An axiomatic theory for comonotonicity-based risk sharing

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1. Introduction

- Consider a pool of individual random future losses.
- **Decentralized risk-sharing**:
  - Refers to **risk-sharing** (RS) mechanisms under which the participants in the pool share their risks among each other.
  - Each **participant** in the **risk-sharing pool** is compensated *ex-post* from the pool for his loss.
  - In return, each participant pays an ex-post **contribution** to the pool.
  - These contributions follow from a **risk-sharing rule**, satisfying the *self-financing condition*.
- Decentralized risk-sharing does not require an insurer, but an **administrator**.

2. Risk-sharing and risk-sharing rules

Agents and their losses

- Let χ be an appropriate (sufficiently rich) set of r.v.'s in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, representing random losses at time 1.[1]
- Consider n economic **agents**, numbered $i = 1, 2, \ldots, n$.
- Each agent i faces a **loss** $X_i \in \chi$ at the end of the observation period $[0, 1]$.
- Without insurance or pooling, each agent bears his own loss: at time 1, agent i suffers loss x_i, which is the realization of X_i.

[1] $(\Omega, \mathcal{F}, \mathbb{P})$ is assumed to contain the r.v. U, which is uniformly distributed over the unit interval $[0, 1]$.

- The joint cdf of the **loss vector** $\mathbf{X} = (X_1, X_2, \ldots, X_n)$ is denoted by $F_{\mathbf{X}}$.
- The marginal cdf's of the individual losses are denoted by $F_{X_1}, F_{X_2}, \ldots, F_{X_n}$.
- The **aggregate loss** faced by the n agents with loss vector $\mathbf{X}$ is denoted by $S_{\mathbf{X}} = \sum_{i=1}^{n} X_i$.
- Hereafter, we will often call $\mathbf{X}$ the **pool**, and call each agent a **participant** in the pool.

- **Definition**: For any pool $\mathbf{X} \in \chi^n$ with aggregate loss $S_{\mathbf{X}}$ the set $\mathcal{A}_n(S_{\mathbf{X}})$ is defined by:

$$\mathcal{A}_n(S_{\mathbf{X}}) = \left\{ (Y_1, Y_2, \ldots, Y_n) \in \chi^n \mid \sum_i^n Y_i = S_{\mathbf{X}} \right\}$$

- The elements of $\mathcal{A}_n(S_{\mathbf{X}})$ are called the n-dimensional **allocations** of $S_{\mathbf{X}}$ in χ^n.

2. Risk-sharing and risk-sharing rules

Risk-sharing

- **Definition**: **Risk-sharing** in a pool $\mathbf{X} \in \chi^n$ is a two-stage process.
  - **Ex-ante step** (at time 0):
    The losses X_i in the pool are re-allocated by transforming $\mathbf{X}$ into another random vector $\mathbf{H}_{\mathbf{X}} \in \mathcal{A}_n(S_{\mathbf{X}})$:

    $$\mathbf{H}_{\mathbf{X}} = \left(H_{\mathbf{X},1}, H_{\mathbf{X},2}, \ldots, H_{\mathbf{X},n}\right)$$

  - **Ex-post step** (at time 1):
    - Each participant *i receives from the pool* the realization of his loss X_i.
    - In return, he *pays to the pool* a contribution equal to the realization of his re-allocated loss $H_{\mathbf{X},i}$.

- **Remark**: As $\mathbf{H}_{\mathbf{X}} \in \mathcal{A}_n(S_{\mathbf{X}})$, risk sharing is **self-financing**:

$$\sum_{i=1}^{n} H_{\mathbf{X},i} = \sum_{i=1}^{n} X_i$$

- **Definition**: A **risk-sharing rule** is a mapping $\mathbf{H} : \chi^n \to \chi^n$ satisfying the self-financing condition:

$$\mathbf{X} \to \mathbf{H}_{\mathbf{X}} \in \mathcal{A}_n(S_{\mathbf{X}}), \qquad \text{for any } \mathbf{X} \in \chi^n$$

- **Remarks**: For any participant i in the pool $\mathbf{X} = (X_1, \ldots, X_n)$,
  - X_i is called his **loss**, (paid by the pool).
  - $H_{\mathbf{X},i}$ is called his **contribution**, (paid to the pool).

- **Contribution vector**:

$$\mathbf{H}_{\mathbf{X}} = \left(H_{\mathbf{X},1}, H_{\mathbf{X},2}, \ldots, H_{\mathbf{X},n}\right)$$

- **Notation**: $\mathcal{F}\left(\chi^{n}\right)$ is the class of all cdf's of elements in χ^{n}.
- **Definition**: $\mathbf{H}:\chi^{n}\rightarrow\chi^{n}$ is an **internal RS rule** if there exists a function $\mathbf{h}:\mathbb{R}^{n}\times\mathcal{F}\left(\chi^{n}\right)\rightarrow\mathbb{R}^{n}$ such that the contribution vector $\mathbf{H}_{\mathbf{X}}$ of any $\mathbf{X}\in\chi^{n}$ is given by :

$$\mathbf{H}_{\mathbf{X}}=\mathbf{h}\left(\mathbf{X},F_{\mathbf{X}}\right)$$

- **Remarks**:
  - $\mathbf{h}$ is called an **internal function** of the RS rule $\mathbf{H}$.
  - Hereafter, we only consider internal risk-sharing rules.

- **Definition**: $\mathbf{H} : \chi^n \rightarrow \chi^n$ is an **aggregate RS rule** if there exists a function $\mathbf{h}^{\text{aggr}} : \mathbb{R} \times \mathcal{F}\left(\chi^n\right) \rightarrow \mathbb{R}^n$ such that the contribution vector $\mathbf{H}_{\mathbf{X}}$ of any $\mathbf{X} \in \chi^n$ is given by:

$$\mathbf{H}_{\mathbf{X}} = \mathbf{h}^{\text{aggr}}\left(S_{\mathbf{X}}, F_{\mathbf{X}}\right)$$

- **Property**: Any aggregate RS rule **H** is **internal**, with internal function **h** satisfying:

$$\mathbf{h}\left(\mathbf{X}; F_{\mathbf{X}}\right) = \mathbf{h}^{\text{aggr}}\left(S_{\mathbf{X}}, F_{\mathbf{X}}\right) \qquad \text{for any } \mathbf{X} \in \chi^n$$

- **Definition**: $\mathbf{H} : \chi^n \rightarrow \chi^n$ is a **dependence-free RS rule** if there exists a function $\mathbf{h}^{\text{dep-free}} : \mathbb{R}^n \times (\mathcal{F}(\chi))^n \rightarrow \mathbb{R}^n$ such that the contribution vector $\mathbf{H}_\mathbf{X}$ of any $\mathbf{X} \in \chi^n$ is given by:

$$\mathbf{H}_\mathbf{X} = \mathbf{h}^{\text{dep-free}}\left(\mathbf{X}, F_{X_1}, \ldots, F_{X_n}\right)$$

- **Property**: Any dependence-free RS rule $\mathbf{H}$ is **internal**, with internal function $\mathbf{h}$ satisfying:

$$\mathbf{h}\left(\mathbf{X}; F_\mathbf{X}\right) = \mathbf{h}^{\text{dep-free}}\left(\mathbf{X}, F_{X_1}, \ldots, F_{X_n}\right) \qquad \text{for any } \mathbf{X} \in \chi^n$$

3. Examples of risk-sharing rules

The uniform risk-sharing rule

- **Definition**: The **uniform RS rule** $\mathbf{H}^{\text{uni}}$ is defined by

$$H^{\text{uni}}_{\mathbf{X},i} = \frac{S_{\mathbf{X}}}{n}, \qquad i = 1, 2, \ldots, n,$$

  for any $\mathbf{X} \in \chi^n$.

- **Interpretation**: $\mathbf{H}^{\text{uni}}$ equally distributes the aggregate loss $S_{\mathbf{X}}$ over all agents.

- **Property**:

  $\mathbf{H}^{\text{uni}}$ is **internal**, **aggregate** and **dependence–free.**

3. Examples of risk-sharing rules

The conditional mean risk-sharing rule

- **Definition**[2]: The **conditional mean RS rule** $\mathbf{H}^{\mathrm{cm}}$ is defined by

$$H_i^{\mathrm{cm}}(\mathbf{X}) = \mathbb{E}\left[X_i \mid S_{\mathbf{X}}\right], \qquad i = 1, 2, \ldots, n,$$

  for any $\mathbf{X} \in \chi^n$.

- **Interpretation**: Each participant contributes the expected value of the loss he brings to the pool, given the aggregate loss experienced by the pool.

- **Property**:

  $\mathbf{H}^{\mathrm{cm}}$ is **internal** and **aggregate**, but not dependence-free.

[2] The CMRS rule was given its name in D,D (2012). It has many nice properties, see D,D,R (2022). An axiomatic characterization is given in Jiao, Kou, Liu & Wang (2022).

4. The quantile risk-sharing rule

Comonotonic modification of a pool

- Consider the pool $\mathbf{X} = (X_1, \ldots, X_n)$.
- **Definition**: The **comonotonic modification** $\mathbf{X}^c$ of $\mathbf{X}$ is given by[3]:

$$\mathbf{X}^c = \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U) \right)$$

- **Notations**:

$$S_{\mathbf{X}} = \sum_{i=1}^{n} X_i \quad \text{and} \quad S_{\mathbf{X}}^c = \sum_{i=1}^{n} F_{X_i}^{-1}(U)$$

- **Important remark**: For notational and mathematical convenience, hereafter, we will only consider pools $\mathbf{X}$ of which all **marginals** F_{X_i} are **strictly increasing**.

[3] U is uniformly distributed over $[0, 1]$.

4. The quantile risk-sharing rule

Motivation

- Let $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ be the realization of $\mathbf{X}$.
- There exist probabilities $p_1, \ldots, p_n$ such that

$$\mathbf{x} = \left(F_{X_1}^{-1}(p_1), \ldots, F_{X_n}^{-1}(p_n) \right)$$

- **Question**: How to determine the unique probability level $p_{\mathbf{x}}$ such that the contribution vector is given by

$$\mathbf{h}(\mathbf{x}, F_{\mathbf{X}}) = \left(F_{X_1}^{-1}(p_{\mathbf{x}}), \ldots, F_{X_n}^{-1}(p_{\mathbf{x}}) \right)$$

- **Additivity property**: For any outcome s of $S_{\mathbf{X}}$, one has:

$$\sum_{i=1}^{n} F_{X_i}^{-1}\left(F_{S_{\mathbf{X}}^c}(s)\right) = s$$

- **Solution**: Define $\mathbf{H}^{\text{quant}}$ based on this decomposition of s.

4. The quantile risk-sharing rule

- **Definition**:
  - Under the **quantile RS rule** $\mathbf{H}^{\text{quant}} : \chi^n \to \chi^n$, the contribution vector of $\mathbf{X} \in \chi^n$ is given by

$$\mathbf{H}_{\mathbf{X}}^{\text{quant}} = \mathbf{h}^{\text{quant}} \left(S_{\mathbf{X}}, F_{\mathbf{X}} \right)$$

  - where $\mathbf{h}^{\text{quant}} : \mathbb{R} \times \mathcal{F}\left(\chi^n\right) \to \mathbb{R}^n$ is defined by

$$h_i^{\text{quant}} \left(s, F_{\mathbf{X}} \right) = F_{X_i}^{-1} \left(F_{S_{\mathbf{X}}^c} (s) \right), \qquad i = 1, 2, \ldots, n$$

- **Properties**:
  - $\mathbf{H}^{\text{quant}}$ satisfies the **self-financing condition**.
  - $\mathbf{H}^{\text{quant}}$ is an **aggregate RS rule**.
  - $\mathbf{H}^{\text{quant}}$ is a **dependence-free RS rule**.

5. The 'stand-alone for comonotonic pools' property

- **Definition**: $\mathbf{X} \in \chi^n$ is a **comonotonic pool** in case

$$\mathbf{X} \stackrel{d}{=} \left(F_{X_1}^{-1}(U), \ldots, F_{X_n}^{-1}(U)\right)$$

- **Definition**: A RS rule $\mathbf{H} : \chi^n \to \chi^n$ satisfies the **stand-alone for comonotonic pools property** if for any comonotonic pool $\mathbf{X}^c \in \chi^n$, one has that

$$\mathbf{H}_{\mathbf{X}^c} = \mathbf{X}^c$$

- **Property**: $\mathbf{H}^{\text{quant}}$ satisfies the **stand-alone for comonotonic pools** property.

6. Axiomatic characterization of the quantile RS rule

- **Theorem**:
  - Consider the internal RS rule $\mathbf{H} : \chi^n \to \chi^n$.
  - **H** is the quantile RS rule if, and only if, it satisfies the following axioms:

    (1) **H** is **aggregate**.

    (2) **H** is **dependence-free**.

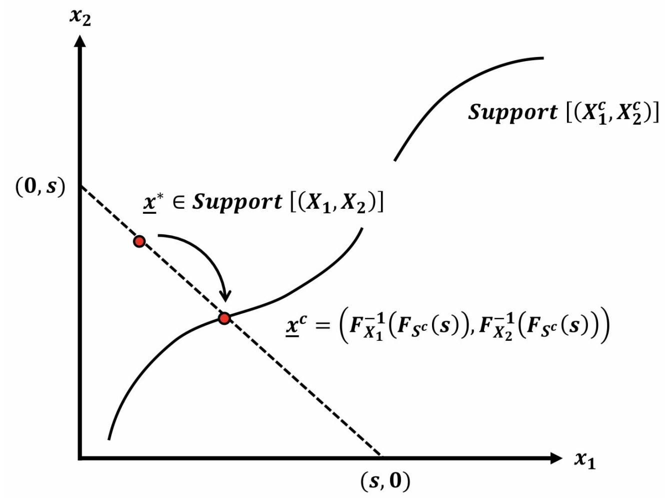
    (3) **H** is **(generalized) stand-alone for comonotonic pools**[4].

- **Proposition**:
  The axioms (1), (2) and (3) are **independent**.

[4] The 'generalized stand-alone for comonotonic pools' property is a slightly stronger property than the 'stand-alone for comonotonic pools' property, see D,R,C,D (2023).

6. Axiomatic characterization of the quantile RS rule

Graphical proof of the theorem (bivariate case)



$$\mathbf{h}\left(\mathbf{x}^*, F_{\mathbf{X}}\right) \overset{\text{axiom 1}}{=} \mathbf{h}\left(\mathbf{x}^{\mathbf{c}}, F_{\mathbf{X}}\right) \overset{\text{axiom 2}}{=} \mathbf{h}\left(\mathbf{x}^{\mathbf{c}}, F_{\mathbf{X}^{\mathbf{c}}}\right) \overset{\text{axiom 3}}{=} \mathbf{x}^{\mathbf{c}}$$

6. Axiomatic characterization of the quantile RS rule

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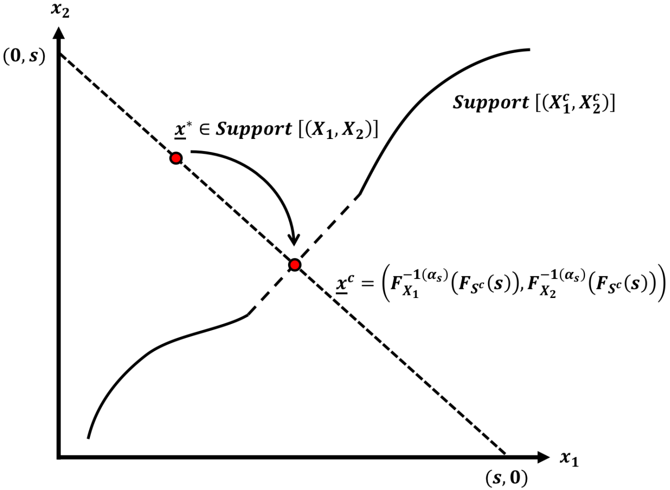

$$\mathbf{h}\left(\mathbf{x}^*, F_{\mathbf{X}}\right) \overset{\text{axiom } 1}{=} \mathbf{h}\left(\mathbf{x}^{\mathbf{c}}, F_{\mathbf{X}}\right) \overset{\text{axiom } 2}{=} \mathbf{h}\left(\mathbf{x}^{\mathbf{c}}, F_{\mathbf{X}^{\mathbf{c}}}\right) \overset{\text{axiom } 3}{=} \mathbf{x}^{\mathbf{c}}$$

7. Example of a non-internal risk-sharing rule

- Consider the RS rule $\mathbf{H} : \chi^n \to \chi^n$, where any $\mathbf{X} \in \chi^n$ is a pool of **health-related costs** of the participants.
- Suppose the participants can be divided in m **age categories**, denoted by $1, 2, \ldots, m$:
  - For any $\mathbf{X} \in \chi^n$, the age category of participant i is denoted by a_i.
- The **RS rule** $\mathbf{H}$ is defined by

$$H_{\mathbf{X},i} = \frac{\sum_{j=1}^{n} X_j \times 1_{a_j=k}}{\sum_{j=1}^{n} 1_{a_j=k}} \qquad \text{if } a_i = k, \quad k = 1, 2, \ldots, m$$

- **Interpretation**: Losses are uniformly shared within each age category.
- **Observations**:
  - $\mathbf{H}$ is $\sigma(\mathbf{X})$-measurable.
  - $\mathbf{H}$ is not an internal RS rule (if we assume that $(\mathbf{x}, F_{\mathbf{X}})$ does not reveal the age category of each participant).

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