under-developed literature on sequential multi-contest campaigns, and lays the analytical and conceptual groundwork for future exploration.

Appendix A
Proofs

Lemma 1. *The leader at $(1, n)$ best responds to a symmetric strategy with the symmetric strategy.*

Proof. For player 1 at $(1, n)$, the probability they lose the series is the probability player 2 wins every contest. For this loss probability, we have that

$$L(i,j) = \frac{g\left(\frac{b_2}{n}\right)}{f(x) + g\left(\frac{b_2}{n}\right)} \cdot \left(\frac{g\left(\frac{b_2}{n}\right)}{f\left(\frac{b_1-x}{n-1}\right) + g\left(\frac{b_2}{n}\right)} \right)^{n-1}$$

and so differentiating with respect to x shows that

$$\begin{aligned}
\frac{\partial L}{\partial x} &= \frac{g\left(\frac{b_2}{n}\right)^n}{\frac{\partial}{\partial x}\left(f(x) + g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right) + g\left(\frac{b_2}{n}\right)\right)^{n-1}} + \frac{g\left(\frac{b_2}{n}\right)^n}{\left(f(x) + g\left(\frac{b_2}{n}\right)\right)\frac{\partial}{\partial x}\left(f\left(\frac{b_1-x}{n-1}\right) + g\left(\frac{b_2}{n}\right)\right)^{n-1}} \\
&= \frac{g\left(\frac{b_2}{n}\right)^n f'\left(\frac{b_1-x}{n-1}\right)\left(\frac{1}{n-1}\right)(n-1)}{\left(f(x) + g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right) + g\left(\frac{b_2}{n}\right)\right)^n} - \frac{g\left(\frac{b_2}{n}\right)^n f'(x)}{\left(f(x) + g\left(\frac{b_2}{n}\right)\right)^2\left(f\left(\frac{b_1-x}{n-1}\right) + g\left(\frac{b_2}{n}\right)\right)^{n-1}} \\
&= C_1\left[f'\left(\frac{b_1-x}{n-1}\right)\left(f(x) + g\left(\frac{b_2}{n}\right)\right) - f'(x)\left(f\left(\frac{b_1-x}{n-1}\right) + g\left(\frac{b_2}{n}\right)\right)\right]
\end{aligned}$$

where

$$C_1 = \frac{g\left(\frac{b_2}{n}\right)^n}{\left(f(x) + g\left(\frac{b_2}{n}\right)\right)^2\left(f\left(\frac{b_1-x}{n-1}\right) + g\left(\frac{b_2}{n}\right)\right)^n} > 0$$

for all $0 \leq x \leq b_1$.

Now, it is clear that when $x = 1/n$ we have that $f(x) = f\left(\frac{b_1-x}{n-1}\right), f'(x) = f'\left(\frac{b_1-x}{n-1}\right)$ and that $\frac{\partial L}{\partial x} = 0$. Now, when $x > 1/n$ we have that $f(x) > f\left(\frac{b_1-x}{n-1}\right)$ and by concavity that $f'(x) < f'\left(\frac{b_1-x}{n-1}\right)$. From this we see that

$$f'\left(\frac{b_1-x}{n-1}\right)\left(f(x) + g\left(\frac{b_2}{n}\right)\right) > f'(x)\left(f\left(\frac{b_1-x}{n-1}\right) + g\left(\frac{b_2}{n}\right)\right)$$

and so that $\frac{\partial L}{\partial x} > 0$. Similarly, we see that when $x < 1/n$ we have that $\frac{\partial L}{\partial x} < 0$.

This shows that, for player 1, the probability of losing the war is minimized by best responding with the symmetric strategy, and thus the probability of winning is maximized. □

Lemma 2. *The lagger at $(1, n)$ best responds to a symmetric strategy with the symmetric strategy.*

Proof. For player 1 to win the war at $(n, 1)$ is the probability they win every remaining match. This is given by

$$W_(i,j) = \frac{f(x)}{f(x)+g\left(\frac{b_2}{n}\right)} \cdot \left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^{n-1}.$$

Differentiating we have that

$$\begin{aligned}
\frac{\partial W}{\partial x} &= \frac{\frac{\partial}{\partial x}\left(f(x)\right) f\left(\frac{b_1-x}{n-1}\right)^{n}}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{n-1}} + \frac{f(x) f\left(\frac{b_1-x}{n-1}\right)^{n-1}}{\frac{\partial}{\partial x}\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{n-1}} \\
&+ \frac{\left(f(x)\right) f\left(\frac{b_1-x}{n-1}\right)^{n}}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\frac{\partial}{\partial x}\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{n-1}} + \frac{f(x)\frac{\partial}{\partial x}\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{n-1}}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{n-1}} \\
&= \left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^{n-1}\left(\frac{f'(x)}{f(x)+g\left(\frac{b_2}{n}\right)} - \frac{f'(x)f(x)}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^2}\right) \\
&+ \frac{f(x)}{f(x)+g\left(\frac{b_2}{n}\right)}\left(\frac{f'\left(\frac{b_1-x}{n-1}\right) f\left(\frac{b_1-x}{n-1}\right)}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^{n}} - \frac{f'\left(\frac{b_1-x}{n-1}\right)}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^{n-1}}\right) \\
&= \frac{f'(x) f\left(\frac{b_1-x}{n-1}\right) g\left(\frac{b_2}{n}\right)}{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{n-1}\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^2} - \frac{f'\left(\frac{b_1-x}{n-1}\right) f(x) g\left(\frac{b_2}{n}\right)}{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{n}\left(f(x)+g\left(\frac{b_2}{n}\right)\right)} \\
&= C_2\left[f'(x) f\left(\frac{b_1-x}{n-1}\right)\left(f(x)+g\left(\frac{b_2}{n}\right)\right) - f'\left(\frac{b_1-x}{n-1}\right) f(x)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)\right]
\end{aligned}$$

where

$$C_2 = \frac{g\left(\frac{b_2}{n}\right)}{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{n}\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^2} > 0$$

for all $0 \leq x \leq b_1$.
As above, it is clear that if $x = 1/n$ then $\frac{\partial W}{\partial x} = 0$. Suppose that $x > 1/n$ so that

$f(x) > f\left(\frac{b_1-x}{n-1}\right)$ and $f'(x) < f'\left(\frac{b_1-x}{n-1}\right)$. Then decomposing

$$\frac{\partial W}{\partial x} = C_2\left[f'(x)f\left(\frac{b_1-x}{n-1}\right)f(x) - f'\left(\frac{b_1-x}{n-1}\right)f(x)f\left(\frac{b_1-x}{n-1}\right)\right] + C_2\left[f'(x)f\left(\frac{b_1-x}{n-1}\right)g\left(\frac{b_2}{n}\right) - f'\left(\frac{b_1-x}{n-1}\right)f(x)g\left(\frac{b_2}{n}\right)\right]$$

we see that each line is negative and so that $\frac{\partial W}{\partial x} < 0$. Similarly we see that if $x < 1/n$ then $\frac{\partial W}{\partial x} > 0$. This demonstrates the symmetric strategy is a best response to an opponent playing the symmetric strategy, as claimed. □

Setup for Proposition 3. Suppose that both players are playing the symmetric strategy and the game is currently at node (i, j). For player 2 to win the game, they must do so when player 1 has won k battles with $0 \leq k \leq i - 1$. The number of paths to victory for player 2 such that the war ends with player 1 needing $i - k$ points at the end of the game is given by

$$\binom{j+k-1}{k}.$$

This is because, in this enumeration, the final battle must be a win for player 2, and so we are arranging k wins for player 1 among $j - 1$ victories for player 2. Thus, the probability of player 2 winning the war from (i, j) where both players are playing a symmetric strategy with effort expended by player i equal to b_i/n is given by

$$W_2(i,j) = \sum_{k=0}^{i-1}\left[\binom{j+k-1}{k}\left(\frac{f\left(\frac{b_1}{n}\right)}{f\left(\frac{b_1}{n}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\left(\frac{g\left(\frac{b_2}{n}\right)}{f\left(\frac{b_1}{n}\right)+g\left(\frac{b_2}{n}\right)}\right)^j\right]. \qquad (*)$$

Then we can see that, assuming player 2 is playing the symmetric strategy from (i, j) onwards, and that player 1 is playing the symmetric strategy at all battles after the current one, the probability of player 1 *losing* the war if they expend effort $0 \leq x \leq b_1$ today is

$$
\begin{aligned}
\pi_2(i,j) &= \left(\frac{f(x)}{f(x)+g\left(\frac{b_2}{n}\right)}\right) W_2(i-1,j) + \left(\frac{g\left(\frac{b_2}{n}\right)}{f(x)+g\left(\frac{b_2}{n}\right)}\right) W_2(i,j-1) \\
&= \frac{\left(g\left(\frac{b_2}{n}\right)\right)^j f(x) \sum_{k=0}^{i-2} \left[\binom{j+k-1}{k} \left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)} \right)^k \right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^j} \\
&+ \frac{\left(g\left(\frac{b_2}{n}\right)\right)^j \sum_{k=0}^{i-1} \left[\binom{j-1+k-1}{k} \left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)} \right)^k \right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{j-1}}.
\end{aligned}
$$

Thus we wish to ultimately show that this function has a global minimum at $x = 1/n$. To do this we will have to manipulate the derivative, (which generally has seven terms in it). To that end, we note the following combinatorial fact to be used throughout (from Gould (1972)).

The following sum can be written

$$
\sum_{k=0}^{n} \binom{r+k}{k} \binom{s+n-k}{n-k} = \binom{r+s+n+1}{n}.
$$

Lemma 3. *The sum of the following three terms in the derivative are equal to*

$$\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j f(x) \sum_{k=0}^{i-2}\left[\binom{j+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\frac{\partial}{\partial x}\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^j}$$

$$+\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j \sum_{k=0}^{i-1}\left[\binom{j-1+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\frac{\partial}{\partial x}\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{j-1}}$$

$$+\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j \frac{\partial}{\partial x} f(x) \sum_{k=0}^{i-2}\left[\binom{j+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^j}$$

$$=\frac{-f'(x)\left(g\left(\frac{b_2}{n}\right)\right)^j\binom{n-1}{i-1}\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{i-1}}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^2\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{n-1}}$$

Proof. Performing the differentiation we have that

$$\frac{\partial D_1}{\partial x}=\frac{-f'(x)\left(g\left(\frac{b_2}{n}\right)\right)^j f(x) \sum_{k=0}^{i-2}\left[\binom{j+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^2\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^j}$$

$$-\frac{f'(x)\left(g\left(\frac{b_2}{n}\right)\right)^j \sum_{k=0}^{i-1}\left[\binom{j-1+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^2\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{j-1}}$$

$$+\frac{f'(x)\left(g\left(\frac{b_2}{n}\right)\right)^j \sum_{k=0}^{i-2}\left[\binom{j+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^j}$$

Adding a factor of $\left(f(x)+g\left(\frac{b_2}{n}\right)\right)$ to the denominator of the third term, and a cancellation reveals gives that

$$\frac{\partial D_1}{\partial x} = \frac{f'(x)\left(g\left(\frac{b_2}{n}\right)\right)^{j+1} \sum_{k=0}^{i-2}\left[\binom{j+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^j}$$

$$- \frac{f'(x)\left(g\left(\frac{b_2}{n}\right)\right)^{j} \sum_{k=0}^{i-1}\left[\binom{j-1+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^2\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{j-1}}.$$

Now inside the summation we want a factor of $1/\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{i-2}$ and $1/\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{i-1}$ in the first and second term respectively. Multiplying top and bottom by these, and collecting the terms in the denominator, we have that

$$\frac{1}{C}\frac{\partial D_1}{\partial x} = g\left(\frac{b_2}{n}\right)\sum_{k=0}^{i-2}\left[\binom{j+k-1}{k}\left(f\left(\frac{b_1-x}{n-1}\right)\right)^k\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{i-2-k}\right]$$
$$-\sum_{k=0}^{i-1}\left[\binom{j-1+k-1}{k}\left(f\left(\frac{b_1-x}{n-1}\right)\right)^k\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{i-1-k}\right]$$

where

$$C = \frac{f'(x)\left(g\left(\frac{b_2}{n}\right)\right)^j}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)^2\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{n-1}} > 0.$$

Each of these terms is a polynomial with terms of the form $\left(g\left(\frac{b_2}{n}\right)\right)^p\left(f\left(\frac{b_1-x}{n-1}\right)\right)^q$ with $p+q=i-1$ where, for the first polynomial, we have $1 \le p \le i-1$, and $0 \le q \le i-2$ whereas for the second polynomial $0 \le p, q \le i-1$. That is, the second polynomial has an extra term of order $\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{i-1}$. Now, all terms except this additional one cancel out.

To see this note that in the first polynomial the coefficient of $\left(g\left(\frac{b_2}{n}\right)\right)^p\left(f\left(\frac{b_1-x}{n-1}\right)\right)^q$ is determined from instances for each $0 \le k \le (i-1)-p$. The first summand in the first polynomial produces terms of the form f^0g^{i-1}, f^1g^{i-2} up to $f^{i-2}g^1$, the second summand produces terms of the form f^1g^{i-2} up to $f^{i-2}g^1$ whilst the $i-$ 2th summand produces terms only of the form$f^{i-2}g^1$. Summing the coefficients from all these instances, we see that the coefficient of $\left(g\left(\frac{b_2}{n}\right)\right)^p\left(f\left(\frac{b_1-x}{n-1}\right)\right)^q$ is

$$\binom{j-1}{0}\binom{i-2}{i-1-p}+\binom{j}{1}\binom{i-3}{i-2-p}+\cdots+\binom{j-1+i-p}{i-p-1}\binom{i-3}{i-2-p}$$

which, via the claim above, is equal to $\binom{j+i-2}{i+1-p}$.

Similarly, the coefficient of $\left(g\left(\frac{b_2}{n}\right)\right)^p\left(f\left(\frac{b_1-x}{n-1}\right)\right)^q$ from the second polynomial is given by

$$\binom{j-2}{0}\binom{i-1}{i-1-p}+\binom{j-1}{1}\binom{i-2}{i-2-p}+\cdots+\binom{j-2+i-p}{i-p-1}\binom{p}{0}$$

which, via the claim above, is also equal to $\binom{j+i-2}{i+1-p}$.

Thus, $\frac{\partial D_1}{\partial x}$ is in fact just equal to the extra term from the second polynomial of index $\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{i-1}$. To compute this, we sum the coefficients in the same way as above, now with $p=0, q=i-1$, and find that the coefficient in question is $-\binom{j+i-2}{i-1}$. Recalling that $n=i+j-1$ finishes the proof. □

Lemma 4. *The sum of the following two terms in the derivative is equal to*

$$\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j\sum_{k=0}^{i-1}\left[\binom{j-1+k-1}{k}\frac{\partial}{\partial x}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{j-1}}$$
$$+\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j\sum_{k=0}^{i-1}\left[\binom{j-1+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\frac{\partial}{\partial x}\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{j-1}}$$
$$=\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j f'\left(\frac{b_1-x}{n-1}\right)\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{i-1}\binom{n-2}{i-1}}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^n}.$$

Proof. First we must write $\frac{\partial}{\partial x}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k$ usefully. Considering this as a product, we have that (ignoring the chain rule component for the moment)

$$\frac{\partial}{\partial x}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k = \frac{-k}{n-1}\frac{\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{k-1}}{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^k} + \frac{k}{n-1}\frac{\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{k}}{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{k+1}}.$$

When $k = 0$ we are differentiating a constant which becomes zero. For $k = 2 \dots i-2$ this is a quasi-telescoping series, with the coefficient on $\frac{\left(f\left(\frac{b_1-x}{n-1}\right)\right)^k}{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{k+1}}$ given by

$$\frac{k}{n-1}\binom{j-2+k}{k} - \frac{k+1}{n-1}\binom{j-2+k+1}{k+1} = \frac{-(j-1)}{n-1}\binom{j-2+k}{k}.$$

Inserting back the terms for $k = 1$ and $k = i-1$, we see that first of the two terms in question becomes

$$\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j \sum_{k=0}^{i-1}\left[\binom{j-1+k-1}{k}\frac{\partial}{\partial x}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{j-1}}$$

$$= -\frac{C}{n-1}\left[\frac{1}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right]$$

$$- C\cdot\frac{j-1}{n-1}\left[\sum_{k=1}^{i-2}\binom{j-2+k}{k}\frac{\left(f\left(\frac{b_1-x}{n-1}\right)\right)^k}{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{k+1}}\right]$$

$$+ C\cdot\frac{i-1}{n-1}\binom{j-2+i-1}{i-1}\frac{\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{i-1}}{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{i}}$$

where

$$C = \frac{\left(g\left(\frac{b_2}{n}\right)\right)^j f'\left(\frac{b_1-x}{n-1}\right)\cdot}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{j-1})}.$$

Now, turning to the second term we began considering, and factoring one instance of $f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)$ from the denominator to the numerator, we see that

$$\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j \sum_{k=0}^{i-1}\left[\binom{j-1+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\frac{\partial}{\partial x}\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{j-1}}$$

$$= C\sum_{k=0}^{i-1}\left[\binom{j-1+k-1}{k}\frac{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^k}{\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{k+1}}\right].$$

Now it is clear that when these two terms are added, everything cancels except the quantity of order $\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{i-1} / \left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^{i}$. The two relevant terms that contribute are

$$C\cdot\frac{i-1}{n-1}\binom{i+j-3}{i-1} \text{ and } C\cdot\frac{j-1}{n-1}\binom{i+j-3}{i-1}$$

from the first and second terms being differentiated respectively. Recalling that $n = i+j-1$ completes the proof.

□

Lemma 5. *The sum of the remaining two terms in the derivative is equal to*

$$\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j f(x)\sum_{k=0}^{i-2}\left[\binom{j+k-1}{k}\frac{\partial}{\partial x}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^j}$$

$$+\frac{\left(g\left(\frac{b_2}{n}\right)\right)^j f(x)\sum_{k=0}^{i-2}\left[\binom{j+k-1}{k}\left(\frac{f\left(\frac{b_1-x}{n-1}\right)}{f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)}\right)^k\right]}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\frac{\partial}{\partial x}\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^j}$$

$$=\frac{f'(x)f(x)\left(g\left(\frac{b_2}{n}\right)\right)^j\left(f\left(\frac{b_1-x}{n-1}\right)\right)^{i-2}\binom{n-2}{i-2}}{\left(f(x)+g\left(\frac{b_2}{n}\right)\right)\left(f\left(\frac{b_1-x}{n-1}\right)+g\left(\frac{b_2}{n}\right)\right)^n}.$$

Proof. Factoring out an $f(x)$ we see that this expression is identical to that from the preceding proposition, except with $i-1$ instead of i and $j-1$ instead of j. With these minor adjustments, the proof above can be directly replicated. □

Lemmata 3, 4 and 5 combine to give the following:

Lemma 6. *The derivative of player 1's win probability at (i, j) is given by*

$$\begin{aligned}\frac{1}{C}\frac{\partial \pi_1(i,j)}{\partial x} &= f'(x) f\left(\frac{b_1 - x}{n-1}\right)\left(f\left(\frac{b_1 - x}{n-1}\right) + g\left(\frac{b_2}{n}\right)\right)\\ &\quad - \frac{j-1}{n-1}\cdot f'\left(\frac{b_1 - x}{n-1}\right) f\left(\frac{b_1 - x}{n-1}\right)\left(f(x) + g\left(\frac{b_2}{n}\right)\right)\\ &\quad - \frac{i-1}{n-1}\cdot f'\left(\frac{b_1 - x}{n-1}\right) f(x)\left(f(x) + g\left(\frac{b_2}{n}\right)\right)\end{aligned}$$

with

$$C = \frac{\left(g\left(\frac{b_2}{n}\right)\right)^j \left(f\left(\frac{b_1 - x}{n-1}\right)\right)^{i-2}\binom{n-1}{j-1}}{\left(f(x) + g\left(\frac{b_2}{n}\right)\right)^2\left(f\left(\frac{b_1 - x}{n-1}\right) + g\left(\frac{b_2}{n}\right)\right)^n} > 0.$$

Proof. From the preceding propositions, noting the basic combinatorial identities $\binom{n-2}{i-1} / \binom{n-1}{i-1} = \frac{j-1}{n-1}$ and $\binom{n-2}{i-1} / \binom{n-1}{i-2} = \frac{i-1}{n-1}$ it is immediate that the derivative of the chance of player 1 *losing* the war from (i, j) is the negative of what is claimed. Then noting that this is a zero sum game, so that the chance of winning is one minus the chance of losing gives the result. $\square$

Proof of Proposition 3.

Proof. Given the expression for the derivative of player 1's win probability in terms of their effort allocation at (i, j) it suffices to show this derivative to be flat at $x = 1/n$ (the symmetric strategy), negative for $b_1 \geq x > 1/n$ and positive for $0 \leq x \leq 1/n$.

Given $C > 0$ for all $0 \leq x \leq b_1$, the sign of $\frac{\partial \pi_1(i,j)}{\partial x}$ is equal to the sign of $S = \frac{1}{C}\frac{\partial \pi_1(i,j)}{\partial x}$. Now, at $x = 1/n$ we have that $f(x) = f\left(\frac{1-x}{n-1}\right) = f$ and similarly for $f'\left(\frac{b_1 - x}{n-1}\right) = f'(x) = f'$ we have:

$$\frac{\partial W}{\partial x} = f' f(f + g) - \frac{j-1}{n-1}\cdot f' f(f + g)\frac{i-1}{n-1}\cdot f' f(f + g) = 0$$

as needed (recalling that $n - 1 = i + j - 2$).

Now, suppose player 1 is overplaying today, such that $x > 1/n$ meaning $f(x) > f\left(\frac{1-x}{n-1}\right)$ and (by concavity) $f'(x) < f'\left(\frac{1-x}{n-1}\right)$. Then we have that

$$
\begin{aligned}
S &= f'(x)f\left(\frac{1-x}{n-1}\right)\left(f\left(\frac{1-x}{n-1}\right)+g(b/n)\right)\\
&-\frac{j-1}{n-1}\cdot f'\left(\frac{1-x}{n-1}\right)f\left(\frac{1-x}{n-1}\right)(f(x)+g(b/n))\\
&-\frac{i-1}{n-1}\cdot f'\left(\frac{1-x}{n-1}\right)f(x)(f(x))+g(b/n))\\
&< f'(x)f\left(\frac{1-x}{n-1}\right)\left(f\left(\frac{1-x}{n-1}\right)+g(b/n)\right)\\
&-\frac{j-1}{n-1}\cdot f'\left(\frac{1-x}{n-1}\right)f\left(\frac{1-x}{n-1}\right)(f(x)+g(b/n))\\
&-\frac{i-1}{n-1}\cdot f'\left(\frac{1-x}{n-1}\right)f\left(\frac{1-x}{n-1}\right)(f(x))+g(b/n))\\
&= f'(x)f\left(\frac{1-x}{n-1}\right)\left(f\left(\frac{1-x}{n-1}\right)+g(b/n)\right)-f'\left(\frac{1-x}{n-1}\right)f\left(\frac{1-x}{n-1}\right)(f(x)+g(b/n))\\
&= f\left(\frac{1-x}{n-1}\right)\left[f'(x)\cdot\left(f\left(\frac{1-x}{n-1}\right)+g(b/n)\right)-f'\left(\frac{1-x}{n-1}\right)(f(x)+g(b/n))\right]\\
&< 0
\end{aligned}
$$

where the first inequality is due to the fact that $f(x) > f\left(\frac{b_1-x}{n-1}\right)$ and the second from the fact that f is increasing and concave.

Conversely, suppose player 1 is underplaying today, such that $x < 1/n$ meaning $f(x) < f\left(\frac{1-x}{n-1}\right)$ and (by concavity) $f'(x) > f'\left(\frac{1-x}{n-1}\right)$. Then we have that

$$
\begin{aligned}
S &= f'(x)f\left(\frac{1-x}{n-1}\right)\left(f\left(\frac{1-x}{n-1}\right)+g(b/n)\right)\\
&-\frac{j-1}{n-1}\cdot f'\left(\frac{1-x}{n-1}\right)f\left(\frac{1-x}{n-1}\right)(f(x)+g(b/n))\\
&-\frac{i-1}{n-1}\cdot f'\left(\frac{1-x}{n-1}\right)f(x)(f(x))+g(b/n))\\
&> f'(x)f\left(\frac{1-x}{n-1}\right)\left(f\left(\frac{1-x}{n-1}\right)+g(b/n)\right)\\
&-\frac{j-1}{n-1}\cdot f'\left(\frac{1-x}{n-1}\right)f\left(\frac{1-x}{n-1}\right)(f(x)+g(b/n))\\
&-\frac{i-1}{n-1}\cdot f'\left(\frac{1-x}{n-1}\right)f\left(\frac{1-x}{n-1}\right)(f(x))+g(b/n))\\
&= f'(x)f\left(\frac{1-x}{n-1}\right)\left(f\left(\frac{1-x}{n-1}\right)+g(b/n)\right)-f'\left(\frac{1-x}{n-1}\right)f\left(\frac{1-x}{n-1}\right)(f(x)+g(b/n))\\
&= f\left(\frac{1-x}{n-1}\right)\left[f'(x)\cdot\left(f\left(\frac{1-x}{n-1}\right)+g(b/n)\right)-f'\left(\frac{1-x}{n-1}\right)(f(x)+g(b/n))\right]\\
&> 0
\end{aligned}
$$

where the first inequality follows from the fact that $f(x) < f\left(\frac{b_1-x}{n-1}\right)$ and the second from the fact that f is increasing and concave. This proves the claim.

□

Proof of Corollary 1:

Proof. **Stage Campaign.** The proof is via induction over the number of stages remaining in the game. The base case is when there is exactly one stage remaining. That is, there is one battle left to be played and although it is possible that this battle will actually be decisive, it is also possible (perhaps likely, depending on strategies) that one player has already reached n wins and this is a dead rubber. But the player's do not know the results of past battles at this stage. So each player must play a single strategy at this stage. Now, conditional on earlier strategy, each player will have some beliefs (determined mechanically) about which particular node they are at within the stage they currently face. But it is clear that at this final stage the players should each spend all of their remaining budget, which is a symmetric strategy in a vacuous sense.

Suppose we find ourselves at the kth last stage of the campaign. Suppose further that the symmetric strategies (over stages) equilibrium from the $k-1$th stage onwards form an equilibrium, and that at the current stage player 2 is playing symmetrically. It remains to show that player 1 should still play symmetrically. Suppose there are N_K nodes in the kth last stage. Of these, say the first $0 \leq N_{K_W} < N_K$ are such that player 1 has already won the campaign (but doesn't know it yet) and the nodes numbered N_{K_L} to N_K with $N_{K_W} < N_{K_L} \leq N_K$. Thus the intermediate nodes, numbered from N_{K_W} to N_{K_L} are the 'live nodes' where strategy might actually affect payoffs. Denote the payoff of deploying effort x today if in fact the game is at node N_I by V_{N_i}, where we are assuming for induction that the game is played symmetrically from this point onwards.

Thus the total payoff to player 1 of deploying effort x at the kth last stage is given by

$$\Pi = \sum_{i=1}^{N_{K_W}} \mu_i \cdot 1 + \sum_{i=N_{K_W}+1}^{N_{K_L}} \mu_i V_i(x) + \sum_{N_{K_L}+1}^{N_K} \mu_i \cdot 0.$$

But then it is clear from Proposition 1 that since the optimal thing to do at any node $N_{K_W}, \ldots, N_{K_L}$ is to play symmetrically, we see that regardless of beliefs the sequentially optimal strategy is to play symmetrically at the k-th last stage. By induction this demonstrates existence. Although this looks like whole spectrum of sequential/perfect Bayesian equilibria, in fact given these equilibrium strategies the beliefs are uniquely pinned down.

Simultaneous Campaign. Recall that the simultaneous campaign is defined by taking the stage campaign and removing from each player's set of state variables knowledge of their opponents historical or current budgets, and as such of their budget choices at past stages. Thus the sequentiality with which the player allocates their budget is purely artificial, since they receive no further information after each choice and as such this is identical to simultaneous allocation.
Supposing one's opponent is playing symmetrically, it is clear that the best response must also be the symmetric strategy. If not, then the best response in the stage campaign (where the player could, for whatever reason, ignore the extra information possessed there) would also be asymmetric.
Uniqueness in both cases follows from identical zero-sum arguments to that in the proof of Proposition 1.

□

Proof of Proposition 5:

Proof. Applying Lemma 4 with $f = g$, $i = j = m$ (noting that the n, which is one more game than might be played 'today' in Lemma 4, becomes $2m$ here) and the variable being differentiated transformed from being the x in $b_1 - x$ to just b_1 (noting that Lemma 4 deals with the probability of player 1 losing the war, so the negatives cancel) we see that the necessary first order condition for player 1 is that

$$\frac{m\binom{2m-1}{m} \cdot f'\left(\frac{b_1}{2m-1}\right) f\left(\frac{b_1}{2m-1}\right)^{m-1} f\left(\frac{b_2}{2m-1}\right)^{m}}{(2m-1)\left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)^{2m}} = c'(b_i)$$

and similarly for player 2.
Dividing these two equations we get that

$$\frac{f'\left(\frac{b_1}{2m-1}\right) f\left(\frac{b_2}{2m-1}\right)}{f'\left(\frac{b_2}{2m-1}\right) f\left(\frac{b_1}{2m-1}\right)} = \frac{c'(b_1)}{c'(b_2)}.$$

Clearly these are satisfied by $b_1 = b_2$. Noting the convexity of $c(\cdot)$,and that $f(\cdot)$ is increasing and concave and uniquely so because $c(\cdot)$ is increasing and convex. Differentiating the payoff function once more with respect to b_i, we get a second order derivative for player 1 equal to

$$
\begin{aligned}
\frac{\partial^2 \Pi_i(b_1,b_2,m)}{\partial b_1^2} &= -\frac{m\binom{2m-1}{m} f'\left(\frac{b_1}{2m-1}\right)^2 \left(\frac{f\left(\frac{b_1}{2m-1}\right)f\left(\frac{b_2}{2m-1}\right)}{\left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)^2}\right)^m \left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)}{(2m-1)^2 f\left(\frac{b_1}{2m-1}\right)^2 \left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)} \\
&\quad - \frac{m^2\binom{2m-1}{m} f'\left(\frac{b_1}{2m-1}\right) \left(\frac{f\left(\frac{b_1}{2m-1}\right)f\left(\frac{b_2}{2m-1}\right)}{\left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)^2}\right)^m \left(f\left(\frac{b_1}{2m-1}\right)-f\left(\frac{b_2}{2m-1}\right)\right)}{(2m-1)^2 f\left(\frac{b_1}{2m-1}\right)^2 \left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)} \\
&\quad + \frac{m\binom{2m-1}{m} f''\left(\frac{b_1}{2m-1}\right) f\left(\frac{b_1}{2m-1}\right)^{m-1} f'\left(\frac{b_2}{2m-1}\right)^m}{(2m-1)^2 \left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)^{2m}} - c''(b_1) \\
&= -\frac{m\binom{2m-1}{m} f'\left(\frac{b_1}{2m-1}\right)^2 \left(\frac{f\left(\frac{b_1}{2m-1}\right)f\left(\frac{b_2}{2m-1}\right)}{\left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)^2}\right)^m}{(2m-1)^2 f\left(\frac{b_1}{2m-1}\right)^2 \left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)} \\
&\quad \times \left[f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)+m\left(f\left(\frac{b_1}{2m-1}\right)-f\left(\frac{b_2}{2m-1}\right)\right)\right] \\
&\quad + \frac{m\binom{2m-1}{m} f''\left(\frac{b_1}{2m-1}\right) f\left(\frac{b_1}{2m-1}\right)^{m-1} f'\left(\frac{b_2}{2m-1}\right)^m}{(2m-1)^2 \left(f\left(\frac{b_1}{2m-1}\right)+f\left(\frac{b_2}{2m-1}\right)\right)^{2m}} - c''(b_1).
\end{aligned}
$$

Now, putting $b_1 = b_2$, noting the convexity of $c(\cdot)$ and the concavity of $f(\cdot)$, we conclude that the second order condition for a maximum is satisfied.
Hence, combined with Theorem 1, this demonstrates that if a subgame perfect Nash equilibrium with symmetric play in the second is to exist, it will feature symmetric budget choice in the first stage. □

Proof of Proposition 6:

Proof. The first order conditions from the proof of proposition 5 ensure that the hypothesized equilibrium budgets do not have any local profitable unilateral deviations. However, as noted, that proof (through the division by the budget variables) did not account for the possibility of a deviation to a budget of zero. That is, given the proposed symmetric budgets, a player might prefer to not participate in the game at all, and get a payoff of zero with certainty. Thus, given the expected payoff to playing the symmetric budget is $1/2$, and the cost is $c(b)$, the players will play b and a symmetric pure strategy equilibrium will exist if and only if $c(b) \leq 1/2$ as claimed.

□

Proof of Proposition 7:

Proof. The corresponding first order conditions, analagous to the proof of proposition 5 reduce to

$$\frac{b_1}{b_2} = \frac{\gamma_2}{\gamma_1} = \frac{1}{r}.$$

Thus, if a pure strategy Nash equilibria is to exist, the constituent strategies will be

$$b_i^* = \frac{m}{\gamma_i}\left(\frac{r}{(1+r)^2}\right)^m \binom{2m-1}{m}.$$

Now, in the section 4, we had $\gamma_1 = \gamma_2 = r = 1$ meaning $r/(1+r)^2 = 1/4$, and, noting the (exponentially tight) approximation by Stirling's formula,

$$\binom{2m-1}{m} = \frac{1}{2}\binom{2m}{m} \sim \frac{4^m}{2\sqrt{\pi m}},$$

we had that

$$b_i^* \sim \frac{\sqrt{m}}{2\sqrt{\pi}}.$$

This increases in m, eventually beyond the value of the prize being competed for, which is why we found the temptation that lead to the eventual non-existence of a pure strategy equilibria was the profitable deviation to abandoning the game with a budget of zero.

Now, though, with $r \neq 1$, we see that $r/(1+r)^2 = \rho < 1/4$ and so

$$b_i^* \sim \frac{\sqrt{m}}{2\sqrt{\pi}\gamma_i}(4\rho)^m \to 0 \text{ as } m \to \infty.$$

□

Proof of Proposition 8:

Proof. It is clear that when $b_1 = b_2$, $W_1 = 1/2$ for all m, and so the limit is also $1/2$. Further, it is clear that when $b_1 = 0$ then $W_1 = 0$ and when $b_1 \to \infty$ then $W_1 \to 1$. As shorthand, we write $W_1(\infty) = 1$. Throughout take b_2 to be fixed.

Now consider the derivatives of W, also as sequence in m. Write $b_1 + \delta = b_2$ for some δ. Recalling the approximation of the central binomial coefficient by

Stirling's formula, we can write an entry in the sequence of derivatives as,

$$\left\{\frac{dW_1}{db_1}\right\}_m = \frac{m(b_1+\delta)^m b_1^{m-1}\binom{2m-1}{m}}{(2b_1+\delta)^{2m}} \sim \frac{4^m\sqrt{m}}{2\sqrt{\pi}}\left(\frac{b_1(b_1+\delta)}{(2b_1+\delta)^2}\right)^m. \tag{A.1}$$

Now, when $b_1 = b_2$, and $\delta = 0$ the sequence of derivatives grows without bound, at rate $\sqrt{m}$. That is, the slope of the win function near $b_1 = b_2$ is getting infinitely steep. When $\delta \neq 0$ we have

$$\delta^2 > 0 \iff 4b_1^2 + 4\delta b_1 < 4b_1^2 + 4\delta b_1 + \delta^2 \iff \frac{b_1^2 + \delta b_1}{4b_1^2 + 4\delta b_1 + \delta^2} = \frac{b_1(b_1+\delta)}{(2b_1+\delta)^2} < \frac{1}{4}.$$

Thus, at any $b_1 \neq b_2$ the value of dW_1/db_1 tends to zero pointwise. Denote this pointwise limit by $\{dW_1/db_1\}_\infty$.

Intuitively, we can now see that the slope function is tending toward the zero function almost everywhere, and getting large at $b_1 = b_2$. Thus, the original function W_1 is tending toward being flat on $[0, b_2) \cup (b_2, \infty)$ and getting infinitely steep at $b_1 = b_2$.

To formalize this, we run through a constructive version of Egoroff's Theorem below. First note,the second-order condition from the proof of proposition 5, becomes, when $f = g = id$,

$$\frac{\partial^2 \Pi_i(b_1, b_2, m)}{\partial b_1^2} = -\frac{m\left(\frac{b_1 b_2}{(b_1+b_2)^2}\right)^m (b_1 + b_2 + m(b_1 - b_2))\binom{2m-1}{m}}{b_1^2(b_1+b_2)}.$$

From this we see that the sign of the second derivative d^2W_1/db_1^2 is equal to the sign of $-(b_1 + b_2 + m(b_1 - b_2))$. Thus, the first derivative dW_1/db_1 increases until $b_1 = b_2 \cdot \frac{m-1}{m+1}$ and then decreases afterwards.

Now, take some $0 < R < b_2$. The above implies that for any such R there exists an m beyond which dW_1/db_1 is increasing on $[0, R]$. Thus, eventually, the largest value of dW_1/db_1 on $[0, R]$ occurs at $b_1 = R$. Since $dW_1/db_1(R)$ tends to zero, it follows that dW_1/db_1 is uniformly convergent to the zero function on $[0, R]$. Uniform convergence (and well behaved continuous, differentiable functions) are sufficient for limits to be moved through integrals, and so we have

Then, on this interval $[0, r]$ we have

$$\{W_k(r) - W_k(0)\}_m = \int_{[0,r]} \left\{\frac{\partial W}{\partial x}\right\}_m db_1 \to \int_{[0,r]} \left\{\frac{\partial W}{\partial x}\right\}_\infty db_1 = 0.$$

Then, using the positivity, continuity and fact that W is increasing from $W(0) = 0$

we conclude $W_k(r) \to 0$ as $k \to \infty$. This is true for any for any $0 < r < b_2$. Similarly for any $r > b_2$ we run the proof on $[r, \infty)$ get uniform convergence and conclude similarly that $\{W_k(\infty) - W_k(r)\}_m \to 0$. This demonstrates almost everywhere convergence of W to the step function that is zero on $[0, b_2)$ and one on (b_2, ∞). But we have already argued that $W_1(b_2) = \frac{1}{2}$ for all m. This completes the proof.

□

Proof of Corollary 2.

Proof. When the cost functions are symmetric, recall that if G is the CDF of the distribution from which the mixed strategy is drawn, then we have that

$$1 \cdot G(b) = c(b) \text{ for all } b \in [0, v].$$

Integrating by parts, the expected bid per player is

$$E[b_i] = \int_0^{v_i} bc'(b)\, db = v_i - \int_0^{v_i} c(b)\, db. \tag{A.2}$$

On the other hand, when the cost functions are assymetric, player 1 plays the atom-less mixed strategy $G_1(b) = c_2(b)$ for all $b \in [0, v_2]$ and player two plays the mixed strategy with atom at zero with weight $c_1(v_1) - c_1(v_2)$ and continuously over $[0, v_2]$ drawing from $G_2(b_2) = c_1(b_2)$.
In this case, the expected bids per player are

$$E[b_1] = v_2 - \int_0^{v_2} c_2(b)\, db \text{ and } E[b_2] = v_2 c_1(v_2) - \int_0^{v_2} c_1(b)\, db. \tag{A.3}$$

As can be seen by comparing (A.2) and (A.3), and noting $c_1(v_2) < 1$, the sum of the expected bids is lower under asymmetric costs. Further, the atom with positive mass in player two's strategy means that the probability of non-participation in the actual campaign is positive, as opposed to zero when costs are symmetric. This proves the corollary. □

Bibliography

Baye, M. R., D. Kovenock, and C. G. de Vries (1996, Jun). The all-pay auction with complete information. *Economic Theory 8*(2), 291–305.

Bevia, C. and L. C. Corchon (2013). Endogenous strength in conflicts. *International Journal of Industrial Organization 31*(3), 297 – 306. Tournaments, Contests and Relative Performance Evaluation.

Blackett, D. W. (1954). Some blotto games. *Naval Research Logistics Quarterly 1*(1), 55–60.

Borel, E. (1921). La thorie du jeu et les quations intgrales noyau symtrique gauche. *Comptes Rendus de lAcademie*.

Duffy, J. and A. Matros (2015). Stochastic asymmetric blotto games: Some new results. *Economics Letters 134*(Supplement C), 4 – 8.

Friedman, L. (1958). Game-theory models in the allocation of advertising expenditures. *Operations Research 6*(5), 699–709.

Fu, Q. and J. Lu (2009, May). The beauty of ”bigness”: On optimal design of multi-winner contests. *Games and Economic Behavior 66*(1), 146–161.

Fu, Q. and J. Lu (2012, Oct). The optimal multi-stage contest. *Economic Theory 51*(2), 351–382.

Fu, Q., J. Lu, and Y. Pan (2015, July). Team contests with multiple pairwise battles. *American Economic Review 105*(7), 2120–40.

Fudenberg, D., R. Gilbert, J. Stiglitz, and J. Tirole (1983). Preemption, leapfrogging and competition in patent races. *European Economic Review 22*(1), 3 – 31. Market Competition, Conflict and Collusion.

Gould, H. (1972). *Combinatorial identities: a standardized set of tables listing 500 binomial coefficient summations*. Morgantown, W Va.

Gross, O. and R. Wagner (1950). A continuous colonel blotto game.

Harris, C. and J. Vickers (1987). Racing with uncertainty. *The Review of Economic Studies 54*(1), 1–21.

J. Clark, D. and K. Konrad (2007, 06). Contests with multi-tasking. *109*, 303 – 319.

Kalla, J. L. and D. E. Broockman (2017). The minimal persuasive effects of campaign contact in general elections: Evidence from 49 field experiments. *American Political Science Review*, 119.

Klumpp, T. and K. Konrad (2018, May). Sequential Majoritarian Blotto Games. (2018-8).

Klumpp, T. and M. K. Polborn (2006). Primaries and the new hampshire effect. *Journal of Public Economics 90*(6-7), 1073–1114.

Konrad, K. A. (2012). Dynamic contests and the discouragement effect. *Revue d'conomie politique 122*(2), 233–256.

Konrad, K. A. (2016). Budget and effort choice in sequential colonel blotto campaigns. *Working Paper*.

Konrad, K. A. and D. Kovenock (2009). Multi-battle contests. *Games and Economic Behavior 66*(1), 256 – 274.

Osborne, M. J. and A. Rubinstein (1994). *A course in game theory*. Cambridge, USA: The MIT Press. electronic edition.

Roberson, B. (2006, Sep). The colonel blotto game. *Economic Theory 29*(1), 1–24.

Robson, A. (2005). Multi-item contests. *IDEAS Working Paper Series from RePEc*, 27. Copyright - Copyright FEDERAL RESERVE BANK OF ST LOUIS 2005; Last updated - 2015-10-03.

Siegel, R. (2007). All-pay contests. *Working Paper* (1).