# Some recent results on the axiomatic theory of risk measures

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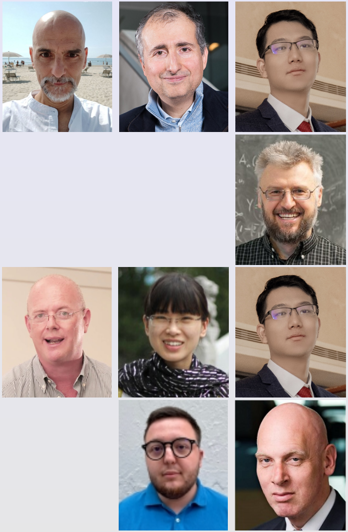
# Content



- Maccheroni/Marinacci/W./Wu
  Risk aversion and hedging motives
  Working paper, 2023
- W./Zitikis
  An axiomatic foundation for the Expected Shortfall
  Management Science, 2021
- Bellini/Mao/W./Wu
  Duet expectile preferences
  Working paper, 2023
- Principi/Wakker/W.
  Antimonotonicity for preference axioms: The natural counterpart to comonotonicity
  arxiv:2307.08542, 2023

# Agenda


1. Risk measures
2. Additivity
3. Comonotonicity
4. Risk concentration
5. Solvency synchronization
6. Antimomonotonicity

# Risk measures

- Fix an atomless probability space $(\Omega, \mathcal{F}, \mathbb{P})$
- $\mathcal{X}$: the set of bounded random variables, representing losses
- A risk measure is $\rho : \mathcal{X} \to \mathbb{R}$ satisfying
  - Monotonicity: $\rho(X) \le \rho(Y)$ whenever $X \le Y$
  - Normalization: $\rho(0) = 0$ and $\rho(1) = 1$
- $\rho$ maps a risk (via a model) to a number
  - regulatory capital calculation
  - insurance pricing
  - decision making, optimization, portfolio selection, ...
  - performance analysis and capital allocation

# General framework

**LI.** (Law invariance) $\rho(X) = \rho(Y)$ if $X \stackrel{\mathrm{d}}{=} Y$, where $\stackrel{\mathrm{d}}{=}$ means equality in distribution under $\mathbb{P}$

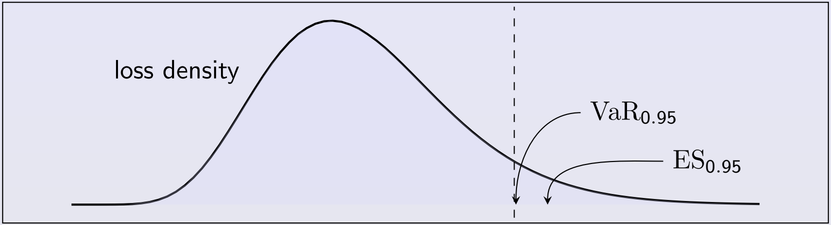
A risk measure is coherent if Artzner/Delbaen/Eber/Heath'99 MF

**TI.** (Translation invariance) $\rho(X+m) = \rho(X) + m$ for $X \in \mathcal{X}$ and $m \in \mathbb{R}$.

**PH.** (Positive homogeneity) $\rho(\lambda X) = \lambda\rho(X)$ for $X \in \mathcal{X}$ and $\lambda > 0$.

**S.** (Subadditivity) $\rho(X+Y) \le \rho(X) + \rho(Y)$ for $X, Y \in \mathcal{X}$.

# VaR and ES




**Value-at-Risk (VaR), $p \in (0,1)$**

$\mathrm{VaR}_p : L^0 \to \mathbb{R}$,

$$\mathrm{VaR}_p(X) = F_X^{-1}(p) = \inf\{x \in \mathbb{R} : \mathbb{P}(X \le x) \ge p\}.$$

(left-quantile)

**Expected Shortfall (ES), $p \in (0,1)$**

$\mathrm{ES}_p : L^1 \to \mathbb{R}$,

$$\mathrm{ES}_p(X) = \frac{1}{1-p}\int_p^1 \mathrm{VaR}_q(X)\mathrm{d}q$$

(also: TVaR/CVaR/AVaR)

# Expectiles

For $\alpha \in (0,1)$ and $X \in \mathcal{X}$, the $\alpha$-expectile $\mathrm{ex}_\alpha(X)$ is the unique number $y$ such that

$$\alpha \mathbb{E}\left[(X-y)_+\right] = (1-\alpha)\mathbb{E}\left[(y-X)_+\right]$$

Expectiles are

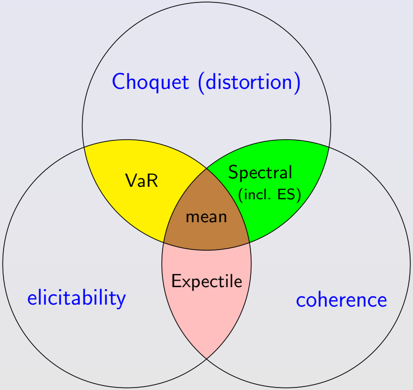
- introduced in asymmetric least squares — Newey/Powell'87 ECMA

$$\mathrm{ex}_\alpha(X) = \arg\min_{y\in\mathbb{R}} \mathbb{E}\left[\alpha(X-y)_+^2 + (1-\alpha)(y-X)_+^2\right]$$

- coherent if $\alpha \geq 1/2$ — Bellini/Klar/Müller/Rosazza Gianin'14 IME
- elicitable — Ziegel'16 MF

# The diagram of law-invariant risk measures

- (Choquet) comonotonic additivity
- (Coherence) **TI** + **PH** + **S**



# Axiomatic theory of risk functionals

- expected utility theory — von Neumann/Morgenstein'44
- subjective expected utility — Savage'54
- rank dependent utility — Qinggin'82 JEBO
- dual utility — Yaari'87 ECMA
- Choquet expected utility — Schmeilder'89 ECMA
- insurance premium — Wang/Young/Panjer'97 IME
- coherent risk measures — Artzner/Delbaen/Eber/Heath'99 MF
- convex risk measures — Föllmer/Schied'02 FS; Frittelli/Rosazza Gianin'02 JBF

# Additivity

# Additivity

Additivity:

$$\rho(X+Y)=\rho(X)+\rho(Y) \text{ for all } X,Y\in\mathcal{X}$$

**Theorem 1**

*A risk measure $\rho:\mathcal{X}\to\mathbb{R}$ is additive if and only if*

$$\rho(X)=\mathbb{E}^{Q}[X],\quad X\in\mathcal{X}$$

*for some probability $Q$. If $\rho$ is further law invariant, then $\rho=\mathbb{E}^{\mathbb{P}}$.*

- Hahn-Banach theorem
- Bookmaking — de Finetti'31
- Risk-neutral pricing

# General framework

Additivity under dependence $\mathcal{D}$

$$\rho(X+Y)=\rho(X)+\rho(Y) \text{ for } (X,Y)\in\mathcal{D}$$

- The set $\mathcal{D}$ represents some dependence
- The choice of $\mathcal{D}$ pins down different classes of risk measures
- Interpretation: $\mathcal{D}$ leads to no diversification benefit
  - this interpretation is the best with subadditivity

# Comonotonicity

# Comonotonicity

Two random variables $X$ and $Y$ are comonotonic if

$$(X(\omega) - X(\omega'))(Y(\omega) - Y(\omega')) \geq 0 \text{ almost surely wrt } \mathbb{P} \times \mathbb{P}$$

- Most positive dependence

e.g., Denneberg'94; Dhaene/Denuit/Goovaerts/Kaas/Vynche'02

- Equivalent definition: For some increasing functions $f$ and $g$,

$$X = f(X+Y) \text{ and } Y = g(X+Y) \text{ almost surely}$$

Capacity

Choquet'54

$$\nu : \mathcal{F} \to [0,1] \text{ increasing with } \nu(\varnothing) = 0$$

Choquet integral

$$\int X \mathrm{d}\nu = \int_0^\infty \nu(X > x)\mathrm{d}x + \int_{-\infty}^0 (\nu(X > x) - \nu(\Omega))\mathrm{d}x$$

# Comonotonicity

**Theorem 2 (Schmeidler'86; Yaari'87)**

*A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is additive for comonotonic risks if and only if*

$$\rho(X) = \int X \mathrm{d}\nu, \quad X \in \mathcal{X}$$

*for some capacity $\nu$ with $\nu(\Omega) = 1$. If $\rho$ is further law invariant, then $\nu = g \circ \mathbb{P}$ for some increasing $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$.*

- Non-additive integral — Schmeidler'86 PAMS; '89 ECMA
- Dual utility theory — Yaari'87 ECMA
- Distortion premium/risk measures — Wang/Young/Panjer'97 IME

# Risk concentration

# Risk concentration

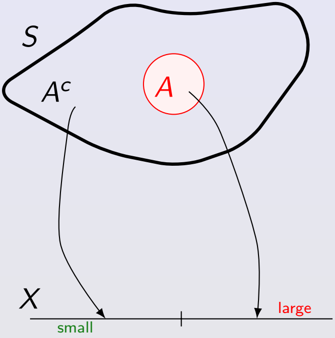
**Definition 1 (Tail events)**

A tail event of $X$ is $A \in \Sigma$ such that

a) $0 < \mathbb{P}(A) < 1$

b) $X(\omega) \geq X(\omega')$

for a.e. all $\omega \in A$ and $\omega' \in A^c$




- tail event $\Longrightarrow$ most severe loss

# Risk concentration

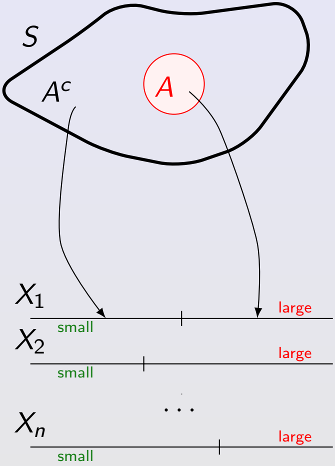
Undesirable dependence

concentrated portfolio $\Longleftrightarrow$

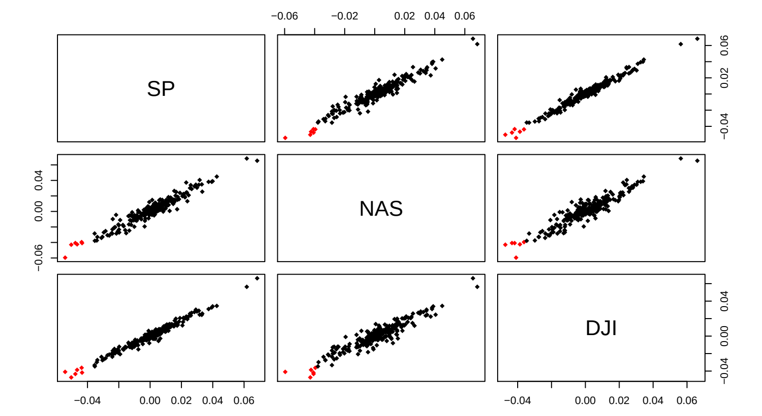
severe losses occur simultaneously

on a stress event

- $A$: a stress event specified by the regulator

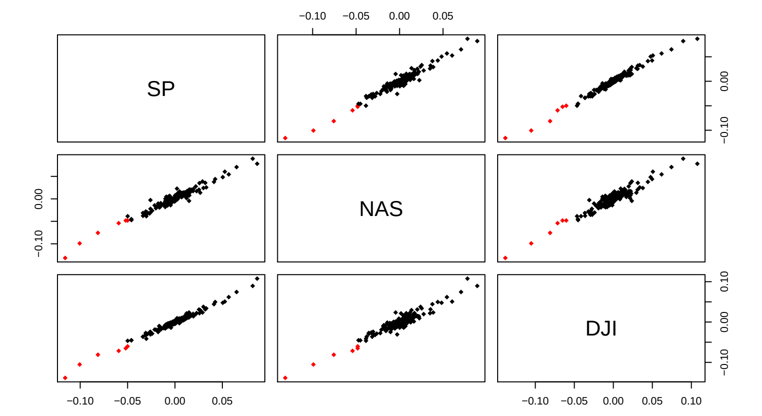


# Risk concentration in 2009




S&P 500, NASDAQ and Dow Jones daily returns, Jan 2, 2009 - Dec 31, 2009

# Risk concentration in 2019 - 2020




S&P 500, NASDAQ and Dow Jones daily returns, Jul 1, 2019 - Jun 30, 2020

# Axiomatizing ES

**No reward for concentration**

**NRC.** (No reward for concentration) There exists an event $A \in \mathcal{F}$ such that $\rho(X+Y) = \rho(X) + \rho(Y)$ holds for all risks $X$ and $Y$ sharing the tail event $A$.

- **NRC**: additivity for concentrated risks

# Axiomatizing ES

**LC.** (Lower semicontinuity) $\liminf_n \rho(X_n) \geq \rho(X)$ whenever $X_n \to X$ point-wise.

- The loss is modeled truthfully (e.g., consistent estimators) $\Longrightarrow$ estimated risk $\geq$ true risk asymptotically

**Theorem 3 (W./Zitikis'21)**

*A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ satisfies* **LI**, **LC** *and* **NRC** *if and only if it is $\mathrm{ES}_p$ for some $p \in (0, 1)$.*

- Additivity for risk concentration characterizes ES!
- $\mathrm{ES}_p$ is coherent and Choquet

# Solvency synchronization

# Solvency synchronization

**Solvency-synced dependence**

Two random variables $X$ and $Y$ are $\rho$-solvency-synced if $\{X > \rho(X)\} = \{Y > \rho(Y)\}$.

**No reward for solvency-synchronization**

**NRS.** (No reward for solvency-sync) $\rho(X + Y) = \rho(X) + \rho(Y)$ if $X$ and $Y$ are $\rho$-solvency-synced.

- Disappointment aversion — Gul'91 ECMA
  - Disappointment: $X$ is worse than its certainty equivalent $\rho(X)$

# Axiomatizing expectiles

**SC.** (Sup-norm continuity) $\rho(X_n) \to \rho(X)$ whenever $X_n \to X$ in sup-norm.

**Theorem 4 (Bellini/Mao/W./Wu'23)**

*A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ satisfies* **LI**, **SC** *and* **NRS** *if and only if it is $\mathrm{ex}_\alpha$ for some $\alpha \in (0, 1)$.*

- Additivity for solvency-synced risks characterizes expectiles!
- An expectile is coherent for $\alpha \geq 1/2$ but not Choquet

# Antimomonotonicity

# Antimomonotonicity

- Two random variables $X$ and $Y$ are antimomonotonic if $X$ and $-Y$ are comonotonic
- Also known as counter-monotonicity
- Most negative dependence e.g., Puccetti/W.'15 STS

**Theorem 5 (Principi/Wakker/W.'23)**

*A risk measure $\rho : \mathcal{X} \to \mathbb{R}$ is additive for antimonotonic risks if and only if*

$$\rho(X) = \mathbb{E}^Q[X], \quad X \in \mathcal{X}$$

*for some probability $Q$. If $\rho$ is further law invariant, then $\rho = \mathbb{E}^{\mathbb{P}}$.*

- Antimonotonic additivity $\iff$ additivity

# Antimonotonicity

Proof for a finite $\Omega = \{\omega_1, \dots, \omega_n\}$.

- We will show antinomonotonic additivity (AA) $\Longrightarrow$ additivity
- (AA) $\Longrightarrow 0 = \rho(X - X) = \rho(X) + \rho(-X) \Longrightarrow \rho(-X) = -\rho(X)$
- $X$ and $Y$ are comonotonic $\Longrightarrow$ $X + Y$ and $-Y$ are antimonotonic
  $\Longrightarrow I(X) = I(X + Y - Y) = I(X + Y) + I(-Y) = I(X + Y) - I(Y)$
- $\Longrightarrow$ comonotonic additivity (CA) holds
- For general $X$, $Y$, write $X = X^\uparrow + X^\downarrow$ with $X^\uparrow(\omega_i)$ increasing and $X^\downarrow(\omega_i)$ decreasing in $i$, and $Y = Y^\uparrow + Y^\downarrow$ similar
- Putting the above together,

$$\begin{aligned} I(X+Y) &\overset{\text{(def)}}{=\!=} I(X^\uparrow + X^\downarrow + Y^\uparrow + Y^\downarrow) \\ &\overset{\text{(AA)}}{=\!=} I(X^\uparrow + Y^\uparrow) + I(X^\downarrow + Y^\downarrow) \\ &\overset{\text{(CA)}}{=\!=} I(X^\uparrow) + I(Y^\uparrow) + I(X^\downarrow) + I(Y^\downarrow) \\ &\overset{\text{(AA)}}{=\!=} I(X^\uparrow + X^\downarrow) + I(Y^\uparrow + Y^\downarrow) = I(X) + I(Y) \end{aligned}$$

# Conclusion

Additivity under dependence

- characterizes law-invariant risk measures
  - arbitrary dependence: mean
  - comonotonicity: Choquet (distortion) risk measures
  - concentration via tail events: ES
  - solvency-synced dependence: expectiles
  - antimonotonicity: mean
- leads to many new mathematics

# Conclusion

Future directions

- Characterizing other risk measures such as VaR
  - Comonotonic additivity + convex level sets — Kou/Peng'16 OR
    - (without monotonicity) — Wang/W.'20 MF
  - Tail relevance + elicitability — Liu/W.'21 MOR
  - Ordinality + continuity — Chambers'09 MF
    - (without monotonicity/continuity) — Fadina/Liu/W.'23 SIFIN
- Preferences for dependence structures
- Ambiguity and uncertainty (relaxing law-invariance)

# Thank you

## Thank you for your attention

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