Collective risk models with FGM dependence

Etienne Marceau (with Christopher Blier-Wong and Hélène Cossette)

École d'actuariat
Université Laval, Québec, Canada
Member of CIMMUL, Quantact, CIRCERB, and CRDM
Associated member of IID and Research Lab SAF

23-28 July 2023
AFRIC 2023
Victoria Falls, Zimbabwe
Contributed Talk Parallel Session 6.2, Risk Modeling
Friday 28 July 2023, 10:00-10:40 am

Duration ≈ 15 minutes

The classical Collective Risk Model (CRM) is the main model in non-life actuarial science.

Definition of the aggregate claim amount rv S for an insurance portfolio :

- $S =$ sum of a random number (*number of claims*) of random variables (*claim amounts*)

$$S = \begin{cases} \sum_{j=1}^{N} X_j & , \quad N > 0 \\ 0 & , \quad N = 0 \end{cases} \tag{1}$$

- Discrete rv N : number of claims, with probablity mass function (pmf)

$$f_N(n) = \Pr(N = n), \quad n \in \mathbb{N}_0 = \{0,1,2,\dots\}$$

- Random variable X_j : amount of the jth claim, $j \in \mathbb{N}_1 = \{1,2,\dots\}$.
- Alternative definition of S :

$$S = \sum_{j=1}^{\infty} X_j \times \mathbb{1}_{\{N \geq j\}} \tag{2}$$

Advantage of the definition : S is a function of two distinct components

- frequency (number of claims rv N)
- severity (claim amounts rvs X_1, X_2, ...)

UNIVERSITÉ LAVAL

Three assumptions of the classical CRM :

A1 $\underline{X} = \{X_j,\ j \in \mathbb{N}_1\}$: sequence of identically distributed strictly positive rvs

- *Interpretation : the claim amounts have the same marginal distribution*
- $X_i \overset{\mathcal{D}}{=} X$ with cumulative distribution function (cdf) F_X, $i \in \mathbb{N}_1$

A2 $\underline{X}$: sequence of independent strictly positive rvs

- *Interpretation : the claim amounts are independent one from each other*

A3 $\underline{X}$ and claim number rv N are independent

- *Interpretation : the amount of a claim does not depend on the number of claims N*
- *Also : the number of claims does not depend on the amount(s) of claim(s)*

Consequences of the three assumptions :

1. Simple expression for the expectation of S :

$$E[S] = E[N] \times E[X] \tag{3}$$

2. Simple expression of the Laplace–Stieltjes transform (LST) of S :

$$\mathcal{L}_S(t) = E[\mathrm{e}^{-St}] = \mathcal{P}_N(\mathcal{L}_X(t)), \quad t \geq 0, \tag{4}$$

   where
   - LST of X : $\mathcal{L}_X(t) = E[\mathrm{e}^{-Xt}], \quad t \geq 0$
   - probability generating function (pgf) of N : $\mathcal{P}_N(s) = E[s^N], \quad |s| \leq 1$

3. Plenty of tools for computing F_S and risk measures defined in function of S
   - Chapters 6 and 10 of [Panjer and Willmot, 1992]
   - Chapter 4 of [Rolski et al., 1999], Chapter 2 of [Wuthrich, 2022]

4. Simplified estimation of the distributions for N and for X :
   - [Parodi, 2023], [Wuthrich, 2022], [Klugman et al., 2018]

Introduction

Classical CRM = scientific basis for non-life insurance pricing :

- [Parodi, 2023], [Wuthrich, 2022], [Albrecher et al., 2017].

In practice, assumptions A2 and A3 are not always verified :

- A3 not verified, [Gschlößl and Czado, 2007] : the authors found that the number and the size of claims are significantly dependent, in car insurance data set.
- A3 not verified, [Kousky and Cooke, 2009] : authors highlight dependency between flood damage and wind damage, using catastrophic loss data.

Therefore :

- One has to consider a larger class of CRMs . . .
  - . . . allowing for dependence relations between the claim amounts ;
  - . . . allowing for dependence relations between the claims and the claim number.
- Larger class = class of CRMs with dependence.

Agenda

Main objective :

- Study the class of CRMs with FGM dependence, a subset of the class of CRMs with dependence.
- FGM dependence = multivariate distributions defined with Farlie-Gumbel-Morgenstern copulas

Revised assumptions :

A1 $\underline{X} = \{X_j,\ j \in \mathbb{N}_1\}$: sequence of identically distributed strictly positive rvs

- *Interpretation : the individual claim amounts have the same marginal distribution*
- $X_j \stackrel{\mathcal{D}}{=} X$ with cdf F_X, $j \in \mathbb{N}_1$

newA2 $\underline{X} = \{X_j,\ j \in \mathbb{N}_1\}$: sequence of dependent strictly positive claim amount rvs

- *Interpretation : the claim amounts are dependent one from each other*

newA3 $\underline{X}$ and claim number rv N are dependent

- *Interpretation : the amount of a claim and the number of claims N are dependent*

fgmA4 for any $k \in \mathbb{N}_1$, the joint distribution $(N, X_1, \ldots, X_k)$ is defined with a $(k+1)$ variate FGM copula

Remarks :

1. A1 + newA2 + newA3 = class of CRMs with dependence
2. A1 + newA2 + newA3 + fgmA4 = class of CRMs with FGM dependence

Agenda

Some challenges with the class of CRMs with FGM dependence :

1. Find $E[S]$ and $\mathcal{L}_S$.
2. Explore strategies for computing F_S, the cdf of the aggregate claim amount rv S
3. Study the dependence properties of the class of CRMs with FGM dependence
   - Impact of newA2 + newA3 + fgmA4 on $E[S]$ and F_S
4. Numerical illustration

Brief literature review about CRMs with dependence :

1. Probabilistic perspective : [Liu and Wang, 2017], [Cossette et al., 2019]
2. Non-Life Actuarial Pricing : [Oh et al., 2021], [Nadarajah et al., 2021], etc.
3. The American Statistician : [Cohen, 2019]

Framework and notation

Definition :

- $\aleph = \aleph(F_N, F_X)$: class of CRMs with dependence, where the distribution of N and the distribution of X are fixed
  - Assumptions : A1 + newA2 + newA3
- A CRM with dependence is defined with a pair $(N, \underline{X}) \in \aleph$
- $(N, \underline{X}) \in \aleph$ corresponds to a CRM with a specific dependence structure governing the interactions between all the components of the pair $(N, \underline{X})$

Notation :

- $(N^{\perp}, \underline{X}^{(\perp,\perp)}) \in \aleph$: the classical CRM
  - Assumptions : A1 + A2 + A3
- $\aleph^{FGM} \subset \aleph$: class of CRMs with FGM dependence
  - Assumptions : A1 + newA2 + newA3 + fgmA4

Class of FGM copulas

Natural representation of a d–variate FGM copula C :

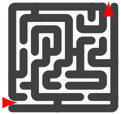
$$C(\boldsymbol{u}) = \prod_{m=1}^{d} u_m \left(1 + \sum_{k=2}^{d} \sum_{1 \le j_1 < \cdots < j_k \le d} \theta_{j_1 \ldots j_k} \overline{u}_{j_1} \overline{u}_{j_2} \ldots \overline{u}_{j_k}\right), \quad \boldsymbol{u} \in [0,1]^d, \tag{5}$$

where $\overline{u}_j = 1 - u_j$, $j \in \{1, \ldots, d\}$.

- C has $2^d - d - 1$ dependence parameters θ.
- To be admissible, the dependence parameters must satisfy 2^d constraints.

Now, next tasks :

1. How do we find $E[S]$ and $\mathcal{L}_S$?
2. Can we analyze the properties of S ?



Let us try the direct approach.

CRMs with FGM dependence

Consider a CRM with FGM dependence :

1. Joint cdf of two claim amounts :

$$F_{X_{j_1},X_{j_2}}(x_{j_1},x_{j_2}) = C(F_{X_{j_1}}(x_{j_1}),F_{X_{j_2}}(x_{j_2})),$$

   with

$$C(u_{j_1},u_{j_2}) = u_{j_1}u_{j_2} \times (1 + \theta_{j_1 j_2}\overline{u}_{j_1}\overline{u}_{j_2}),$$

   where $j_1 < j_2 \in \mathbb{N}_1$.

2. Joint cdf of three claim amounts :

$$F_{X_{j_1},X_{j_2},X_{j_3}}(x_{j_1},x_{j_2},x_{j_3}) = C(F_{X_{j_1}}(x_{j_1}),F_{X_{j_2}}(x_{j_2}),F_{X_{j_3}}(x_{j_3})),$$

   with

$$C(u_{j_1},u_{j_2},u_{j_3}) = u_{j_1}u_{j_2}u_{j_3} \times (1 + \theta_{j_1 j_2}\overline{u}_{j_1}\overline{u}_{j_2} + \theta_{j_1 j_3}\overline{u}_{j_1}\overline{u}_{j_3} + \theta_{j_2 j_3}\overline{u}_{j_2}\overline{u}_{j_3} + \theta_{j_1 j_2 j_3}\overline{u}_{j_1}\overline{u}_{j_2}\overline{u}_{j_3})$$

   where $j_1 < j_2 < j_3 \in \mathbb{N}_1$.

3. Etc.

CRMs with FGM dependence

Consider a CRM with FGM dependence (cont'd) :

1. Joint cdf of N and one claim amount :

$$F_{N,X_{j_1}}(n,x_{j_1}) = C(F_N(n),F_{X_{j_1}}(x_{j_1})),$$

with

$$C(u_0,u_{j_1}) = u_0 u_{j_2} \times (1 + \theta_{0j_1}\overline{u}_0\overline{u}_{j_1}),$$

where $j_1 \in \mathbb{N}_1$.

2. Joint cdf of N and two claim amounts :

$$F_{N,X_{j_1},X_{j_2}}(n,x_{j_1},x_{j_2}) = C(F_N(n),F_{X_{j_1}}(x_{j_1}),F_{X_{j_2}}(x_{j_2}),F_{X_{j_3}}(x_{j_3})),$$

with

$$C(u_0,u_{j_1},u_{j_2}) = u_0 u_{j_1} u_{j_2} \times (1 + \theta_{0j_1}\overline{u}_0\overline{u}_{j_1} + \theta_{0j_2}\overline{u}_0\overline{u}_{j_2} + \theta_{j_1j_2}\overline{u}_{j_1}\overline{u}_{j_2} + \theta_{0j_1j_2}\overline{u}_0\overline{u}_{j_1}\overline{u}_{j_2}),$$

where $j_1 < j_2 \in \mathbb{N}_1$.

3. Etc.

CRMs with FGM dependence

Problems with the direct approach :

1. Finding the expression of $E[S]$ and $\mathcal{L}_S$ rapidly becomes difficult to manage.
2. Computing F_S rapidly becomes difficult to manage.
3. Studying the properties of S becomes difficult to manage.

Solution :

1. Find a stochastic representation for S
2. Inspired from [Blier-Wong et al., 2022], [Blier-Wong et al., 2024] and [Blier-Wong et al., 2023]
3. $\Rightarrow$ 2 ingredients

CRMs with FGM dependence

Ingredient 1 : We define a countable sequence of uniformly distributed rvs with FGM dependence by $\boldsymbol{U} = \{U_j, j \in \mathbb{N}_0\}$, where

1. Uniform marginals :
$$U_j \sim Unif(0,1), \quad i \in \mathbb{N}_0;$$
2. FGM dependence structure : for any $d \in \{2,3,\dots\}$ and $j_1 < \dots < j_d \in \mathbb{N}_0$, we have
$$F_{U_{j_1},\dots,U_{j_d}}(u_{j_1},\dots,u_{j_d}) = C(u_{j_1},\dots,u_{j_d}),$$
where $C \in \mathcal{C}_d^{FGM}$.

We denote by $\mathcal{U}^{FGM}$ as the set of all countable sequences of uniformly distributed rvs with FGM dependence

UNIVERSITÉ LAVAL

Ingredient 2 : Stochastic representation (adapted from [Blier-Wong et al., 2022]) :

1. Let $\boldsymbol{I} = \{I_j, j \in \mathbb{N}_0\}$, with $I_j \sim Bern(\frac{1}{2})$, $j \in \mathbb{N}_0$, be a countable sequence of symmetric Bernoulli rvs.
2. We denote the set of all countable sequences of symmetric Bernoulli rvs by $\mathcal{I}$.
3. Then, there is a one-to-one correspondence between $\mathcal{I}$ and $\mathcal{U}^{FGM}$: for a specific $\boldsymbol{I} \in \mathcal{I}$, there is one and only one $\boldsymbol{U} \in \mathcal{U}^{FGM}$ such that, for any $d \in \mathbb{N}_1$, we have

$$U_0 \stackrel{\mathcal{D}}{=} (1 - I_0)V_{[1],0} + I_0 V_{[2],0}$$

$$U_j \stackrel{\mathcal{D}}{=} (1 - I_j)V_{[1],j} + I_j V_{[2],j}, \quad j \in \mathbb{N}_1,$$

where

- $\{(V_{j,1}, V_{j,2}), j \in \mathbb{N}_0\}$ = sequence of iid pairs of uniformly distributed rvs ;
- $V_{[1],j} = \min(V_{j,1}; V_{j,2}) \sim Beta(1,2)$, $i \in \mathbb{N}_0$;
- $V_{[2],j} = \max(V_{j,1}; V_{j,2}) \sim Beta(2,1)$, $i \in \mathbb{N}_0$;
- sequences $\boldsymbol{I}$ and $\{(V_{[1],j}, V_{[2],j}), j \in \mathbb{N}_1\}$ are independent.

Important to remember :

- Setting the dependence structure of $\boldsymbol{I}$ induces the FGM dependence structure of $\boldsymbol{U}$.

CRMs with FGM dependence

Definition 1 (CRMs with FGM dependence)

Here is the construction method of a CRM with FGM dependence, defined with $(N,\underline{X}) \in \aleph^{FGM} \subset \aleph$:

- For any $k \in \mathbb{N}_1$, the multivariate cdf of $(N,X_1,\dots,X_k)$ is defined by

$$F_{N,X_1,\dots,X_k}(n,x_1,\dots,x_k) = C(F_N(n),F_X(x_1),\dots,F_X(x_k)), \quad (n,x_1,\dots,x_k) \in \mathbb{N}_0 \times \mathbb{R}^k \tag{6}$$

  where C is a $(k+1)$ variate FGM copula, $k \in \mathbb{N}_1$.

- Then, the aggregate claim amount rv S abmits the representation

$$S = \sum_{j=1}^{\infty} F_X^{-1}(U_j)\mathbb{1}_{\{F_N^{-1}(U_0)\geq j\}}, \quad \text{where } \boldsymbol{U} \in \mathcal{U}^{FGM}. \tag{7}$$

Recipe :

- Mix ingredients 1 and 2 with Definition 1.
- Bake in 375°F (190°C) oven for 20 minutes.
- Let it cool before serving the next result.
- Enjoy ! ☺

Main result No.1 - Stochastic Representation

UNIVERSITÉ LAVAL

Theorem 2 (Stochastic representation of the aggregate claim amount rv S)

- *Consider a collective risk model with FGM dependence, as described in Definition 1.*
- *Then, the components of $(N, \underline{X})$ admits the representation*

$$N \overset{\mathcal{D}}{=} F_N^{-1}(U_0) \overset{\mathcal{D}}{=} (1 - I_0)N_{[1]} + I_0 N_{[2]}$$

$$X_j \overset{\mathcal{D}}{=} F_X^{-1}(U_j) \overset{\mathcal{D}}{=} (1 - I_j)X_{[1],j} + I_j X_{[2],j}, \quad j \in \mathbb{N}_1,$$

where sequence $\boldsymbol{I}$, pair $(N_{[1]}, N_{[2]})$, and sequence $\{(X_{[1],j}, X_{[2],j}), j \in \mathbb{N}_1\}$ are independent

- *Then, the rv S admits the representation*

$$S = \sum_{j=1}^{\infty} ((1 - I_j)X_{[1],j} + I_j X_{[2],j}) \times 1_{\{(1-I_0)N_{[1]} + I_0 N_{[2]} \geq i\}} \tag{8}$$

or

$$S = \begin{cases} \sum_{j=1}^{(1-I_0)N_{[1]}+I_0 N_{[2]}} ((1 - I_j)X_{[1],j} + I_j X_{[2],j}) & , \quad (1 - I_0)N_{[1]} + I_0 N_{[2]} > 0 \\ 0 & , \quad (1 - I_0)N_{[1]} + I_0 N_{[2]} = 0 \end{cases}. \tag{9}$$

Main result No.1 - Stochastic Representation

Advantages of (8) and (9) in Theorem 2 :

1. Provides a deeper understanding of the CRMs with FGM dependence
2. Facilitates the computation of the expectation of S and any quantities related to S

Algorithm for simulating samples of S :

1. Set $S = 0$
2. Sample I_0 and the pair $(N_{[1]}, N_{[2]})$
3. Calculate $n = (1 - I_0)N_{[1]} + I_0 N_{[2]}$
4. If $n > 0$:
   - Sample n pairs $(X_{[1],1}, X_{[2],1})$, . . ., $(X_{[1],n}, X_{[2],n})$
   - Sample $(I_1, \dots, I_n)$ given I_0
   - Calculate $S = \sum_{j=1}^{n}((1 - I_j)X_{[1],j} + I_j X_{[2],j})$
5. Return to Item 1

Important to remember :

- Setting the dependence structure of $\boldsymbol{I}$ induces the FGM dependence structure of $(N, \underline{X})$.

Main result No.2 - Expectation

UNIVERSITÉ LAVAL

Theorem 3 (Expectation of S)

- *Let* $(N,\underline{X}) \in \aleph^{FGM}$*, with* $E[N] < \infty$ *and* $E[X] < \infty$.
- *Assume* $(I_0, I_j) \stackrel{\mathcal{D}}{=} (I_0, I_1)$ *with* $f_{I_0,I_j} = f_{I_0,I_1}$ *for all* $j \in \mathbb{N}_1$.
- *Then, using the stochastic representation of* S *in Theorem 2, we find*

$$E[S] = \sum_{(i_0,i_1)\in\{0,1\}^2} f_{I_0,I_1}(i_0,i_1)\mu_{N_{[1+i_0]}}\mu_{X_{[1+i_1]}} \tag{10}$$

Interpretation of (10) :

- $E[S]$ only depends on the dependence relation between the claim number and the amount of claim.
- The 4 values of f_{I_0,I_1} modulate the weights given to the following 4 values :

$$\mu_{N_{[1]}}\mu_{X_{[1]}} \quad \mu_{N_{[1]}}\mu_{X_{[2]}} \quad \mu_{N_{[2]}}\mu_{X_{[1]}} \quad \mu_{N_{[2]}}\mu_{X_{[2]}}$$

Main result No.2 - Expectation

Alternative expression of $E[S]$

- In [Blier-Wong et al., 2022], it is shown that

$$f_{I_0,I_1}(i_0,i_1) = \frac{1+(-1)^{i_0+i_1}\theta_{01}}{4}, \quad (i_0,i_1) \in \{0,1\}^2, \quad j,k \in \mathbb{N}_1$$

- Then, (10) becomes

$$E[S] = \underbrace{E[N]E[X]}_{E[S^{(\perp,\perp)}]} + \underbrace{\frac{\theta_{01}}{4} \times \sum_{(i_0,i_1)\in\{0,1\}^2} (-1)^{i_0+i_1} \mu_{N_{[1+i_0]}} \mu_{X_{[1+i_1]}}}_{\text{additional term due to FGM dependence}}, \quad \theta_{01} \in [-1,1]. \tag{11}$$

Interpretation of (11) :

- Explicit impact on $E[S]$ of the dependence relation between the claim number rv N and the claim amount rvs $\underline{X}$.
- Independence between N and $\underline{X}$ $\Rightarrow$ dependence parameter $\theta_{01} = 0 \Rightarrow E[S] = E[N]E[X]$.

Main result No.3 - LST

Define the rv $K_n = I_1 + \cdots + I_n$, for $n \in \mathbb{N}_1$:

- K_n is a random variable with support $\{0, \ldots, n\}$.
- $E[K_n] = \frac{n}{2}$.

LST of the aggregate claim amount rv S :

$$\mathcal{L}_S(t) = \frac{1}{2}\left(\Pr(N_{[1]} = 0) + \sum_{n=1}^{\infty} \Pr(N_{[1]} = n) \sum_{k=0}^{n} \Pr(K_n = k | I_0 = 0) \mathcal{L}_{X_{[1]}}(t)^{n-k} \mathcal{L}_{X_{[2]}}(t)^k\right) + \frac{1}{2}\left(\Pr(N_{[2]} = 0) + \sum_{n=1}^{\infty} \Pr(N_{[2]} = n) \sum_{k=0}^{n} \Pr(K_n = k | I_0 = 1) \mathcal{L}_{X_{[1]}}(t)^{n-k} \mathcal{L}_{X_{[2]}}(t)^k\right), \quad t \geq 0. \quad (12)$$

LST of $S^{(\perp,\perp)}$, the aggregate claim amount rv within the classical CRM :

$$\mathcal{L}_{S^{(\perp,\perp)}}(t) = \Pr(N = 0) + \sum_{n=1}^{\infty} \Pr(N = k) \mathcal{L}_X(t)^n = \mathcal{P}_N(\mathcal{L}_X(t)) \quad (13)$$

Example No.1

Assume that, for $n \in \mathbb{N}_1$,

$$\Pr(K_n = k | I_0 = 0) = \binom{n}{k} (1-\alpha)^k \alpha^{n-k}, \quad k \in \{0,1,\ldots,n\}$$
$$\Pr(K_n = k | I_0 = 1) = \binom{n}{k} \alpha^k (1-\alpha)^{n-k}, \quad k \in \{0,1,\ldots,n\},$$

where $\alpha \in (0,1)$.

LST of the aggregate claim amount rv, denoted by $S^{(\alpha)}$, is

$$\mathcal{L}_{S^{(\alpha)}}(t) = \frac{1}{2}\mathcal{P}_{N_{[1]}}\left(\alpha \mathcal{L}_{X_{[1]}}(t) + (1-\alpha)\mathcal{L}_{X_{[2]}}(t)\right) + \frac{1}{2}\mathcal{P}_{N_{[2]}}\left((1-\alpha)\mathcal{L}_{X_{[1]}}(t) + \alpha \mathcal{L}_{X_{[2]}}(t)\right), \quad t \geq 0. \quad (14)$$

Expectation of the aggregate claim amount rv $S^{(\alpha)}$:

$$E[S^{(\alpha)}] = \frac{1}{2}\mu_{N_{[1]}}\left(\alpha \mu_{X_{[1]}} + (1-\alpha)\mu_{X_{[2]}}\right) + \frac{1}{2}\mu_{N_{[2]}}\left((1-\alpha)\mu_{X_{[1]}} + \alpha \mu_{X_{[2]}}\right) \quad (15)$$

Example No.2

Consider the CRM assuming that the components of $\boldsymbol{I}$ are comonotonic :

$$I_0 \stackrel{\mathcal{D}}{=} I, \quad I_j \stackrel{\mathcal{D}}{=} I, \quad j \in \mathbb{N}_1, \qquad \text{where } I \sim Bern(\frac{1}{2}).$$

Non-zero values of the joint pmf $f_{I_0,I_1,\ldots,I_k}$ for any $k \in \mathbb{N}_1$:

i_0	i_1	$\cdots$	i_k	$f_{I_0,I_1,\ldots,I_k}(i_0,i_1,\ldots,i_k)$
0	0	$\cdots$	0	$\frac{1}{2}$
1	1	$\cdots$	1	$\frac{1}{2}$

Dependence structure of $(N,\underline{X})$: for any $k \in \mathbb{N}_1$, the $(k+1)$ variate FGM copula is

$$C(\boldsymbol{u}) = \prod_{j=0}^{k} u_j \left(1 + \sum_{m=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \sum_{1 \leq j_1 < \cdots < j_{2m} \leq d} \overline{u}_{j_1} \cdots \overline{u}_{j_{2m}} \right), \quad \boldsymbol{u} = (u_0,u_1,\ldots,u_k) \in [0,1]^{k+1}, \qquad (16)$$

where $\lfloor y \rfloor$ is the floor function.

1. Dependence relation between the claims amounts : positive
2. Dependence relation between the number of claims and a claim amount : positive

Example No.2

Using Theorem 2, the representation of the aggregate claim amount rv in (8) becomes

$$S^{(\triangle,\triangle)} = (1-I)\sum_{j=1}^{\infty} X^{[1]}\mathbb{1}_{\{N^{[1]}\geq j\}} + I\sum_{j=1}^{\infty} X^{[2]}\mathbb{1}_{\{N^{[2]}\geq j\}} \tag{17}$$

Interpretation of (17) :

- $I = 0 \quad \Rightarrow \quad S^{(\triangle,\triangle)}$ = sum of a small number of small claims

  or

- $I = 1 \quad \Rightarrow \quad S^{(\triangle,\triangle)}$ = sum of a large number of large claims

Expectation and LST of $S^{(\triangle,\triangle)}$:

$$E[S^{(\triangle,\triangle)}] = \frac{1}{2}\mu_{N^{[1]}}\mu_{X^{[1]}} + \frac{1}{2}\mu_{N^{[2]}}\mu_{X^{[2]}}$$

$$\mathcal{L}_{S^{(\triangle,\triangle)}}(t) = \frac{1}{2}\mathcal{P}_{N^{[1]}}\left(\mathcal{L}_{X^{[1]}}(t)\right) + \frac{1}{2}\mathcal{P}_{N^{[2]}}\left(\mathcal{L}_{X^{[2]}}(t)\right), \quad t \geq 0.$$

Example No.3

Consider the CRM assuming that the components of $\boldsymbol{I}$ are such that

$$I_0 \overset{\mathcal{D}}{=} 1 - I, \quad I_j \overset{\mathcal{D}}{=} I, \quad j \in \mathbb{N}_1, \qquad \text{where } I \sim Bern(\frac{1}{2}).$$

Non-zero values of the joint pmf $f_{I_0,I_1,\ldots,I_k}$ for any $k \in \mathbb{N}_1$:

i_0	i_1	$\cdots$	i_k	$f_{I_0,I_1,\ldots,I_k}(i_0,i_1,\ldots,i_k)$
1	0	$\cdots$	0	$\frac{1}{2}$
0	1	$\cdots$	1	$\frac{1}{2}$

Dependence structure of $(N,\underline{X})$: for any $k \in \mathbb{N}_1$, the $(k+1)$ variate FGM copula is

$$C(\boldsymbol{u}) = \prod_{j=0}^{k} u_j \left(1 - \sum_{m=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \sum_{1 \le j_1 < \cdots < j_{2m} \le d} \overline{u}_{j0} \overline{u}_{j_1} \cdots \overline{u}_{j_{2m-1}} + \sum_{m=1}^{\left\lfloor \frac{k+1}{2} \right\rfloor} \sum_{1 \le j_1 < \cdots < j_{2m} \le d} \overline{u}_{j_1} \cdots \overline{u}_{j_{2m}} \right), \quad \boldsymbol{u} \in [0,1]^{k+1}. \tag{18}$$

1. Dependence relation between the claims amounts : positive
2. Dependence relation between the number of claims and a claim amount : negative

Example No.3

Using Theorem (2), the representation of the aggregate claim amount rv in (8) becomes

$$S^{(\nabla,\triangle)} = (1-I)\sum_{j=1}^{\infty} X^{[2]} \mathbb{1}_{\{N^{[1]} \geq j\}} + I \sum_{j=1}^{\infty} X^{[1]} \mathbb{1}_{\{N^{[2]} \geq j\}}, \tag{19}$$

Interpretation of (19) :

- $I = 0 \quad \Rightarrow \quad S^{(\nabla,\triangle)}$ = sum of a small number of large claims

or

- $I = 1 \quad \Rightarrow \quad S^{(\nabla,\triangle)}$ = sum of a large number of small claims

Expectation and the LST of $S^{(\nabla,\triangle)}$ are given by

$$E[S^{(\nabla,\triangle)}] = \frac{1}{2}\mu_{N^{[1]}}\mu_{X^{[2]}} + \frac{1}{2}\mu_{N^{[2]}}\mu_{X^{[1]}}$$

$$\mathcal{L}_{S^{(\nabla,\triangle)}}(t) = \frac{1}{2}\mathcal{P}_{N^{[1]}}\left(\mathcal{L}_{X^{[2]}}(t)\right) + \frac{1}{2}\mathcal{P}_{N^{[2]}}\left(\mathcal{L}_{X^{[1]}}(t)\right), \quad t \geq 0.$$

Extremal element of the class of CRMs with FGM dependence

Dependence properties :

1. For any $k \in \mathbb{N}_1$, the CRM of Example No.2 is the extremal element under the supermodular order $\preceq_{sm}$ of the class of CRMs with FGM dependence :

$$(N, X_1, \ldots, X_k) \preceq_{sm} (N^{\triangle}, X_1^{(\triangle,\triangle)}, \ldots, X_k^{(\triangle,\triangle)}), \quad \forall (N, \underline{X}) \in \aleph. \tag{20}$$

2. The inequality in (20) implies that

$$S \preceq_{icx} S^{(\triangle,\triangle)}, \quad \forall (N, \underline{X}) \in \aleph, \tag{21}$$

where $\preceq_{icx}$ is the increasing convex order.

3. Examples of implications of (21) :
   - TVaR :
   $$TVaR_{\kappa}(S) \leq TVaR_{\kappa}(S^{(\triangle,\triangle)}), \quad \kappa \in (0,1), \quad \forall (N, \underline{X}) \in \aleph.$$
   - stop-loss function :
   $$\pi_S(x) \leq \pi_{S^{(\triangle,\triangle)}}(x), \quad x \geq 0, \quad \forall (N, \underline{X}) \in \aleph.$$

Numerical illustration

Assumptions :

- Claim number : $N \sim NBinom(r,q)$, $r = \frac{1}{2}$, $q = \frac{1}{11}$, $E[N] = 5$
- Claim amounts : $X_j \sim X \sim MixedErlang$, $j \in \mathbb{N}_1$, with

$$\mathcal{L}_X(t) = 0.4\left(\frac{0.01}{0.01+t}\right) + 0.3\left(\frac{0.01}{0.01+t}\right)^2 + 0.2\left(\frac{0.01}{0.01+t}\right)^3 + 0.1\left(\frac{0.01}{0.01+t}\right)^4, \quad t \geq 0,$$

and $E[X] = 200$

Numerical results :

	Neg Dep btw N and claims	Classical CRM	Extremal element of $\aleph^{FGM}$
	$S^{(\triangledown,\triangle)}$	$S^{(\perp,\perp)}$	$S^{(\triangle,\triangle)}$
$E[S]$	697	1000	1303
$\sqrt{Var(S)}$	976	1725	2194
$VaR_{0.99}(S)$	4446	8345	10210
$TVaR_{0.99}(S)$	5629	11116	13042
$\Pr(S=0) = \Pr(N=0)$	0.3015	0.3015	0.3015

Numerical illustration

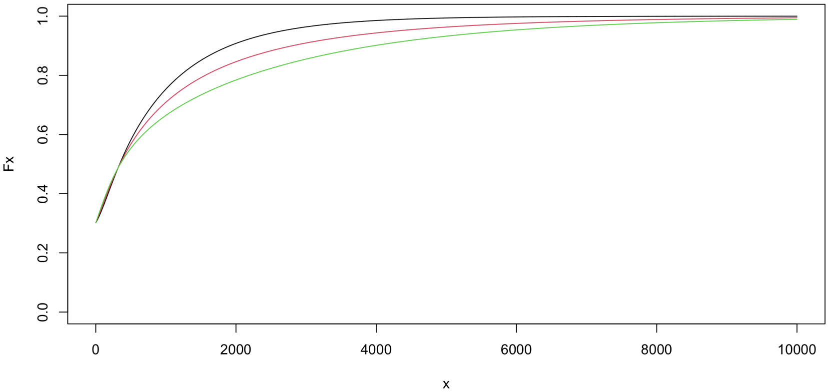


Illustration – Values of $F_{S(\triangledown,\triangle)}$, $F_{S(\perp,\perp)}$, and $F_{S(\triangle,\triangle)}$

Conclusion

Main results about the class of CRMs with FGM dependence :

- Find a stochastic representation for the aggregate claim amount rv S.
- Derive the expectation and the LST of S.
- Study the properties of the class of CRMs with FGM dependence.

In our paper, we also derive the following results :

1. We find closed-expressions when N follows a Geometric distribution and X follows an Exponential distribution.
2. We show that, if X follows a mixed Erland distribution, than S also follows a mixed Erland distribution.
3. We describe strategies to compute numerical approximations of F_S based on discretization technics and FFT algorithm.
4. We introduce a class of CRMs with FGM dependence based on bivariate random factors
5. We explore multivariate extensions of the class of CRMs with FGM dependence.

Conclusion

Many thanks to Jonathan, Maree, and their colleages for this magnificient conference !

Thank you for your attention !

Appendix - Textbooks

UNIVERSITÉ LAVAL

Description and analysis of the classical risk model, as defined via $(N^{\perp},\underline{X}^{\perp}) \in \aleph$:

- Chapter 4 of [Gerber, 1979]
- Section 2.9, chapter 6, and chapter 10 of [Panjer and Willmot, 1992]
- Chapter 4 of [Rolski et al., 1999]
- Chapter 3 of [Kaas et al., 2009]
- Chapter 2 of [Wuthrich, 2022]

[Panjer and Willmot, 1992] :

- In Section 2.9, the authors examine the properties of the distribution of random sum $S^{(\perp,\perp)}$ defined within the classical risk model associated to $(N^{\perp},\underline{X}^{\perp}) \in \aleph$.
- The authors devote the entire Chapter 6 to the study of the collective risk model under different choices of distributions for N and X (78 pages).
- In Chapter 10, the authors investigate the tail behavior of the distribution of $S^{(\perp,\perp)}$ under the collective risk model under different choices of distributions for N and X (18 pages).

Appendix - A sum of a random number of random variables

References on the topic "A sum of a random number of random variables" out of the traditional textbooks in Actuarial Science :

- [Robbins, 1948]
- [Rényi, 1957]
- [Melamed, 1989]
- [Thomas, 1972]
- [Klebanov et al., 2012]
- [Cohen, 2019]
- etc.

References I

Albrecher, H., Teugels, J. L., and Beirlant, J. (2017).
Reinsurance : Actuarial and Statistical Aspects.
John Wiley & Sons.

Blier-Wong, C., Cossette, H., and Marceau, E. (2022).
Stochastic representation of FGM copulas using multivariate Bernoulli random variables.
Computational Statistics & Data Analysis, 173.

Blier-Wong, C., Cossette, H., and Marceau, E. (2023).
Risk aggregation with fgm copulas.
Insurance : Mathematics and Economics, 111 :102–120.

References II

Blier-Wong, C., Cossette, H., and Marceau, E. (2024).
Exchangeable FGM copulas.
Advanced in Applied Probability.

Cohen, J. E. (2019).
Sum of a random number of correlated random variables that depend on the number of summands.
The American Statistician, 73(1) :56–60.

Cossette, H., Marceau, E., and Mtalai, I. (2019).
Collective risk models with dependence.
Insurance : Mathematics and Economics, 87 :153–168.

Gerber, H. U. (1979).
An Introduction to Mathematical Risk Theory.
S. S. Huebner Foundation for Insurance Education.

References III

Gschlößl, S. and Czado, C. (2007).
Spatial modelling of claim frequency and claim size in non-life insurance.
Scandinavian Actuarial Journal, 2007(3) :202–225.

Kaas, R., Goovaerts, M. J., Dhaene, J., and Denuit, M., editors (2009).
Modern Actuarial Risk Theory : Using R.
Springer, Berlin, 2. ed., corr. print edition.

Klebanov, L. B., Kakosyan, A. V., Rachev, S., and Temnov, G. (2012).
On a class of distributions stable under random summation.
Journal of Applied Probability, 49(2) :303–318.

References IV

Klugman, S. A., Panjer, H. H., and Willmot, G. E. (2018).
Loss Models : From Data to Decisions.
Wiley Series in Probability and Statistics. Society of Actuaries ; John Wiley and Sons, Inc, [Schaumburg, IL] : Hoboken, NJ, fifth edition edition.

Kousky, C. and Cooke, R. M. (2009).
The Unholy Trinity : Fat Tails, Tail Dependence, and Micro-Correlations.
SSRN Electronic Journal.

Liu, H. and Wang, R. (2017).
Collective Risk Models with Dependence Uncertainty.
ASTIN Bulletin : The Journal of the IAA, 47(2) :361–389.

References V

Melamed, J. (1989).
Limit theorems in the set up of sumnation of a random number of independent identically distributed random variables.
Stability Problems for Stochastic Models, pages 194–228.

Nadarajah, S., Kwong, H. S., and Tank, F. (2021).
Compound sum distributions with dependence.
Statistics, 55(2) :409–425.

Oh, R., Jeong, H., Ahn, J. Y., and Valdez, E. A. (2021).
A multi-year microlevel collective risk model.
Insurance : Mathematics and Economics, 100 :309–328.

Panjer, H. H. and Willmot, G. E. (1992).
Insurance risk models.
Society of Actuaries.

Parodi, P. (2023).
Pricing in general insurance (2nd edition).
CRC press.

Rényi, A. (1957).
On the asymptotic distribution of the sum of a random number of independent random variables.
Acta Math, 8 :193–199.

Robbins, H. (1948).
The asymptotic distribution of the sum of a random number of random variables.
Bulletin of the American Mathematical Society, 54(12) :1151–1161.

Rolski, T., Schmidli, H., Schmidt, V., and Teugels, J. L. (1999).
Stochastic Processes for Insurance and Finance.
John Wiley & Sons.

References VII

Thomas, D. (1972).
On limiting distributions of a random number of dependent random variables.
The Annals of Mathematical Statistics, pages 1719–1726.

Wuthrich, M. V. (2022).
Non-life insurance : mathematics & statistics.
Available at SSRN 2319328.

Abstract

Collective risk models, in which the aggregate claim amount of a portfolio is defined in terms of as a sum of a random number (frequency) of random claim amounts (severities), play a crucial role. In these models, the classical approach is to assume that the random number of claims and their amounts are independent, even if this might not always be the case. We consider a class of collective risk models, in which the dependence structure of the random number of claims and the individual claim amounts is defined in terms a multivariate Farlie-Gumbel-Morgenstern (FGM) copula. By leveraging a one-to-one correspondence between the family of FGM copulas and the family of multivariate symmetric Bernoulli random vectors, we find closed-form expressions for the moments and Laplace-Stieltjes transform of the aggregate claim amount. We examine the dependence properties of the proposed class of collective risk models. Even if the Farlie-Gumbel-Morgenstern copula may only induce moderate dependence, we show through numerical examples that the cumulative effect of dependence can generate large ranges of values for the expectation, the variance, and risk measures (such as the Tail-Value-at-Risk and the entropic risk measure) of the aggregate claim amount. We present applications of the proposed class of collective risk models in various contexts of non-life insurance.

Keywords

Keywords : Stochastic representation, Mixed Erlang distribution, random sum, FGM copulas