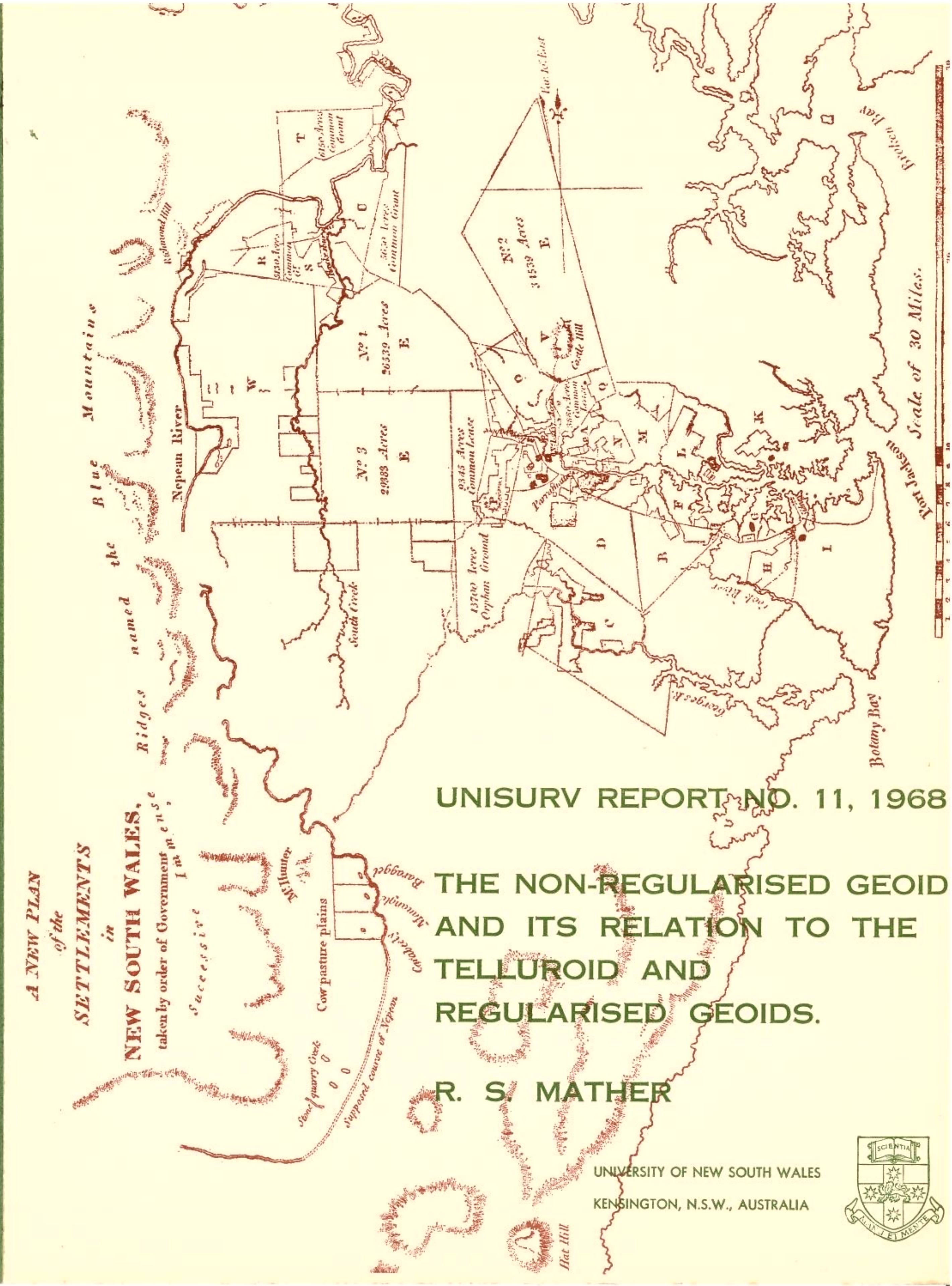

UNISURV REPORT NO. 11, 1968

THE NON-REGULARISED GEOID AND ITS RELATION TO THE TELLUROID AND REGULARISED GEOIDS.

R. S. MATHER

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KENSINGTON, N.S.W., AUSTRALIA

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Q

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S Green Hills

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SUMMARY

The assumptions made in solving the boundary value problem are critically examined and revised expressions derived for the complete definition of the non-regularised geoid to the order of the flattening. The Free Air Geoid is shown to be a good approximation to the non-regularised geoid. The relation between regularised geoids and the non-regularised geoid is also established and the latter is shown to be no more difficult to compute than any regularised geoid.

Expressions are developed for the computation of the height anomaly from free air anomalies and elevations to the order of the flattening. The significance of the zero order term is re-examined and its implications in assigning revised values of parameters to the reference system are studied.

THE NON-REGULARISED GEOID AND ITS RELATION TO THE TELLUROID AND REGULARISED GEOIDS

by

R. S. Mather

1. INTRODUCTION

In a recent study (Mather, 1968, 330 et seq), the correction (N_{ip}) which constituted the indirect effect for the Free Air Geoid (N_f), given by the use of free air anomalies (Δg_f) in Stokes' Integral (Heiskanen & Moritz, 1967, 94), was computed from relations derived for a non-regularised earth. The values of N_{ip} obtained were two orders larger than generally accepted estimates of its magnitude, which is held (ibid, 145) to be equivalent to that arising from the second condensation reduction of Helmert.

This reduction is the process of regularisation where all the topography exterior to the geoid is removed and condensed, column-wise, as a surface layer of density τ on the geoid, where

$$\tau = \int_0^h \rho \, dz \quad \ldots\ldots\ldots\ldots\ldots \quad (1)$$

where ρ is the density of matter over the element of elevation dz, h being the total elevation of the topography. N_{ip}, in this particular case of regularisation, is held to be of the order of 1 metre. Hence, the Free Air Geoid, obtained by the use of the equation

$$N_f = \frac{R_m}{4\pi\gamma_m} \int_0^{\sigma=4\pi} f(\psi)\ \Delta g_f\ d\sigma \qquad \text{...........(2)},$$

where $f(\psi)$ is Stokes' function,
$d\sigma$ is the element of surface area on a unit sphere, R_m the mean radius of the earth and γ_m the mean value of gravity over a model of the earth used in computing the free air anomaly, is usually referred to as the "geoid" (e.g., Lundquist & Veis, 1966, III 136). The earlier determination referred to was derived from the work of Moritz (1965, 18) who, in the same paper (ibid, 27), shows that the basic integral for the non-regularised geoid reduces to any of the common methods of regularisation.

This has also been proved by Molodenskii (1962, 60 et seq.) who shows that the solutions for the non-regularised geoid attributed to Moiseev and Malkin are both equivalent to regularised geoids. The equation given by Moritz was reduced to a form which was approximately equal to that attributed to Malkin (ibid, 65) and the final form for the separation of the non-regularised geoid and the spheroid (N_p) was deduced as (Mather, 1968, 76)

$$N_p = \frac{2\,\phi_{ep}}{\gamma_m} + \frac{R_m}{4\pi\gamma_m}\int_0^{\sigma=4\pi} f(\psi)\,\Delta g_o\, d\sigma + o\{Nf\} \quad ..(3),$$

where ϕ_{ep} is the potential at a point P on the geoid of all matter exterior to it

and Δg_o the gravity anomaly at the geoid.

All other quantities are as defined in equation (2), f being the flattening of the reference spheroid. Δg_o was shown (ibid, 76 - 81) to be related to the free air anomaly by the relation

$$\Delta g_o = \Delta g_f + \left(\frac{\partial \phi_e}{\partial h}\right)_S - \left(\frac{\partial \phi_e}{\partial h}\right)_G + o\,\{f\,\Delta g\} \quad ..(4),$$

where the suffixes S and G refer to evaluation at the physical surface of the earth and geoid respectively. Δg_o can also be approximately represented by the Prey anomaly (Heiskanen & Moritz, 1967, 164).

Equation (3) was based on the following assumptions :-

(a) The potential of the geoid (W_o) was equal to that assigned to the reference spheroid (U_o).

(b) The gravity anomaly gradient $\left|\frac{\partial \gamma}{\partial h}\right|$ was given by the equation

$$\left|\frac{\partial \gamma}{\partial h}\right| = \frac{2\gamma}{R_m} \qquad \text{............. (5),}$$

where γ is the value of normal gravity at a point on an equipotential surface where the mean curvature is R_m^{-1}.

(c) The disturbing potential (V_{dp}) at any point P on a surface S can be expressed by the equation

$$V_{dp} = \iint_S \frac{\tau}{r}\, dS \qquad \text{......... (6),}$$

where r is the distance of the surface element dS from P and τ is a surface harmonic. It was further assumed that equation (6) could be expanded by adopting the harmonic expansion for r^{-1} (Jeffreys, 1962, 634)

$$\frac{1}{r} = \frac{1}{r}\sum_{n=0}^{\infty} \left(\frac{R_p}{R}\right)^n P_n(\cos\psi) \qquad \text{.........(7),}$$

where $P_n(\cos\psi)$ is the Legendre Zonal Harmonic and ψ is the angular distance from dS to P. As τ is a surface harmonic and the product (ibid, 646)

$$\iint_S \tau_n P_n(\cos\Psi)\, dS = A_n' \qquad \text{......(8),}$$

it is possible to represent V_{dp} by the expression

$$V_{dp} = \sum_{n=2}^{\infty} \frac{A_n}{R^{n+1}} \qquad \text{.................. (9).}$$

Such a series will be convergent only if $A_n < R^{n+1}$.

This condition is satisfied if the surface S is a sphere, though it does not necessarily hold for a spheroidal model. The mathematical model defined in equation (9) is used purely as an intermediary to enable V_d to be related to the gravity anomaly.

(d) A_o and A_1 have been put equal to zero in equation (9). The adoption of the first equality is equivalent to assuming that the mass of the reference spheroid is equal to that of the existent earth. The second equality, in the case of a uniformly stratified reference spheroid, effectively locates the centre of the spheroid at the centre of mass of the existent earth.

The values of N_{ip} obtained by the use of the relation

$$N_{ip} = \frac{2\phi_{ep}}{\gamma_m} + \frac{R_m}{4\pi\gamma_m} \int_0^{\sigma=4\pi} f(\psi)\ \Delta g_{co}\ d\sigma \quad \ldots\ldots(10),$$

where

$$\Delta g_{co} = \left(\frac{\partial \phi_e}{\partial h}\right)_S - \left(\frac{\partial \phi_e}{\partial h}\right)_G \quad \ldots\ldots\ldots\ldots(11)$$

$$\doteqdot - 4\pi k\rho h ;$$

k being the gravitational constant, are as large as 700 metres. This is to be expected as ϕ_e can be expressed by the relation

$$\phi_e = \phi_{ee} + \phi_{ei} \quad \ldots\ldots\ldots\ldots\ldots(12),$$

where ϕ_{ee} is the potential due to an outer area over which the former can be expressed by a relation of the form

$$\phi_{ee} = \iint_S \frac{k\rho h}{r} \, dS \quad \ldots\ldots\ldots\ldots\ldots \text{(13)},$$

and ϕ_{ei} is the contribution of an inner zone comprising departures from the expression set out in equation (13). As $h \ngtr 10$ km and potential is a scalar, ϕ_{ei} can be expressed by the relation (Mather, 1968, 106-109)

$$\phi_{ei} = \iint_S \frac{3k\rho h^2}{4rR} \, dS + 2\pi k\rho h r_o \left(1 - \frac{h}{2r_o} + \frac{h^2}{6r_o^2} - \frac{h^4}{40r_o^4}\right) \quad \ldots\ldots\text{(14)},$$

where the inner zone of the improper integral is replaced by a cylindrical model of elevation h and radius r_o. As $\frac{h}{R} \ngtr 2 \times 10^{-6}$, the first term can be considered to be negligible for most practical purposes. The second (ibid, 110), for a value of $r_o = 10$ km, is of the order of 1 kgal metre per km of station elevation.

Equation (13) shows that ϕ_{ee} is the major contributor to the value of ϕ_e, the contribution being positive always and of magnitude varying from 100 - 300 kgal metres as a function of position on the earth's surface. The second term in equation (10) has a variable sign as

Δg_{co} is always negative while $f(\psi)$ (Lambert and Darling, 1936, 114) takes both positive and negative values on global summation. In fact, the contributions of the two constituent terms of equation (10) are almost equal in magnitude but of opposite sign over the inner zones of the computation but tend to accumulate over the distant zones (Mather, 1968, 331).

It is obvious that the assumption at (a) above cannot produce the abnormally large variations in the calculated values of N_{ip} though it certainly does effect the zero order term in the solution. The Free Air Geoid (ibid, 56) is a good approximation to the telluroid which coincides with the reference spheroid over ocean areas if $W_o = U_o$. On the other hand, the geoid as obtained from the use of equation (3), has considerable divergence from the Free Air Geoid (ibid, 363 et seq.).

The divergence of the solutions requires re-investigation in that

(i) regularisation postulates that the Free Air Geoid is a good approximation of the regularised geoid ;

(ii) the Free Air Geoid is also a good approximation of the telluroid.

A fallacy, if any, will have to exist in one or more of the assumptions listed as (a) to (d) above. The admission of the general possibility that $W_o \neq U_o$ upsets Hirvonen's original definition of the telluroid (Hirvonen, 1960, 40). He defined the telluroid as the locus of points P_R, (see fig (i)) at which the potential on the reference system (U_Q) equalled that on the existent system (W_P) at P on the physical surface of the earth, P_R, being located on the normal through P. W_P cannot be defined unless W_o is known. As this is not so, it is impossible to define U_o. A much more practical definition of the telluroid, so far as normal displacements go, is the definition of the points P_R, as those on the reference system having the same difference of potential with reference to U_o (on the spheroid) as the difference in geopotential (ΔW) between P and the geoid.

A reference surface of complete definition would be the locus of the points Q shown in fig (1), defined by

$$\left.\begin{aligned} \phi_Q &= \phi_{A_P} \\ \lambda_Q &= \lambda_{A_P} \\ U_Q - U_o &= W_P - W_o \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (15),$$

where ϕ_{A_P} and λ_{A_P} are the astronomically determined latitude and longitude of P.

This new definition suggested for the telluroid is an adaptation of Moritz' *normal surface* (Moritz, 1965, 12). Such a definition has the advantage that the separation vector $\underline{d}$ is completely represented in space by $\dot{P}Q$. If an x_i (i=1,3) axis system is chosen with the x_3 axis coincident with the

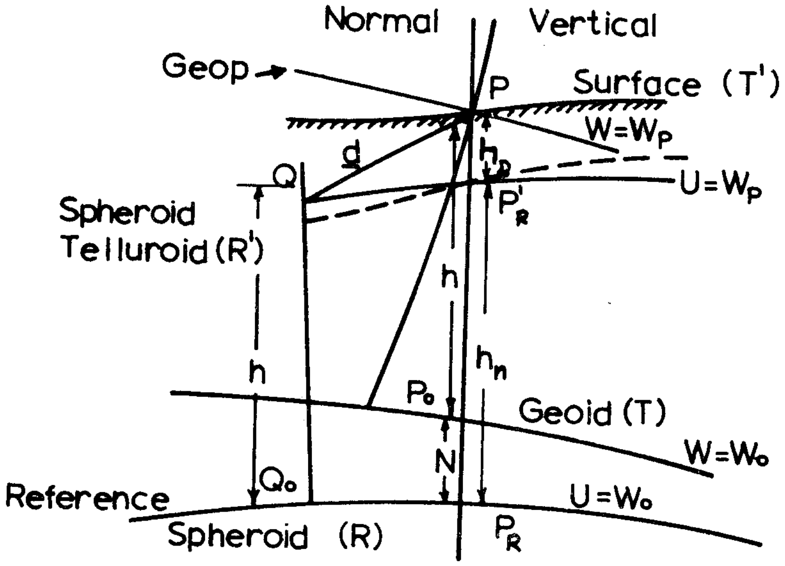
Normal
Vertical
Geop
P
Surface (T')
W=WP
d
hp
Q
U=WP
P'R
Spheroid
Telluroid (R')
h
h
hn
P0
Geoid (T)
W=W0
N
Q0
Reference
U=W0
Spheroid (R)
PR

FIG. 1

local spherop normal at Q and the x_1 axis oriented north,

$$\underline{d} = \sum_{i=1}^{2} R\, \xi_i \underline{i} + h_d\, \underline{3} \quad ....(16),$$

where ξ_i (i=1,2) are the components of the deflection of the local vertical in the meridian and prime vertical respectively. h_d is the height anomaly equal to the displacement $PP_{R'}$. The quantities $\underline{i}$ (i=1,3) are unit vectors along the x_i (i=1,3) axes.

The second assumption is the adoption of equation (5) for the normal gravity gradient (e.g., Heiskanen and Moritz, 1967, 87). A more acceptable expression for this quantity is obtained from Bruns' formula (ibid, 53) which can readily be transposed to the form

$$\left|\frac{\partial \gamma}{\partial h}\right| = \frac{2\gamma}{a} - 4\,\pi k \rho + o\{\, f\, \frac{\partial \gamma}{\partial h}\} \quad ......(17),$$

where ρ is the density of matter in the region in which the normal gravity gradient is evaluated. a and f are parameters of the reference spheroid. At points on the reference system which lie in regions not occupied by matter, $\rho = 0$. For the purpose of generalisation of formulae, the possibility that $\rho \neq 0$ will be admitted.

The solutions to be derived in the following sections will not make the assumptions defined in (a) to (d) above.

2. RELATIONS FUNDAMENTAL TO ALL SOLUTIONS.

All derivations stem from Green's third identity (Heiskanen & Moritz, 1967, 11) which is obtained by the application of Green's theorem to the scalars r^{-1} and $\emptyset$. If r is the distance of the relevant element of area or volume from a point P situated on the surface S which bounds the volume V_i,

$$\iiint_{V_i} \frac{1}{r} \underline{\nabla}^2 \emptyset \, dV_i = -2\pi \, \emptyset_P + \iint_S \left\{ \frac{1}{r} \underline{\nabla}.\underline{N}\emptyset - \emptyset \, \underline{\nabla}.\underline{N} \, \frac{1}{r} \right\} dS \qquad \text{............ (18),}$$

where $$\underline{\nabla} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \underline{i} \qquad \text{.............(19),}$$

$\underline{i}$ is the unit vector along the x_i axis,

$\underline{N}$ the unit normal vector and the suffix P referring to evaluation at the point P.

Jeffreys (1962, 194 et seq.) sets out the conditions for the existence of a solution for equation (18) as

(a) $\underline{\nabla}^2\emptyset$ exists everywhere within the volume V_i and

(b) $\underline{\nabla}.\underline{N}$ $\emptyset$ exists everywhere on the surface S.

If $\emptyset$ is the potential due to a system of masses lying partly within S and rotating with angular velocity ω, Poisson's equation for a rotating body holds within the volume V_i. Thus

$$\underline{\nabla}^2 \phi = -4\pi k\rho + 2\omega^2 \quad \ldots\ldots\ldots\ldots\ldots (20).$$

Continuity of the density function (ibid, 696) is a necessary and, in most cases, sufficient condition for equation (20) to hold. A sufficient condition would be the existence of integrable derivatives of ρ. If such conditions were assumed to exist, the application of equation (18) to the non-regularised earth of potential W, setting S in coincidence with the geoid, gives

$$\iiint_{V_i} \frac{1}{r} \underline{\nabla}^2 W \, dV_i = -2\pi W_p + \iint_S \{\frac{1}{r} \underline{\nabla}.\underline{N}W - W \underline{\nabla}.\underline{N} \frac{1}{r}\} \, dS \quad \ldots\ldots\ldots\ldots\ldots\ldots (21).$$

The application of equation (20) to the volume integral in equation (21) gives

$$\iiint_{V_i} \frac{1}{r} \underline{\nabla}^2 W \, dV_i = -4\pi \iiint_{V_i} \frac{k\rho}{r} \, dV_i + 2\omega^2 \iiint_{V_i} \frac{1}{r} \, dV_i$$

$$= -4\pi\phi_{ip} + 2\omega^2 \iiint_{V_i} \frac{1}{r} \, dV_i \quad \ldots (22),$$

where ϕ_{ip} is the potential of matter within the geoid. If ϕ_{ep} is the potential at P of matter exterior to the geoid,

$$W_p = \phi_{ip} + \phi_{ep} + \phi_{rp} \quad \ldots\ldots\ldots\ldots (23),$$

where ϕ_{rp} is the rotational potential at P. The elimination of ϕ_{ip} in equation (22) by the use of equation (23), together with the use of the result in equation

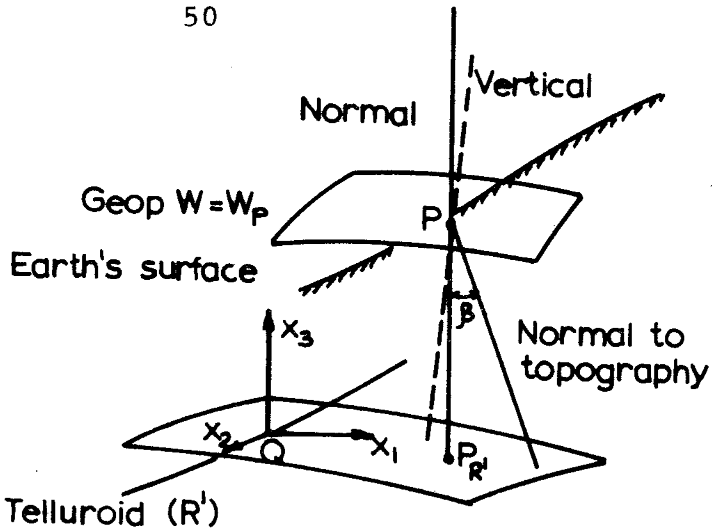
Vertical
Normal
Geop W = W_P
P
Earth's surface
x_3
β
Normal to topography
x_2
Q
x_1
P_R'
Telluroid (R')

FIG. 2

(21) gives

$$W_p = 2\phi_{ep} + 2\phi_{rp} + \frac{\omega^2}{\pi}\iiint_{V_i} \frac{1}{r}\, dV_i \; -$$

$$- \frac{1}{2\pi}\iint_S \{\frac{1}{r}\, \underline{\nabla}.\underline{N}\, W - W\, \underline{\nabla}.\underline{N}\, \frac{1}{r}\}\, dS \quad ..(24).$$

Equation (24)is identical with that obtained in a slightly different manner by Moritz (1965, 10) whose solution was independent of any conditions regarding the integrability of density functions. This equation is linearised by superimposing a hypothetical reference system afforded by the gravitational field of a rotating oblate spheroid with a bounding equipotential

$$U = U_o \,.$$

If this spheroid has the *same volume* as the geoid, it constitutes a fixed closed surface in space such that the geoid spheroid separation N has no zero order harmonic. The application of equation (18) to this system *with the surface S still being the geoid*, gives

$$\iiint_{V_i} \frac{1}{r}\, \underline{\nabla}^2 U\, dV_i = -2\pi U_p + \iint_S \{\, \frac{1}{r}\, \underline{\nabla}.\underline{N}\, U - U\, \underline{\nabla}.\underline{N}\, \frac{1}{r}\}\, dS$$

$$\ldots\ldots\ldots\ldots\ldots \quad (25).$$

If the spheroid has the same rotational characteristics as the existent earth, manipulation similar to that used in the derivation of equation (24) gives

$$U_p = 2\phi_{up} + 2\phi_{rp} + \frac{\omega^2}{\pi}\iiint_{V_i} \frac{1}{r}\, dV_i -$$

$$- \frac{1}{2\pi}\iint_S \{\frac{1}{r}\, \underline{\nabla}.\underline{N}U - U\, \underline{\nabla}.\underline{N}\, \frac{1}{r}\}dS \quad ....(26),$$

where ϕ_{up} is the potential due to matter of the reference system which is exterior to the geoid. ϕ_{up} would thus be due to matter in those regions where the geoid spheroid separation $N < 0$. ϕ_{up} will have exactly the same characteristics as ϕ_{ep}, defined by equations (12) to (14). On a similar basis,

$$\phi_u = \phi_{ue} + \phi_{ui} \quad .... (27),$$

where

$$\phi_{ue} = -k \iint_S \frac{\rho\ N}{r}\, dS \quad .... (28)$$

and

$$\phi_{ui} = -\ 2\pi k\rho N r_o + o\{f\ \phi_{ui}\} \quad ....(29)$$

In both equations (28) and (29), $\rho = 0$ when $N > 0$. As the geoid spheroid separation is expected to lie in the range $-100 < N^{(met)} < 100$ and the contribution to the above effect over half the earth is zero, the magnitude of ϕ_{ue} is expected to be of the order pf 20 - 50 kgal. metres. ϕ_{ui} will seldom exceed 0.1 kgal. metres.

The disturbing potential (V_{dp}) at points P on the geoid is defined by the equation

$$V_{dp} = W_p - U_p \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (30).$$

The use of equations (24), (26) and (30) gives

$$V_{dp} = 2\phi'_p - \frac{1}{2\pi}\iint_S \{\frac{1}{r}\,[\underline{\nabla}.\underline{N}\, W - \underline{\nabla}.\underline{N}\, U] - V_d\, \underline{\nabla}.\underline{N}\, \frac{1}{r}\}\, dS \ldots (31),$$

where

$$\phi'_p = \phi_{ep} - \phi_{up} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (32).$$

If the potential on the geoid is W_o and that on the reference spheroid is U_o (Mather, 1968, 309),

$$V_{dp} = W_o - U_o + \gamma N + o\{fV_d\} \quad \ldots\ldots\ldots (33),$$

where γ is normal gravity at P and N the geoid spheroid separation. As the geoid is a level surface,

$$\underline{\nabla}.\underline{N} = \frac{\partial}{\partial h} .$$

The first set of quantities within the integral in equation (31) are obtained by the differentiation of equation (30) when

$$\frac{\partial V_d}{\partial h} = \frac{\partial W}{\partial h} - \frac{\partial U}{\partial h} = -g_o + \gamma_o - |\frac{\partial \gamma}{\partial h}|\, N + o\{f\, \frac{\partial V_d}{\partial h}\} \,..$$

Defining the gravity anomaly at the geoid as

$$\Delta g_o = g_o - \gamma_o \quad . \quad \ldots\ldots\ldots\ldots\ldots (34),$$

where g_o is the value of observed gravity at the geoid and γ_o is the value of normal gravity at the equivalent point on the reference spheroid, the earlier equation becomes

$$\frac{\partial V_d}{\partial h} = - \; [\Delta g_o + |\frac{\partial \gamma}{\partial h}| N \,] \; + o\{f \frac{\partial V_d}{\partial h}\} \qquad \ldots\ldots\ldots (35).$$

It must be emphasised that g_o is the value of gravity at the geoid of the non-regularised earth. Equation (35) is dependent on the value of $\frac{\partial \gamma}{\partial h}$, given by equation (17), and refers to the gradient on the reference system. Equation (35) can therefore be simplified by the use of equations (17) and (33) to give

$$\frac{\partial V_d}{\partial h} = - \; [\; \Delta g_o + \frac{2V_d}{R} - \frac{2(W_o - U_o)}{R} - 4\pi k \rho N \;] + o\{f \frac{\partial V_d}{\partial h}\} \qquad \ldots\ldots\ldots\ldots (36),$$

where R, to the required order of accuracy, is the mean radius of the earth and $\rho = 0$ when $N > 0$. ρ is non-zero when $N < 0$, a circumstance which occurs over 50% of the earth's surface *if the geoid has the same volume as the reference spheroid.* Adoption of a spherical approximation for the earth gives (Moritz, 1965, 22)

$$\underline{\nabla} \cdot \underline{N} \, \frac{1}{r} = - \frac{1}{2Rr} \qquad \ldots\ldots\ldots\ldots\ldots (37).$$

The use of equations (36) and (37) in equation (31) gives

$$V_{dp} = 2\phi'_p + \frac{1}{2\pi} \iint_S \frac{1}{r} \, [\; \Delta g_o + \frac{3V_d}{2R} - \frac{2(W_o - U_0)}{R} - 4\pi k \rho N] dS \; +$$

$$+ \; o\{fV_d\} \qquad \ldots\ldots\ldots (38).$$

Other relations of interest are expressions for $\frac{\partial \phi_e}{\partial h}$

and $\frac{\partial \phi_u}{\partial h}$. The differentiation of equation (12) with respect to h gives

$$\frac{\partial \phi_e}{\partial h} = \frac{\partial \phi_{ee}}{\partial h} + \frac{\partial \phi_{ei}}{\partial h} \qquad \ldots\ldots\ldots\ldots(39).$$

These terms can be interpreted as being the gravitational attraction of matter which lies exterior to the geoid. The former term (Mather, 1968, 91 & 246) is approximately two orders smaller than the latter and only those regions within $1\frac{1}{2}^{o}$ of the computation point need be taken into account in the evaluation. For this limited zone (see fig (3)), on adopting a spherical approximation for the earth, the general formulae (Mather, 1968, 86 et seq.) reduce to

$$r_o = \ell_o\left(1 + \frac{z}{R} \right) \qquad \ldots\ldots\ldots\ldots\ldots \; (40)$$

$$r = \ell_o\left(1 + \frac{z + h_p}{2R}\right) \qquad \ldots\ldots\ldots\ldots\ldots \; (41)$$

$$\cos \beta_o = - \frac{z}{\ell_o} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(42)$$

$$\cos \beta = - \frac{z - h_p}{\ell_o} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(43)$$

In equations (40) to (43), ℓ_o is the distance of dS from the computation point along the earth's surface. Thus

$$\left(\frac{\partial \phi_{ee}}{\partial h}\right)_G = - k\iint \rho \, dS \int_0^h \frac{-z \; dz}{\ell_o^3} + o\{f\frac{\partial \phi_{ee}}{\partial h}\} \quad ,$$

where $dS = \ell_o d\alpha \; d\ell_o$; $d\alpha$ being the increment in azimuth.

$$\left(\frac{\partial \phi_{ee}}{\partial h}\right)_G = \tfrac{1}{2}k \iint \frac{\rho h^2}{\ell_o^3} \, dS \qquad \text{.......... (44).}$$

Similarly,

$$\left(\frac{\partial \phi_{ee}}{\partial h}\right)_S = \tfrac{1}{2}k \iint \rho \frac{h(h-2h_p)}{\ell_o^3} \, dS + o\{f \, \frac{\partial \phi_{ee}}{\partial h}\} \qquad \text{..(45),}$$

where h is the elevation of topography at the element of surface area dS. ϕ_{ei} is the effect of an inner zone within a radius r_o of the computation point. The gravitational attraction of this zone can be evaluated by adopting a cylindrical assumption (see fig (4)), the cylinder having a mean elevation h_m equal to the mean elevation of the region. A simple integration gives

$$\left(\frac{\partial \phi_{ei}}{\partial h}\right)_G = 2\pi k\rho \, [h_m + r_o - (h_m^2 + r_o^2)^{\frac{1}{2}}]$$

$$= 2\pi k\rho h_m [1 - \frac{h_m}{2r_o} + \frac{h_m^3}{8r_o^3} - \frac{h_m^5}{16r_o^5} + ..] \qquad \text{.(46).}$$

On a similar basis,

$$\left(\frac{\partial \phi_{ei}}{\partial h}\right)_S = -2\pi k\rho h_m [1 + \frac{(h_p - h_m)^2}{2h_m r_o} - \frac{h_p^2}{2h_m r_o} +$$

$$+ \frac{h_p^3}{8r_o^3} - \frac{h_p^5}{16r_o^5} +] \qquad \text{..........(47),}$$

where h_p is taken equal to h_m in the smaller

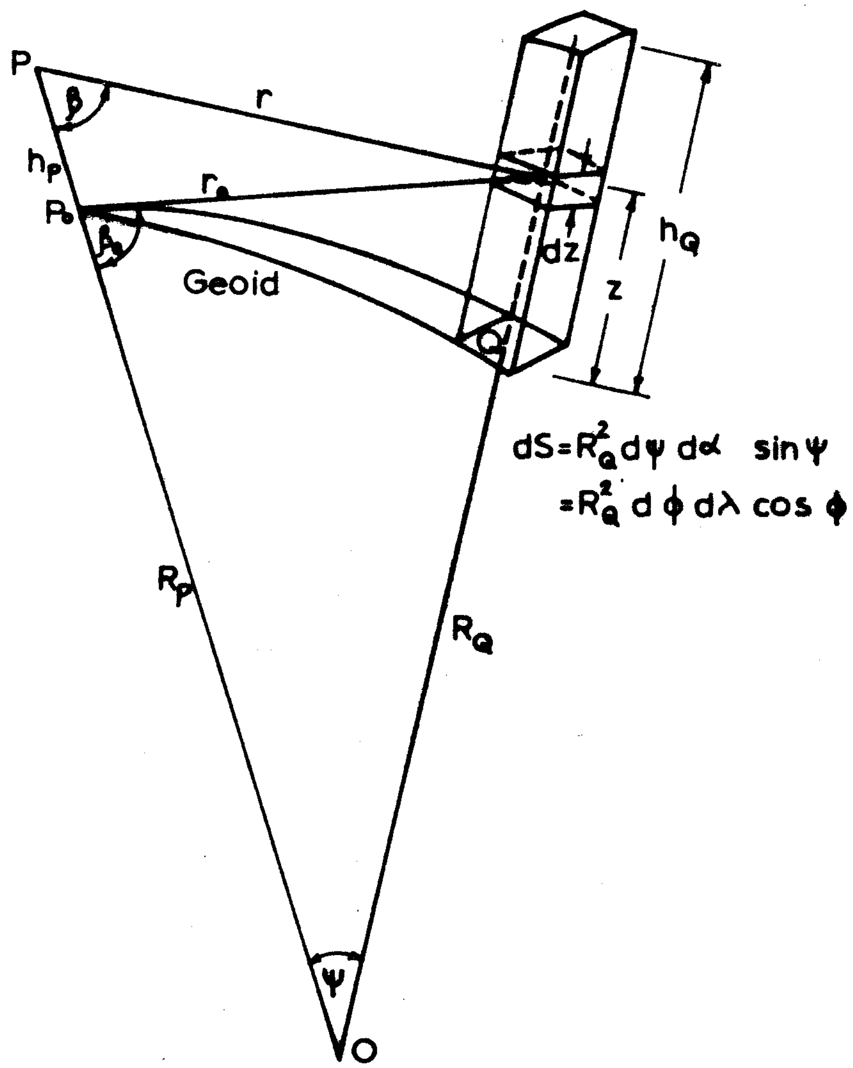
P
β
r
h_P
P_0
r_0
β_0
Geoid
dz
h_Q
z
Q
$dS = R_Q^2\, d\psi\, d\alpha \sin\psi$
$= R_Q^2\, d\phi\, d\lambda \cos\phi$
R_P
R_Q
ψ
O

FIG. 3

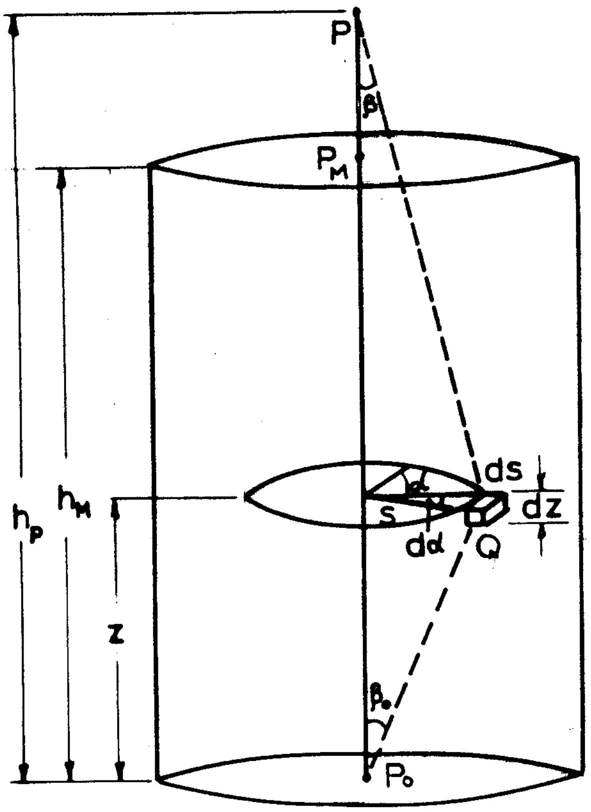

FIG. 4

order terms.

The equivalent terms for the reference system have to be considered in the light of N never exceeding 100 metres in magnitude. As

(a) only those areas with negative N values are considered and

(b) the attractive effects have the same characteristics as those quantities defined in equations (44) to (47),

$$\left(\frac{\partial \phi_{ue}}{\partial h}\right) = \tfrac{1}{2}k\rho \iint \frac{N^2}{\ell_o^3} \, dS \qquad \ldots\ldots\ldots\ldots \quad (48)$$

and

$$\left(\frac{\partial \phi_{ui}}{\partial h}\right) = -2\pi k\rho N\left[1 + \frac{N}{2r_o} + o\{f\}\right] \ldots\ldots\ldots \quad (49).$$

It should be noted that equations (48) and (49) apply only at the geoid. The contribution of $\frac{\partial \phi_{ue}}{\partial h}$ to $\frac{\partial \phi_u}{\partial h}$ is very small and can be neglected for most practical purposes. The density function ρ in equation (49) is non-zero only when $N < 0$.

3: THE SOLUTION FOR THE NON-REGULARISED GEOID.

Equation (38) is solved conventionally by the use of Chasles' theorem (Heiskanen & Moritz, 1967, 13) which asserts that V_{dp} can be expressed by an equation of the form set out in equation (6). As V_{dp} is given by equation (38) this requires that both Δg_o and V_d should be capable of representation by a set of surface harmonics. It is also necessary that both ϕ_e and ϕ_u be capable of expression by equations of the form

$$\phi_{e(u)} = \iint \frac{\alpha}{r}\, dS \ .$$

It can be seen from a study of equations (12) to (14) and (27) to (29) that equation (38) can be accurately expressed as

$$V_{dp} = 2\phi'_{ip} + \iint_S \frac{\tau}{r}\, dS \qquad \text{........ (50)},$$

where

$$\phi'_{ip} = \phi_{eip} - \phi_{uip} \qquad \text{......... (51)}$$

and τ is a surface harmonic. Thereafter, the conventional procedure as set out in equations (7) to (9) can be applied to the surface integral in equation (50) when

$$V_d = 2\phi'_i + \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}} \quad , \ n \neq 1 \ \text{....... (52)}$$

and

$$\left(\frac{\partial V_d}{\partial h}\right) = 2\left(\frac{\partial \phi'_i}{\partial h}\right) - \sum_{n=0}^{\infty} (n+1)\frac{A_n}{R^{n+2}} \ , \ n \neq 1 \ \text{... (53)}.$$

The use of equation (36) gives

$$\Delta g_o = \sum_{n=0}^{\infty} (n-1) \frac{A_n}{R^{n+2}} - 2\left(\frac{\partial \phi'_i}{\partial h}\right) - 4 \frac{\phi'_i}{R} + 2\frac{W_o - U_o}{R} + $$

$$+ 4\pi k \rho N, \; n \neq 1 \; .... \; (54).$$

Defining the corrected gravity anomaly (Δg_{oc}) at the geoid by the relations

$$\Delta g_{oc} = \sum_{n=0}^{\infty} g_{nm} S_{nm} = \sum_{n=0}^{\infty} (n-1) \frac{A_n}{R^{n+2}} \; , \; n \neq 1$$

$$= \Delta g_o + 2\left(\frac{\partial \phi'_i}{\partial h}\right) + 4\frac{\phi'_i}{R} - 2 \frac{W_o - U_o}{R} - 4\pi k\rho \; N \; ..(55)$$

where $g_{nm}S_{nm}$ represents a set of surface harmonics, the use of equations (52) and (54) in (38) gives

$$V_{dp} = 2\phi'_p + \frac{1}{2\pi} \iint_S \frac{1}{r} \left\{ \sum_{n=0}^{\infty} \frac{2n+1}{2} \frac{A_n}{R^{n+2}} - 2\left(\frac{\partial \phi'_i}{\partial h}\right) - \frac{\phi'_i}{R} \right\} dS \; , \; n \neq 1.$$

The use of equation (55) gives

$$V_{dp} = 2\phi'_p + \frac{1}{2\pi}\iint_S \frac{1}{r} \sum_{n=0}^{\infty} \frac{2n + 1}{2(n-1)} g_{nm}S_{nm} \; dS \; - $$

$$- \frac{1}{2\pi} \iint_S \frac{1}{r} \left[2\left(\frac{\partial \phi'_i}{\partial h}\right) + \frac{\phi'_i}{R} \right] dS, \; n \neq 1 \quad (56).$$

The adoption of the expansion for r^{-1} given in equation (7) and the use of the property that the surface integral of pairs of surface harmonics satisfies the relation

$$\iint_S S_n \; S_m \; dS = 0 \quad \text{if } m \neq n \; ,$$

transforms the first integral in equation (56) into the expression for Stokes' function in terms of Legendre zonal harmonics which can thereafter be transformed to the standard form of Stokes' integral by the familiar routine. Thus equation (56), on taking into account the zero order term (Mather, 1968, 313) gives

$$V_{dp} = 2\phi'_p - R\,M\{\Delta g_{oc}\} + \frac{1}{4\pi}\iint_S f(\psi)\,\Delta g_{oc}\,dS -$$

$$- \frac{1}{2\pi}\iint_S \frac{1}{r}\left[2\left(\frac{\partial \phi'_i}{\partial h}\right) + \frac{\phi'_i}{R}\right] dS \quad \ldots.(57),$$

where $M\{\Delta g_{oc}\}$ is the global mean value of the corrected gravity anomaly at the geoid. Δg_{co} as given in equation (55) can be modified by the use of equation (4) and written as

$$\Delta g_{oc} = \Delta g_f + \left\{\left(\frac{\partial \phi_e}{\partial h}\right)_S - \left(\frac{\partial \phi_e}{\partial h}\right)_G + 2\left(\frac{\partial \phi_{ei}}{\partial h}\right)\right\} -$$

$$- \frac{4}{R}[\phi_{ei} - \phi_{ui}] - \left\{2\left(\frac{\partial \phi_{ui}}{\partial h}\right) + 4\pi k\rho N\right\} -$$

$$- 2\,\frac{W_o - U_o}{R} .$$

From equations (39), (46) and (47) and the related discussion, it can be seen that the first group of terms on the right hand side of the above equation are very nearly zero, as the effect of the distant zones on the gradients of ϕ_e is very small. The second group of terms contribute less than 1 mgal to Δg_{oc} for reasons given in the discusfollowing equation (14). The third group of terms are directly evaluated from equation (49) when it is easily seen that

$$2\left(\frac{\partial \phi_{ui}}{\partial h}\right) + 4\pi k\rho N = - 4\pi k\rho N \left(\frac{N}{2r_o}\right) + o\{4\pi k\rho N f\} \quad \ldots\ldots\ldots (58)$$

For N = - 100 metres, $4\pi k\rho N$ = - 22.4 mgal and the term within the bracket is of order - 5 x 10^{-3}. Thus the contribution of the term defined in equation (58) is of magnitude 0.1 mgal. The effect on Δg_{oc} is obviously systematic though always less than 0.1 mgal. The term containing $(W_o - U_o)$ is a zero order term and has no effect on surface integration as (Lambert & Darling, 1936, 117)

$$\int_0^{\pi} f(\psi) \sin\psi \, d\psi = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (59).$$

Thus the formula

$$\Delta g_{oc} = \Delta g_f + \left(\frac{\partial \phi_e}{\partial h}\right)_S - \left(\frac{\partial \phi_e}{\partial h}\right)_G + 2\left(\frac{\partial \phi_{ei}}{\partial h}\right) - $$

$$- \frac{4}{R}\left[\phi_{ei} - \phi_{ui}\right] \quad \ldots\ldots\ldots\ldots (60)$$

holds to the nearest 0.1 mgal. The formula

$$\Delta g_{oc} = \Delta g_f$$

would hold to 4 - 5 mgal in non-mountainous terrain. The use of equation (60) together with equation (33) and the zero order effect of $(W_o - U_o)$ transforms equation (57) to

$$N_p = \frac{W_o - U_o}{\gamma m} + 2\,\frac{\phi_{res}}{\gamma_m} - R_m \frac{M\{\Delta g_{oc}\}}{\gamma_m} +$$

$$+ \frac{1}{4\pi\gamma_m}\iint f(\psi)\ \Delta g_{oc}\ dS \quad .... \ (61),$$

where

$$\phi_{res} = \phi_e - \frac{1}{4\pi}\iint \frac{1}{r}\ [\ 2(\frac{\partial \phi_{ei}}{\partial h}) + \frac{\phi_{ei}}{R}\ ]\ dS$$

$$\doteqdot \phi_{ei} + o\ \{1 \text{ kgal metre}\} \quad .... \ (62),$$

as $\frac{\phi_{ei}}{R} \approx 20.2$ mgal per km. elevation of inner zone and can be ignored in all but very mountainous regions. The ϕ_u terms have a negligible effect on the result.

The indirect effect for the Free Air Geoid obtained from a non-regularised earth is

$$N_{ip} = \frac{W_o - U_o}{\gamma_m} + 2\,\frac{\phi_{res}}{\gamma_m} - R_m \frac{M\{\Delta g_{oc}\}}{\gamma_m} +$$

$$\frac{1}{4\pi\gamma_m}\iint_S f(\psi)\ [\ \Delta g_{oc} - \Delta g_f]\ dS \quad ....(63),$$

where the quantity $[\Delta g_{oc} - \Delta g_f]$ should seldom exceed 4 to 5 mgal.

Thus the Free Air Geoid constitutes an excellent approximation of the non-regularised geoid, provided $(W_o - U_o)$ is not too large.

4. THE DETERMINATION OF N AFTER REGULARISATION.

The process of regularisation transfers matter from locations exterior to the geoid according to certain postulates to positions within it. General relations can be obtained without recourse to the exact rules governing the regularisation technique. Thus :-

(i) All topography on the existent system which lies exterior to the geoid is removed. This is equivalent to reducing the potential at all points in space from W to $(W - \phi_e)$.

(ii) This matter is now re-introduced according to a set of mathematical rules which is usually equivalent to some simple geophysical or geometrical concept. The resulting effect is the increase of the potential at all points in space from $(W - \phi_e)$ to $(W - [\phi_e - \phi])$, where ϕ is the potential due to the re-arranged form of the topography which was originally exterior to the geoid but has now been transferred to a location within it.

The application of Green's third identity (equation (18)) to such a system gives

$$\iiint_{V_i} \frac{1}{r} \underline{\nabla}^2 (W - [\phi_e - \phi]) dV_i = - 2\pi (W - [\phi_e - \phi]) +$$

$$+ \iint_S \{\frac{1}{r} \underline{\nabla}.\underline{N} (W - [\phi_e - \phi]) - (W - [\phi_e - \phi]) \underline{\nabla}.\underline{N} \frac{1}{r}\} dS.$$

The surface S in the above equation is, once again, the true geoid, the volume V_i being exactly the same as that defined in equation (24). The application of equation (20) to the above equation gives, on the same lines as before,

$$-4\pi(W_p-[\phi_{ep}-\phi_p]) + 4\pi\phi_{rp} + 2\omega^2\iiint_{V_i} \frac{1}{r}\, dV_i = -2\pi(W_p-[\phi_{ep} - \phi_p]) + \iint_S \{\frac{1}{r}\, \underline{\nabla}.\underline{N}\, (W-[\phi_e-\phi]) - (W-[\phi_e-\phi])\, \underline{\nabla}.\underline{N}\, \frac{1}{r}\} dS.$$

Defining the potential at points in space on the regularised system by the equation

$$W' = W - [\phi_e - \phi] \quad \ldots\ldots\ldots\ldots\ldots\ldots (64),$$

the above equation reduces to

$$W'_p = 2\phi_{rp} + \frac{\omega^2}{\pi}\iiint_{V_i} \frac{1}{r}\, dV_i - \frac{1}{2\pi}\iint_S \{\frac{1}{r}\, \underline{\nabla}.\underline{N}\, W' - W'\, \underline{\nabla}.\underline{N}\, \frac{1}{r}\} dS \quad \ldots\ldots\ldots\ldots\ldots\ldots (65).$$

The disturbing potential (V'_d) on the new system is given by

$$V'_{dp} = W'_p - U_p \quad \ldots\ldots\ldots\ldots\ldots\ldots (66).$$

The linearisation of equation (65) by the use of equation (26) gives

$$V'_{dp} = -2\phi_{up} - \frac{1}{2\pi}\iint_S \frac{1}{r}[\underline{\nabla}.\underline{N}\, W' - \underline{\nabla}.\underline{N}\, U] - V'_d\, \underline{\nabla}.\underline{N}\, \frac{1}{r}\, dS \ldots (67).$$

From equations (33), (64) and (66),

$$V'_{dp} = (W_o - U_o) - (\phi_{ep} - \phi) + \gamma N + o\{fV'_d\} \quad ....(68),$$

where W_o, U_o and N have exactly the same definition as for the non-regularised earth. N is therefore the true geoid spheroid separation. As S is still an equipotential surface, the relation immediately following equation (33) holds and hence the first pair of quantities within the integral in equation (67) is given , on lines analogous to the derivation of equation (35),by

$$\frac{\partial W'}{\partial h} - \frac{\partial U}{\partial h} = \frac{\partial V'_d}{\partial h} = - [\ \Delta g_o + |\frac{\partial \gamma}{\partial h}|\ N] - \frac{\partial}{\partial h}(\phi_e - \phi) \quad ...(69),$$

where Δg_o is the gravity anomaly at the non-regularised geoid, being defined by equations (4) and (34) as before. The use of equations (68) and (69) in (67) gives

$$V'_{dp} = -2\phi_{up} + \frac{1}{2\pi}\iint_S \frac{1}{r}\{\Delta g_o + \frac{3V'_d}{2R} - 2\frac{W_o - U_o}{R} + $$
$$+ 2\frac{\phi_e - \phi}{R} + \frac{\partial}{\partial h}(\phi_e - \phi) - 4\pi k\rho N\}\ dS + o\{fV'_d\}..$$
$$.............(70).$$

For the same reasons as set out in deriving equation (50), the disturbing potential on the regularised system can be expressed by an equation of the form

$$V'_{dp} = - 2\phi_{uip} + \iint_S \frac{\tau}{r}\ dS \quad ....(71),$$

which can be transformed to an equation of the form

$$V'_{dp} = -2\phi_{uip} + \sum_{n=o}^{\infty} \frac{A'_n}{R^{n+1}} \quad , n \neq 1 \ \ldots . \ (71).$$

The introduction of the condition $n \neq 1$ for the existent earth with a non-symmetrical distribution of the topography exterior to the geoid, which topography has now been moved by varying amounts to positions closer to the centre of mass, will obviously give rise to a change in the absolute position of the centre of the reference spheroid on an earth based system. This will arise as a consequence of the regularised earth having a new centre of mass together with the introduction of the condition $A_1 = 0$ which has the effect of fixing the centre of the symmetrical reference spheroid at the new centre of mass. However, this change in position of the reference spheroid is small enough to be ignored.

From equations (68), (69) and (71), development on lines similar to the procedure used in the derivation of equation (54) gives

$$\Delta g_o = \sum_{n=0}^{\infty} (n-1) \frac{A'_n}{R^{n+2}} + 2\frac{W_o - U_o}{R} + 4\pi k \rho N -$$

$$- 2 \frac{\phi_e - \phi}{R} - \frac{\partial}{\partial h}(\phi_e - \phi) + 2\frac{\partial \phi_{ui}}{\partial h} + 4\frac{\phi_{ui}}{R} \quad , n \neq 1$$

$$\ldots\ldots\ldots \ (72).$$

Defining the regularised gravity anomaly (Δg_{or}) by the equation

$$\Delta g_{or} = \sum_{n=0}^{\infty} g'_{nm} S_{nm} = \sum_{n=0}^{\infty} (n-1) \frac{A_n}{R^{n+2}} \quad , n \neq 1$$

$$= \Delta g_o + \frac{\partial}{\partial h}(\phi_e - \phi) + 2 \frac{\phi_e - \phi}{R} - [4\pi k \rho N +$$

$$+ 2 \frac{\partial \phi_{ui}}{\partial h}] - 4\frac{\phi_{ui}}{R} - 2 \frac{W_o - U_o}{R} \quad \ldots\ldots\ldots(73),$$

equation (70) becomes

$$V'_{dp} = -2\phi_{up} + \frac{1}{2\pi}\iint_S \frac{1}{r} \{ \sum_{n=0}^{\infty} \frac{(2n+1)}{2(n-1)} g'_{nm} S_{nm} +$$

$$+ \frac{\phi_{ui}}{R} + 2 \frac{\partial \phi_{ui}}{\partial h} \} dS \; , n \neq 1 \quad \ldots\ldots\ldots(74).$$

The above equation is similar in structure to equation (56), the solution being

$$N_p = \frac{W_o - U_o}{\gamma_m} - \frac{\phi_{ep} - \phi_p}{\gamma_m} - \frac{2\phi_{up}}{\gamma_m} - R_m \frac{M\{\Delta g_{or}\}}{\gamma_m} +$$

$$+ \frac{1}{4\pi\gamma_m}\iint_S f(\psi) \; \Delta g_{or} dS + \frac{1}{2\pi}\iint_S \frac{1}{r}\{ \frac{\phi_{ui}}{R} + \frac{\partial \phi_{ui}}{\partial h}\} \; dS,$$

From a study of equations (58) to (62), it can be seen that, for all practical purposes, the above equation reduces to

$$N_p = \frac{W_o - U_o}{\gamma_m} - \frac{\phi_{ep} - \phi_p}{\gamma_m} - R_m \frac{M\{\Delta g_r\}}{\gamma_m} + \frac{1}{4\pi\gamma_m}\iint_S f(\psi) \; \Delta g_r dS \quad \ldots(75),$$

where the gravity anomaly (Δg_r) on the regularised geoid is given by the use of equations (4) and (73) as

$$\Delta g_r = \Delta g_f + \left(\frac{\partial \phi_e}{\partial h}\right)_S - \left(\frac{\partial \phi}{\partial h}\right)_G + 2\,\frac{\phi_e - \phi}{R} + \text{Res} \quad ..(76),$$

where Res is less than 0.2 mgal for the same reasons as set out in the earlier section. The exact amount by which Δg_r deviates from Δg_f is dependent on the method used for effecting the regularisation. In the case of the Helmert second method of condensation described in section (1), it is obvious from a study of equations (13) and (14) that the terms ϕ_e and ϕ are completely dependent on the effect of the close zones, the principal terms being given in all but very mountainous regions by

$$\phi_e - \phi = \iint_S \frac{3k\rho h^2}{4rR}\, dS - \pi k\rho h^2 \left[1 - \frac{h}{3r_o} + \frac{h^3}{20r_o^3}\right] \quad ....(77).$$

In equation (77), the effect of the surface integral diminishes very rapidly and only requires evaluation in the immediate vicinity of the computation point. The second term is of order $6 \times 10^{-8} [h^{(met)}]^2$ kgal. metres.

The second and third terms of equation (76) are also a consequence of near zone effects, as can be seen from a study of equations (40) to (45).

$$\left(\frac{\partial \phi_e}{\partial h}\right)_S - \left(\frac{\partial \phi}{\partial h}\right)_G = -k\rho \iint_S \frac{h}{\ell_o^2}\left[\frac{2h_p - h}{\ell_o} - \frac{\ell_o}{2R}\right] dS \quad ...(78).$$

In the inner zone, the use of the planar approximation and equation (47) gives

$$\left(\frac{\partial \phi_{ei}}{\partial h}\right)_S - \left(\frac{\partial \phi_i}{\partial h}\right)_G = -2\pi k\rho h_m\left[1 + \frac{(h_p - h_m)^2}{2h_m r_o} - \frac{h_p^2}{2h_m r_o} + o\{\frac{h^3}{r^3}\}\right]$$

.........(79).

Thus equations (75) and (76) completely define any regularisation procedure. From a purely geodetic point of view, there appears to be no advantage at all in effecting any form of regularisation *if it is intended to make an exact evaluation of the geoid spheroid separation*, as the expressions in the latter case are no less complex. All indirect effects will have, in addition to the zero order term, both a potential dependent term as well as one which is Stokesian in character.

5. THE SOLUTION FOR THE PHYSICAL SURFACE OF THE EARTH.

The telluroid has already been defined in section (1). In the following development, it will be assumed that the telluroid has the same volume as the physical surface of the earth, being the locus of points in space which have the same difference in potential on the reference system with respect to the spheroid as the equivalent points on the earth's surface have with respect to the geoid. As pointed out earlier, this is slightly different from Hirvonen's original definition, though not at variance with any aspect of the practical significance of his definition. It has the advantage of not assuming W_o as equal to U_o and hence generalises the solution.

Equations (24) and (26) still hold even when S is taken as the physical surface of the earth. The term in ϕ_{up} in equation (26) will still have to be considered as the assumption of equal volume will imply that the height anomaly (h_d) is negative over 50% of the earth and as 70% of the surface area is ocean, there will exist areas where matter on the reference system can be external to the physical surface of the earth. If P is a point on the earth's surface, equations (24), (26) and (30) give

$$V_{dp} = -2\phi_{up} - \frac{1}{2\pi}\iint_S \{\frac{1}{r}[\underline{\nabla}.\underline{N}\ W - \underline{\nabla}.\underline{N}\ U] - V_d\ \underline{\nabla}.\underline{N}\ \frac{1}{r}\}\ dS \quad (80),$$

where S is the physical surface of the earth.

Equations (33) to (36) become

$$V_{dp} = (W_o - U_o) + \gamma h_d + o\{fV_d\} \quad \ldots\ldots\ldots\ldots (81)$$

$$\frac{\partial V_d}{\partial h} = -\ [\Delta g_f + |\frac{\partial \gamma}{\partial h}|\ h_d] + o\{f\ \frac{\partial V_d}{\partial h}\} \quad \ldots\ldots\ldots\ldots (82),$$

where (see fig (1))

$$\Delta g_f = g - \gamma_Q \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (83).$$

In practice, it is assumed that the difference in potential on both the existent and reference systems is equivalent to the same normal displacement (orthometric height) when Δg_f becomes the free air anomaly. Equation (82) can be re-written as

$$\frac{\partial V_d}{\partial h} = - \left[\Delta g_f + 2\frac{V_d}{R} - 2\frac{W_o - U_o}{R} - 4\pi k\rho h_d\right] \quad .. \ (84),$$

where $\rho = 0$ if $\left[h_d + \frac{|\Delta W|}{\gamma}\right] > 0$.

In this case, the main problem lies in the evaluation of the normal derivatives of the various functions in the surface integral. The relation between the local vertical and the normal to the physical surface of the earth is , for all practical purposes, the same as that between the spherop normal and the normal to the telluroid. As shown in fig (2),

$$\frac{\partial V_d}{\partial h} \neq \underline{\nabla}.\underline{N}\, W - \underline{\nabla}.\underline{N}\, U \quad (= \underline{\nabla}.\underline{N}\, V_d) \quad ......(85).$$

The relationship between the vertical derivative of the disturbing potential and the derivative normal to the physical surface of the earth can be effected by the choice of a locally oriented three dimensional cartesian frame (x_i, i=1,3 system) with the x_3 axis coincident with the local spherop normal and the x_1 axis oriented north. The angle β between the spherop normal and the telluroid normal is also the maximum slope of the topography. Any simple surface in the vicinity (Moritz, 1965, 15) can be represented by an expression of the form

$$x_3 = \Delta h(x_i,\ i=1,2) \ ,$$

where Δh is the increment in normal height. If such a point lies on the telluroid given by the equation

$$U(x_i,\ i=1,2\ ,\ \Delta h(x_i,\ i=1,2)\ ) = 0,$$

then

$$\frac{\partial U}{\partial x_i} + \frac{\partial U}{\partial x_3}\frac{\partial h}{\partial x_i} = 0 \quad i=1,2 \ \ldots\ldots\ldots\ldots\ (86).$$

From equation (86) and elementary differential geometry it can be seen that the direction cosines of the telluroid normal are proportional to

$$-\frac{\partial h}{\partial x_1}\ ,\ -\frac{\partial h}{\partial x_2}\ ,\ 1.$$

Hence

$$\underline{N} = \left(-\sum_{i=1}^{2}\frac{\partial h}{\partial x_i}\underline{i} + \underline{3}\right)\cos\beta \quad \ldots\ldots\ldots\ (87).$$

Thus

$$\underline{\nabla}.\underline{N}\ V_d = \cos\beta\left(\frac{\partial V_d}{\partial h} - \sum_{i=1}^{2}\frac{\partial V_d}{\partial x_i}\frac{\partial h}{\partial x_i}\right) \ldots\ldots (88).$$

The horizontal gradients of the disturbing potential are given (Moritz, 1965, 17) by

$$\frac{\partial V_d}{\partial x_i} = -\gamma\xi_i\ ,\quad i=1,2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ (89),$$

where $\xi_i\,(i=1,2)$ has the same definition as in equation

(16). Also

$$\frac{\partial h}{\partial x_i} = \tan \beta_i \quad , \; i=1,2 \qquad \text{..........(90)},$$

where β_1, β_2 represent the ground slope rising north, east respectively.

The other derivative to be evaluated in the surface integral is

$$\underline{\nabla} \frac{1}{r} = - \sum_{i=1}^{3} \frac{x_i}{r^3} \underline{i} \quad ,$$

where r, shown in fig (5), is given by

$$r = \left(\sum_{i=1}^{3} x_i^2 \right)^{\frac{1}{2}}$$

The combination of this result with equation (87) gives

$$\underline{N} \cdot \underline{\nabla} \frac{1}{r} = \cos \beta \left(\sum_{i=1}^{2} \frac{x_i}{r^3} \frac{\partial h}{\partial x_i} - \frac{x_3}{r^3} \right) \qquad \text{........ (91)}.$$

The following relations hold for a spherical approximation of the earth of radius R.

$$\left.\begin{aligned} r &= 2R \sin \tfrac{1}{2}\psi \\ x_1 &= R \sin \psi \cos A \\ x_2 &= R \sin\psi \sin A \\ x_3 &= 2R \sin^2 \tfrac{1}{2}\psi \end{aligned}\right\} \qquad \text{.......... (93)},$$

where ψ is the angular distance of the computation point from the element of surface area dS and A is the azimuth of the former from the latter. Thus equation (91) becomes

$$\underline{N}.\underline{\nabla}\,\frac{1}{r} = -\cos\beta\,\left[\frac{1}{2Rr} - \frac{f(\beta,\psi)}{r}\right] \ldots\ldots\ldots\ldots (93),$$

where

$$f(\beta,\psi) = \frac{\sin\psi}{2R(1-\cos\psi)}\,[\cos A \tan\beta_1 + \sin A \tan\beta_2].(94).$$

For distant zones, $f(\beta,\psi)$ is at least one order smaller than $(2R)^{-1}$ and is obviously zero for all ocean and flat areas. It does however, have a significant effect in the immediate vicinity of computation points in non-level areas. To retain the simplicity of solution, the effect of this term is considered to be an order smaller than h_d and hence could be obtained as a small order term in an iterated evaluation. On this basis, the use of equations (88), (89),(90) and (93) in equation (80) gives

$$V_{dp} = -2\phi_{up} + \frac{1}{2\pi}\iint_S \frac{1}{r}\Big(\Delta g_f + \frac{3V_d}{2R} - \frac{2(W_o - U_o)}{R} - 4\pi k\rho h_d - \gamma\sum_{i=1}^{2}\xi_i \tan\beta_i + f(\beta,\psi)\,V_d\Big)\cos\beta\, dS \ldots (95).$$

On lines similar to the derivation of equations (70) and (71) , V_d can be represented by an equation identical with (71). From equation (84),

$$\Delta g_f = \sum_{n=0}^{\infty}(n-1)\frac{A_n}{R^{n+2}} + 2\frac{\partial\phi_{ui}}{\partial h} + 4\frac{\phi_{ui}}{R} + 2\frac{W_o - U_o}{R} + 4\pi k\rho h_d\ , \ n \neq 1, \rho = 0 \text{ when}\{h_d + \frac{|\Delta W|}{\gamma}\} > 0 \ldots\ldots (96).$$

Defining the corrected free air anomaly (Δg_{fc}) by the equations

$$\Delta g_{fc} = \sum_{n=0}^{\infty} g_{nm} S_{nm} = \sum_{n=0}^{\infty} (n-1) \frac{A_n}{R^{n+2}} \cos \beta \quad , n \neq 1$$

$$= \left(\Delta g_f - 2 \frac{\partial \phi_{ui}}{\partial h} - 4 \frac{\phi_{ui}}{R} - 2 \frac{W_o - U_o}{R} - 4\pi k \rho h_d \right) \cos \beta$$

$$\ldots\ldots\ldots\ldots \quad (97),$$

it is possible to adapt equation (95) to obtain a Stokesian term. The use of the modified form of equation (71) and equations (96) and (97) for the evaluation of Δg_f and V_d in in all but the smaller order terms of equation (95) gives

$$V_{dp} = \frac{1}{2\pi} \iint_S \frac{1}{r} \sum_{n=0}^{\infty} \frac{2n+1}{2(n-1)} g_{nm} S_{nm} \, dS \; -$$

$$-2\phi_{up} + \frac{1}{2\pi} \iint_S \frac{1}{r} \left\{ 2 \frac{\partial \phi_{ui}}{\partial h} + \frac{\phi_{ui}}{R} \right\} \cos \beta \, dS \; -$$

$$- \frac{1}{2\pi} \iint_S \frac{1}{r} \cos \beta \, \gamma \sum_{i=1}^{2} \xi_i \tan \beta_i \, dS \; +$$

$$+ \frac{1}{2\pi} \iint_S \frac{1}{r} f(\beta, \psi) \, V_d \cos \beta \, dS \quad , n \neq 1 \quad .(98).$$

The first row of equation (98) is the term which, on development, gives Stokes' integral. The effect of the second row is very nearly zero as β will be equal to zero or or negligible in those regions which contribute to the constituent terms. Thus, for normal.use, this equation can be expressed on lines similar to those adopted in the derivation of equation (75) when combination with equation (81) gives

$$h_d = \frac{W_o - U_o}{\gamma_m} - R_m \frac{M\{\Delta g_{fc}\}}{\gamma_m} + \frac{1}{4\pi\gamma_m}\iint_S f(\psi)\ \Delta g_{fc} dS -$$

$$- \frac{1}{2\pi\gamma_m}\iint_S \frac{\cos\ \beta}{r} \left(\gamma \sum_{i=1}^{2} \xi_i \tan\ \beta_i\ -\ f(\beta,\psi) V_d\right) dS \ldots (99),$$

where

$$\Delta g_{fc} = \Delta g_f \cos\ \beta\ + o\{0.1 \text{ mgal}\} \ \ldots\ldots (100).$$

The terms requiring solution by iteration are placed together in the second integral in equation (99). The principal contribution is from the Stokesian term where the anomaly to be used is the derivative of the free air anomaly normal to the physical surface. Except in mountainous regions this term, to order f Δg_f, should be the free air anomaly itself and hence the free air geoid constitutes a good first approximation to the telluroid. The extent of deviation of the latter from the former is dependent on the local topography as the influence of the second surface integral and the cos β scaling factor is zero over oceans and small for distant zones.

For near zones, the cos β scaling factor differs from 1 by amounts exceeding f when β is approximately $4\frac{1}{2}^{\circ}$, and should be considered when evaluating the effects for these regions. The value of β can be evaluated from the values of β_1 and β_2 by the standard property of all direction

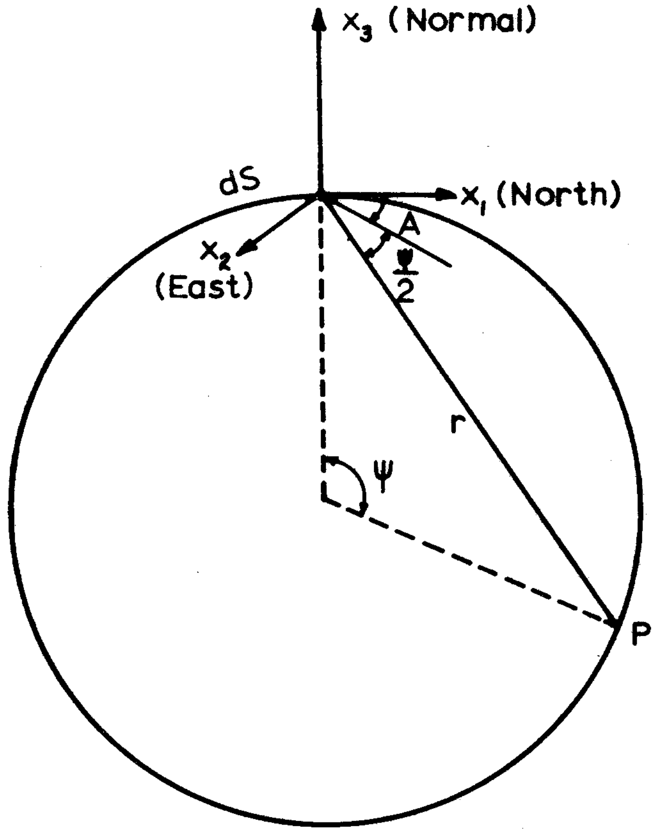

FIG. 5

cosines when applied to the telluroid normal.

$$\left(\sum_{i=1}^{2} \tan^2\beta_i\right) \cos^2\beta + \cos^2\beta = 1.$$

Hence,

$$\beta = \tan^{-1}\left(\left[\sum_{i=1}^{2} \tan^2 \beta_i \right]^{\frac{1}{2}} \right) \qquad \ldots\ldots\ (101).$$

The first obvious step in solving for the physical surface of the earth is the determination of the Free Air Geoid, obtained by the use of free air anomalies in the Stokes' and Vening Meinesz integrals. The second stage is the computation of the correction term for the near zones using the anomaly Δg_f versine β in Stokes' integral if the local ground slopes exceed 4^o or 1 in 15. The third stage is the evaluation of the second integral using the Free Air Geoid as a source for the values of h_d, ξ_1 and ξ_2. It is expected that a second iteration for the evaluation of the small order terms will only be necessary in mountainous regions when it would be easiest to compute directly in terms of V_d In all evaluations of the second integral, it is unlikely that the contribution of the outer zones will exceed 20 cm to the value of h_d and thus the second integral requires evaluation for near zones only.

6. THE ZERO ORDER TERM.

As shown in a previous work (Mather, 1968, 293 et seq), changes in normal gravity consequent to changing the parameters of the reference spheroid produce changes in the deflections of the vertical, computed from the Vening Meinesz formulae, which are exactly equivalent to purely geometrical changes. Such changes also produce significant changes in the potential of the reference spheroid. Thus the value of U_o is dependent on

(a) the parameters of the spheroid of reference ;

(b) the value adopted for kM.

It would therefore be imprecise to state that the amount by which U_o departs from the potential of the existent geoid (W_o) is a negligible quantity.

The concept of a reference spheroid with a bounding equipotential is geodetic fiction and not meant to be representative of geophysical fact. Defining the parameters a and f for a particular spheroid along with the conditions that the spheroid is centred at the earth's centre of gravity and the x_3 rotation axis coincidence fixes a very definite' locus in earth based space. Continuing in this vein of definite spatial displacements, equations (59), (75) and (99) represent linear displacements from such a reference spheroid and/or related surfaces: These equations are not dependent on the equal volume condition though this was specified in defining the conditions of solution to show why the ϕ_u term *may have been* of significance.

Gravimetric geodesy has not proved very successful in providing the parameters for the best reference system. The equatorial radius of a spheroid which best fits the geoid is preferably obtained by scale measurements and astro-geodetic levelling using a representative global sample. The flattening of this spheroid of best fit can be obtained by a number of methods, the most reliable of which appears to be satellite geodesy

Thus non-gravimetric data provide the dimensions of a reference spheroid which has the same volume as the geoid. If this is so, M{N} must have a value zero on world wide summation, provided such a spheroid were used as the reference figure for the earth. This enables a value to be assigned for W_o which is necessary for the complete evaluation of h_d. W_o is obtained from manipulation of the zero order terms in equation (59) as

$$W_o = U_o + R_m \, M\{\Delta g_{oc}\} - 2 \, M\{\phi_{res}\} - M\{N_{oc}\} \quad ..(102),$$

where M { } refers to world wide means and N_{oc} is the Stokesian contribution obtained by the use of the anomaly Δg_{oc} in Stokes' integral.

Once a value has been assigned for W_o by the use of equation (102), it is of interest to consider whether a change is warranted in the value of U_o and hence, in the related parameters a, f and kM. Any change in the value of f can only be justified if definite systematic effects exist, on a global basis in ξ_1 but not in ξ_2. If this is not so, the change in U_o can be interpreted as being changes in one or both of a and kM. A change in a which is not supported by evidence from scale determinations (astro-geodetic levelling) would certainly not be warranted. Hence any correction to U_o to make it equal to the computed value of W_o will have to be distributed between corrections to a and kM on the basis of correlation with external evidence. Such a line of action is warranted as the geoid defined in equation (59) is defined on an earth space system without ambiguity, *with the proviso* that an adequate model is available for the topography exterior to the geoid

The existence of an uncertainty in the density (ρ) of the topography which, even when a variable density formula similar to that suggested by de Graaff-Hunter (1966) is used, could be of the order of 4% to 10%, will not cause the degree of uncertainty in the value of N as was first estimated (Mather, 1968, 113) in view of the fact that the topography terms contribute only 1% to 10% of the total value. As this uncertainty in the density contributes less than 1 metre to the final value of N, any deductions from the gravimetric solution as laid down in the previous paragraph, can be considered to be as good as the gravity field used in effecting the solution and not on any assumptions made regarding the density or stratification of the topography.

7. CONCLUSIONS.

Equations (59) and (75) define exactly the same quantity N which is the separation of the geoid and the spheroid. From the structure of the equations involved, it can be concluded that the computations for determining N for the non-regularised geoid are no more complex than for the *complete* solution using any of the well known methods of regularisation. There is no advantage as such, from a geodetic point of view, in using regularisation techniques though they may have some geophysical significance. In fact it is rather misleading to refer to the co-geoids obtained by the use of any anomaly except the free air anomaly as being a good approximation to the geoid.

A study of equations (59) and (99) shows that the Free Air Geoid obtained by the use of free air anomalies in Stokes' integral is a good approximation of the non-regularised geoid as well as the solution for the physical surface of the earth provided no zero order terms occur. In the strictest sense, the free air anomaly must be computed using potential differences and not elevations (Hirvonen, 1960, 21).

The effect of matter on the reference system which lies exterior to the surface being mapped,has been considered in the solution and has been shown to have a negligible effect on the final result. The writer is unaware of any serious attempts to postulate any stratification of matter within the reference spheroid. All derivations in this paper have made no assumptions regarding stratification of matter within the bounding equipotential except that of symmetry and hence the above conclusion can be held to be of general application.

Equation (99) shows that the solution for the physical surface of the earth is incomplete without a value for W_o. Conventionally this is assumed to be equal to U_o on the basis that such an effect is merely due to an error in the parameters of the reference spheroid. On the other hand, if independent evidence indicates that the spheroid of reference has the same volume as the geoid, the adoption of this line of reasoning could only be interpreted in terms of the existence of an error in the adopted value of kM.

Those advocating solution at the physical surface of the earth have in their favour the fact that the slope of the surface defined by h_d can be directly compared with quantities related to the local vertical. A further advantage, if only an apparent one, is the fact that all quantities constituting the correction terms to the Free Air Geoid are capable of determination, being functions of the ground slope.

On the other hand, a solution for the geoid itself, while *nominally* restricted by uncertainties in the model adopted for the topography exterior to the geoid, is nevertheless of considerable importance in studying the characteristics of the reference spheroid and the complete definition of points on the earth's surface on a three dimensional system fixed with respect to the earth.

At this stage it would be prudent to operate with generalised formulae bearing in mind that the free air geoid is a good approximation to both the geoid and the physical surface of the earth/telluroid system, exclusive of any zero order effects. Each of the individual solutions could then be generated from the first order solution (Free Air Geoid) as explained in sections (3) and (5). All other co-geoids are of no geodetic significance *unless* the relevant correction terms are fully defined.

The final solutions in each case and Stokes' integral continue to hinge on the validity of the development expressed in equations (6) to (9). The crux is whether it is valid to useas an intermediary in the solution a series which at times is apparently divergent. There is no doubt that both the gravity anomaly and the disturbing potential on the geoid can be represented with an accuracy of the order of the flattening by a set of surface harmonics. In addition, both the physical surface of the earth and the geoid can be represented by a sphere to this same order. Thus S in equation (6) can be a sphere with τ a surface harmonic. The resulting definition of V_d will be correct to the order of the flattening. In the case of the spherical approximation there is no doubt about the convergence of the series defined in equation (9) and used as an intermediary. Hence the existence of the solutions is defined.

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RON MATHER was educated at the University of Ceylon, Christ's College, Cambridge, and the University of New South Wales. On graduating with a Bachelor of Science degree in 1955 he joined the Ceylon Survey Department in which he served until 1962. During this period he was for a while in charge of the Ceylon Survey Training School. After a spell as lecturer at the South Australian Institute of Technology, he joined the University of New South Wales in 1966 where he is at present a senior lecturer.

Dr. Mather has published papers on the propagation of errors, extension of gravity fields and other aspects of physical geodesy. He is currently preparing a map of the non-regularised geoid for Australia and is working on the orientation of the Australian Geodetic Datum using gravity data.

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