# Optimal reinsurance with model uncertainty

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- Reinsurance is an insurance that insurance company purchases to reduce its risk exposure.
- We study $\mathrm{VaR}$ and Range-Value-at-Risk (RVaR) optimal reinsurance problem for worst-case scenario under dependence uncertainty,
  - that is, we consider dependence structure that gives rise to the largest level of risk for any reinsurance contracts.
- Due to the limited analysis of robust risk aggregation for VaR and RVaR in some cases, we impose some constraints on the ceded loss functions.

# Reinsurance

| | Primary insurance | Reinsurance |
|---|---|---|
| Life and health | 2500 | 65 |
| Non-life | 2000 | 170 |

Table: Global premium volume 2015 (in US$ billions). Data from Swiss Re.

For reinsured risk,

- small volume
- complicated to handle

Benefits;

- reduce the risk exposure
- stabilize the profit over a period
- underwrite more insurance contracts

# Reinsurance - Single risk

An insurance company faces a risk $X$ over a period.
Reinsurance:

- $f(X) \to$ reinsurer
- $R_f(X) = X - f(X) \to$ insurer
- The insurer pays premium $\pi(f(X))$ to the reinsurer
- Total risk exposure $S^f(X) = X - f(X) + \pi(f(X))$

To minimize

$$\rho(S^f(X))$$

Three factors

- $\rho$ the optimization objective on $S^f(X)$
- $\pi$ is a the premium principle
- $f$ is the ceded loss function

# Risk measures

- Value-at-Risk: $\mathrm{VaR}_\alpha(X) = (F_X)_L^{-1}(\alpha)$ (Solvency II);
- Expected Shortfall (ES): (Swiss Solvency Test) For $\alpha \in [0,1)$,

$$\mathrm{ES}_\alpha(X) = \frac{1}{1-\alpha}\int_\alpha^1 \mathrm{VaR}_t(X)dt;$$

- Range-Value-at-Risk (RVaR) (Cont-Deguest-Scandolo'10 QF): For $0 \le \beta < \beta + \alpha \le 1$,

$$R_{\beta,\alpha}(X) = \frac{1}{\alpha}\int_\beta^{\beta+\alpha} \mathrm{VaR}_{1-t}(X)dt,$$

Clearly, $R_{0,\alpha}(X) = \mathrm{ES}_\alpha(X)$ and $\lim_{\alpha\to 0} R_{\beta,\alpha}(X) = \mathrm{VaR}_{1-\beta}(X)$.

# Example of premium principles

- Expectation principle:

$$\pi(X) = (1+\theta)\mathbb{E}(X)$$

for $X \in \mathcal{X}$ with $\theta > 0$;

- Standard deviation principle:

$$\pi(X) = \mathbb{E}(X) + \lambda\sqrt{Var(X)}$$

for $X \in \mathcal{X}$ with $\lambda > 0$;

- Wang's principle:

$$\pi_g(X) = \int_0^\infty g(\mathbb{P}(X > x))dx$$

for $X \in \mathcal{X}_g$, where $g : [0,1] \to [0,1]$ with $g(0) = 0$ and $g(1) = 1$, and $g$ is increasing.

Book: Young' 04, eleven widely used premium principles.

# Loss function

Both $f$ and $R_f$ are non-negative and increasing on $[0, \infty) \Longrightarrow$ Lipschitz-continuous, i.e.,

$$0 \leq f(y) - f(x) \leq y - x, \; 0 \leq x \leq y, \; 0 \leq f(x) \leq x, \; x \geq 0.$$

Examples:

- Quota-share: $f(x) = ax$ with $0 \leq a \leq 1$;
- Stop-loss: $f(x) = (x - c)_+$ with $c > 0$;
- Limited stop-loss:
  $f(x) = (x - a)_+ - (x - b)_+ = \min((x - a)_+, b - a)$ with $0 \leq a \leq b$.

(Cai-Chi'20 STRF: review)

# Multiple risks

An insurance company usually has many lines of business and each line generates a risk $X_i$.
Life insurance and non-life insurance.

- Reinsurance for each business: $X_i - f_i(X_i) + \pi_i(f_i(X_i))$
- The total risk: $S^{\mathbf{f}} = \sum_{i=1}^{n} X_i - f_i(X_i) + \pi_i(f_i(X_i))$, where $\mathbf{f} = (f_1, \ldots, f_n)$
- The task is *to minimize* $\rho(S^{\mathbf{f}})$.
  - In many cases, individual risks are not completely independent, as they tend to be driven by common events, such as natural catastrophes, financial crises, man-made disasters and longevity risk.
  - Such events may create different levels of dependence among individual losses.

# Literature

- Cai-Wei'12 IME: $\rho(X) = \mathbb{E}(u(X))$, $\pi_i(X) = (1+\theta_i)\mathbb{E}(X)$, and $(X_1, \ldots, X_n)$ are positive dependence through stochastic ordering
- Cheung-Sung-Yam'14 JRI: $\rho$ : convex risk measure, $\pi_i(X) = (1+\theta_i)\mathbb{E}(X)$, $(X_1, \ldots, X_n)$ are comonotonic (the worst case scenario)
- Bernard-Liu-Vanduffel'20 JEBO: $\rho(X) = \mathbb{E}(u(X))$, general premium principle, and some specific dependence structure, $f_i(x) = a_i x$ Quota-Share policy

# Dependence uncertainty

In practice, it is difficult to know the dependence structure.

$$\mathcal{E}_n(\mathbf{F}) = \{(X_1, \dots, X_n) : X_i \sim F_i,\ i = 1, \dots, n\}.$$

We consider the worst-case scenario:

$$\sup_{(X_1,\dots,X_n)\in\mathcal{E}_n(\mathbf{F})} \rho(S_n^{\mathbf{f}}(X_1, \dots, X_n)).$$

The objective is to find

$$\arg\inf_{\mathbf{f}\in\mathcal{D}^n} \sup_{(X_1,\dots,X_n)\in\mathcal{E}_n(\mathbf{F})} \rho(S_n^{\mathbf{f}}(X_1, \dots, X_n)),$$

where $\mathbf{f} = (f_1, \dots, f_n)$ and

$$\mathcal{D}^n = \{\mathbf{f} :\ f_i \text{ and } R_{f_i} \text{ are non-negative and increasing on } [0, \infty)\ \}.$$

$$\mathcal{D}^n(P) = \left\{\mathbf{f} \in \mathcal{D}^n : \sum_{i=1}^{n} \pi_i(f_i(X_i)) \leq P\right\}.$$

# Conditions on premium principle

We impose the following conditions on $\pi_i$:

(i) Distribution invariance: For $Y, Z \in \mathcal{X}$, $\pi_i(Y) = \pi_i(Z)$ if $F_Y = F_Z$;

(ii) Risk loading: $\pi_i(Y) \geq \mathbb{E}(Y)$ for $Y \in \mathcal{X}$;

(iii) Reserving the stop-loss order: For $Y, Z \in \mathcal{X}$, $\pi_i(Y) \leq \pi_i(Z)$ if $\mathbb{E}[(Y-u)_+] \leq \mathbb{E}[(Z-u)_+]$ for all $u \in \mathbb{R}$.

(iv) Continuity: For $Y_n, Y \in \mathcal{X},\ n \geq 1$, $\lim_{n\to\infty} \pi_i(Y_n) = \pi_i(Y)$ provided that $\lim_{n\to\infty} \operatorname{ess\,sup} |Y_n - Y| = 0$. Moreover, $\lim_{d\to\infty} \pi_i(Y \wedge d) = \pi_i(Y)$.

Throughout the talk, we assume that $\pi_i$ satisfies $(i) - (iii)$.

Limited stop loss policy: $l_{a,b}(x) := \min((x-a)_+, b-a)$ where $0 \leq a \leq b \leq \infty$.

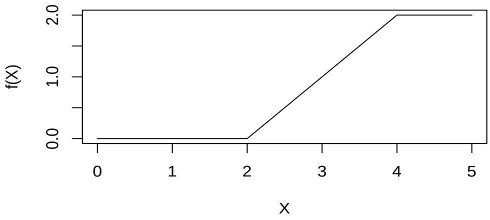



Figure: $f(x) = (x-2)_+ - (x-4)_+$.

$\mathbf{l_{a,b}} = (l_{a_1,b_1}, \ldots, l_{a_n,b_n})$ with the domain of the parameters

$$\mathcal{A}(P) = \{(\mathbf{a}, \mathbf{b}) : \mathbf{l_{a,b}} \in \mathcal{D}^n(P), 0 \leq a_i \leq b_i \leq \infty, \; i = 1, \ldots, n\}.$$

**Theorem**

*For $n = 2$, suppose that $F_1^{-1}$ and $F_2^{-1}$ are continuous over $(0, 1)$, then*

$$\inf_{(f_1,f_2)\in\mathcal{D}^2} \sup_{(X_1,X_2)\in\mathcal{E}_2(\mathbf{F})} \mathrm{VaR}_\alpha(S_2^{\mathbf{f}}(X_1, X_2)) = \inf_{(a_1,a_2,b_1,b_2)\in\mathcal{A}(P)} \inf_{t\in[0,1-\alpha]} L_1(a_1, a_2, b_1, b_2, t),$$

*where*

$$L_1(a_1, a_2, b_1, b_2, t) = \mathrm{VaR}_{\alpha+t}(X_1 - l_{a_1,b_1}(X_1)) + \mathrm{VaR}_{1-t}(X_2 - l_{a_2,b_2}(X_2)) + \pi_1(l_{a_1,b_1}(X_1)) + \pi_2(l_{a_2,b_2}(X_2)).$$

*Moreover, supposing $\pi_1$ and $\pi_2$ are continuous, $(l_{a_1,b_1}, l_{a_2,b_2})$ is the optimal ceded loss function for the worst case scenario provided*

$$(a_1, a_2, b_1, b_2) \in \arg \inf_{(a_1,a_2,b_1,b_2)\in\mathcal{A}(P)} \left\{ \inf_{t\in[0,1-\alpha]} L_1(a_1, a_2, b_1, b_2, t) \right\}.$$

Theorem 2 (i) of Blanchet-Lam-Liu-Wang 20'

We extend the result of Blanchet-Lam-Liu-Wang 20' as below. Let $\Delta_n = \left\{(\gamma_0, \gamma_1, \ldots, \gamma_n) \in (0,1) \times [0,1)^n : \sum_{i=0}^n \gamma_i = 1\right\}$.

**Proposition**

*For $\alpha \in (0,1)$,*

$$\sup_{(X_1,\ldots,X_n)\in\mathcal{E}_n(\mathbf{F})} \mathrm{VaR}_\alpha^+\left(\sum_{i=1}^n X_i\right) = \inf_{\boldsymbol{\gamma}\in(1-\alpha)\Delta_n} \sum_{i=1}^n R_{\gamma_i,\gamma_0}(X_i)$$

*holds if one of the following statements is true:*

(i) *each of $F_1, \ldots, F_n$ is convex beyond its $\alpha$-quantile;*

(ii) *each of $F_1, \ldots, F_n$ is concave beyond its $\alpha$-quantile.*

- If $X$ has decreasing density, $\min(X, a)$ may not have decreasing density.
- Dependence structure: combination of mutually exclusive and joint mixability. (negative dependence)
- Exponential, Pareto, Gamma.

# Convex distributions on tail part

To guarantee that $X - f(X)$ has a convex distribution on its tail part,

$$\mathcal{D}_1^n = \{\mathbf{f} = (f_1, \ldots, f_n) : \ \mathbf{f} \in \mathcal{D}, \ f_i \text{ is convex for } i = 1, \ldots, n\}.$$

$$h_{a,b,c,d}(x) := c(x-a)_+ + d(x-b)_+,$$

where $0 \leq a \leq b \leq \infty$ $0 \leq c, d \leq c + d \leq 1$.

$$\mathcal{A}_1 = \{(\mathbf{a}, \mathbf{b}, \mathbf{c}, \mathbf{d}) : \mathbf{h}_{\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}} \in \mathcal{D}_1^n(P)\}.$$

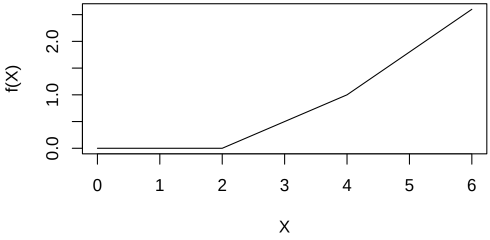



Figure: $f(x) = 0.5(x-2)_+ + 0.3(x-4)_+$.

# Convex tail distributions

**Theorem**

*Suppose $F_1^{-1}(\cdot), \ldots, F_n^{-1}(\cdot)$ are all continuous over $(0,1)$ and $\alpha \in (0,1)$. If each of $F_1, \ldots, F_n$ is convex beyond its $\alpha$-quantile, then*

$$\inf_{\mathbf{f} \in \mathcal{D}_1^n} \sup_{(X_1,\ldots,X_n) \in \mathcal{E}_n(\mathbf{F})} \mathrm{VaR}_\alpha(S_n^{\mathbf{f}}(X_1,\ldots,X_n)) = \inf_{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \in \mathcal{A}_1} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} H(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\boldsymbol{\gamma}),$$

*where*
$H(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\boldsymbol{\gamma}) = \sum_{i=1}^n \{R_{\gamma_i,\gamma_0}(X_i) - R_{\gamma_i,\gamma_0}(h_{a_i,b_i,c_i,d_i}(X_i)) + \pi_i(h_{a_i,b_i,c_i,d_i}(X_i))\}$.

*Additionally, if $\pi_i$ are continuous, $(h_{a_1,b_1,c_1,d_1}, \ldots, h_{a_n,b_n,c_n,d_n})$ is the optimal ceded loss functions for the worst case scenario provided*

$$(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) = \arg \inf_{(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d}) \in \mathcal{A}_1} \left\{ \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} H(\mathbf{a},\mathbf{b},\mathbf{c},\mathbf{d},\boldsymbol{\gamma}) \right\}$$

## Concave distributions on tail part

To guarantee that $X - f(X)$ has a concave distribution on its tail part,

$$\mathcal{D}_2^n = \{\mathbf{f} = (f_1, \ldots, f_n) : \ \mathbf{f} \in \mathcal{D}, \ f_i \text{ is concave for } i = 1, \ldots, n\}$$

$$g_{a,b}(x) := a \min(x, b),$$

where $0 \leq a \leq 1$ and $b \geq 0$ and

$$\mathcal{A}_2 = \{(\mathbf{a}, \mathbf{b}) : \mathbf{g_{a,b}} \in \mathcal{D}_2^n(P)\}.$$

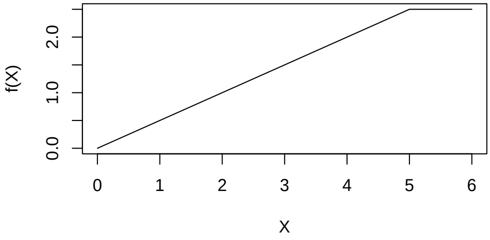



Figure: $f(x) = 0.5 \min(x, 5)$.

# Concave tail distributions

## Theorem

*Suppose $F_1^{-1}(\cdot), \ldots, F_n^{-1}(\cdot)$ are all continuous over $(0,1)$ and $\alpha \in (0,1)$. If each of $F_1, \ldots, F_n$ is concave beyond its $\alpha$-quantile, then*

$$\inf_{\mathbf{f} \in \mathcal{D}_2^n} \sup_{(X_1,\ldots,X_n) \in \mathcal{E}_n(\mathbf{F})} \mathrm{VaR}_\alpha(S_n^{\mathbf{f}}(X_1, \ldots, X_n))$$
$$= \inf_{(\mathbf{a},\mathbf{b}) \in \mathcal{A}_2} \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} G(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}),$$

*where $G(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) = \sum_{i=1}^n \{R_{\gamma_i,\gamma_0}(X_i) - R_{\gamma_i,\gamma_0}(g_{a_i,b_i}(X_i)) + \pi_i(g_{a_i,b_i}(X_i))\}$.*

*Additionally, if $\pi_i$ are continuous, $(g_{a_1,b_1}, \ldots, g_{a_n,b_n})$ is the optimal ceded loss functions for the worst case scenario provided*

$$(\mathbf{a}, \mathbf{b}) = \arg \inf_{(\mathbf{a},\mathbf{b}) \in \mathcal{A}_2} \left\{ \inf_{\boldsymbol{\gamma} \in (1-\alpha)\Delta_n} G(\mathbf{a}, \mathbf{b}, \boldsymbol{\gamma}) \right\}$$

# RVaR

- We extend the result (Theorem 1) of Blanchet-Lam-Liu-Wang 20' on convolution bounds on RVaR aggregation from marginal with decreasing densities in the tail part to those with concave distribution in the tail part.
- We obtain similar results on the optimal reinsurance problems.

We solve

$$\min_{\mathbf{f}\in\mathcal{D}^2} \max_{(X_1,X_2)\in\mathcal{E}_2} \mathrm{VaR}_\alpha \left(S_2^{\mathbf{f}}(X_1, X_2)\right), \tag{1}$$

The optimisation problem in (1) can be written as

$$\min_{(a_1,a_2,b_1,b_2)\in\mathcal{A}(P)} \min_{t\in[0,1-\alpha]} L_1\left(a_1, a_2, b_1, b_2, t\right) \tag{2}$$

where

$$\begin{aligned} L_1(a_1, a_2, b_1, b_2, t) \quad = \quad & \mathrm{VaR}_{\alpha+t}(X_1 - l_{a_1,b_1}(X_1)) + \mathrm{VaR}_{1-t}(X_2 - l_{a_2,b_2}(X_2)) \\ & + \pi_1(l_{a_1,b_1}(X_1)) + \pi_2(l_{a_2,b_2}(X_2)) \end{aligned}$$

and $\pi_i(X_i) = (1+\theta_i)\mathbb{E}(X_i)$ where $\theta_i > 0$ represents the risk-loading factor for $i = 1, 2$.

# Grid search

Problem (2) is non-convex and we use grid search.

- Discretise $X_i$ into $\mathbf{x}_i := (x_{i1}, x_{i2}, \ldots, x_{in})^T$, for $i = 1, 2$
- $I_{a_i, b_i}(X_i)$ is represented by $\mathbf{y}_i := (y_{i1}, y_{i2}, \ldots, y_{in})^T$
- Suppose $x_{ij}$ is collected in an ascending order for $i = 1, 2$, i.e. $x_{i1} \leq x_{i2} \leq \cdots \leq x_{in}$
- $y_{i1} \leq y_{i2} \leq \cdots \leq y_{in}$

# Optimization

For any given value of $t \in [0, 1-\alpha]$, $t^*$, the local optimum of the optimisation problem (2) can be found by solving the following linear minimisation problem:

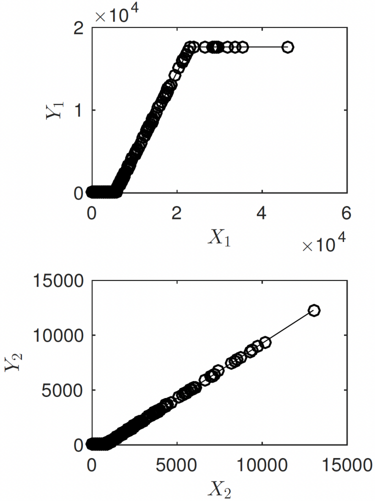
$$\min_{(\mathbf{y}_1,\mathbf{y}_2)\in\mathbb{R}_+^n\times\mathbb{R}_+^n} -y_{1q_1} - y_{2q_2} + \frac{(1+\theta_1)}{n}\mathbf{1}^T\mathbf{y}_1 + \frac{(1+\theta_2)}{n}\mathbf{1}^T\mathbf{y}_2 \quad (3)$$
$$\text{s.t. } \mathbf{0} \leq \mathbf{A}\mathbf{y}_i \leq \mathbf{A}\mathbf{x}_i, \quad \forall i = 1, 2$$

where $q_1 = \lceil n(\alpha + t^*) \rceil$, $q_2 = \lceil n(1 - t^*) \rceil$ and $\mathbf{A}$ is an $n$-by-$n$ matrix.

The global optimum of the optimisation problem (2) can be found by using Global Search over the value of $t \in [0, 1-\alpha]$.

# Example: Exponential marginals

- $X_i \sim Exp(\lambda_i)$ with $E(X_i) = \lambda_i > 0$
- $\lambda_1 = 8000$, $\lambda_2 = 3000$, $\theta_1 = 0.8$, $\theta_2 = 0.3$, $\alpha = 0.95$ and $n = 200$



# Summary

- Our main results show that finding the optimal ceded loss functions for the worst case reinsurance models with dependence uncertainty boils down to finding the minimiser of a deterministic function.

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Thank You!