Model uncertainty and applications in insurance design

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Actuarial, Finance, Risk and Insurance Congress
July 25, 2023, Victoria Falls, Zimbabwe

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Insurance 101

Insurance is an effective risk management tool used to protect against contingent losses of market participants.

$$X$$

$$\underbrace{I(X)}_{\text{Reimbursement}} \qquad \underbrace{X - I(X)}_{\text{Insured's retained loss}}$$

$$\underbrace{I(X) - \underbrace{\pi(I(X))}_{\text{premium}}}_{\text{Insurer's loss or benefit}} \qquad \underbrace{X - I(X) + \underbrace{\pi(I(X))}_{\text{premium}}}_{\text{Insured's total loss}}$$

where $I \in \mathcal{I}$ is an admissible indemnity function, and π is a premium principle.

Classical optimization problems in insurance

Popular optimal (re-)insurance design problems:

1. Maximize expected utility:

$$\max_{I \in \mathcal{I}} \mathbb{E}\left[v(w - X + I(X) - \pi(I(X)))\right].$$

   - Arrow (1963): optimality of a stop-loss contract.
   - Gerber(1979), Young (1999), Kaluszka (2001,2005), etc.

2. Minimize risk measure:

$$\min_{I \in \mathcal{I}} \rho\left(X - I(X) + \pi(I(X))\right).$$

   - Cai et al. (2008), Kaluszka and Okolewki (2008), Bernard and Tian (2009), Cheung (2010), etc.

All problems are considered under the assumption that **the distribution of X is known.** Can we take this assumption for granted?

Uncertainty

From data to models

- Parameter uncertainty
  Estimation error, simulation error, etc
- Model uncertainty
  Choice of models, complexity of models, etc.

Distributional uncertainty

- Only partial information about the true distribution are observed from the historical data.
- Changes of the underlying risks
- In a conservative decision, the **worst-case distribution** is important

Worst-case scenario

- Suppose an agent faces an underlying risk X
  - ℓ is the loss function/strategy the agent adopts.
  - ρ is the risk measure used to quantify the agent's risk exposure
  - $\mathcal{S}$ is the uncertainty set includes all distributions of alternative risks considered
- From the perspective of risk management, the **worst-case scenario** in which the agent has the largest risk exposure is of special interests.
- The agent's optimization problem with model uncertainty can be formulated as

$$\min_{\ell} \underbrace{\sup_{F \in \mathcal{S}} \rho(\ell(X^F)),}_{\text{worst-case scenario}} \qquad X^F \sim F.$$

Literature

In a financial market, under the mean-variance constraints

- Theorem 1 in El Ghaoui et al. (2003) solves the worst-case VaR where $\mathrm{VaR}_u(X^F) = F^{-1}(u)$
- Theorem 2.9 in Chen et al. (2011) solves the worst-case ES where $\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_u(X)\mathrm{d}u$
- Li (2018) determines the closed-from solutions for worst-case law invariant coherent risk measures

Under both the mean-variance and Wasserstein distance constraints

- Bernard et al. (2020b) Consider both the worst-case and the best-case scenarios:

$$\sup_{F\in\mathcal{S}} \rho(X^F), \qquad \text{and} \qquad \inf_{F\in\mathcal{S}} \rho(X^F)$$

  for a distortion risk measure ρ.

Literature

In the literature of insurance

- Asimit et al. (2017): for $\rho = \mathrm{VaR}, \mathrm{ES}$,

$$\begin{cases} \min_{(I,P)\in\mathcal{I}\times\mathbb{R}} \max_{k\in\mathcal{M}} \{\rho_{\mathcal{P}_k}(X - I(X) + P)\}, \\ \text{s.t. } \omega_0 + (1+\theta)\mathbb{H}_{\mathcal{P}_k}(I(X)) \le P \le \bar{P}, \forall k \in \mathcal{M}. \end{cases}$$

where $\mathcal{P}_k$, $k \in \mathcal{M}$ includes finite many probability measures.

- Birghila and Pflug (2019)

$$\min_{I\in\mathcal{I}} \max_{F\in\mathcal{C}} \{\rho(X^F - I(X^F) + \pi(I(X^F))\}, \quad \text{s.t. } \pi(I(X^F)) \le B$$

where $\mathcal{C}$ is the convex cone of n reference distributions.

- Liu and Mao (2021): for $\rho = \mathrm{VaR}, \mathrm{ES}$,

$$\min_{d\ge 0} \sup_{F\in\mathcal{S}(\mu,\sigma)} \rho(X^F \wedge d + (1+\theta)\mathbb{E}^F[(X^F - d)_+]).$$

where $\mathcal{S}(\mu, \sigma)$ gives first & second moments constraints.

In this talk, we focus on the **worst-case scenario** for an agent

$$\sup_{F \in \mathcal{S}} \rho_h(\ell(X^F)), \qquad X^F \sim F$$

where

- ρ_h is a **distortion risk measure** (e.g. Dhaene et al. (2012)):

$$\rho_h(X^F) = -\int_{-\infty}^{0} h(F(x))\mathrm{d}x + \int_{0}^{\infty} 1 - h(F(x))\mathrm{d}x = \int_{0}^{1} \gamma(u) F^{-1}(u)\mathrm{d}u,$$

  where $h : [0,1] \mapsto [0,1]$ is non-decreasing (convex) with $h(0) = 0$ and $h(1) = 1$, and $\gamma(u) = h'(u)$, $0 < u < 1$
- $\mathcal{S}$ is the **uncertainty set** defined by Wasserstein distance constraints
- ℓ is the **loss function/strategy** the agent adopts.

Uncertainty set with Wasserstein distance constraint

- For $X \sim F$ and $Y \sim G$, for $k \geq 1$, the **Wasserstein distance** is

$$W_k(X,Y) = W_k(F,G) = \left(\int_0^1 \left| F^{-1}(x) - G^{-1}(x) \right|^k \right)^{1/k}.$$

- The uncertainty set with Wasserstein distance constraint

$$\mathcal{S} = \{\text{r. v. } Y : W_k(Y,X) \leq \varepsilon\} = \{\text{distribution } G : W_k(G,F) \leq \varepsilon\}$$

where

  - $X \sim F$ is the reference distribution
  - ε is the tolerant bound for the Wasserstein distance

- Consider **worst-case scenario**

$$\begin{aligned}\sup_{G \in \mathcal{S}} \rho_h(X^G) &= \sup \left\{ \rho_h(X^G) : W_k(G,F) \leq \varepsilon \right\} \\ &= \sup \left\{ \int_0^1 \gamma(u) G^{-1}(u) \mathrm{d}u : W_k(G,F) \leq \varepsilon \right\}\end{aligned}$$

Uncertainty set with Wasserstein distance constraint

Theorem (Proposition 4 in Liu et al. (2022))

For a continuous and convex distortion function h,

$$\sup\left\{\rho_h(X^G) : W_k(G,F) \le \varepsilon\right\} = \rho_h(X^F) + \varepsilon\|\gamma\|_q,$$

where $q = (1 - 1/k)^{-1}$ *with the convention* $0^{-1} = \infty$, *and* $||\cdot||_q$ *is the* $\mathcal{L}_q$*-norm.*

For $k > 1$, *the above maximum value is attained by the worst-case distribution*

$$G^{-1}(t) = F^{-1}(t) + \varepsilon\frac{(\gamma(t))^{q-1}}{\|\gamma\|_q^{q/k}}, \qquad 0 < t < 1.$$

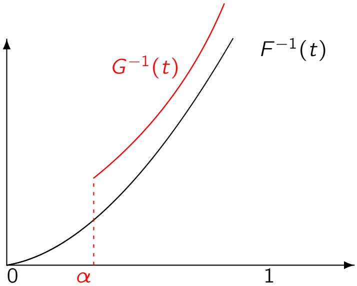
Example – Expected shortfall (ES)

Take $\rho = \mathrm{ES}_\alpha$ for $\alpha \in (0,1)$, then $\rho(X) = \int_0^1 \mathrm{VaR}_t(X)\mathrm{d}h(t)$, where

$$h(t) = \frac{1}{1-\alpha}(t-\alpha)^+ \quad \text{and} \quad \gamma(t) = \frac{1}{1-\alpha}\mathbb{1}_{[\alpha,1]}.$$

The worst-case value is

$$\sup\left\{\mathrm{ES}_\alpha(X^G) : W_k(G,F) \le \varepsilon\right\} = \mathrm{ES}_\alpha(X^F) + \varepsilon \cdot (1-\alpha)^{-1/k}.$$



Uncertainty set with Wasserstein distance constraint

- Uncertainty set is

$$\mathcal{S} = \{G : W_k(G, F) \leq \varepsilon\}$$

where $X^F \sim F$ is considered as a reference distribution, and ε is the tolerant bound for the Wasserstein distance.

- Consider the worst-case scenario:

$$\sup_{G \in \mathcal{S}} \rho_h(\ell(X^G)) = \sup \left\{\rho_h(\ell(X^G)), W_k(G, F) \leq \varepsilon\right\},$$

with two types of **loss functions**:

  - Stop-loss function: (optimal to the utility maximization)

$$\ell(x) = (x - d)^+$$

  - Limited-loss function: (optimal to the VaR minimization)

$$\ell(x) = \min\{x, M\}$$

Stop-loss function

- Take $\ell_1(x) = (x-d)^+$ for $d > \text{ess-inf}(X)$
- Worst-case risk measure

$$\sup\left\{\rho_h((X^G - d)^+) : W_k(G,F) \leq \varepsilon\right\}$$

- For $\beta \in [0,1]$, define $\gamma_{1,\beta} := \gamma \cdot \mathbb{I}_{[\beta,1]}$ which is again a non-negative and increasing function.

$$\begin{aligned}
\sup_{G\in\mathcal{S}} \rho_h\left((X^G - d)^+\right) &= \sup_{G\in\mathcal{S}} \int_{G(0)}^1 \gamma(u)\left(G^{-1}(u) - d\right)\mathrm{d}u \\
&= \sup_{G\in\mathcal{S}} \max_{\beta\in[0,1]} \int_{\beta}^1 \gamma(u)\left(G^{-1}(u) - d\right)\mathrm{d}u \\
&= \sup_{\beta\in[0,1]} \underbrace{\sup_{G\in\mathcal{S}} \int_0^1 \gamma_{1,\beta}(u)\left(G^{-1}(u) - d\right)\mathrm{d}u}_{\text{worst-case without transform}},
\end{aligned}$$

Wasserstein distance constraint and stop-loss transform

Theorem (Cai et al. (2022b))

Take $k \geq 1$ and $q = (1 - 1/k)^{-1}$.

(i) *The worst-case risk measures value is*

$$\sup\left\{\rho_h((X^G - d)^+) : W_k(G, F) \leq \varepsilon\right\}$$
$$= \max_{\beta \in [0,1]} \left\{\int_0^1 \gamma_{1,\beta}(u) F^{-1}(u) \mathrm{d}u + \varepsilon \|\gamma_{1,\beta}\|_q - d\|\gamma_{1,\beta}\|_1\right\}.$$

(ii) *The worst-case distribution is given by*

$$G^{-1}(t) = F^{-1}(t) + \varepsilon \cdot \frac{(\gamma_{1,\beta^*}(t))^{q-1}}{\|\gamma_{1,\beta^*}\|_q^{q/k}}, \quad 0 < t < 1.$$

where β^* is the maximizer in (i).

Example - Expected shortfall

Take $\rho = \mathrm{ES}_\alpha$ for some $\alpha \in (0,1)$.

(i) The worst-case value is

$$\begin{aligned} &\sup\left\{\mathrm{ES}_\alpha((X^G - d)^+) : W_k(G,F) \le \varepsilon\right\} \\ &= \frac{1}{1-\alpha} \max_{\beta\in[\alpha,1]} \left\{(1-\beta)\left(\mathrm{ES}_\beta(X^{\hat{F}}) - d\right) + \varepsilon(1-\beta)^{1/\bar{k}}\right\}. \end{aligned}$$

(ii) The worst-case distribution is

$$G^{-1}(t) = F^{-1}(t) + \varepsilon \cdot \frac{(\gamma_{1,\beta^*}(t))^{q-1}}{\|\gamma_{1,\beta^*}\|_q^{q/k}}$$

where $\gamma_{1,\beta^*} = \frac{1}{1-\alpha}\mathbb{I}_{[\alpha\vee\beta^*,1]}$ and β^* is the solution to the maximization problem in (i).

Example - Wang's premium

- **Wang's premium:**

$$\rho_h(X) = \int_0^\infty (1 - h(F(x)))\,\mathrm{d}u$$

where

$$h(u) = 1 - \Phi(\Phi^{-1}(1-u) + 0.5), \quad 0 \le u \le 1$$

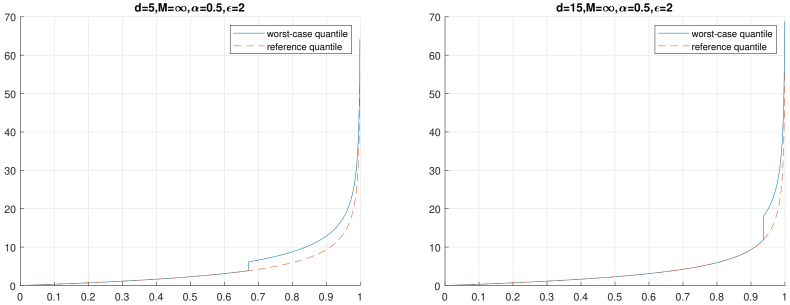
- Take Pareto loss distribution: (heavy tail, non-negative, etc)

$$F(x) = 1 - \left(\frac{12}{x+12}\right)^4, \quad x \ge 0$$

- $\varepsilon = 2$ and $k = 2$, i.e., $\sup\left\{\rho_h(\ell(X^G)), W_2(G,F) \le 2\right\}$

Example - Wang's premium

Figure: Worst-case distributions with stop-loss function.



Limited-loss function

- Take $\ell_2(x) = \max\{x, M\} = x \wedge M$ for $M < \text{ess-sup}(X)$
- Worst-case risk measure

$$\sup\left\{\rho_h(X^G \wedge M) : W_k(G, F) \leq \varepsilon\right\}$$

- Define $\gamma_{2,\beta} = \gamma \cdot \mathbb{I}_{[0,\beta]}$ which is not an increasing function

$$\begin{aligned}\sup_{G\in\mathcal{S}} \rho_h(X^G \wedge M) &= M + \sup_{G\in\mathcal{S}} \int_0^{G(d)} \gamma(u)\left(G^{-1}(u) - M\right)\mathrm{d}u \\ &= M + \sup_{G\in\mathcal{S}} \min_{\beta\in[0,1]} \int_0^1 \gamma_{2,\beta}(u)\left(G^{-1}(u) - M\right)\mathrm{d}u \\ &= M + \min_{\beta\in[0,1]} \sup_{G\in\mathcal{S}} \int_0^1 \gamma_{2,\beta}(u)\left(G^{-1}(u) - M\right)\mathrm{d}u,\end{aligned}$$

by the Min-Max theorem (e.g., Sion et al. (1958))

Wasserstein distance constraint and limited-loss transform

Theorem (Cai et al. (2022b))

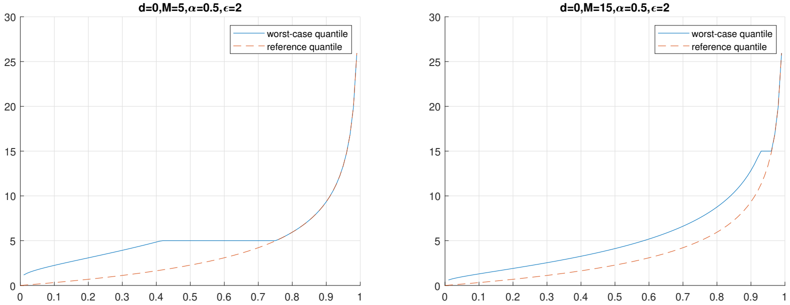
Let $k = 2$. The worst-case distribution is given by

$$(F^*)^{-1}(u) = \begin{cases} F^{-1}(u) + \lambda^* \gamma(u), & \text{for } 0 < u \leq \theta^*, \\ M, & \text{for } \theta^* < u \leq F(M), \\ F^{-1}(u), & \text{for } F(M) < u < 1 \end{cases}$$

where $\lambda^* > 0$ and $\theta^* \in (0, F(M))$ satisfies $W_2(F^*, F) = \varepsilon$.

Example - Wang's premium (cont')

Figure: Worst-case distributions with limited loss function.



Wasserstein distance constraint and limited stop-loss transform

- Wang's premium ρ_h with $h(u) = 1 - \Phi(\Phi^{-1}(1-u) + 0.5)$.
- Exponential reference $F_1(x) = 1 - e^{-x/4}$, $x \geq 0$
- Pareto reference $F_2(x) = 1 - \left(\frac{12}{x+12}\right)^4$
- Limited stop-loss function

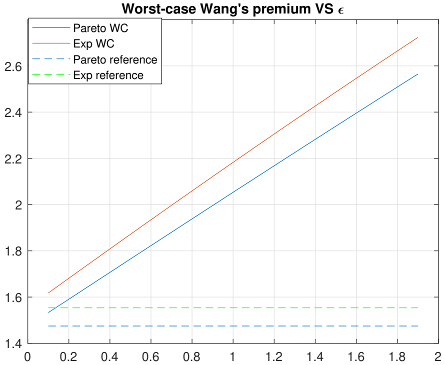
$$\ell(x) = \max\left\{(x-d)^+, M\right\}$$

- Wang's premium in the worst-case:

$$\sup\left\{\rho_h\left(\max\left\{(X^G - d)^+, M\right\}\right), W_2(G, F_i) \leq \varepsilon\right\}, \quad i = 1, 2.$$

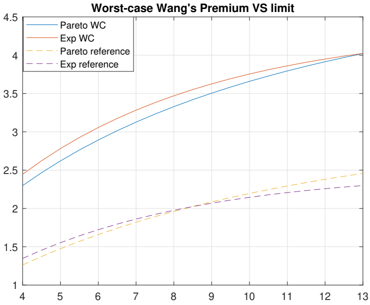
Example - Worst-case Wang's premium VS ε

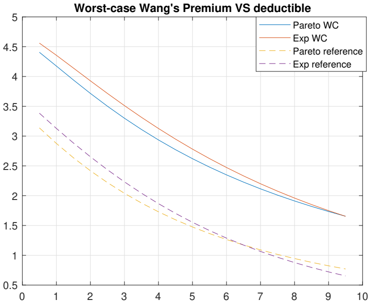
- Fix $d = 5$ and $M = 5$



Example - Worst-case Wang's premium VS deductible and limit

- Fix $\varepsilon = 2$





Uncertainty set with Wasserstein distance and moments constraints

- Take 2nd-order Wasserstein distance. Let $X \sim F$ with $\mathbb{E}[X] = \mu_F$ and $\mathrm{var}(X) = \sigma_F^2$. The uncertainty set is

$$\mathcal{S} = \left\{ G : W_2(F, G) \leq \varepsilon,\ \mathbb{E}[Y^G] = \mu,\ \mathrm{var}(Y^G) = \sigma^2 \right\},$$

- It is not necessary to assume $\mu_F = \mu$ and $\sigma_F^2 = \sigma^2$. Note

$$\varepsilon^2 \geq (\mu - \mu_F)^2 + (\sigma - \sigma_F)^2 \quad \Rightarrow \quad \mathcal{S} \neq \emptyset.$$

Isotonic Projection: For $h \in \mathcal{L}^2(0,1)$, let

$$h^{\uparrow} = \arg\min_{k \in \mathcal{K}} ||h - k||^2,$$

$$\text{where } \mathcal{K} = \left\{ k : (0,1) \mapsto \mathbb{R} \,\middle|\, \int_0^1 k(u)^2 \mathrm{d}u < \infty, k \text{ non-decreasing} \right\}.$$

Notation

- Denote $\gamma_{1,\beta}(u) := \gamma(u)\,\mathbb{1}_{[\beta,1]}(u)$, for $u \in [0,1]$, and the isotonic Projection for $\gamma_{1,\beta} + \lambda F^{-1}$ for some $\lambda \geq 0$ as

$$h^{\uparrow}_{1,\beta,\lambda} = \arg\min_{h \in \mathcal{K}} ||h - \gamma_{1,\beta} - \lambda F^{-1}||_2.$$

- Denote $\gamma_{2,\beta}(u) := \gamma(u)\,\mathbb{1}_{[0,\beta]}(u)$, for $u \in [0,1]$, and the isotonic Projection for $\gamma_{2,\beta} + \lambda F^{-1}$ for some $\lambda \geq 0$ as

$$h^{\uparrow}_{2,\beta,\lambda} = \arg\min_{h \in \mathcal{K}} ||h - \gamma_{2,\beta} - \lambda F^{-1}||_2.$$

Wasserstein distance plus moments constraints and stop-loss transform

Theorem (Cai et al. (2022a))

Consider the worst-case problem $\sup_{G\in\mathcal{S}} \rho_h\left((Y^G - d)_+\right)$.
The quantile function of the worst-case distribution is

$$G_{\beta^*}^{-1}(u) = \mu + \sigma\left(\frac{h^{\uparrow}_{1,\beta^*,\lambda}(u) - a_{\beta^*,\lambda}}{b_{\beta^*,\lambda}}\right), \quad 0 < u < 1,$$

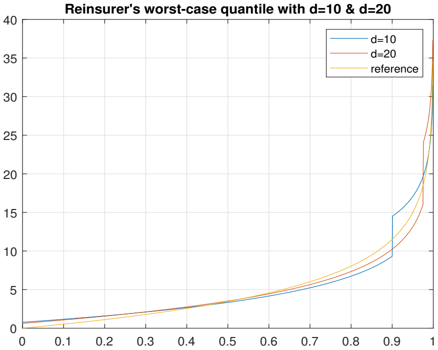
where $a_{\beta^*,\lambda} = \mathbb{E}[h^{\uparrow}_{1,\beta^*,\lambda}(U)]$, $b_{\beta^*,\lambda} = \sqrt{\mathrm{var}(h^{\uparrow}_{1,\beta^*,\lambda}(U))}$, $\lambda > 0$ *is determined uniquely by the distance constraint* $W_2(F, G_{\beta^*}) = \varepsilon$, *and*

$$\beta^* = \underset{\beta\in[0,1]}{\arg\max} \int_0^1 \gamma_{1,\beta}(u)\left(G_\beta^{-1}(u) - d\right)\mathrm{d}u.$$

Example – Expected shortfall

Assume the reference distribution is $F(x) = 1 - e^{-x/5}$, $\mu = \sigma = 5$, $\varepsilon = 1$, and $\rho_h = \mathrm{ES}_{0.9}$.

d	β^*
10	$[0, 0.9]$
20	0.9751



Wasserstein distance plus moments constraints and limited-loss transform

Theorem (Cai et al. (2022a))

Consider the worst-case problem $\sup_{G\in\mathcal{S}} \rho_h\left(Y^G \wedge M\right)$.
The quantile function of the worst-case distribution is

$$F_{\beta^*}^{-1}(u) = \mu + \sigma\left(\frac{h^{\uparrow}_{2,\beta^*,\lambda}(u) - a_{\beta^*,\lambda}}{b_{\beta^*,\lambda}}\right), \quad 0 < u < 1,$$

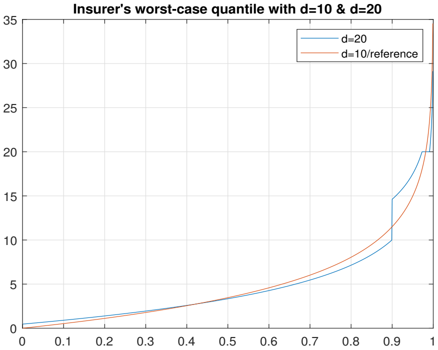
where $a_{\beta^*,\lambda} = \mathbb{E}[h^{\uparrow}_{2,\beta^*,\lambda}(U)]$, $b_{\beta^*,\lambda} = \sqrt{\mathrm{var}(h^{\uparrow}_{2,\beta^*,\lambda}(U))}$, $\lambda > 0$ *is determined uniquely by the distance constraint* $W_2(F, F_{\beta^*}) = \varepsilon$, *and*

$$\beta^* = \underset{\beta\in[0,1]}{\arg\min} \int_0^\beta \gamma_{2,\beta}(u)\left(F_\beta^{-1}(u) - d\right)\mathrm{d}u.$$

Example – Expected shortfall

Assume the reference distribution is $F(x) = 1 - e^{-x/5}$, $\mu = \sigma = 5$, $\varepsilon = 1$, and $\rho_h = \mathrm{ES}_{0.9}$.

d	β^*
10	$[0, 0.9]$
20	0.9835



Summary

In this talk we discuss multiple model uncertainty models

- Distortion risk measure
- With or without transform
  - Stop-loss, limited-loss
- Wasserstein distance, moments contraints

Future works

- Other risk measures
- General transformation
- Various uncertainty sets: likelihood ratio, KL-divergent, etc.
- Novel techniques to characterize worst-case distribution and worst-case risk measure value

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