Incorporating Information on Robust Quantities into Model Uncertainty Assessment

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A look into model uncertainty

F is LogNormal(μ, σ)

$$\downarrow \rho$$

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- Mean
- Variance
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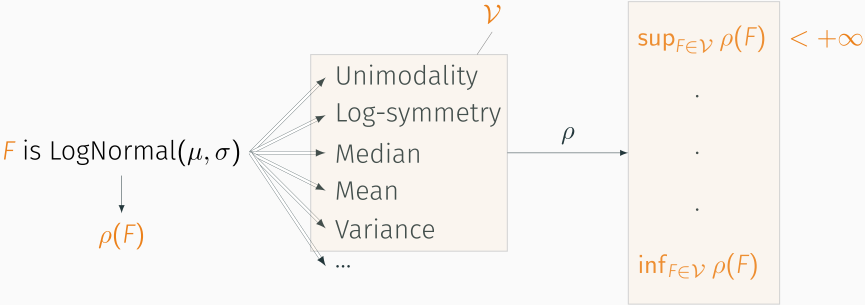
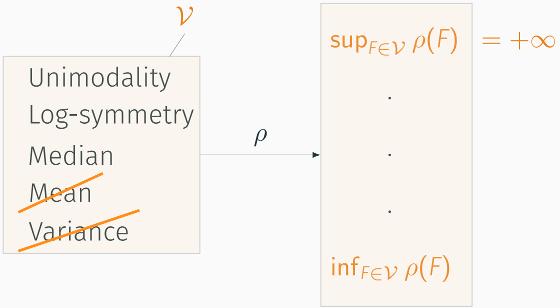


Already dealt with in Bernard, Kazzi, and Vanduffel [2023a], and Bernard, Kazzi, and Vanduffel [2023b].

A look into model uncertainty for heavy-tailed risks



$\Longrightarrow$ We need to incorporate information on some robust quantities to assess model uncertainty in heavy-tailed distributions

Outline

1. Problem formulation
2. A taste of the solution
3. Application to SAS OpRisk dataset

Problem formulation

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Basic Problem

$$\sup_{F\in\mathcal{V}} \rho(F) \qquad \textit{and} \qquad \inf_{F\in\mathcal{V}} \rho(F)$$

for some measure $\rho : F \to \mathbb{R}$ *and set* $\mathcal{V}$ *where*

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Measures of interest

For some $(\alpha; \beta) \in (0,1) \times (\alpha, 1)$, $(x_1, x_2) \in \mathbb{R} \times (x_1, +\infty)$,

- $VaR_\alpha(F) = F^{-1}(\alpha)$
- $TVaR_\alpha(F)$
- $VaR_\beta(F) - VaR_\alpha(F)$
- $RVaR_{\alpha,\beta}(F) = \frac{1}{\beta - \alpha} \int_\alpha^\beta F^{-1}(p)dp$
- $F(x_2) - F(x_1)$
- $E\left[g(F)\right]$ *for some* $g(.)$

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Information that can be incorporated

- Unimodality
- Symmetry
- T-unimodality
- T-symmetry

- Non-negativity / Support
- Moments on the original distribution
- Moments on the transformed distribution
- Robust and quantile-based measures

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Examples of robust and quantile-based measures

For $0 < \alpha_1 < \alpha_2 < 1$,

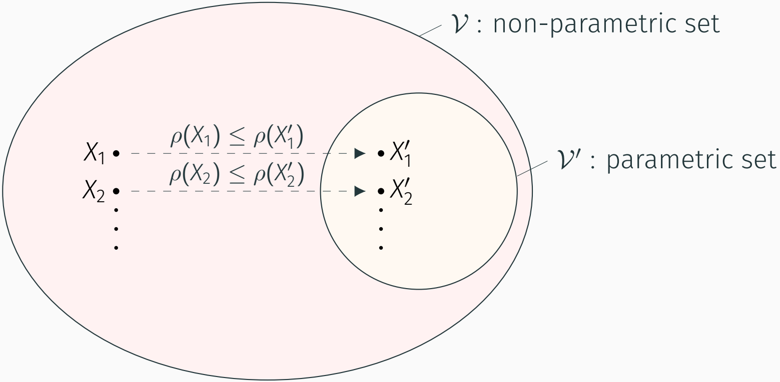
- A specific quantile, e.g., $F^{-1}(0.75)$
- Interpercentile range: $F^{-1}(\alpha_2) - F^{-1}(\alpha_1)$
- Truncated/trimmed moments: $\frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} h\left(F^{-1}(p)\right) dp$ for some function h

  E.g., $\frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} F^{-1}(p)dp$ and $\frac{1}{\alpha_2-\alpha_1} \int_{\alpha_1}^{\alpha_2} \left(F^{-1}(p)\right)^2 dp$
- Moor's kurtosis: $\frac{F^{-1}(7/8)-F^{-1}(5/8)+F^{-1}(3/8)-F^{-1}(1/8)}{F^{-1}(6/8)-F^{-1}(2/8)}$
- ...

General approach

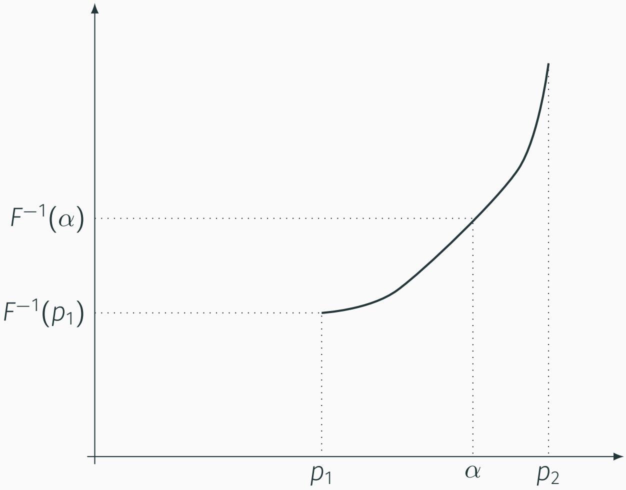
Mathematical challenge: The optimization is non-parametric

Solution: Reduce it to a parametric optimization via stochastic ordering

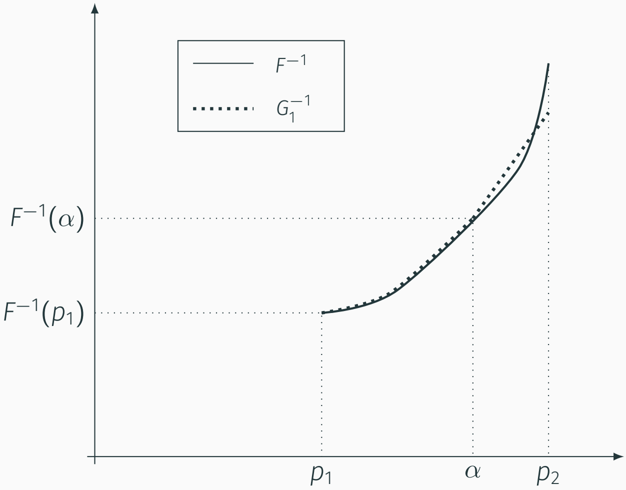


A taste of the solution

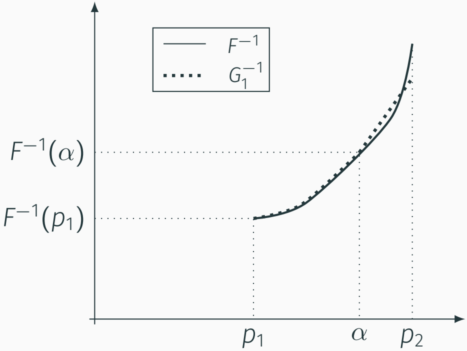
Arbitrary element F of $\mathcal{V}$



Construction of G_1^{-1}



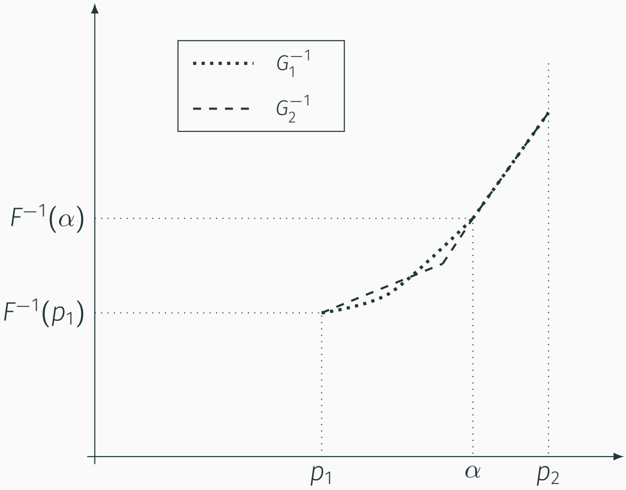
G_1 vs F



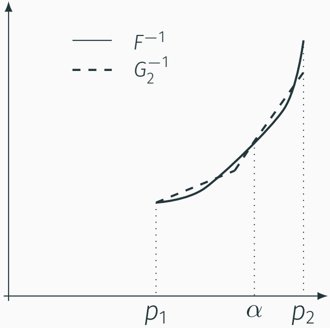
$\int_{p_1}^{p_2} G_1^{-1}(p)dp = \int_{p_1}^{p_2} F^{-1}(p)dp$
and F^{-1} up-crosses G_1^{-1} exactly once on $[p_1, p_2]$

$$\implies \int_{p_1}^{p_2} h\left(G_1^{-1}(p)\right) dp \le \int_{p_1}^{p_2} h\left(F^{-1}(p)\right) dp \text{ for any convex } h$$

Similarly for G_2 vs G_1



For every $F \in \mathcal{V}$, there exists G_2 such that



- G_2 is parametric on $[p_1, p_2]$
- $G_2^{-1}(p_1) = F^{-1}(p_1)$
- $\mathrm{VaR}_\alpha(G_2) = \mathrm{VaR}_\alpha(F)$
- $\mathrm{RVaR}_{\alpha,p_2}(G_2) = \mathrm{RVaR}_{\alpha,p_2}(F)$
- $\int_{p_1}^{p_2} G_2^{-1}(p)dp = \int_{p_1}^{p_2} F^{-1}(p)dp$
- $\int_{p_1}^{p_2} h\left(G_2^{-1}(p)\right) dp \le \int_{p_1}^{p_2} h\left(F^{-1}(p)\right) dp$ for any convex h

Application to SAS OpRisk dataset

SAS OpRisk Global dataset

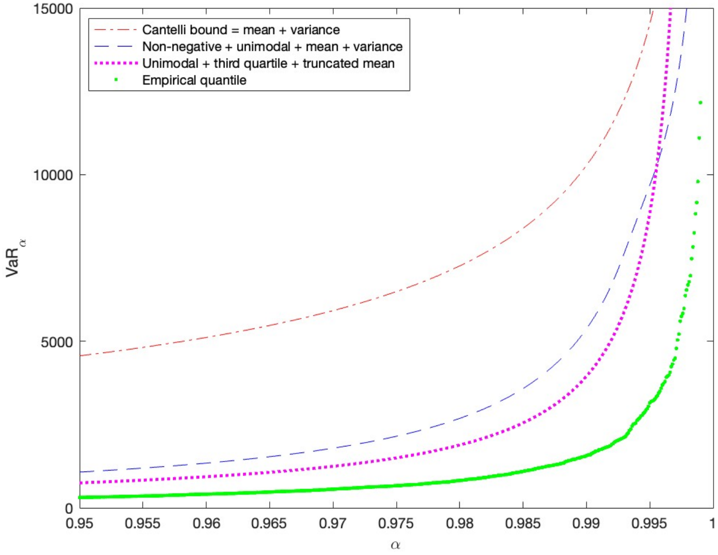
- The dataset contains 39,359 operational losses exceeding \$0.1 million, recorded from March 1971 to April 2023 worldwide.
- The losses are adjusted for inflation and expressed in millions of USD.
- Mean $\approx$ 107, std.dev. $\approx$ 1,022, and 75^{th} percentile $\approx$ 30.
- Truncated moments between 75^{th} and 99.9^{th} percentiles: mean $\approx$ 313 and std.dev. $\approx$ 798.

VaR upper bound: unimodal + lower quantile + truncated mean

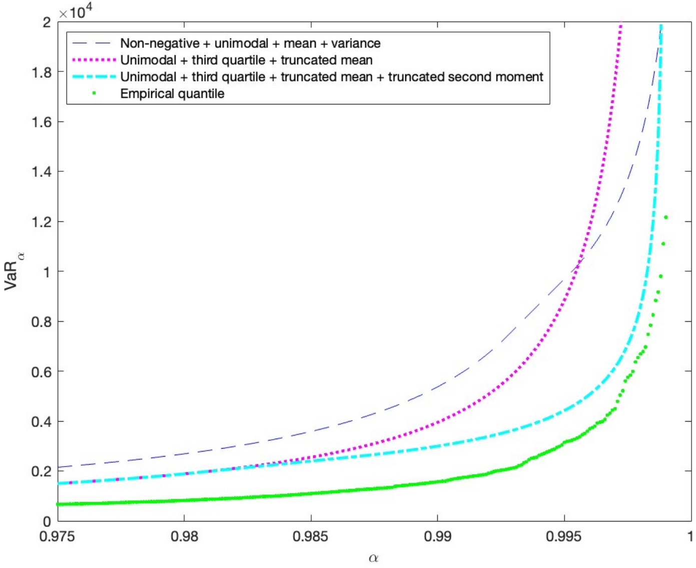
For $0.5 \leq \frac{p_1+p_2}{2} < \alpha < p_2 < 1$ and $q_1, \mu_{1,t} \in \mathbb{R}^+$, we have that

$$\sup_{\substack{F \text{ unimodal} \\ F^{-1}(p_1)=q_1 \\ \int_{p_1}^{p_2} F^{-1}(p)dp \leq \mu_{1,t}}} \mathrm{VaR}_\alpha(F) = q_1 \frac{p_2 + p_1 - 2\alpha}{2(p_2 - \alpha)} + \mu_{1,t} \frac{p_2 - p_1}{2(p_2 - \alpha)}$$

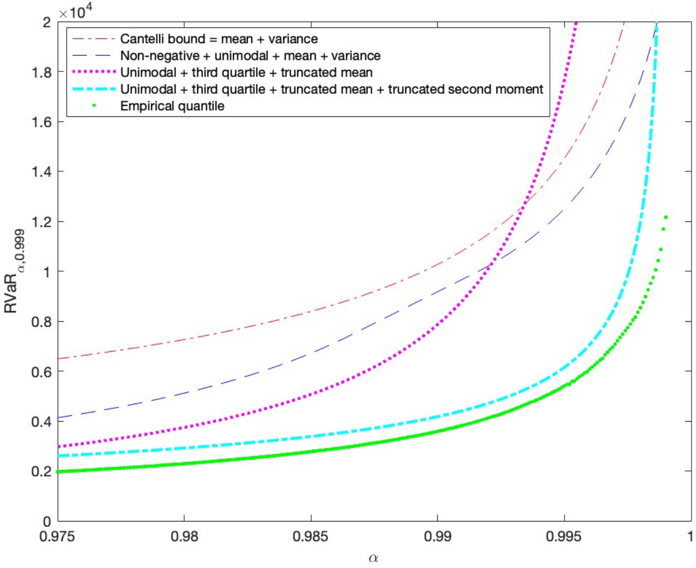
Comparision of VaR upper bounds - 1



Comparision of VaR upper bounds - 2



Comparision of RVaR upper bounds



The main takeaway

Assessing model uncertainty in heavy-tailed distributions is possible.

THANK YOU!

References

Carole Bernard, Rodrigue Kazzi, and Steven Vanduffel. Impact of model misspecification on the Value-at-Risk of unimodal T-symmetric distributions. *Working Paper*, 2023a.

Carole Bernard, Rodrigue Kazzi, and Steven Vanduffel. Model uncertainty assessment for symmetric and right-skewed distributions. *Available at SSRN 4468467*, 2023b.