UiO : Department of Mathematics
University of Oslo

Change of measure in a Heston-Hawkes stochastic volatility model

Actuarial, Finance, Risk and Insurance Congress
Victoria Falls, Zimbabwe

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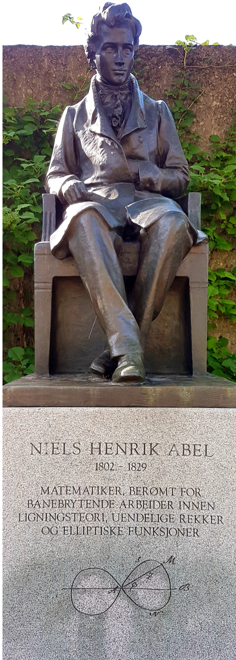
Joint work with David R. Baños and Salvador Ortiz-Latorre

SCROLLER project



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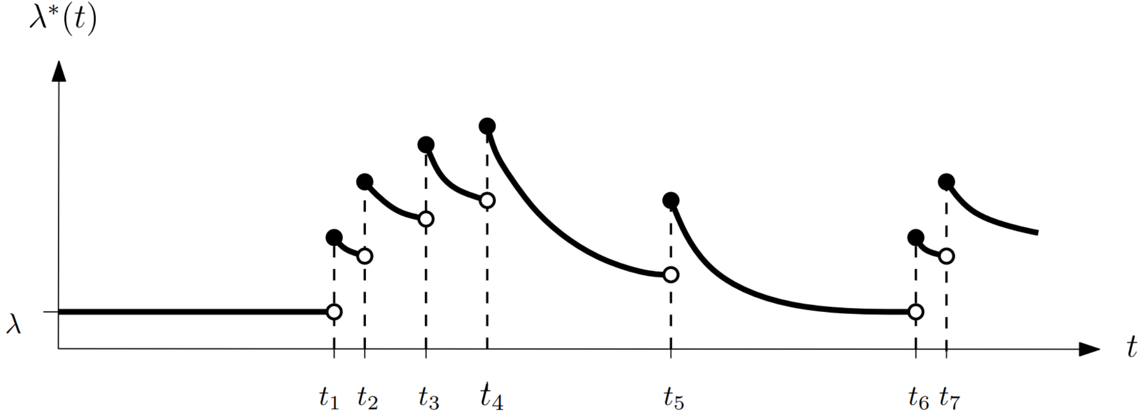
- $(\Omega, \mathcal{A}, \mathbb{P})$ complete probability space.
- (B, W) two-dimensional standard Brownian motion.
- $N = \{N_t, t \in [0, T]\}$ Hawkes process: **self-exciting** counting process with stochastic intensity and exponential kernel given by

$$\lambda_t = \lambda_0 + \alpha \int_0^t e^{-\beta(t-s)} dN_s, \qquad \text{equivalently,} \qquad d\lambda_t = -\beta(\lambda_t - \lambda_0)dt + \alpha dN_t,$$

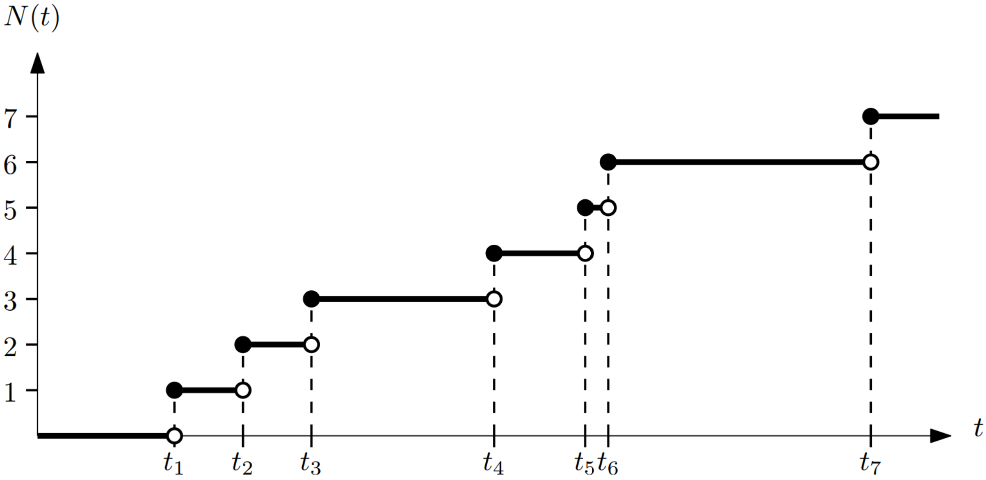
where $\lambda_0, \alpha, \beta > 0$ and the **stability condition** is satisfied

$$\alpha \int_0^\infty e^{-\beta s} ds = \frac{\alpha}{\beta} < 1.$$

Intensity of a Hawkes process



Hawkes process



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- (N, λ) is a **Markov process.**
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- We consider the joint and minimally augmented filtration

$$\mathcal{F} = \{\mathcal{F}_t = \mathcal{F}_t^{(B,W)} \vee \mathcal{F}_t^L, t \in [0,T]\}.$$

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$$\frac{dS_t}{S_t} = \mu_t dt + \sqrt{v_t}\left(\sqrt{1-\rho^2}dB_t + \rho dW_t\right)$$
$$dv_t = -\kappa\left(v_t - \bar{v}\right)dt + \sigma\sqrt{v_t}dW_t + \eta dL_t,$$

  where $S_0, v_0, \kappa, \bar{v}, \sigma, \eta > 0$, $\rho \in (-1, 1)$ and $\mu : [0, T] \to \mathbb{R}$ measurable and bounded.
- $L = \{L_t, t \in [0, T]\}$ is the compound Hakwes process.

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- There are models where a risk neutral probability measure exists only up to an explosion time. See [1] and [2].
- Risk exposures computed under $\mathbb{Q}$ are basically arbitrary. See [3].
- The passage from $\mathbb{P}$ to $\mathbb{Q}$ and vice versa is necessary and not negligible.

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Heston model

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- Let $\widetilde{v} = \{\widetilde{v}_t, t \in [0, T]\}$ be the standard Heston variance, that is,

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- We assume the **Feller condition** $2\kappa\bar{v} \geq \sigma^2 \implies \widetilde{v}$ is a strictly positive process.
- By a slight variation of the comparison theorem,

$$\mathbb{P}\left(\widetilde{v}_t \leq v_t \; \forall t \in [0, T]\right) = 1 \implies v \text{ is a strictly positive process.}$$

Change of measure

- Let $a \in \mathbb{R}$ and define the process $(B^{\mathbb{Q}(a)}, W^{\mathbb{Q}(a)}) = \{(B_t^{\mathbb{Q}(a)}, W_t^{\mathbb{Q}(a)}), t \in [0, T]\}$ by

$$dB_t^{\mathbb{Q}(a)} = dB_t + \theta_t^{(a)} dt \qquad \text{and} \qquad dW_t^{\mathbb{Q}(a)} = dW_t + a\sqrt{v_t} dt.$$

where

$$\theta_t^{(a)} := \frac{1}{\sqrt{1-\rho^2}} \left(\frac{\mu_t - r}{\sqrt{v_t}} - a\rho\sqrt{v_t} \right),$$

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- The dynamics of the stock is given by

$$\frac{dS_t}{S_t} = rdt + \sqrt{v_t} \left(\sqrt{1-\rho^2} dB_t^{\mathbb{Q}(a)} + \rho dW_t^{\mathbb{Q}(a)} \right).$$

Change of measure

- To apply **Girsanov's theorem** we check that the process $X^{(a)} = \{X_t^{(a)}, t \in [0, T]\}$ defined by $X_t^{(a)} := Y_t^{(a)} Z_t^{(a)}$ is a $(\mathcal{F}, \mathbb{P})$-martingale where

$$Y_t^{(a)} := \mathcal{E}_t \left\{ -\int_0^{\cdot} \theta_s^{(a)} dB_s \right\} \qquad \text{and} \qquad Z_t^{(a)} := \mathcal{E}_t \left\{ -a \int_0^{\cdot} \sqrt{v_s} dW_s \right\}.$$

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- Since $X^{(a)}$ is a positive $(\mathcal{F}, \mathbb{P})$-local martingale with $X_0^{(a)} = 1$, it is a $(\mathcal{F}, \mathbb{P})$-supermartingale and it is a $(\mathcal{F}, \mathbb{P})$-martingale if and only if

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- Since $Z_T^{(a)}$ is $\mathcal{F}_T^W \vee \mathcal{F}_T^L$-measurable

$$\mathbb{E}\left[X_T^{(a)}\right] = \mathbb{E}\left[Y_T^{(a)} Z_T^{(a)}\right] = \mathbb{E}\left[\mathbb{E}\left[Y_T^{(a)} Z_T^{(a)} | \mathcal{F}_T^W \vee \mathcal{F}_T^L\right]\right] = \mathbb{E}\left[Z_T^{(a)} \mathbb{E}\left[Y_T^{(a)} | \mathcal{F}_T^W \vee \mathcal{F}_T^L\right]\right].$$

Change of measure

- Recall that

$$\theta_t^{(a)} = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\mu_t - r}{\sqrt{v_t}} - a\rho\sqrt{v_t}\right) \qquad \text{and} \qquad Y_t^{(a)} = \mathcal{E}_t\left\{-\int_0^{\cdot} \theta_s^{(a)} dB_s\right\}.$$

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- We obtain that

$$\mathbb{E}\left[Y_T^{(a)}|\mathcal{F}_T^W \vee \mathcal{F}_T^L\right] = 1 \implies \mathbb{E}\left[X_T^{(a)}\right] = \mathbb{E}\left[Z_T^{(a)}\mathbb{E}\left[Y_T^{(a)}|\mathcal{F}_T^W \vee \mathcal{F}_T^L\right]\right] = \mathbb{E}\left[Z_T^{(a)}\right].$$

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- **Objective:** What values, if any, $c > 0$ satisfy

$$\mathbb{E}\left[\exp\left(c\int_0^T v_u du\right)\right] < \infty?$$

Change of measure

- In [4] it is proved that the variance of the standard Heston model satisfies for $c \leq \frac{\kappa^2}{2\sigma^2}$

$$\mathbb{E}\left[\exp\left(c\int_0^T \widetilde{v}_u du\right)\right] \leq \exp\left(-(\kappa\bar{v})\Phi(0) - v_0\psi(0)\right) < \infty,$$

where Φ and ψ satisfy the following Riccati ODEs

$$\begin{aligned} \psi'(t) &= \frac{\sigma^2}{2}\psi^2(t) + \kappa\psi(t) + c \\ -\Phi'(t) &= \psi(t) \\ \psi(T) &= \Phi(T) = 0. \end{aligned}$$

[4] Wong, B. and Heyde, C. C. ‘On changes of measure in stochastic volatility models’. In: J. Appl. Math. Stoch. Anal. (2006), Art. ID 18130, 13.

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$$M(t) = \exp\left(F(t) + G(t)v_t + H(t)\lambda_t + c\int_0^t v_u du\right),$$

for $F, G, H : [0, T] \to \mathbb{R}$ functions satisfying $F(T) = G(T) = H(T) = 0$.

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- Note that

$$M(T) = \exp\left(c\int_0^T v_u du\right).$$

- $\mathbb{E}[M(T)]$ is exactly the expectation that we want to study.

Change of measure

- If there exist functions F, G and H such that M is a $(\mathcal{F}, \mathbb{P})$-local martingale, since it is non-negative, it will be a $(\mathcal{F}, \mathbb{P})$-supermartingale and then

$$\mathbb{E}\left[\exp\left(c\int_0^T v_u du\right)\right] = \mathbb{E}[M(T)] \leq M(0) = \exp\left(F(0) + G(0)v_0 + H(0)\lambda_0\right).$$

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- Applying Itô formula and equating all the drift terms to 0 we get that F, G and H must solve the following ODEs:

$$\begin{aligned} G'(t) &= -\frac{1}{2}\sigma^2 G^2(t) + \kappa G(t) - c \\ H'(t) &= \beta H(t) - M_J\left(\eta G(t)\right)\exp\left(\alpha H(t)\right) + 1 \\ F'(t) &= -\kappa \bar{v} G(t) - \beta\lambda_0 H(t) \\ G(T) &= H(T) = F(T) = 0. \end{aligned}$$

where M_J is the m.g.f of J_1.

Change of measure

- **Assumption**: There exists $\epsilon_J > 0$ such that M_J is well defined in $(-\infty, \epsilon_J)$ and it is the maximal domain in the sense that

$$\lim_{t \to \epsilon_J^-} M_J(t) = \infty.$$

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- **Idea**: Require that $M_J(\eta G(t))$ is well-defined for all $t \in [0, T]$ and bound the ODE of H from above and below by autonomous ODEs and impose that

$$\sup_{t \in [0,T]} M_J(\eta G(t)) \leq \frac{\beta}{\alpha} \exp\left(\frac{\alpha}{\beta} - 1\right).$$

to ensure existence of the solution on the interval $[0, T]$.

Change of measure

Theorem

For $c \leq \frac{\kappa^2}{2\sigma^2}$*, define* $D(c) := \sqrt{\kappa^2 - 2\sigma^2 c}$*,* $\Lambda(c) := \frac{2\eta c\left(e^{D(c)T}-1\right)}{D(c)-\kappa+(D(c)+\kappa)e^{D(c)T}}$ *and*

$$c_l := \sup\left\{c \leq \frac{\kappa^2}{2\sigma^2} : \Lambda(c) < \epsilon_J \quad \text{and} \quad M_J\left(\Lambda(c)\right) \leq \frac{\beta}{\alpha}\exp\left(\frac{\alpha}{\beta} - 1\right)\right\}.$$

Then, $0 < c_l \leq \frac{\kappa^2}{2\sigma^2}$ *and for* $c < c_l$ *the system of ODEs has a solution in* $[0, T]$ *and*

$$\mathbb{E}\left[\exp\left(c\int_0^T v_u du\right)\right] \leq \exp\left(F(0) + G(0)v_0 + H(0)\lambda_0\right) < \infty.$$

Change of measure

Proposition

Define c_s by

$$c_s = \min\left\{\frac{\kappa\epsilon_J}{2\eta}, \frac{\kappa}{2\eta}M_J^{-1}\left(\frac{\beta}{\alpha}\exp\left(\frac{\alpha}{\beta}-1\right)\right), \frac{\kappa^2}{2\sigma^2}\right\}.$$

Then, $0 < c_s < c_l$*.*

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Example

If $J_1 \sim \text{Exponential}(\mu)$, then

$$c_s = \min\left\{\frac{\kappa\mu}{2\eta}\left(1-\frac{\alpha}{\beta}\exp\left(1-\frac{\alpha}{\beta}\right)\right), \frac{\kappa^2}{2\sigma^2}\right\}.$$

Change of measure

- Recall that

$$dB_t^{\mathbb{Q}(a)} = dB_t + \theta_t^{(a)} dt \qquad \text{and} \qquad dW_t^{\mathbb{Q}(a)} = dW_t + a\sqrt{v_t} dt.$$

and we needed to check that

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Theorem

The set

$$\mathcal{E} := \left\{ \mathbb{Q}(a) \;\; \textit{given by} \;\; \frac{d\mathbb{Q}(a)}{d\mathbb{P}} = X_T^{(a)} \;\; \textit{with} \;\; |a| < \sqrt{2c_l} \right\}$$

is a set of equivalent local martingale measures.

Change of measure

- Let $\mathbb{Q}(a) \in \mathcal{E}$, the dynamics of S and v are given by

$$\frac{dS_t}{S_t} = rdt + \sqrt{v_t}\left(\sqrt{1-\rho^2}dB_t^{\mathbb{Q}(a)} + \rho dW_t^{\mathbb{Q}(a)}\right),$$
$$dv_t = -\kappa^{(a)}\left(v_t - \bar{v}^{(a)}\right)dt + \sigma\sqrt{v_t}dW_t^{\mathbb{Q}(a)} + \eta dL_t,$$

where $\kappa^{(a)} = \kappa + a\sigma$ and $\bar{v}^{(a)} = \frac{k\bar{v}}{k+a\sigma}$.

Change of measure

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Theorem

If $\rho^2 < c_l$, the set

$$\mathcal{E}_m := \left\{\mathbb{Q}(a) \in \mathcal{E} : |a| < \min\left\{\frac{\sqrt{2c_l}}{2}, \sqrt{c_l - \rho^2}\right\}\right\}$$

is a set of equivalent martingale measures.
The market is ***arbitrage-free*** *and* ***incomplete****.*

Conclusions

- We propose an extension of the well-known Heston model that incorporates the **volatility clustering** feature by adding a compound **Hawkes process** to the volatility.

Conclusions

- We propose an extension of the well-known Heston model that incorporates the **volatility clustering** feature by adding a compound **Hawkes process** to the volatility.
- We have proved that the model is **arbitrage-free and incomplete** by finding a **family of risk neutral probability measures** using the tractability of the exponential Hawkes.

Thank you for your attention!

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SCROLLER project

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Actuarial, Finance, Risk and Insurance Congress

Victoria Falls, Zimbabwe