The valuation and assessment of retirement income products: A unified Markov chain Monte Carlo framework

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Overview

- Motivation
- The multi-asset market model
- Retirement income products
- Markov chain Monte Carlo methods
- Numerical results
- Conclusion

Motivation

- The retirement income covenant (Treasury, 2021) requires providers to assist their membership in:
  - understanding their potential income in retirement,
  - maximising income,
  - managing and understanding longevity risk, financial and inflation risks.
- Australian retirees are drawing down 17% less income from their superannuation than what is 'optimal'[1].
- Lack of clearer drawdown strategies for retirement income.
- Complexity on valuation and risk management in high dimensions.
  - High-dimensional integration problem.

[1] *https://www.yourlifechoices.com.au/retirement/push-to-make-retirees-withdraw-more-of-their-super-every-year/*

Literature review – pricing techniques

- Simulation-based approaches:
  - Monte Carlo.
  - Least Squares Monte Carlo.
  - Quasi-Monte Carlo.
- Other numerical approaches:
  - Partial differential equation.
  - Tree-based methods.
  - Stochastic control approach.
  - Fourier Space Time-stepping algorithm.
  - Fourier-cosine method.
  - …

A common assumption: The underlying fund invests in one stock.

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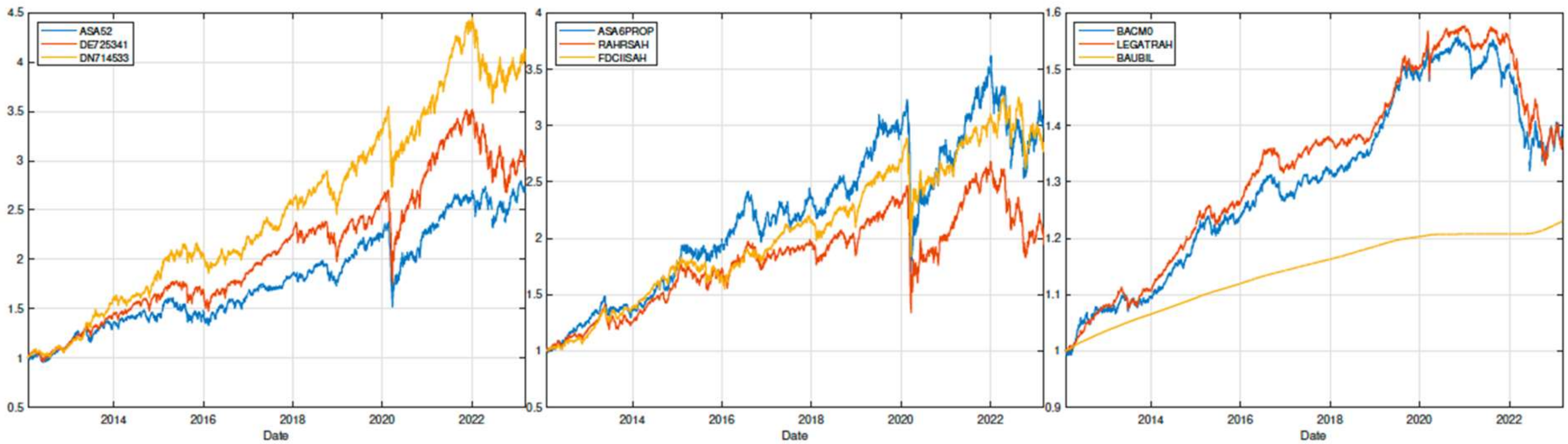
Research questions

- How to efficiently value retirement income products when the underlying investment fund consists of **multiple asset classes**?
  - Markov chain Monte Carlo (MCMC) algorithm.
- How to devise a framework for **retirement income product comparison**:
  - Longevity risk protection.
  - Income volatility.
  - Bequest.

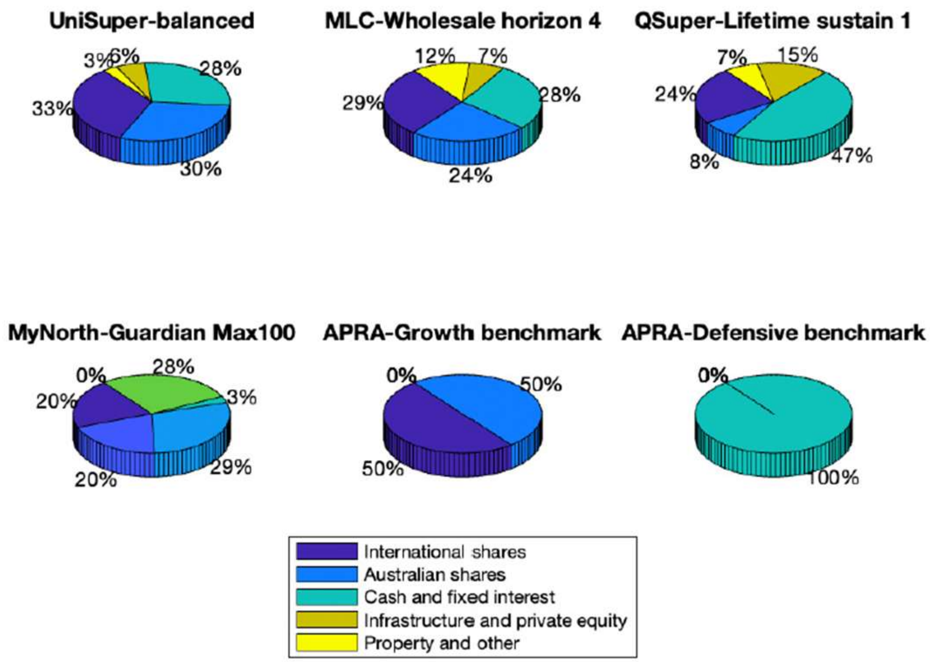


(a) Equity. (b) Property and infrastructure. (c) Cash and fixed interest.

Market indices (APRA 2022)

Figure 1: Price observations of nine market indices[2] from February 2012 to March 2023. The prices at the beginning are normalised to one.

[2] *The Australian Prudential Regulation Authority (APRA) choose nine indices to determine benchmark investment return for MySuper products (APRA, 2022).*



The underlying fund invests in multiple asset classes

Figure 2: Some typical asset allocations of superannuation trustees in Australia (Source: providers' website).

The financial market

- There are d assets in the market.
- We assume that asset prices follow the geometric Brownian motion (GBM) process.
- Economic uncertainty: the regime-switching framework (Ignatieva et al., 2016).
  - Risk-free interest rate.
  - Inflation rate.
  - Asset return and volatility.

The financial market

- The demeaned continuously compounded return of the assets $\boldsymbol{R}(\Delta t)$ given a current financial market state $\boldsymbol{X}(t)$ follows a **multivariate normal distribution** with probability density function

$$p_{\mathbf{R}(\Delta t)|\mathbf{X}(t)}(\mathbf{R}) = (2\pi)^{-\frac{d}{2}}(\det \boldsymbol{\Sigma}(t))^{-\frac{1}{2}} \exp\left[-\frac{1}{2}\mathbf{R}^\top \boldsymbol{\Sigma}(t)^{-1}\mathbf{R}\right],$$

where $\boldsymbol{\Sigma}(t) = \boldsymbol{\sigma}(t)\boldsymbol{\sigma}(t)^\top(t)\Delta t$, and $\boldsymbol{\sigma}(t)$ is the volatility coefficient in the GBM model.

The investment fund

- Fund value before fee:

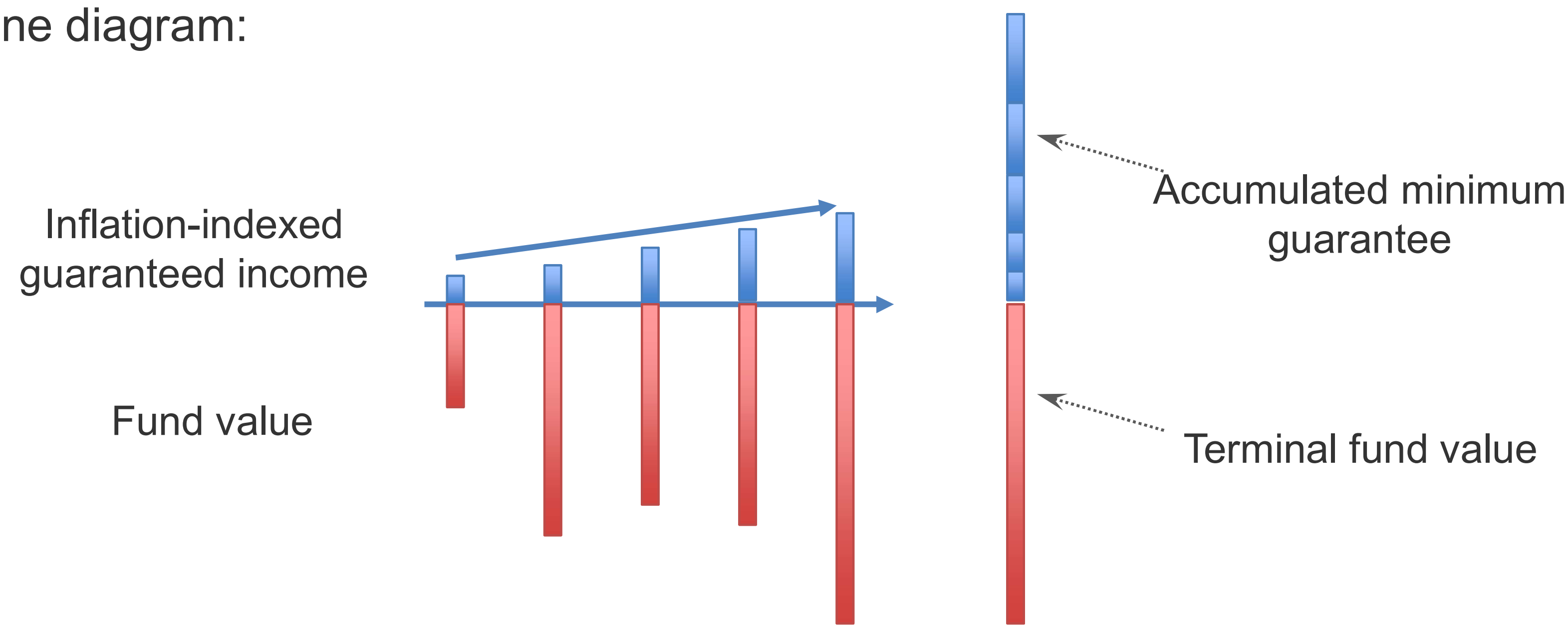
$$F(t) = F(0)\boldsymbol{\alpha}^\top \mathbf{S}(t),$$

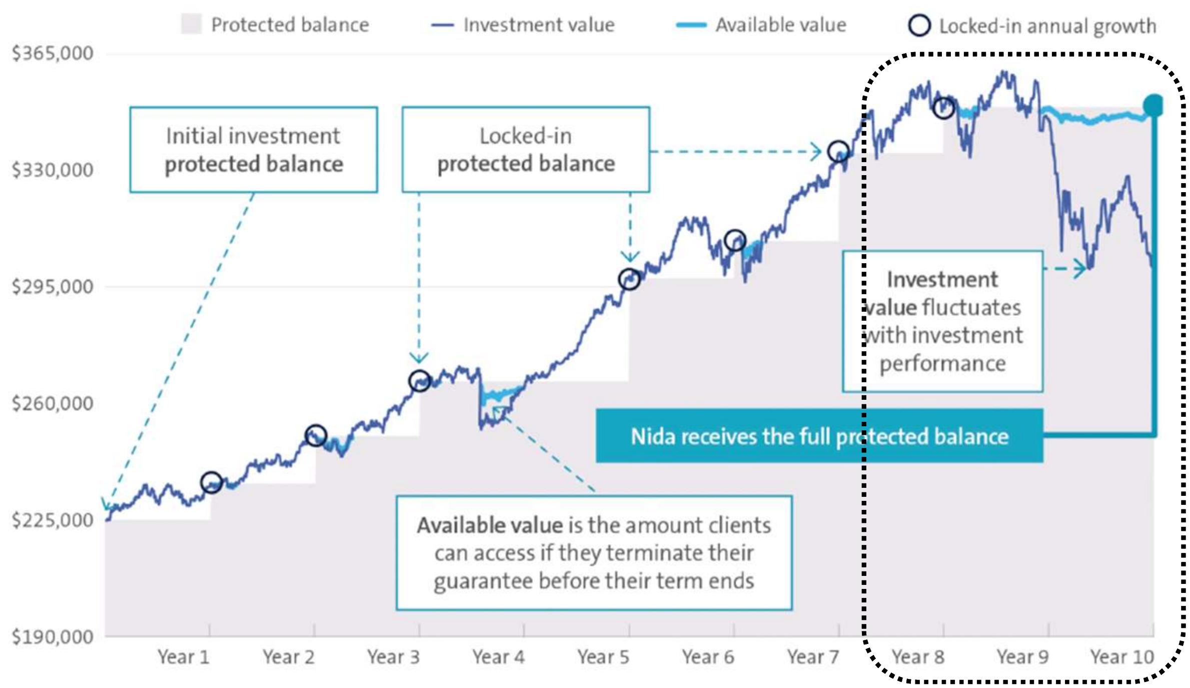
- From the policyholder's perspective:

$$\tilde{F}(t) := e^{-\zeta t} F(t).$$

Guaranteed Minimum Income Benefit (GMIB)

- Payoff = max(terminal fund value, accumulated minimum guarantee).
- GMIB payment: Terminal fund value > accumulated minimum guarantee.
- Timeline diagram:



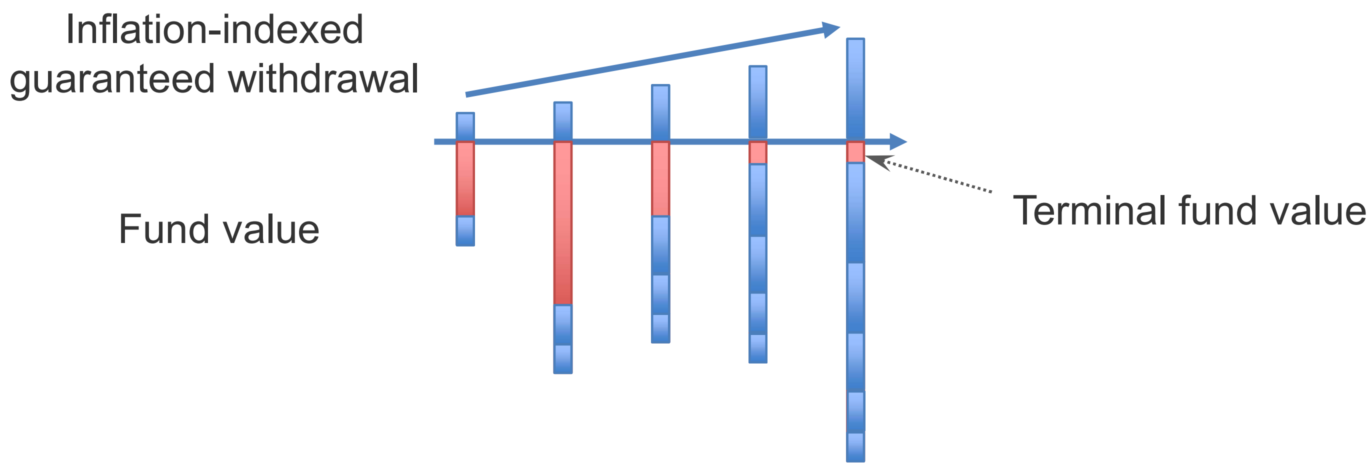


Growth lock-in feature for GMIB

Figure 3: An example of annual growth lock-in feature (MyNorth, 2022).

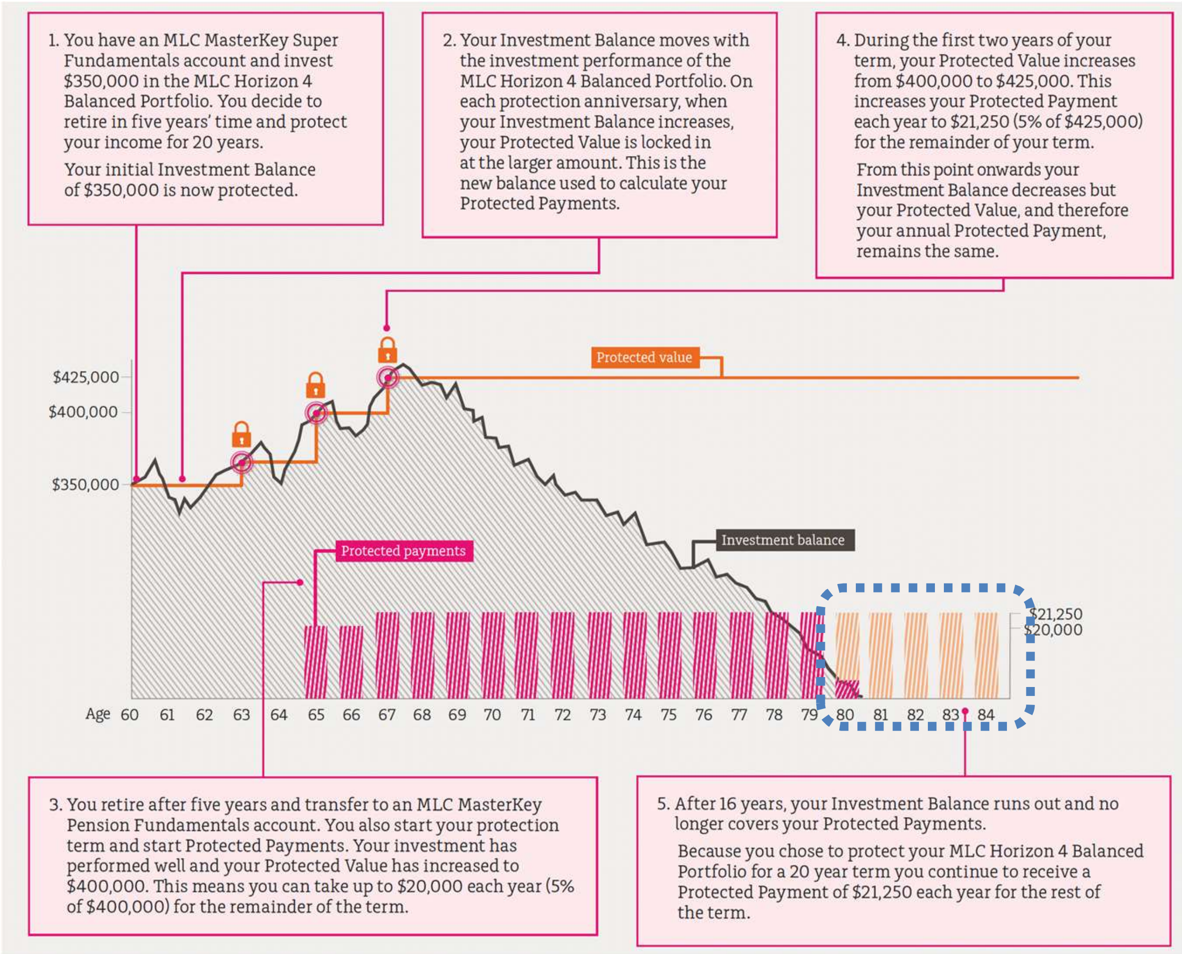
Guaranteed Minimum Withdrawal Benefit (GMWB)

- Payoff = Periodically withdrawal + Terminal fund value.
- We assume:
  - The policyholder follows the static withdrawal approach.
  - The policyholder does not surrender.
- Timeline diagram:



GMWB in the market

Figure 4: An example of GMWB (MLC, 2022).



GMWB with spouse benefit option (MLC, 2022)

- The spouse can continue making periodic withdrawals if the policyholder passes away.
- Joint life model.

Fees for variable annuities

- Risk-neutral pricing: Expected payoff equals the initial investment.
- We use the bisection method[3] to solve for the fair management fee.

[3]*Finding the root of a continuous function is frequently accomplished using the bisection approach. First, it locates an interval with the start and end of the interval having opposing function value signs. To identify the root, the approach then repeatedly bisects the interval with opposed function value signs at the start and end.*

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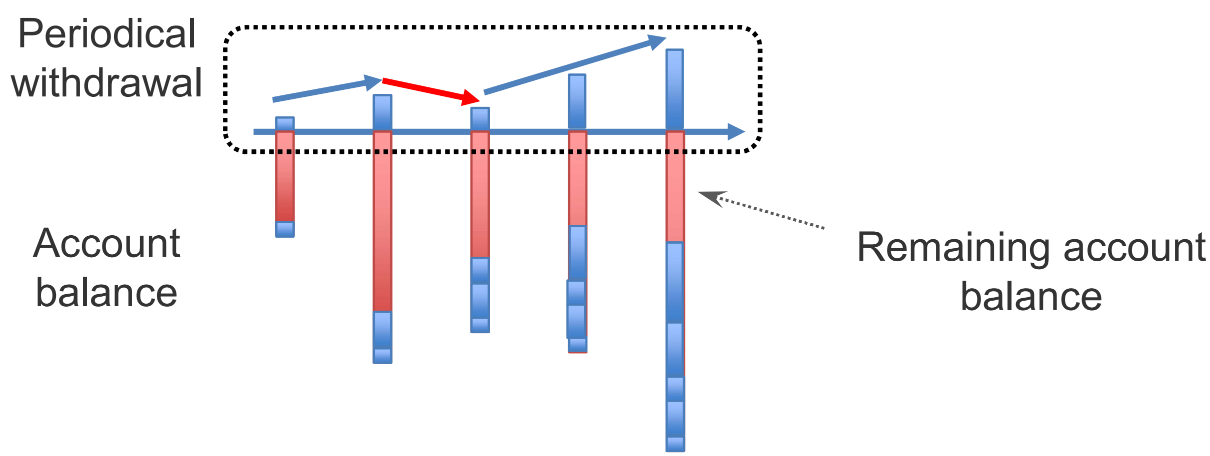
Account-based pension

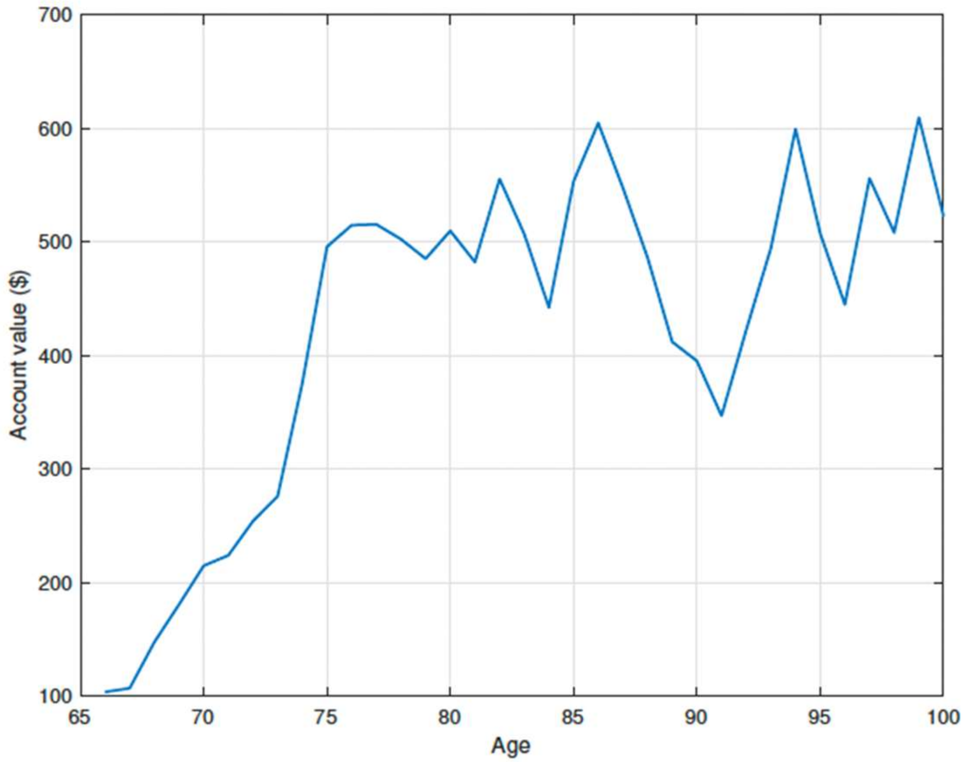
- Similar to a GMWB, except that there is no guarantee.
- The minimum drawdown rates[4].
- Timeline diagram:



[4]*Withdrawal funds in line with the minimum drawdown rate is one dominant withdrawal strategy among Australian retirees (Chomik et al., 2018). For the minimum drawdown rates, see https://www.ato.gov.au/Rates/Key-superannuation-rates-and-thresholds/?page=9.*

Account-based pension

Figure 5: A simulated account-based pension balance path that follows the minimum drawdown rate. The initial account value is $100.



Group self-annuitisation (GSA)

- Risk-pooling product.
- Life-long living benefits.
- Mortality credit.
- Dynamic pooling strategy.
- Random investment return.
- The living benefit to a member:

$$Living\ benefit = \frac{Fund\ value\ belongs\ to\ a\ member}{Annuity\ factor\ with\ future\ mortality\ improvement}.$$

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GSA with spouse protection feature

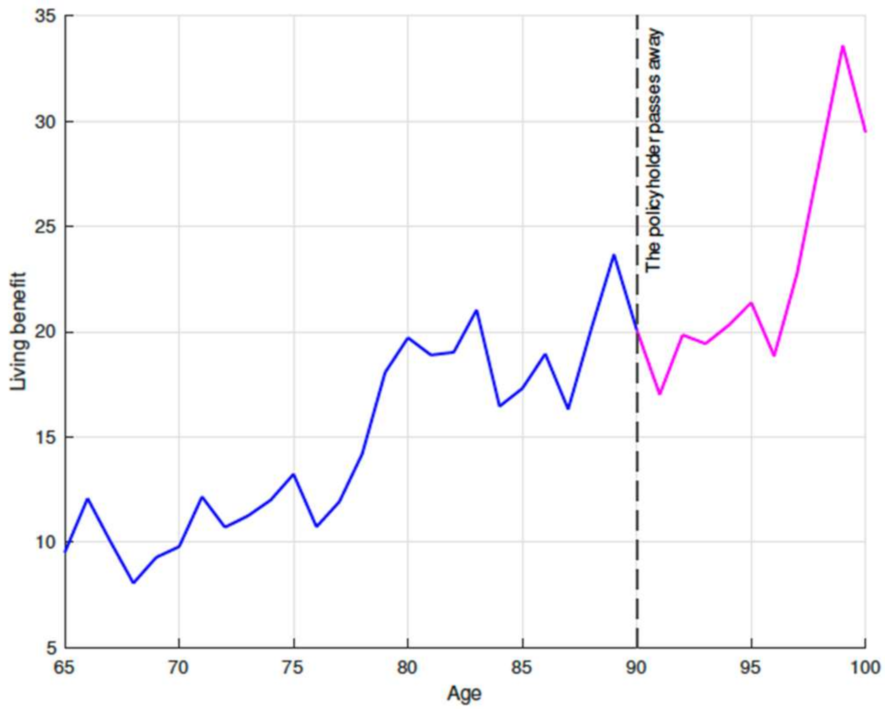
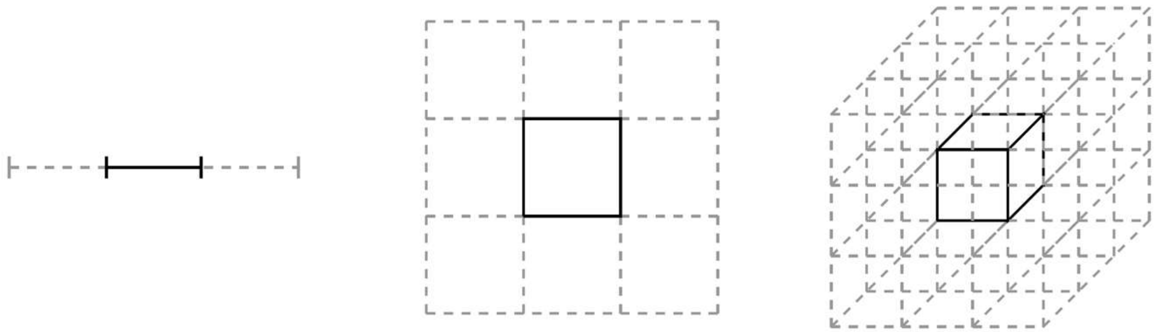


Figure 6: A GSA with the spouse protection feature continues paying living benefits to the spouse if the member passes away. (QSuper, 2022)



(a) $\frac{1}{3}$

(b) $\frac{1}{9}$

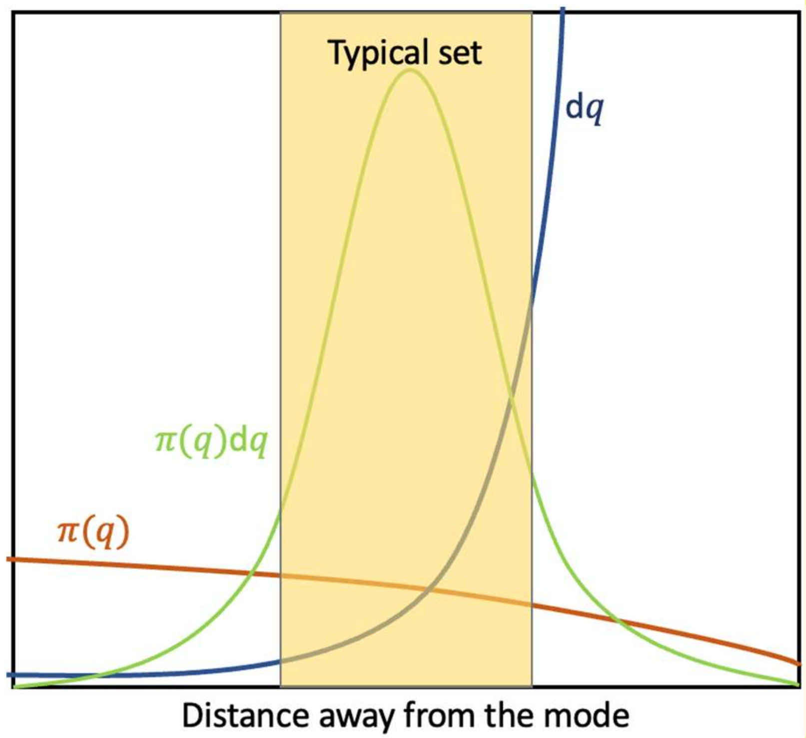
(c) $\frac{1}{27}$

Volume of the parameter space

Figure 7: Consider a rectangular partitioning centred around a distinguished point, such as the mode.

Concentration of measure

- Figure 8: In high dimensions, $\pi(q)$ will concentrate around its mode, but the volume over which we integrate that density, dq, is much larger away from the mode.



Assumption

- We consider a male policyholder aged 65, whose spouse is also aged 65.
- The economy is in State 1 when risk-free interest rate is less than 3% and in State 2 otherwise.
- We use the 9 indices[5] chosen by Australian Prudential Regulation Authority (APRA) and the initial weights[6] in the balanced investment option of UniSuper to construct the portfolio.
- Our MCMC framework accommodates other asset classes whose distributional properties are known.
- Human mortality follows the stochastic GoMa model (Qiao and Sherris, 2013).

[5]The Bloomberg ticker of the 9 indices we use are ASA52, DE725341, DN714533, ASA6PROP, RAHRSAH, FDCIISAH, BACM0, LEGATRAH, BAUBIL.

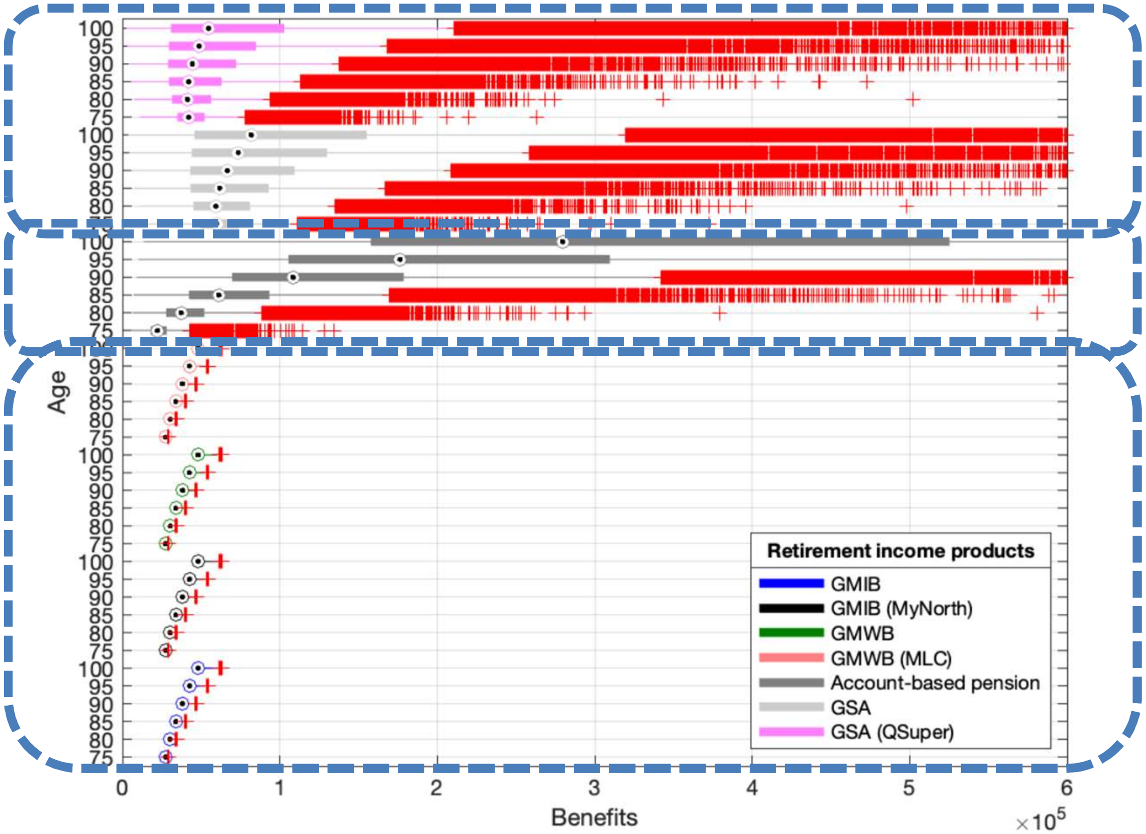
[6]The weights for Australian and international equity, cash and fixed interest, listed infrastructure, and listed property are 30%, 33%, 28%, 6%, and 3%, respectively. We assume the indices are of equal weight in each asset category.

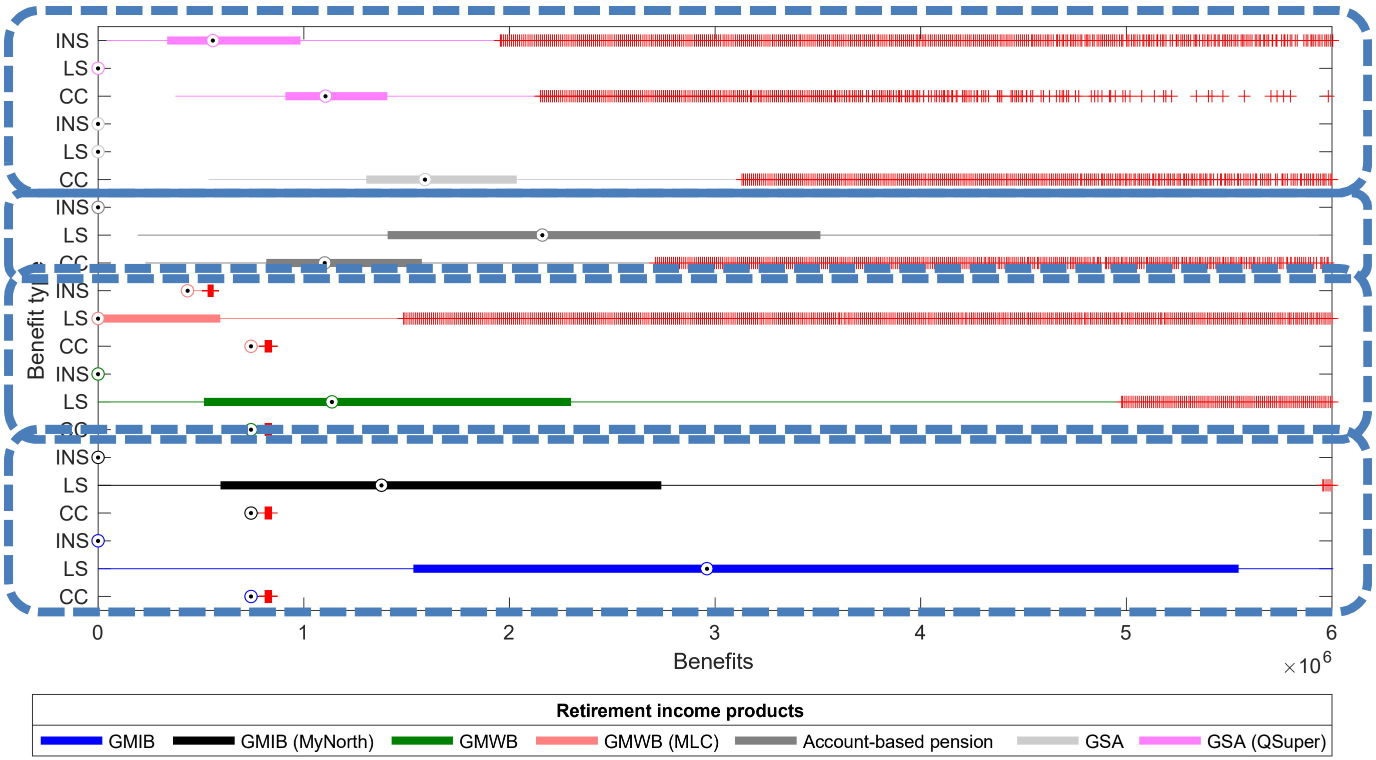
Retirement income products

- Initial investment is $500,000.
- Variable annuities: 35 years of maturity, the minimum guarantee is 4.5% per annum.
- The fair management fees for the GMIB, GMIB with growth lock-in feature, GMWB and GMWB with spouse benefit option are **0.85%, 3.64%, 1.61%, and 4.75%,** respectively.
- Account-based pension: minimum drawdown rates set by Australian Taxation Office.
- The GSA pool consists of 1,000 members with no death benefit; later, each year 1,000 new members aged 65 join the pool.

A comparison of longevity risk protection

Figure 9: Box-plot of the retirement income products' annual living benefit at selected ages from 75 to 100.





Consumption vs. bequest

Figure 10: Box-plot of each component in the retirement income products. We assume the policyholder passes away at age 90, and the beneficiary survives to age 100.

Summary of the products

- Longevity risk protection:
  - GSA offers the highest longevity risk protection.
- Volatility of the policyholder's income:
  - Account-based pension > GSAs > VAs.
- Bequest motive:

Income product	Bequest type	
	Income stream	Lump sum payment
GMIB	×	✓
GMIB (MyNorth)	×	✓
GMWB	×	✓
GMWB (MLC)	✓	✓
Account-based pension	×	✓
GSA	×	×
GSA (QSuper)	✓	×

Conclusion

In this research, we:

- Devise an MCMC framework to efficiently value retirement income products in high dimensions.
- Conduct product comparison to reveal some insights.
- Extend the GSA design to allow for investment return adjustment.

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THANK YOU!

APPENDIX

MCMC algorithms in general

To simulate a Markov chain $\{x_0, x_1, \dots, x_n\}$:

Initialize x0; #The initial starting point of the Markov chain.

repeat {

propose changes to x; #Propose a next state given the current one.

accept or reject the proposal; #To generate Markov transition to preserve the target distribution.

output x; #If accept, the chain moves to the proposed state; if not, the chain stays at the current state.

}

Convergence process of a Markov chain

- Stage (a): first the Markov chain converges to the typical set but suffers from initial but ultimately transient biases.
- Stage (b): once the Markov chain finds the typical set and makes the first sojourn through it, this initial bias rapidly vanishes.
- Stage (c): as the Markov chain continues it explores more details of the typical set, and our estimation becomes more and more accurate.

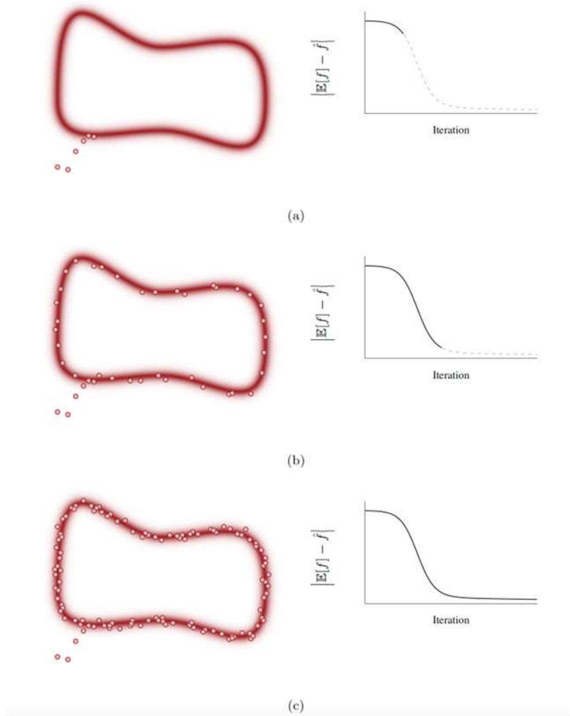

Figure 17: Converge process to the typical set (red) of the Markov chain. (Betancourt, 2017)

Random walk Metropolis-Hastings (RWM) sampling

- Random walk Metropolis-Hastings (RWM) sampling is a commonly used MCMC method in practice.
- RWM proposes a new state by random guess and does not work well in high-dimensional simulation.
- The Hamiltonian Monte Carlo (HMC) algorithm is a way to overcome this difficulty by proposing the new state according to the Hamiltonian dynamics (Neal, 2011).

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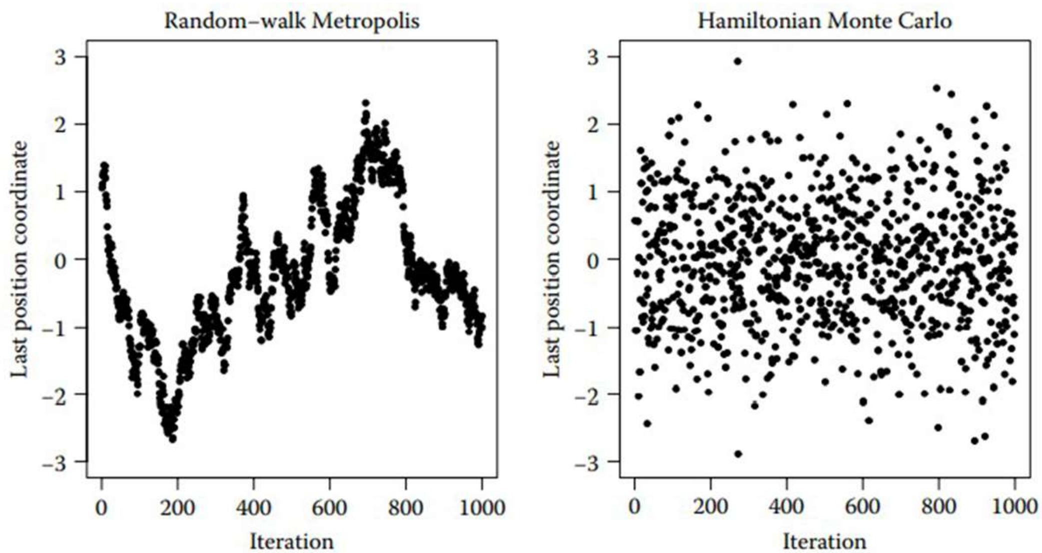
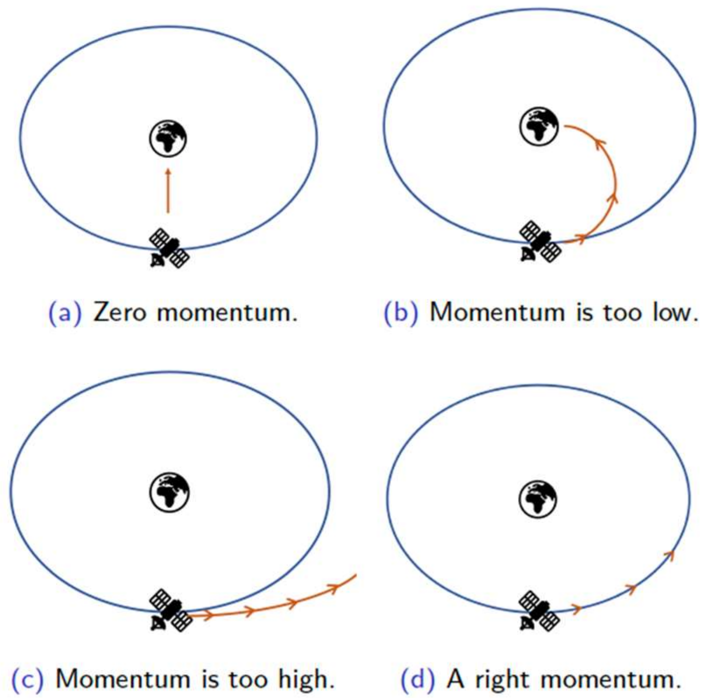


Drawbacks of the RWM method

Figure 18: An example in Neal (2011) where the authors simulate a 100-dimensional Gaussian distribution.

Differential geometry in the HMC algorithm

Figure 19: The mode, gradient, and typical set are equivalent to the Earth, a gravitational field, and an orbit. Panel (a): if we only consider the gradient information and there is no momentum, the satellite will directly crash into the surface of the Earth. Panel (b): if the momentum is too small, the satellite will also crash into the surface of the Earth. Panel (c): if the momentum is too large, the satellite will escape the gravitational attraction. Panel (d): when we introduce the right amount of momentum, the satellite will move along the orbit.



HMC algorithm

- Origin from quantum physics (Alder and Wainwright, 1959).
- The Hamiltonian function (Neal, 2011):

$$H(\mathbf{R},\mathbf{P}) = \underbrace{U(\mathbf{R})}_{\text{Potential energy}} + \underbrace{K(\mathbf{P})}_{\text{Kinetic energy}},$$

  where $\boldsymbol{P}$ is an auxiliary d-dimensional momentum vector and $\boldsymbol{U}(R) \coloneqq -\log\left(p_{\boldsymbol{R}(\Delta t)}(\boldsymbol{R})\right) + (any\ convenient\ constant)$.
- $\boldsymbol{K}(P)$ is also defined as:

$$K(\mathbf{P}) = \frac{1}{2}\mathbf{P}^{\top}\mathbf{M}^{-1}\mathbf{P},$$

  where $\boldsymbol{M}$ is is a symmetric, positive-definite matrix.
- The Hamiltonian dynamics (Newton, 1846):

$$\frac{d\mathbf{R}_i}{dt} = \frac{\partial H}{\partial \mathbf{P}_i}, \quad \frac{d\mathbf{P}_i}{dt} = -\frac{\partial H}{\partial \mathbf{R}_i},$$

  for $i = 1,2,\dots,d$.

HMC sampling

The main idea of HMC algorithm:

- Start from the current state $\bar{R}_0$ on energy set E_1;
- Run the Hamiltonian dynamics for a duration λ;
- Use the state $\bar{R}_1$ on E_1 at the end of the trajectory as the new state;
- Randomly draw a new momentum vector, the chain jumps to E_3;
- Repeat the previous steps.

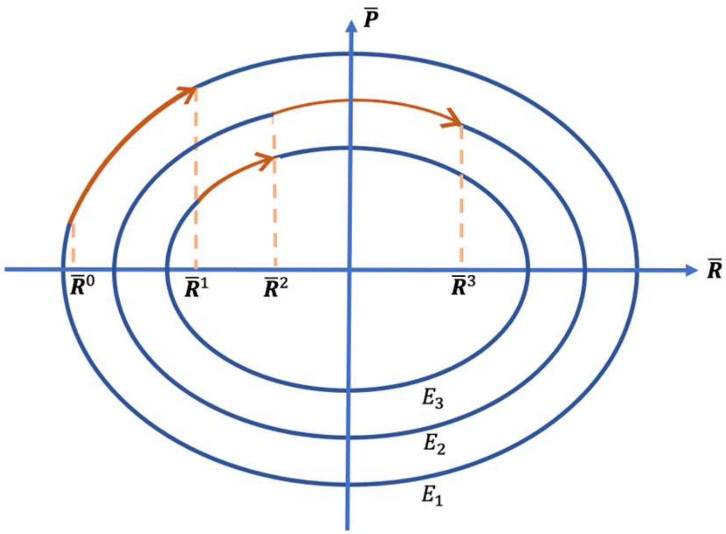


Figure 20: A sample path of the HMC algorithm. E_1, E_2 and E_3 are three energy sets. The horizontal axis is the target random variable, and the vertical axis is the auxiliary momentum vector we introduce.

Exact randomised Hamiltonian Monte Carlo (RHMC) algorithm

- Set $\boldsymbol{M}^{-1}$ to $\boldsymbol{\Sigma}$ (this is the optimal one over the family of Euclidean-Gaussian kinetic energies), and the Hamiltonian function becomes

$$H(\mathbf{R}, \mathbf{P}) = \frac{1}{2}\mathbf{R}^{\top}\boldsymbol{\Sigma}^{-1}\mathbf{R} + \frac{1}{2}\mathbf{P}^{\top}\boldsymbol{\Sigma}\mathbf{P}.$$

- Determine the Cholesky factorization to the symmetric positive definite matrix $\boldsymbol{\Sigma}^{-1}$: $\boldsymbol{\Sigma}^{-1} = \boldsymbol{L}\boldsymbol{L}^{\intercal}$, where $\boldsymbol{L}$ is an upper triangular matrix.
- Determine the eigendecomposition to the matrix $\boldsymbol{L}^{-1}\boldsymbol{\Sigma}^{-1}\ \boldsymbol{L}^{-\intercal}$: $\boldsymbol{Q}^{\intercal}\boldsymbol{L}^{-1}\boldsymbol{\Sigma}^{-1}\ \boldsymbol{L}^{-\intercal}\boldsymbol{Q} = \boldsymbol{\Omega}^{2}$, where $\boldsymbol{Q}$ is orthogonal and $\boldsymbol{\Omega}^{2}$ is diagonal with diagonal entries ω_i, for $i = 1,2, \dots, d$.

Exact randomised Hamiltonian Monte Carlo (RHMC) algorithm

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Proposition:

The change of variables $\boldsymbol{R} = \boldsymbol{L}^{-\intercal}\boldsymbol{Q}\overline{\boldsymbol{R}}, \mathbf{P} = \mathbf{LQ\Omega}\overline{\boldsymbol{R}}$, where $\boldsymbol{\Omega}$ is a diagonal matrix with diagonal entries $\sqrt{\omega_i}$, decouples the depended Hamiltonian function into the following independent one:

$$H(\overline{\boldsymbol{R}}, \overline{\boldsymbol{P}}) = U(\overline{\boldsymbol{R}}) + K(\overline{\boldsymbol{P}}) = \frac{1}{2}\overline{\boldsymbol{R}}^{\intercal}\boldsymbol{\Omega}^2\overline{\boldsymbol{R}} + \frac{1}{2}\overline{\boldsymbol{P}}^{\intercal}\boldsymbol{\Omega}^2\overline{\boldsymbol{P}},$$

and the Hamiltonian dynamics becomes a collection of d harmonic oscillators:

$$\frac{d\overline{\boldsymbol{R}}_i}{dt} = \omega_i\overline{\boldsymbol{P}}_i, \qquad \frac{d\overline{\boldsymbol{P}}_i}{dt} = -\omega_i\overline{\boldsymbol{R}}_i.$$

After running the Hamiltonian dynamics for a duration λ, the solutions to the above harmonic oscillators at the end of the trajectory are:

$$\begin{bmatrix}\overline{\boldsymbol{R}}_i^{\lambda} \\ \overline{\boldsymbol{P}}_i^{\lambda}\end{bmatrix} = \begin{bmatrix}\cos(\omega_i\lambda) & \sin(\omega_i\lambda) \\ -\sin(\omega_i\lambda) & \cos(\omega_i\lambda)\end{bmatrix}\begin{bmatrix}\overline{\boldsymbol{R}}_i^{0} \\ \overline{\boldsymbol{P}}_i^{0}\end{bmatrix}.$$

Here, in our exact RHMC algorithm, the duration λ in each iteration follows an exponential distribution with mean λ^e.

Exact randomised Hamiltonian Monte Carlo (RHMC) algorithm

Key benefits of our exact RHMC algorithm:

- **Computational efficiency**: we derive explicit form for the Hamiltonian dynamics. In practice, people usually use the leapfrog algorithm to numerically approximate the solution to the Hamiltonian dynamics, which will introduce approximation error.
- **Sampling efficiency**: there are no approximation errors associated with the Hamiltonian dynamics, we can accept all proposals at the end of each iteration of the Hamiltonian dynamics. With the leapfrog algorithm, the optimal acceptance rate is 65% (Beskos et al., 2013).
- **Faster exploration of the typical set**: the duration of each iteration follows an exponential distribution, which reduces the autocorrelation among the samples and decreases the chance of a U-turn.

Convergence test of the Markov chain

- We use the multivariate potential scale reduction factor (MPSRF) in Brooks and Gelman (1998) to monitor the convergence.
- MPSRF measures the difference between mixture-of-sequences covariance and within-sequence covariance and is given by

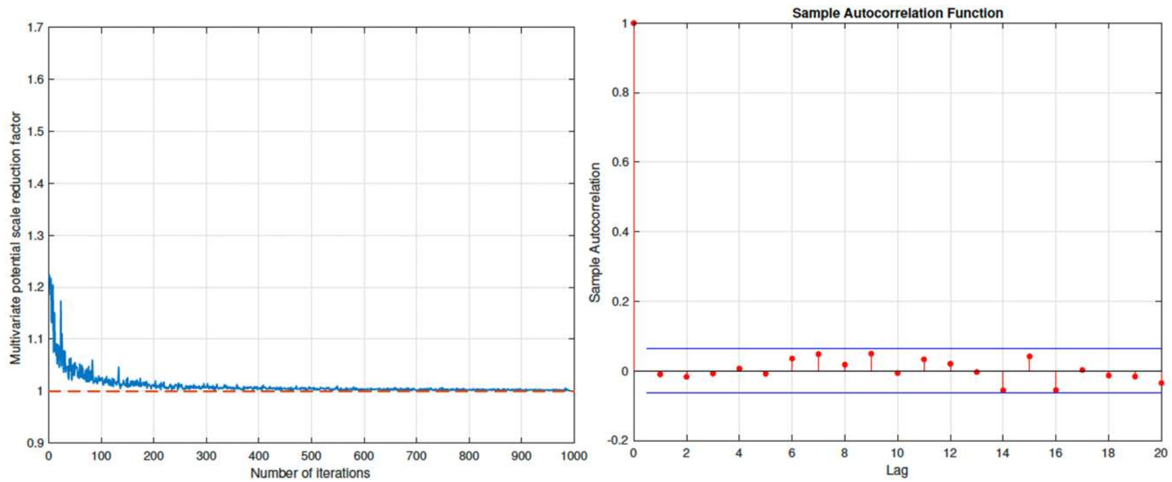
$$MPSRF = \frac{n-1}{n} + \frac{m+1}{m}\lambda_1,$$

where λ_1 is the largest eigenvalue of the matrix $\frac{1}{n}V_W^{-1}V_B$, n is the number of samples in each chain, and there are m chians in total. Here, V_W is the within-chain variance and V_B is the between-chain variance.

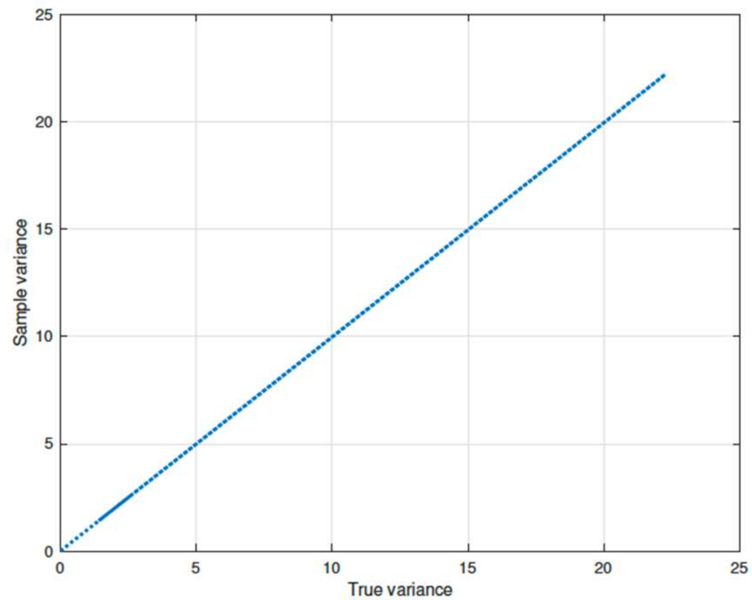
- When $n \to \infty$, the MPSRF should decline to 1.

Convergence test of the exact RHMC algorithm

Figure 21: Convergence test of the exact randomised Hamiltonian Monte Carlo (RHMC) algorithm. Panel (a) shows the estimated multivariate potential scale reduction factor (MPSRF) within 1,000 iterations. We simulate 5 Markov Chains with different initial starting points to estimate the MPSRF. The dimension d is 9. Panel (b) shows the sample auto-correlation of one stock return in one simulated Markov Chain.



(a) MPSRF. (b) Auto-correlation.



Convergence test of the exact RHMC algorithm

Figure 22: True variances and sample variances of the 9 demeaned index returns. The sample size is 1×10^5.

Accuracy test of the exact RHMC algorithm

Table 1: The mean squared error (MSE) for the crude Monte Carlo and the exact RHMC algorithm in numerically computing $\mathbb{E}(\mathbf{1}_d \mathbf{Z}^\top)$. Here, $\mathbf{Z}$ is a d-dimensional random vector which follows a multivariate normal distribution with mean $\boldsymbol{\mu}_Z$ and variance $\boldsymbol{\Sigma}_Z$. The last column is the ratio of the MSE of the crude Monte Carlo relative to that of the exact RHMC algorithm. We use a sample size of 1×10^5.

Dimension	MSE		Ratio
	Crude Monte Carlo	Exact RHMC algorithm	
1	5.3585×10^{-6}	5.1517×10^{-6}	1.0402
2	9.0953×10^{-6}	9.2038×10^{-6}	0.9882
4	2.4241×10^{-5}	2.4166×10^{-5}	1.0031
8	1.0788×10^{-4}	8.5297×10^{-5}	1.2647
16	4.6196×10^{-4}	3.8421×10^{-4}	1.2024
32	4.5238×10^{-3}	3.7209×10^{-3}	1.2158
64	1.1513×10^{-2}	6.2262×10^{-3}	1.8491

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Mortality model

- We use the stochastic dynamic GoMa model in Qiao and Sherris (2013) to model human mortality.
- Let $\mu_x(t)$ be the mortality rate of a life aged x at time t, the mortality rate is given by:

$$\mu_x(t) = Y_1(t) + Y_2(t)c^x,$$
$$dY_1(t) = a_1 dt + \sigma_1 dZ_1^{\mathbb{Q}}(t),$$
$$dY_2(t) = a_2 dt + \sigma_2 dZ_2^{\mathbb{Q}}(t),$$
$$dZ_1^{\mathbb{Q}}(t) dZ_2^{\mathbb{Q}}(t) = \rho_m dt,$$

where we assume $Z_1^{\mathbb{Q}}(t)$ and $Z_2^{\mathbb{Q}}(t)$ are independent of the financial market.

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