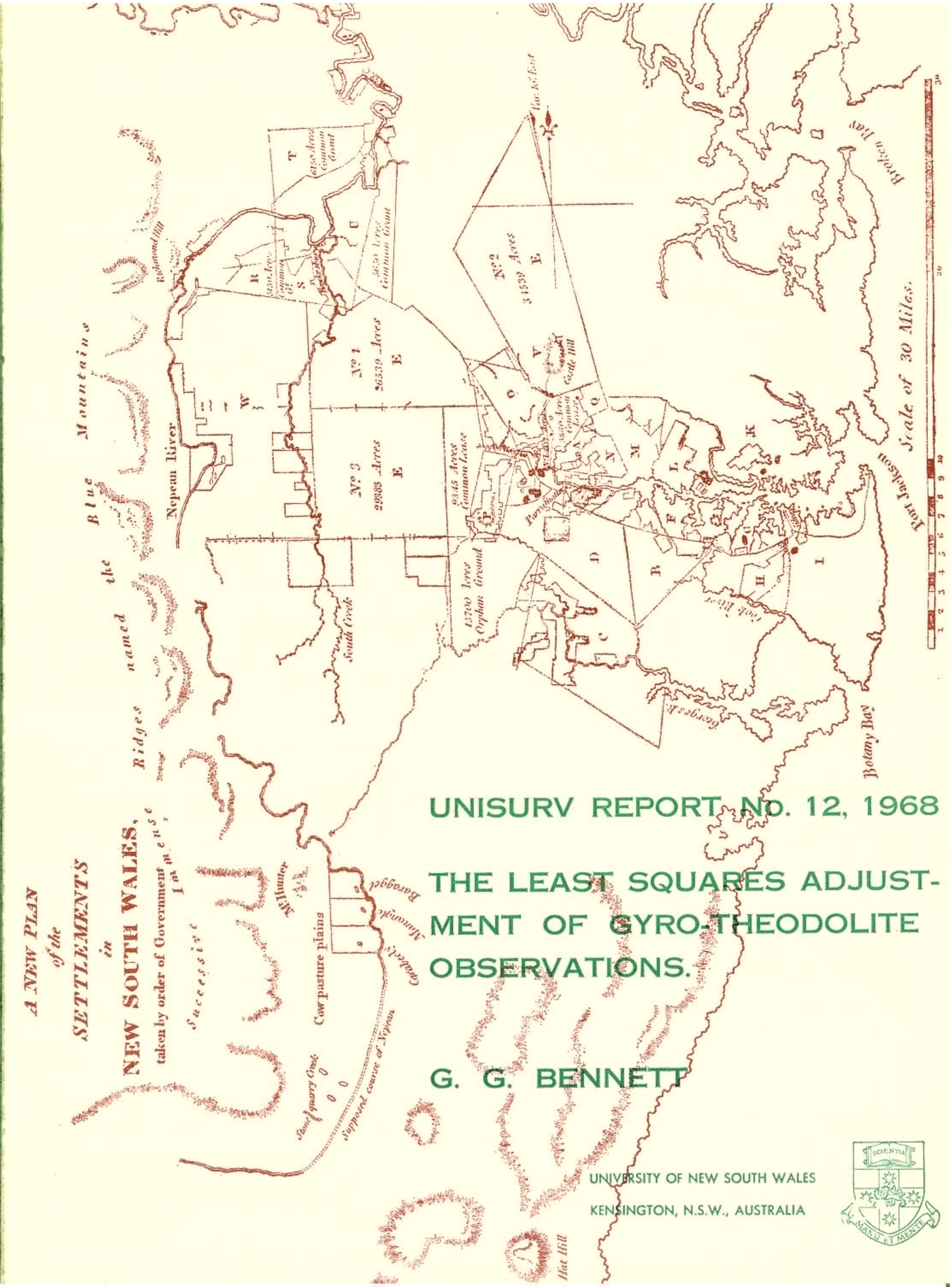

UNISURV REPORT No. 12, 1968

THE LEAST SQUARES ADJUSTMENT OF GYRO-THEODOLITE OBSERVATIONS.

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THE LEAST SQUARES ADJUSTMENT OF GYRO-THEODOLITE OBSERVATIONS.

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Received 9th September, 1968.

SUMMARY: The direction of the meridian as derived from a number of successive observations of the "turning points" of a pendulous gyroscope has been investigated by many writers. A rigorous solution of this problem has been proposed by Professor G.B. Lauf in 1963. However, it is of practical advantage to consider alternative approximate solutions. In particular the approach of Professor Max Schuler has been popular over many years, undoubtedly due to the simplicity of applying his formulae. Writers such as F. Kohlrausch, Professor G.B. Lauf and Dr. T.L. Thomas have contributed much to the understanding of this latter problem.

The purpose of this report is to analyse and evaluate aspects of a lightly damped simple harmonic motion in relation to the gyro-theodolite. The linear damping case has been treated in some detail and a new approach has been made to the interpretation of the Schuler and related Means. This latter investigation has revealed that a simple combination of the Schuler Means will give a least squares solution of the direction of the line of mean oscillation. Examples and tables to assist in numerical reduction are given.

THE LEAST SQUARES ADJUSTMENT OF GYRO-THEODOLITE OBSERVATIONS.

THE LINEAR DAMPED MODEL.

Lauf (1963) has shown that the precession angle about the meridian of a pendulous gyroscope is of the form

$$\theta = Be^{-\alpha t} \cos (\beta t + \gamma) \qquad (1)$$

in which, according to Williams and Belling (1967)

- B is the half-amplitude of the oscillation at some initial instant,
- e is the base of natural logarithms, and
- α and β are constants for a given instrument for a particular latitude when the angular momentum of the gyro-rotor is constant.
- γ is an arbitary phase-angle

It has also been shown by Lauf (1967) that a good approximation to the precession angle at the "turning point" is when $(\beta t + \gamma) = 0, \pi, 2\pi, 3\pi$, etc. For convenience we may change the form of equation (1) to

$$\theta = -Be^{-2\alpha' \tau} \cos 2\pi\tau$$

where τ represents the fractional period elapsed.

"Turning points" will occur when $\tau = 0, \frac{1}{2}, 1$ ... etc. For the gyro-theodolite the damping is usually slight and the exponential term may be expanded retaining only linear damping terms giving

$$\theta = -B(1 - 2\alpha'\tau) \cos 2\pi\tau$$

Now at the "turning points", $\tau = \frac{i}{2}$, where $i = 1, 2, \ldots n$, corresponding to the number of the "turning point". Therefore

$$\theta_i = B(1 - i\alpha')(-1)^{i-1}$$

The measurements of θ_i are made on a circle whose orientation with respect to the meridian is unknown and defined by θ_o. If these measurements are defined as y_i and their corresponding corrections by v_i then the previous equation takes the form

$$v_i = B(1 - i\alpha') (-1)^{i-1} + \theta_o - y_i \tag{2}$$

or, in extenso

$$v_1 = B - \alpha' B + \theta_o - y_1$$
$$v_2 = -B + 2\alpha' B + \theta_o - y_2$$
$$v_3 = B - 3\alpha' B + \theta_o - y_3$$
$$v_4 = -B + 4\alpha' B + \theta_o - y_4$$

etc.

Which are the required correction equations.

With regard to the following adjustment procedure, the underlying assumptions are that the observations are stochastically independent and are made with equal precision. It will be noted that no hypothesis is made concerning the normality of the distribution, merely that a variance exists for this distribution. The least squares method seeks to satisfy the relationship (vv) = minimum, and requires no assumption regarding the distribution other than the existence of a variance. For elaboration of this point see Sunter (1966).

The correction equations in their present form are not suitable for a direct solution because they are not linear. Lauf (1967) has used the familiar technique of introducing approximate values of the unknown parameters α and B and then expanding the non-linear term by Taylor's series to solve this problem. An alternative method of separating the unknown parameters is to substitute in the correction equations a new set of parameters such that the equations become linear in respect of these new values.

If we put $a = \alpha' B$, $b = \theta_o + B$, and $c = \theta_o - B$, then the general form of correction equations becomes

$$v_i = i(-1)^i a + \left(\frac{1 + (-1)^{i-1}}{2} \right) b + \left(\frac{1 + (-1)^i}{2} \right) c - y_i \qquad (3)$$

$$i = 1, 2, \ldots\ldots n$$

$$n \geqslant 3$$

or, in extenso

$$\begin{aligned} v_1 &= -a + b \qquad - y_1 \\ v_2 &= 2a \qquad + c - y_2 \\ v_3 &= -3a + b \qquad - y_3 \\ v_4 &= 4a \qquad + c - y_4 \end{aligned}$$

etc.

Our main interest will lie with the original parameters, which will be given by

$$\theta_o + \frac{b + c}{2}, \quad B = \frac{b - c}{2} \quad \text{and} \quad \alpha' = \frac{a}{B} \quad \text{or} \quad \alpha' = \frac{2a}{b - c} \qquad (4)$$

The geometrical interpretation of the adjustment is shown in Fig. 1. The mathematical model is in fact a pair of straight lines, at the same inclination to the line of mean oscillation, passing through the observations.

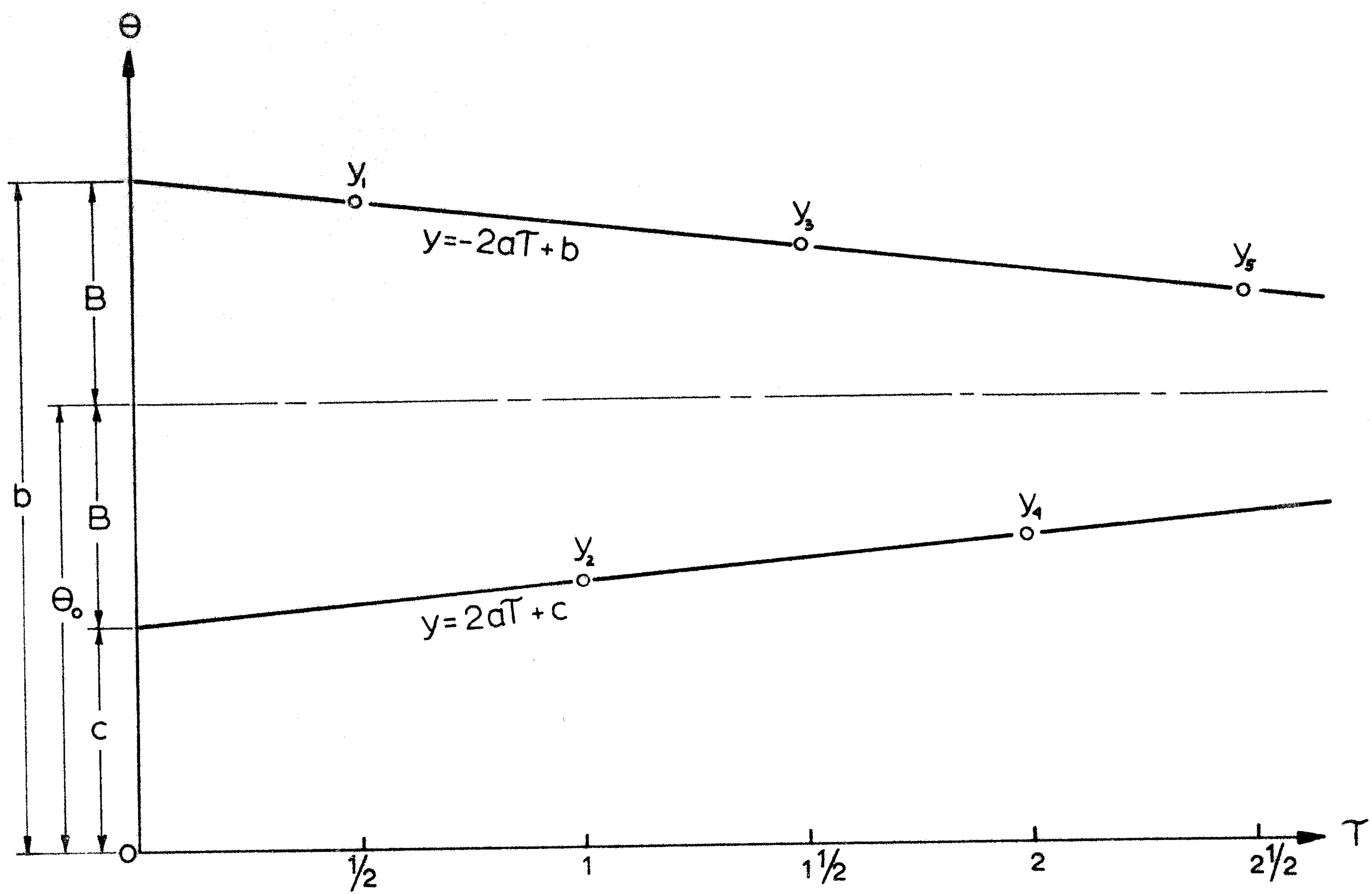

FIG. 1

Forming normal equations* we obtain

a	b	c	Absolute Term	
$\sum_{i=1}^{n} i^2$	$\sum_{i=1}^{n} \frac{i(-1)^i\left(1+(-1)^{i-1}\right)}{2}$	$\sum_{i=1}^{n} \frac{i(-1)^i\left(1+(-1)^{i}\right)}{2}$	$-\sum_{i=1}^{n} i(-1)^i y_i$	$= 0$
	$\sum_{i=1}^{n} \left(\frac{1+(-1)^{i-1}}{2}\right)^2$	0	$-\sum_{i=1}^{n} \left(\frac{1+(-1)^{i-1}}{2}\right) y_i$	$= 0$
		$\sum_{i=1}^{n} \left(\frac{1+(-1)^{i}}{2}\right)^2$	$-\sum_{i=1}^{n} \left(\frac{1+(-1)^{i}}{2}\right) y_i$	$= 0$

It has been found to be convenient in the following reduction process to treat two types of observation, one containing an even number and the other an odd number of observations.

The normal equations for n even are

a	b	c	Absolute Term		
$\frac{n(n+1)(2n+1)}{6}$	$\frac{-n^2}{4}$	$\frac{n}{4}(n+2)$	$\sum_{i=1}^{n} i^0 y^0_i - \sum_{i=1}^{n} i^E y^E_i$	$= 0$	
	$\frac{n}{2}$	0	$-\sum_{i=1}^{n} y^0_i$	$= 0$	(5)
		$\frac{n}{2}$	$-\sum_{i=1}^{n} y^E_i$	$= 0$	

and for n odd are

a	b	c	Absolute Term		
$\frac{n(n+1)(2n+1)}{6}$	$-\left(\frac{n+1}{2}\right)^2$	$\frac{n^2-1}{4}$	$\sum_{i=1}^{n} i^0 y^0_i - \sum_{i=1}^{n} i^E y^E_i$	$= 0$	
	$\frac{n+1}{2}$	0	$-\sum_{i=1}^{n} y^0_i$	$= 0$	(6)
		$\frac{n-1}{2}$	$-\sum_{i=1}^{n} y^E_i$	$= 0$	

* For symmetrical matrices the lower triangular portion has been omitted.

where the superscript 0 and E denotes odd and even values only are to be taken, respectively.

From the practical viewpoint one will seldom be concerned with $n > 8$ and therefore equations (5) and (6) have been solved for $n = 3, 4 \ldots \ldots 8$. The results of these solutions are given in Table 1. The values of $\frac{b + c}{2}$ and $\frac{b - c}{2}$ have also been found, because these represent the principal unknowns θ_o and B. The values of θ_o for n = 4 and 8 agree precisely with those obtained by Lauf (1967) who has been concerned mainly with those values of n which are a multiple of 4.

Two important relationships can be derived immediately from the sets of normal equations (5) and (6). The sum of the second and third normal equations of (5) is

$$\frac{n}{2} a + \frac{n}{2} b + \frac{n}{2} c = \sum_{i=1}^{n} y_i^0 + \sum_{i=1}^{n} y_i^E$$

therefore

$$\frac{b + c}{2} = \theta_o = \frac{1}{n} \sum_{i=1}^{n} y_i - \frac{a}{2} \tag{7}$$

Also, from the set of normal equations (6), if we divide the second normal equation by n + 1 and the third normal equation by n - 1 and add, we obtain

$$\frac{b + c}{2} = \theta_o = \frac{1}{n^2 - 1} \left\{ (n - 1) \sum_{i=1}^{n} y_i^0 + (n + 1) \sum_{i=1}^{n} y_i^E \right\} \tag{8}$$

which may be evaluated very simply by taking the mean of the odd numbered observations, the mean of the even numbered observations and then taking a grand mean.

It is worthwhile comparing the results of these derivations with those obtained by Williams and Belling (1967). These authors state - "It must be recognised nevertheless that unless a better criterion can be found, [vv]* is as acceptable as any. Therefore, any simple reduction procedure which agrees very closely with the least squares result is as acceptable as this latter, and is indeed preferable to it if it offers a more general solution." They derive the following formulae on a basis of "symmetry" and "the principle of least displacement of random lines ....." :-

for n even

$$\theta_o = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{4(n-2)} (y_1 - y_2 - y_{n-1} + y_n)$$

and for n odd

$$\theta_o = \frac{1}{n+1}\sum_{i=1}^{n} y_i^0 + \frac{1}{n-1}\sum_{i=1}^{n} y_i^E$$

The agreement between these formulae and (7) and (8) is only partial. For n odd the agreement is exact but for n even $\frac{a}{2}$ is approximated by $\frac{1}{4(n-2)} (y_1 - y_2 - y_{n-1} + y_n)$. It is a fact that this approximation is reasonable because from the geometrical interpretation of the least squares solution it can be seen that $\frac{a}{2}$ can be derived relatively efficiently from the end observations only. These authors further state regarding their formulae that "The advantage of their application over the use of more complicated formulae derived via Gaussian theory is based on their simplicity of form and the consequent fact that θ_o can be rapidly obtained under field conditions." In fact, however, there is only a marginal difference in

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* This should probably be "[vv] is a minimum."

complexity between the rigorous least squares solution and that offered by these authors. The least squares solution is to be preferred. It will be shown further that an even simpler form of rigorous reduction is available from consideration of the Schuler means.

Besides the solution of the estimates of the adjusted parameters, the calculation of their variances and co-variances is necessary in order to estimate the performance of both the mathematical model and the observations. A useful summary of the least squares adjustment techniques, using matrix algebra, has been prepared by Allman (1967) and has been used extensively in the following derivations.

The matrix of the weight coefficients of the adjusted parameters is given by the inverse matrix of the normal equations after removal of the absolute terms.

For n even the inverse matrix is

$$\begin{bmatrix} \frac{12}{n(n^2-4)} & \frac{6}{(n^2-4)} & \frac{-6}{n(n-2)} \\ & \frac{5n^2-8}{n(n^2-4)} & \frac{-3}{(n-2)} \\ & & \frac{5n+2}{n(n-2)} \end{bmatrix}$$

retaining the same order of parameters as in the normal equations,

and for n odd

$$\begin{bmatrix} \frac{12}{n(n^2-1)} & \frac{6}{n(n-1)} & \frac{-6}{n(n-1)} \\ & \frac{5n^2+4n+3}{n(n^2-1)} & \frac{-3(n+1)}{n(n-1)} \\ & & \frac{5n+3}{n(n-1)} \end{bmatrix}$$

If the estimate of the variance of the adjusted observations is $\hat{S}_y^2 = \frac{[vv]}{r}$ where r is the number of redundancies, then the estimates of the variance of the adjusted parameters a, b and c is given by the following -

$$\hat{S}_a^2 = Q_{aa}\,\hat{S}_y^2$$

$$\hat{S}_b^2 = Q_{bb}\,\hat{S}_y^2$$

$$\hat{S}_c^2 = Q_{cc}\,\hat{S}_y^2$$

As stated before, our interest lies with the original parameters

$$\theta_o = \frac{b+c}{2}, \qquad B = \frac{b-c}{2} \quad \text{and} \quad \alpha' = \frac{a}{B}$$

Applying the general law of propagation of variances to these relationships we have

$$\hat{S}^2_{\theta_o} = \frac{1}{4}\left\{Q_{bb} + 2Q_{bc} + Q_{cc}\right\}\hat{S}^2_y$$

$$\hat{S}^2_B = \frac{1}{4}\left\{Q_{bb} + 2Q_{bc} + Q_{cc}\right\}\hat{S}^2_y$$

$$\hat{S}^2_{\alpha'} = \frac{4}{(b-c)^4}\left\{(b-c)^2Q_{aa}+a^2(Q_{bb}-2Q_{bc}+Q_{cc})-2a(b-c)(Q_{ab}-Q_{ac})\right\}\hat{S}^2_y$$

After substitution of the appropriate values from the inverse matrices we have for n even

$$\hat{S}^2_{\theta_o} = \frac{n^2-1}{n(n^2-4)} \cdot \hat{S}^2_y$$

$$\hat{S}^2_B = \frac{4n^2+6n-1}{n(n^2-4)} \cdot \hat{S}^2_y$$

$$\hat{S}^2_{\alpha'} = \frac{12}{n(n^2-4)B^2}\left\{1 + \frac{\alpha'^2}{24}(5n+4)(n+1)-\alpha'(n+1)\right\}\hat{S}^2_y$$

If n is not large as is the case in practice then this last expression may be approximated by

$$\hat{S}^2_{\alpha'} = \frac{12}{n(n^2-4)B^2}\hat{S}^2_y$$

and for n odd

$$\hat{S}^2_{\theta_o} = \frac{n}{n^2-1}\hat{S}^2_y$$

$$\hat{S}^2_B = \frac{4n^2+6n+3}{n(n^2-1)}\hat{S}^2_y$$

$$\hat{S}^2_{\alpha'} = \frac{12}{n(n^2-1)B^2}\left\{1 + \frac{\alpha'^2}{12}(4n^2+6n+3)-\alpha'(n+1)\right\}\hat{S}^2_y$$

which again may be approximated by

$$\hat{S}^2_{\alpha'} = \frac{12}{n(n^2-1)B^2} \cdot \hat{S}^2_y$$

If the origin of the τ axis is moved to the centre of the observations as Lauf (1967) has done for his derivation, then a comparison is possible for the even values of n, thus if we substitute

$$b = b' + \frac{(n+1)}{2} a$$

and

$$c = c' - \frac{(n+1)}{2} a$$

in equations (5) and (6) we get

for n even

a	b'	c'	Absolute Term			
$\frac{n(n^2-1)}{12}$	$\frac{-n^2}{4}$	$\frac{n}{4}(n+2)$	$\sum_{i=1}^{n} i^0 y^0_i - \sum_{i=1}^{n} i^E y^E_i$	=	0	
$\frac{n}{4}$	$\frac{n}{2}$	0	$-\sum_{i=1}^{n} y^0_i$	=	0	(9)
$\frac{n}{4}$	0	$\frac{n}{2}$	$-\sum_{i=1}^{n} y^E_i$	=	0	

and for n odd

$\underline{a}$	$\underline{b'}$	$\underline{c'}$	Absolute Term		
$\frac{n(n^2-1)}{12}$	$-\left(\frac{n+1}{2}\right)^2$	$\frac{(n^2-1)}{4}$	$\sum_{i=1}^{n} i^O y_i^O - \sum_{i=1}^{n} i^E y_i^E$	$= 0$	
0	$\frac{(n+1)}{2}$	0	$-\sum_{i=1}^{n} y_i^O$	$= 0$	(10)
0	0	$\frac{(n-1)}{2}$	$-\sum_{i=1}^{n} y_i^E$	$= 0$	

The solutions of these equations for "a" and $\frac{b+c}{2}$ will remain unchanged from those given for the solutions of equations (5) and (6). However, simpler expressions can be obtained for the "middle amplitude", B' where

$$B' = \frac{b'-c'}{2}$$

If we subtract the third from the second equation in (9) we obtain

for n even

$$B' = \frac{b'-c'}{2} = \frac{1}{n}\left(\sum_{i=1}^{n} y_i^O - \sum_{i=1}^{n} y_i^E\right) \qquad (11)$$

Also, if we do likewise in (10) we obtain

for n odd

$$B' = \frac{b'-c'}{2} = \frac{1}{n^2-1}\left((n-1)\sum_{i=1}^{n} y_i^O - (n+1)\sum_{i=1}^{n} y_i^E\right) \qquad (12)$$

which may be evaluated very simply by taking half the difference between the means of the odd and even numbered observations

The inverse matrices of (9) and (10) are

for n even

$$\begin{bmatrix} \frac{12}{n(n^2-4)} & \frac{6}{n^2-4} & \frac{-6}{n(n-2)} \\ \frac{-6}{n(n^2-4)} & \frac{2n^2-3n-8}{n(n^2-4)} & \frac{3}{n(n-2)} \\ \frac{-6}{n(n^2-4)} & \frac{-3}{n^2-4} & \frac{2n-1}{n(n-2)} \end{bmatrix}$$

and for n odd

$$\begin{bmatrix} \frac{12}{n(n^2-1)} & \frac{6}{n(n-1)} & \frac{-6}{n(n-1)} \\ 0 & \frac{2}{n+1} & 0 \\ 0 & 0 & \frac{2}{n-1} \end{bmatrix}$$

The estimate of the variance of the adjusted parameter θ_o will remain unchanged. Applying the general law of propagation of variances to B' and α'' where

$$B' = \frac{b'-c'}{2} \qquad \text{and} \qquad \alpha'' = \frac{2a}{b'-c'}$$

we obtain

$$\hat{S}^2_{B'} = \frac{1}{4}\left(Q_{b'b'}-Q_{b'c'}-Q_{c'b'}+Q_{c'c'}\right)\hat{S}^2_y$$

and

$$\hat{S}^2_{\alpha''} = \frac{4}{(b'-c')^4}\left\{(b'-c')^2Q_{aa}+a^2(Q_{b'b'}-Q_{b'c'}-Q_{c'b'}+Q_{c'c'}) + a(b'-c')(Q_{ac'}+Q_{c'a}-Q_{ab'}-Q_{b'a})\right\}\hat{S}^2_y$$

After substitution of the appropriate values from the inverse matrices we have

for n even

$$\hat{S}^2_{\theta_o} = \frac{n^2-1}{n(n^2-4)} \hat{S}^2_y$$

$$\hat{S}^2_{B'} = \frac{1}{n} \hat{S}^2_y$$

$$\hat{S}^2_{\alpha''} = \frac{12}{n(n^2-4)B'^2} \left\{ 1 + \frac{\alpha''^2}{12} (n^2-4) - \frac{\alpha''}{2} (n+1) \right\} \hat{S}^2_y$$

which again may be approximated by

$$\hat{S}^2_{\alpha''} = \frac{12}{n(n^2-4)B'^2} \hat{S}^2_y \qquad (13)$$

and for n odd

$$\hat{S}^2_{\theta_o} = \frac{n}{n^2-1} \hat{S}^2_y$$

$$\hat{S}^2_{B'} = \frac{n}{n^2-1} \hat{S}^2_y$$

$$\hat{S}^2_{\alpha''} = \frac{12}{n(n^2-1)B'^2} \left\{ 1 + \frac{\alpha''^2 n^2}{12} - \frac{\alpha''}{2} (n+1) \right\} \hat{S}^2_y$$

which again may be approximated by

$$\hat{S}^2_{\alpha''} = \frac{12}{n(n^2-1)B'^2} \hat{S}^2_y$$

The values for n even, agree precisely with those obtained by Lauf (1967) after taking into account the difference in notation.

With the exception of a formula for the calculation of "a", simple general expressions for a least squares solution have been derived for any value of n, odd or even. Returning to equations (5) and

(6) or (9) and (10) we can eliminate the unknowns b and c or b' and c' and obtain

for n even

$$a = \frac{12}{n(n^2-4)}\left(\sum_{i=1}^{n} i^E y_i^E - \sum_{i=1}^{n} i^0 y_i^0 + \frac{n}{2}\sum_{i=1}^{n} y_i^0 - \frac{(n+2)}{2}\sum_{i=1}^{n} y_i^E\right)$$

and for n odd (14)

$$a = \frac{12}{n(n^2-1)}\left(\sum_{i=1}^{n} i^E y_i^E - \sum_{i=1}^{n} i^0 y_i^0 + \frac{(n+1)}{2}\left[\sum_{i=1}^{n} y_i^0 - \sum_{i=1}^{n} y_i^E\right]\right)$$

PROPERTIES OF THE SCHULER AND RELATED MEANS.

For the turning or reversal point method with a pendulous gyroscope, it has been customary and convenient to derive the mean direction of the meridian from an approximate relationship given by Professor Max Schuler, namely,

$$S_1 = \frac{y_1+2y_2+y_3}{4}, \quad S_2 = \frac{y_2+2y_3+y_4}{4}, \quad S_3 = \frac{y_3+2y_4+y_5}{4} \quad \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots \quad S_{n-2} = \frac{y_{n-2}+2y_{n-1}+y_n}{4}$$

and

$$\bar{S} = \frac{S_1 + S_2 + S_3 + \ldots\ldots\ldots\ldots S_{n-2}}{n-2}$$

The numerical calculation is simple and in the field the value of $\bar{S}$ can be obtained almost immediately after the completion of the observations. Convenience is not the only characteristic possessed by $\bar{S}$ which has led to its almost universal adoption but that the difference between $\bar{S}$ and the least squares estimate is seldom significantly big. Disadvantages of the technique are that :-

(a) It may not be known if the difference between the least squares estimate and $\bar{S}$ is significant.

(b) Because the original observations are combined, a poor observation or mistake may be hidden in the Schuler Means, thus biasing $\bar{S}$.

(c) A variance estimate based on the individual Schuler Means is invalid if these are treated as independent quantities. See Lauf (1967) for Schuler's original derivation.

(d) Neither the errors of the original observations nor the remaining parameters and their variance estimates are disclosed.

The Schuler Mean is a special case of a series of Means which can be formed from the original observations. The necessary and sufficient number of observations required will be three, producing a minimum of two types of Mean in the series. For each additional observation a further Mean in the series may be found. Only the first three types of Mean will be considered here, which will be designated the First Mean, the Schuler Mean and the Thomas Mean. The third Mean has been named after Dr. T.L. Thomas who suggested its use and that of higher order Means in 1965. He named them the 3 point mean (Schuler Mean), 4 point mean, 5 point mean etc. The First Mean or 2 point mean was not considered. The process is summarised as follows:-

Observations	First Mean	Schuler Mean	Thomas Mean
y	S^1	S^2	S^3
y_1			
	$S^1_1 = \frac{y_1+y_2}{2}$		
		$S^2_1 = \frac{S^1_1+S^1_2}{2} = \frac{y_1+2y_2+y_3}{4}$	
y_2			
	$S^1_2 = \frac{y_2+y_3}{2}$		$S^3_1 = \frac{S^2_1+S^2_2}{2} = \frac{y_1+3y_2+3y_3+y_4}{8}$
		$S^2_2 = \frac{S^1_2+S^1_3}{2} = \frac{y_2+2y_3+y_4}{4}$	
y_3			
	$S^1_3 = \frac{y_3+y_4}{2}$		$S^3_2 = \frac{S^2_2+S^2_3}{2} = \frac{y_2+3y_3+3y_4+y_5}{8}$
		$S^2_3 = \frac{S^1_3+S^1_4}{2} = \frac{y_3+2y_4+y_5}{4}$	
y_4			
	$S^1_4 = \frac{y_4+y_5}{2}$		
y_5			

The coefficients of the original observations in each Mean may be obtained from the Binomial expansion $(a+b)^n$ and the denominator is the sum of these coefficients, according to Thomas (1965). Alternatively these values may be obtained more readily from the Pascal triangle, see Table 2. The denominator is 2^{n-1}.

It is instructive to examine the mathematical reasoning behind this process of taking means. Equation (2) is an approximation of the rigorous expression

$$y_i = B(-1)^{i-1} e^{-\alpha i} + \theta_o - v_i \quad *$$

which in extenso gives :-

$$y_1 = B(1-\alpha + \frac{1}{2!}\alpha^2 - \frac{1}{3!}\alpha^3 + \frac{1}{4!}\alpha^4 \ldots\ldots\ldots\ldots) + \theta_o - v_1$$

$$\vdots$$

$$y_{j-2} = B(-1)^{j-3}\left\{1-(j-2)\alpha + \frac{1}{2!}\left[(j-2)\alpha\right]^2 - \frac{1}{3!}\left[(j-2)\alpha\right]^3 + \frac{1}{4!}\left[(j-2)\alpha\right]^4 \ldots\right\} + \theta_o - v_{j-2}$$

$$y_{j-1} = B(-1)^{j-2}\left\{1-(j-1)\alpha + \frac{1}{2!}\left[(j-1)\alpha\right]^2 - \frac{1}{3!}\left[(j-1)\alpha\right]^3 + \frac{1}{4!}\left[(j-1)\alpha\right]^4 \ldots\right\} + \theta_o - v_{j-1}$$

$$y_j = B(-1)^{j-1}\left\{1-j\alpha + \frac{1}{2!}(j\alpha)^2 - \frac{1}{3!}(j\alpha)^3 + \frac{1}{4!}(j\alpha)^4 \ldots\ldots\ldots\ldots\right\} + \theta_o - v_j$$

$$y_{j+1} = B(-1)^{j}\left\{1-(j+1)\alpha + \frac{1}{2!}\left[(j+1)\alpha\right]^2 - \frac{1}{3!}\left[(j+1)\alpha\right]^3 + \frac{1}{4!}\left[(j+1)\alpha\right]^4 \ldots\right\} + \theta_o - v_{j+1}$$

$$y_{j+2} = B(-1)^{j+1}\left\{1-(j+2)\alpha + \frac{1}{2!}\left[(j+2)\alpha\right]^2 - \frac{1}{3!}\left[(j+2)\alpha\right]^3 + \frac{1}{4!}\left[(j+2)\alpha\right]^4 \ldots\right\} + \theta_o - v_{j+2}$$

$$\vdots$$

* For convenience α instead of α' has been used in the following analysis

$$y_n = B(-1)^{n-1}\left\{1-n\alpha + \frac{1}{2!}(n\alpha)^2 - \frac{1}{3!}(n\alpha)^3 + \frac{1}{4!}(n\alpha)^4 \ldots\ldots\ldots\ldots\ldots\right\} + \theta_o - v_n$$

$$n \geqslant 3 \qquad j \geqslant 3$$

Taking means of consecutive equations gives :-

<u>First Means.</u>

$$S^1_{j-2} = \frac{y_{j-2}+y_{j-1}}{2} = \frac{B(-1)^{j-2}}{2}\left\{-\alpha + \frac{(2j-3)\alpha^2}{2!} - \left\{(j-1)^3-(j-2)^3\right\}\frac{\alpha^3}{3!} + \left\{(j-1)^4-(j-2)^4\right\}\frac{\alpha^4}{4!} \ldots\ldots\right\} + \theta_o - \frac{v_{j-2}+v_{j-1}}{2}$$

$$S^1_{j-1} = \frac{y_{j-1}+y_j}{2} = \frac{B(-1)^{j-1}}{2}\left\{-\alpha + \frac{(2j-1)\alpha^2}{2!} - \left\{j^3-(j-1)^3\right\}\frac{\alpha^3}{3!} + \left\{j^4-(j-1)^4\right\}\frac{\alpha^4}{4!} \ldots\ldots\right\} + \theta_o - \frac{v_{j-1}+v_j}{2}$$

$$S^1_j = \frac{y_j+y_{j+1}}{2} = \frac{B(-1)^j}{2}\left\{-\alpha + \frac{(2j+1)\alpha^2}{2!} - \left\{(j+1)^3-j^3\right\}\frac{\alpha^3}{3!} + \left\{(j+1)^4-j^4\right\}\frac{\alpha^4}{4!} \ldots\ldots\right\} + \theta_o - \frac{v_j+{}_{j+1}}{2}$$

$$S^1_{j+1} = \frac{y_{j+1}+y_{j+2}}{2} = \frac{B(-1)^{j+1}}{2}\left\{-\alpha + \frac{(2j+3)\alpha^2}{2!} - \left\{(j+2)^3+(j+1)^3\right\}\frac{\alpha^3}{3!} + \left\{(j+2)^4-(j+1)^4\right\}\frac{\alpha^4}{4!}\ldots\right\} + \theta_o - \frac{v_{j+1}+v_{j+2}}{2}$$

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Schuler Means

$$S^2_{j-2} = \frac{S^1_{j-2}+S^1_{j-1}}{2} = \frac{B(-1)^{j-1}}{4}\left\{\alpha^2 - \frac{6(j-1)\alpha^3}{3!} + (12j^2-24j+14)\frac{\alpha^4}{4!} \;\ldots\ldots\ldots\right\} + \theta_o - \frac{v_{j-2}+2v_{j-1}+v_j}{4}$$

$$S^2_{j-1} = \frac{S^1_{j-1}+S^1_{j}}{2} = \frac{B(-1)^{j}}{4}\left\{\alpha^2 - \frac{6j\alpha^3}{3!} + (12j^2+2)\frac{\alpha^4}{4!} \;\ldots\ldots\ldots\ldots\ldots\right\} + \theta_o - \frac{v_{j-1}+2v_{j}+v_{j+1}}{4}$$

$$S^2_{j} = \frac{S^1_{j}+S^1_{j+1}}{2} = \frac{B(-1)^{j+1}}{4}\left\{\alpha^2 - \frac{6(j+1)\alpha^3}{3!} + (12j^2+24j+14)\frac{\alpha^4}{4!} \;\ldots\ldots\ldots\right\} + \theta_o - \frac{v_{j}+2v_{j+1}+v_{j+2}}{4}$$

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Thomas Means.

$$S^3_{j-2} = \frac{S^2_{j-2}+S^2_{j-1}}{2} = \frac{B(-1)^{j}}{8}\left\{-\alpha^3+12(2j-1)\frac{\alpha^4}{4!} \;\ldots\ldots\right\} + \theta_o - \frac{v_{j-2}+3v_{j-1}+3v_j+v_{j+1}}{8}$$

$$S^3_{j-1} = \frac{S^2_{j-1}+S^2_{j}}{2} = \frac{B(-1)^{j+1}}{8}\left\{-\alpha^3+12(2j+1)\frac{\alpha^4}{4!} \;\ldots\right\} + \theta_o - \frac{v_{j-1}+3v_{j}+3v_{j+1}+v_{j+2}}{8}$$

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Mr. M. Maughan of the Department of Surveying, University of New South Wales has shown that the series expressions in each Mean may be written concisely as follows :-

First Means $B(-1)^{i-1}e^{-\alpha i}\left\{\frac{1-e^{-\alpha}}{2}\right\}$

Schuler Means $B(-1)^{i-1}e^{-\alpha i}\left\{\frac{1-e^{-\alpha}}{2}\right\}^2$

Thomas Means $B(-1)^{i-1}e^{-\alpha i}\left\{\frac{1-e^{-\alpha}}{2}\right\}^3$

It is apparent from the series expansions that the process of taking successive means eliminates terms containing powers of α i.e. for the First Means terms containing α^{o} are removed, for the Schuler Means terms containing α are removed, for the Thomas Means terms containing α^2 are removed etc. Thus if we consider that a sufficient approximation to the mathematical model is

$$y_i = B(-1)^{i-1} + \theta_o - v_i$$

then we consider that $\alpha = 0$ and the First Means will contain only θ_o and combinations of the residuals v. For the Schuler Means the model is

$$y_i = B(-1)^{i-1}(1-i\alpha) + \theta_o - v_i \quad \text{etc.}$$

Thus if we accept the average (or some other combination) of the First Means or the average of the Schuler Means etc. as an estimate of θ_o then it is presumed that the quantities

$$B(-1)^{i-1}\, e^{-\alpha i}\left\{\frac{1-e^{-\alpha}}{2}\right\}$$

or

$$B(-1)^{i-1}\, e^{-\alpha i}\left\{\frac{1-e^{-\alpha}}{2}\right\}^2 \qquad \text{etc.}$$

are not numerically significant. This is demonstrated graphically in Fig. 2 for the First and Schuler Means.

An alternative explanation of this process is that if the First Mean is considered sufficient, then damping is ignored and the envelope of the turning points is composed of two parallel straight lines, for the Schuler Mean equally inclined straight lines, for the Thomas Mean quadratic curves and so on. Each of which, in their particular case are considered to be a sufficient approximation to an exponential envelope.

Whichever Mean is considered appropriate to the observations, there remains a serious objection to the second stage of taking an average. Taking the average is sound if we are considering original independent observations but not if the quantities to be adjusted are functions of these observations. Although the average in this case is inadmissable for a least squares adjustment, we may still use the derived quantities in an adjustment process provided that the mathematical correlation is taken into account.

Let the parametric equations under consideration be of the form

$$
\begin{array}{ccccc}
\theta_o & - & S_1 & = & V_1 \\
\theta_o & - & S_2 & = & V_2 \\
\theta_o & - & S_3 & = & V_3 \\
\cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot \\
\cdot & & \cdot & & \cdot \\
\theta_o & - & S_m & = & V_m
\end{array}
$$

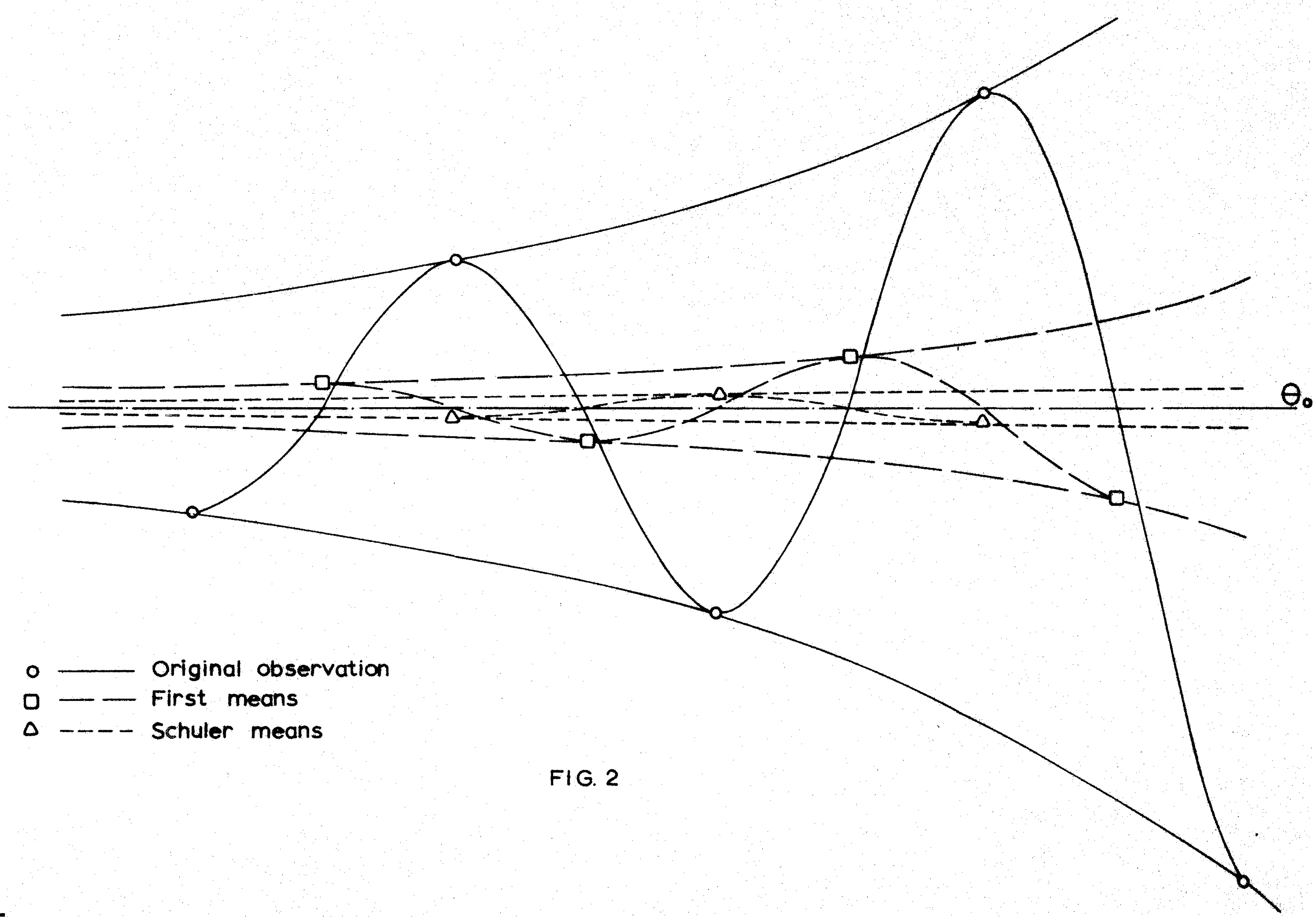


FIG. 2

where for the First Mean $m = n-1$

Schuler Mean $m = n-2$

Thomas Mean $m = n-3$ etc.

or in matrix notation

$$A\,\theta_o - S = V$$

where A is a unit column vector of dimension m, S is the column vector of the individual Means

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \cdot \\ \cdot \\ \cdot \\ S_m \end{bmatrix}$$

V is the column vector of the corrections

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \cdot \\ \cdot \\ \cdot \\ V_m \end{bmatrix}$$

and θ_o is 1 x 1 matrix

The coefficients in the weight coefficient matrix of the individual Means may be found by applying the general law of propagation of variances to

$$S^1_1 = \frac{y_1+y_2}{2}$$

$$S^1_2 = \frac{y_2+y_3}{2}$$

$$S^1_3 = \frac{y_3+y_4}{2} \qquad \text{etc. for the First Mean,}$$

and obtaining

$$g_{S^1_1} = \frac{1}{2} g_{y_1} + \frac{1}{2} g_{y_2}$$

$$g_{S^1_2} = \frac{1}{2} g_{y_2} + \frac{1}{2} g_{y_3}$$

$$g_{S^1_3} = \frac{1}{2} g_{y_3} + \frac{1}{2} g_{y_4} \qquad \text{etc.}$$

and

$$g_{S^1_1 S^1_1} = \frac{1}{4} g_{y_1 y_1} + \frac{1}{4} g_{y_2 y_2} = \frac{1}{2} g_{yy}$$

$$g_{S^1_1 S^1_2} = \frac{1}{4} g_{y_2 y_2} = \frac{1}{4} g_{yy} \qquad \text{etc.}$$

assuming that each observation is independent and of equal variance. Thus the required weight matrix is

$$G^1 = \frac{1}{4} \begin{bmatrix} 2 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ & 1 & 2 & 1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 2 \end{bmatrix}$$

Applying the same technique to the Schuler and Thomas Means gives

$$G^2 = \frac{1}{16}\begin{bmatrix}
6 & 4 & 1 & & & & & & & & \\
4 & 6 & 4 & 1 & & & & & & & \\
1 & 4 & 6 & 4 & 1 & & & & & & \\
 & 1 & 4 & 6 & 4 & 1 & & & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\
 & & & & & 1 & 4 & 6 & 4 & 1 & \\
 & & & & & & 1 & 4 & 6 & 4 & 1 \\
 & & & & & & & 1 & 4 & 6 & 4 \\
 & & & & & & & & 1 & 4 & 6
\end{bmatrix}$$

and for G^3

$$\frac{1}{64}\begin{bmatrix}
20 & 15 & 6 & 1 & & & & & & & & & \\
15 & 20 & 15 & 6 & 1 & & & & & & & & \\
6 & 15 & 20 & 15 & 6 & 1 & & & & & & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & & & & & & \\
 & 1 & 6 & 15 & 20 & 15 & 6 & 1 & & & & & \\
 & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
 & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\
 & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\
 & & & & & 1 & 6 & 15 & 20 & 15 & 6 & 1 & \\
 & & & & & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\
 & & & & & & & 1 & 6 & 15 & 20 & 15 & 6 \\
 & & & & & & & & 1 & 6 & 15 & 20 & 15 \\
 & & & & & & & & & 1 & 6 & 15 & 20
\end{bmatrix}$$ respectively.

The coefficients of these matrices may be conveniently obtained from the Pascal Triangle. The single normal equation is

$$A^T G^{-1} A \, \theta_o + A^T G^{-1} S \quad = \quad 0$$

and its solution

$$\theta_o \quad = \quad -(A^T G^{-1} A)^{-1} \, A^T G^{-1} S$$

This last equation is of little practical value unless general expressions can be found for the coefficients of the individual Means for any number of observations. An attempt was made to solve this problem by inverting the G matrices by pivotal condensation and bordering methods. This proved to be uneconomical by hand for the Schuler and Thomas Means when the number of observations was large. The writer is grateful for the assistance of Mr. M. Maughan who modified an existing IBM 360 structural engineering computer programme for matrix inversion to suit the requirements of these matrices. The inverse matrices for n up to 8 are shown in Table 4. From the computer output general expressions have been derived using the technique of divided differences.

<u>First Mean</u>.

n even

$$\theta_o \quad = \quad \frac{2}{n}\,(S^1_1 + S^1_3 + S^1_5 + S^1_7 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

or

$$\theta_o \quad = \quad \frac{2}{n} \sum_{i=1}^{\frac{1}{2}n} S^1_{2i-1}$$

n odd

$$\theta_o = \frac{4}{n^2-1}\left\{\frac{(n-1)}{2}S^1_1 + S^1_2 + \frac{(n-3)}{2}S^1_3 + 2S^1_4 + \frac{(n-5)}{2}S^1_5 + 3S^1_6 \ldots\ldots\right\}$$

or

$$\theta_o = \frac{4}{n^2-1}\sum_{i=1}^{\frac{1}{2}(n-1)}\left\{\frac{(n-2i+1)}{2}S^1_{2i-1} + iS^1_{2i}\right\}$$

<u>Schuler Mean.</u>

n even

$$\theta_o = \frac{4}{n(n^2-4)}\left\{\begin{array}{ll}(n-2)(n-1)\,S^2_1 & -\ (n-2)(n-7)\,S^2_2 \\ +2(n-4)(n-5)\,S^2_3 & -\ 2(n-4)(n-11)S^2_4 \\ +3(n-6)(n-9)\,S^2_5 & -\ 3(n-6)(n-15)S^2_6 \\ +4(n-8)(n-13)S^2_7 & -\ 4(n-8)(n-19)S^2_8 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}\right\}$$

or

$$\theta_o = \frac{4}{n(n^2-4)}\sum_{i=1}^{\frac{1}{2}(n-2)} i(n-2i)\left\{(n-4i+3)S^2_{2i-1} - (n-4i-3)S^2_{2i}\right\}$$

n odd

$$\theta_o = \frac{4}{(n^2-1)}\left\{\begin{array}{ll}(n-1)\,S^2_1 & -\ (n-3)\,S^2_2 \\ +2(n-3)\,S^2_3 & -\ 2(n-5)\,S^2_4 \\ +3(n-5)\,S^2_5 & -\ 3(n-7)\,S^2_6 \\ +4(n-7)\,S^2_7 & -\ 4(n-9)\,S^2_8 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}\right\}$$

or

$$\theta_o = \frac{4}{(n^2-1)} \left[(n-1)S_1^2 + \sum_{i=1}^{\frac{1}{2}(n-3)} (n-2i-1) \left\{ -iS_{2i}^2 + (i+1)S_{2i+1}^2 \right\} \right]$$

Thomas Mean

n even

$$\theta_o = \frac{8}{n(n^2-4)} \left\{ \begin{array}{lll} (n-2)(n-1)\, S_1^3 & - & 2(n-2)(n-4)\, S_2^3 \\ +2(n-4)(2n-7)\, S_3^3 & - & 6(n-4)(n-6)\, S_4^3 \\ +3(n-6)(3n-17)S_5^3 & - & 12(n-6)(n-8)\, S_6^3 \\ +4(n-8)(4n-31)S_7^3 & - & 20(n-8)(n-10)S_8^3 \\ \vdots & & \vdots \end{array} \right\}$$

or

$$\theta_o = \frac{8}{n(n^2-4)} \left[(n-2)(n-1)S_1^3 + \sum_{i=1}^{\frac{1}{2}(n-4)} \left\{ -i(i+1)(n-2i)(n-2i-2)S_{2i}^3 + (i+1)(n-2i-2) \left\{ (i+1)n-2i^2-4i-1 \right\} S_{2i+1}^3 \right\} \right]$$

n odd

$$\theta_o = \frac{8}{(n^2-1)(n^2-9)} \left\{ \begin{array}{lll} (n-3)(n^2-3n+2)\, S_1^3 & - & 2(n-3)(n^2-9n+14)\, S_2^3 \\ +2(n-5)(2n^2-20n+42)\, S_3^3 & - & 6(n-5)(n^2-15n+46)S_4^3 \\ +3(n-7)(3n^2-49n+170)S_5^3 & - & 12(n-7)(n^2-21n+94)S_6^3 \\ +4(n-9)(4n^2-90n+434)S_7^3 & - & 20(n-9)(n^2-27n+158)S_8^3 \\ +5(n-11)(5n^2-143n+882)S_9^3 & - & 30(n-11)(n^2-33n+238)S_{10}^3 \\ \vdots & & \vdots \end{array} \right\}$$

or

$$\theta_o = \frac{8}{(n^2-1)(n^2-9)} \sum_{i=1}^{\frac{1}{2}(n-3)} \left\{ i(n-2i-1) \left\{ in^2-(3i-2)(2i+1)n+2(2i-1)(2i^2-1) \right\} S^3_{2i-1} - i(i+1)(n-2i-1) \left\{ n^2-3(2i+1)n+2(4i^2+4i-1) \right\} S^3_{2i} \right\}$$

To facilitate numerical calculation the values of the coefficients up to $n = 8$ are given in Table 3.

The estimate of the variance of the adjusted parameter θ_o is given by

$$\hat{S}^2_{\theta_o} = Q_{\theta_o\theta_o} \frac{V^T G^{-1} V}{r}$$

where $Q_{\theta_o\theta_o} = (A^T G^{-1} A)^{-1}$, which is tabulated below, and r is the number of redundancies.

$Q_{\theta_o\theta_o}$

	n odd	n even
First Mean	$\frac{n}{n^2-1}$	$\frac{1}{n}$
Schuler Mean	$\frac{n}{n^2-1}$	$\frac{n^2-1}{n(n^2-4)}$
Thomas Mean	$\frac{n(n^2-4)}{(n^2-1)(n^2-9)}$	$\frac{n^2-1}{n(n^2-4)}$

The process outlined does not overcome all the disadvantages of taking the average of the derived Means,as stated before, but it does remove the principal objection in that it gives the least squaresestimate of θ_o. The method of obtaining this value is extremely simple and is preferred to any previous method. It may be noted that for an even number of observations, the solutions by the Schuler and Thomas Mean are identical and for an odd number of observations the First and Schuler Mean are identical. A suggested technique of calculation which combines the characteristics of the Linear Damped Model and the Schuler Mean has been used in the two numerical examples.

NUMERICAL CALCULATIONS.

The two most important requirements for practical calculations are that :

(1) Mistakes may be readily detected at the time of observation. If any extra observations are then required they may be done with the minimum of inconvenience.

(2) Calculation methods should be compatible with the expected precision of the determination.

For both of these requirements, consideration must also be given to the conditions under which these calculations are performed. In the field, quick and simple methods are preferred and in most cases approximate methods will suffice. However in the office the second requirement above is of paramount importance. Furthermore the choice of method may be governed by the calculation aids. For the high speed digital computer it is immaterial which of the variety of rigorous expressions for the required unknowns is used. However, for a desk machine those expressions which offer speed and simplicity are preferred.

Suggested procedures are as follows:-

(a) Field Calculation.

(1) Calculate Schuler Means progressively as the observations are being made and plot the turning points and Schuler Means at a suitable scale on graph paper. This is a recommended procedure by the Wild Instrument Company in their handbook for the GAKI and is called the "oscillation graph."

(2) Calculate θ_o either approximately from the mean of the Schuler Means or rigorously from the weighted mean of the Schuler Means.

(3) Mistakes or large errors in the original observations may be concealed by this process of taking Schuler Means and may not be readily discernible in the oscillation graph. As stated before, the geometrical interpretation of the least squares adjustment is a pair of straight lines at the same inclination to the line of mean oscillation passing through the observations. As the position of the line of mean oscillation is known, it is a simple matter to transfe either the left or the right observations over this line and then construct a line of best fit through all of these points. This transfer may be done either by (i) folding the graph paper along the line of mean oscillation or (ii) subtracting either the left or right observations from $2\theta_o$ and replotting.

(4) Draw a line of best fit through these points. The deviations from this line will give a good estimate of the errors of the original observations. Instead of estimating the line of best fit by eye, a procedure proposed by Eddington given by Jeffreys (1948) is eminently suitable. Rainsford (1957) describes the method as follows:-

"Divide the data into 3 equal groups : the line joining the mean positions of the first and last groups gives the slope of the line, which is then fixed in position by making it pass through the mean position of all observations. Not only is this solution very simple but, provided that the observations are uniformly spaced, its efficiency is of the order of 8/9 of that of a rigorous least squares solution."

Office Calculation.

(1) Calculate θ_o from the weighted mean of the Schuler Means.

(2) Calculate, for n even, from equation (7)

$$a = 2\left(\frac{1}{n}\sum_{i=1}^{n} y_i - \theta_o\right)$$

and from equation (11)

$$B' = \frac{1}{n}\left(\sum_{i=1}^{n} y_i^O - \sum_{i=1}^{n} y_i^E\right)$$

For n odd, after re-arranging equation (14)

$$a = \frac{12}{n(n^2-1)}\left\{\frac{n-1}{2}(y_1-y_n) - \frac{n-3}{2}(y_2-y_{n-1}) + \frac{n-5}{2}(y_3-y_{n-2}) \;\ldots\ldots \ldots\ldots\ldots\ldots (-1)^{\frac{1}{2}(n+1)}\left(y_{\frac{1}{2}(n-1)}-y_{\frac{1}{2}(n+3)}\right)\right\}$$

and from equation (12)

$$B' = \frac{1}{2}\left\{\frac{\sum_{i=1}^{n} y_i^O}{\frac{1}{2}(n+1)} - \frac{\sum_{i=1}^{n} y_i^E}{\frac{1}{2}(n-1)}\right\}$$

(3) Calculate the variances according to equations (13)

Example A, n even.

Observations

Left	Right	S.M. $359°59'$	$SM - \overline{SM}$ V
358° 24' 18"			
(358 24 45)*	1° 33' 36"	10".5	-2".0
358 25 12	(1 33 15)	13.5	+1.0
(358 25 24)	1 32 54	9.0	-3.5
358 25 36	(1 32 48)	12.0	-0.5
(358 25 54)	1 32 42	18.0	+5.5
358 26 12	(1 32 12)	12.0	-0.5
	1 31 42		
	$\overline{SM}$	12.5	$\sum = 0$

$$\theta_o = 359°\ 59'\ 12''.5 + \frac{7(-2.0 - 0.5) - (1.0 + 5.5) + 4(-3.5 - 0.5)}{20}$$

$$\theta_o = 359°\ 59'\ 12''.5 - 2''.0 = \underline{\underline{359°\ 59'\ 10''.5}}$$

$$\sum y^O = 4(358°\ 20') + 21'\ 18'' \qquad \sum y^E = 4(1°\ 30') + 10'\ 54''$$

$$a = 2\left[\frac{1}{8}\left\{4(359°\ 50') + 32'\ 12''\right\} - 359°\ 59'\ 10''.5\right]$$

$$\underline{\underline{a = -18''}}$$

---

* The figures in brackets are the intermediate steps in the calculation of the Schuler Means.

$$B' = \frac{1}{8}\left[4(358^\circ\ 20') + 21'\ 18'' - 4(1^\circ\ 30') - 10'\ 54''\right]$$

$$\underline{\underline{B' = 358^\circ\ 26'\ 18''}}$$

$$\alpha'' = \frac{a}{B'} = \frac{-18}{-5622} = \underline{\underline{3.20 \times 10^{-3}}}$$

$$B' + \theta_o = 358^\circ\ 25'\ 28''.5$$

$$-B' + \theta_o = 1^\circ\ 32'\ 52''.5$$

$$v_1 = B' + \theta_o - y_1 + \frac{7}{2}a = \begin{array}{rrr} 358^\circ & 25' & 28''.5 \\ -358 & 24 & 18.0 \\ - & 1 & 03.0 \\ \hline & v_1 + & 7.5 \end{array}$$

$$v_2 = -B' + \theta_o - y_2 - \frac{5}{2}a = \begin{array}{rrr} 1 & 32 & 52.5 \\ -1 & 33 & 36.0 \\ + & & 45.0 \\ \hline & v_2 + & 1.5 \end{array}$$

$$v_3 = B' + \theta_o - y_3 + \frac{3}{2}a = \begin{array}{rrr} 358 & 25 & 28.5 \\ -358 & 25 & 12.0 \\ & - & 27.0 \\ \hline & v_3 - & 10.5 \end{array}$$

$$v_4 = -B' + \theta_o - y_4 - \frac{1}{2}a = \begin{array}{rrr} 1 & 32 & 52.5 \\ -1 & 32 & 54.0 \\ & + & 9.0 \\ \hline & v_4 + & 7.5 \end{array}$$

$$v_5 = B' + \theta_o - y_5 - \frac{1}{2}a = \begin{array}{r} 358^\circ\ 25'\ 28''.5 \\ -358\ \ 25\ \ 36.0 \\ +\ \ 9.0 \\ \hline v_5 + \ 1.5 \end{array}$$

$$v_6 = -B' + \theta_o - y_6 + \frac{3}{2}a = \begin{array}{r} 1\ \ 32\ \ 52.5 \\ -1\ \ 32\ \ 42.0 \\ -\ 27.0 \\ \hline v_6 - 16.5 \end{array}$$

$$v_7 = B' + \theta_o - y_7 - \frac{5}{2}a = \begin{array}{r} 358\ \ 25\ \ 28.5 \\ -358\ \ 26\ \ 12.0 \\ +\ 45.0 \\ \hline v_7 + \ 1.5 \end{array}$$

$$v_8 = -B' + \theta_o - y_8 + \frac{7}{2}a = \begin{array}{r} 1\ \ 32\ \ 52.5 \\ -1\ \ 31\ \ 42.0 \\ -\ 1\ \ 03.0 \\ \hline v_8 + \ 7.5 \end{array}$$

Check $\Sigma v^0 = \Sigma v^E = 0$

$\Sigma v^2 = 558.00$

$$\hat{S}^2_y = \frac{558}{5} = 111.6$$

$$\hat{S}^2_{\theta_o} = \frac{21}{160} \cdot \frac{558}{5} = 14.65$$

$$\hat{S}^2_{B'} = \frac{1}{8} \cdot \frac{558}{5} = 13.95$$

$$\hat{S}^2_{\alpha''} = \frac{1}{40(-5622)^2} \cdot \frac{558}{5} = 8.8272 \times 10^{-8}$$

SUMMARY.

$n = 8, \quad r = 5$	$\hat{S}_y$	$= \pm\ 10''.6$
$\theta_o = 359^\circ\ 59'\ 10''.5$	$\hat{S}_{\theta_o}$	$= \pm\ 3.8$
$B' = 358^\circ\ 26'\ 18''$	$\hat{S}_{B'}$	$= \pm\ 3.7$
$\alpha'' = 3.20 \times 10^{-3}$	$\hat{S}_{\alpha''}$	$= \pm\ 0.30 \times 10^{-3}$

Graphical Solution.

See previous calculation for Schuler Means and θ_o

Observations: Left	Observations: Right	Transferred Left ($2\theta_o$ - Left)	Graphical Values (see plot)	v.
358°24'18"		1°34'03"	1°33'52"	+ 11"
	1°33'36"		1 33 35	- 1
358 25 12		1 33 09	1 33 18	- 9
	1 32 54		1 33 01	+ 7
358 25 36		1 32 45	1 32 44	+ 1
	1 32 42		1 32 27	- 15
358 26 12		1 32 09	1 32 10	- 1
	1 31 42		1 31 53	+ 11

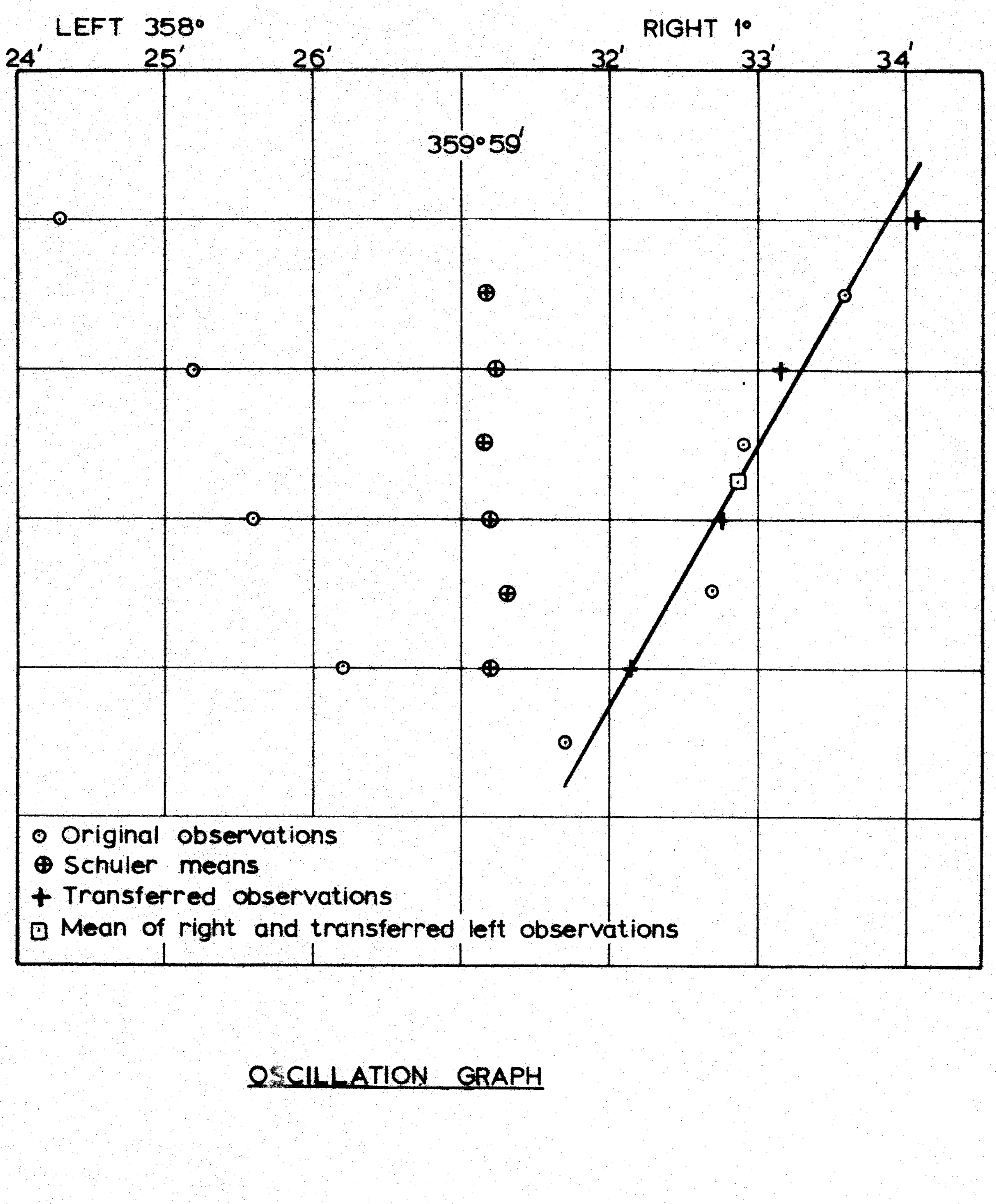
$2\theta_o = 359^\circ\ 58'\ 21''$

Mean of the right and transferred left observations 1° 32' 52".5

Mean of the first three observations 1 33 36

Mean of the last three observations 1 32 11

$$\text{Slope } a = \frac{1^\circ\ 33'\ 36'' - 1^\circ\ 32'\ 11''}{5} = 17'' \text{ per half period.}$$



OSCILLATION GRAPH

Example B, n odd

Observations

Left	Right	SM 359°57'	SM−$\overline{SM}$ V
	1° 11' 18"		
358° 44' 42"	(1 11 06)	54".0	− 7".8
(358 45 09)	1 10 54	61.5	− 0.3
358 45 36	(1 10 36)	66.0	+ 4.2
(358 45 51)	1 10 18	64.5	+ 2.7
358 46 06	(1 10 00)	63.0	+ 1.2
	1 09 42		
	$\overline{SM}$	61.8	$\sum = 0$

$$\theta_o = 359^\circ\ 58'\ 01''.8 + \frac{3(-7.8 + 1.2) - 2(-0.3 + 2.7) + 4 \times 4.2}{6}$$

$$\theta_o = 359^\circ\ 58'\ 01''.8 - 1''.3 = \underline{\underline{359^\circ\ 58'\ 00''.5}}$$

$$\sum y^E = 3(358^\circ\ 44') + 4'\ 24'' \qquad \sum y^O = 4(1^\circ\ 09') + 6'\ 12''$$

$$B' = \frac{1}{2}\left\{\frac{4(1^\circ\ 09') + 6'\ 12''}{4} - \frac{3(358^\circ\ 44') + 4'\ 24''}{3}\right\}$$

$$B' = \frac{1}{2}(1^\circ\ 10'\ 33'' - 358^\circ\ 45'\ 28'') = \underline{\underline{1^\circ\ 12'\ 32''.5}}$$

$$\text{Note } \theta_o = \frac{358^\circ\ 45'\ 28'' + 1^\circ\ 10'\ 33''}{2} = \underline{\underline{359^\circ\ 58'\ 00''.5}}$$

$$a = \frac{1}{28}(3 \times 1'\ 36'' + 2 \times 1'\ 24'' + 36'')$$

$$a = \underline{\underline{17''.57}}$$

$$\alpha'' = \frac{a}{B'} = \frac{17.57}{4352.5} = \underline{\underline{4.04 \times 10^{-3}}}$$

$$B' + \theta_o = 1^\circ\ 10'\ 33''$$

$$-B' + \theta_o = 358^\circ\ 45'\ 28''$$

$$v_1 = B' + \theta_o + 3a - y_1 =$$

1°	10'	33"
-1	11	18
	+	52.71
v_1	+	7.71

$$v_2 = -B' + \theta_o - 2a - y_2 =$$

358	45	28
-358	44	42
	-	35.14
v_2	+	10.86

$$v_3 = B' + \theta_o + a - y_3 =$$

1	10	33
-1	10	54
	+	17.57
v_3	-	3.43

$$v_4 = -B' + \theta_o - y_4 =$$

358	45	28
-358	45	36
v_4	-	8.00

$$v_5 = B' + \theta_o - a - y_5 =$$

1	10	33
-1	10	18
	-	17.57
v_5	-	2.57

$$v_6 = -B' + \theta_o + 2a - y_6 =$$

358	45	28
-358	46	06
	+	35.14
v_6	-	2.86

$$v_7 = B' + \theta_o - 3a - y_7 =$$

1	10	33
-1	09	42
	-	52.72
v_7	-	1.72

Check $\Sigma v^O = \Sigma v^E = 0$

$$\Sigma v^2 = 270.89$$

$$\hat{S}_y^2 = \frac{270.89}{4} = 67.72$$

$$\hat{S}_B^2 = \hat{S}_{\theta_o}^2 = \frac{7}{48} \cdot \frac{270.89}{4} = 9.88$$

$$\hat{S}_{\alpha''}^2 = \frac{1}{28(4352.5)^2} \cdot \frac{270.89}{4} = 0.1277 \times 10^{-6}$$

SUMMARY.

n	=	7, r = 4	$\hat{S}_y$	=	± 8".2
θ_o	=	359° 58' 00".5	$\hat{S}_{\theta_o}$	=	± 3.1
B'	=	1° 12' 32".5	$\hat{S}_{B'}$	=	± 3.1
α''	=	4.04×10^{-3}	$\hat{S}_{\alpha''}$	=	$\pm\ 0.36 \times 10^{-3}$

Graphical Solution.

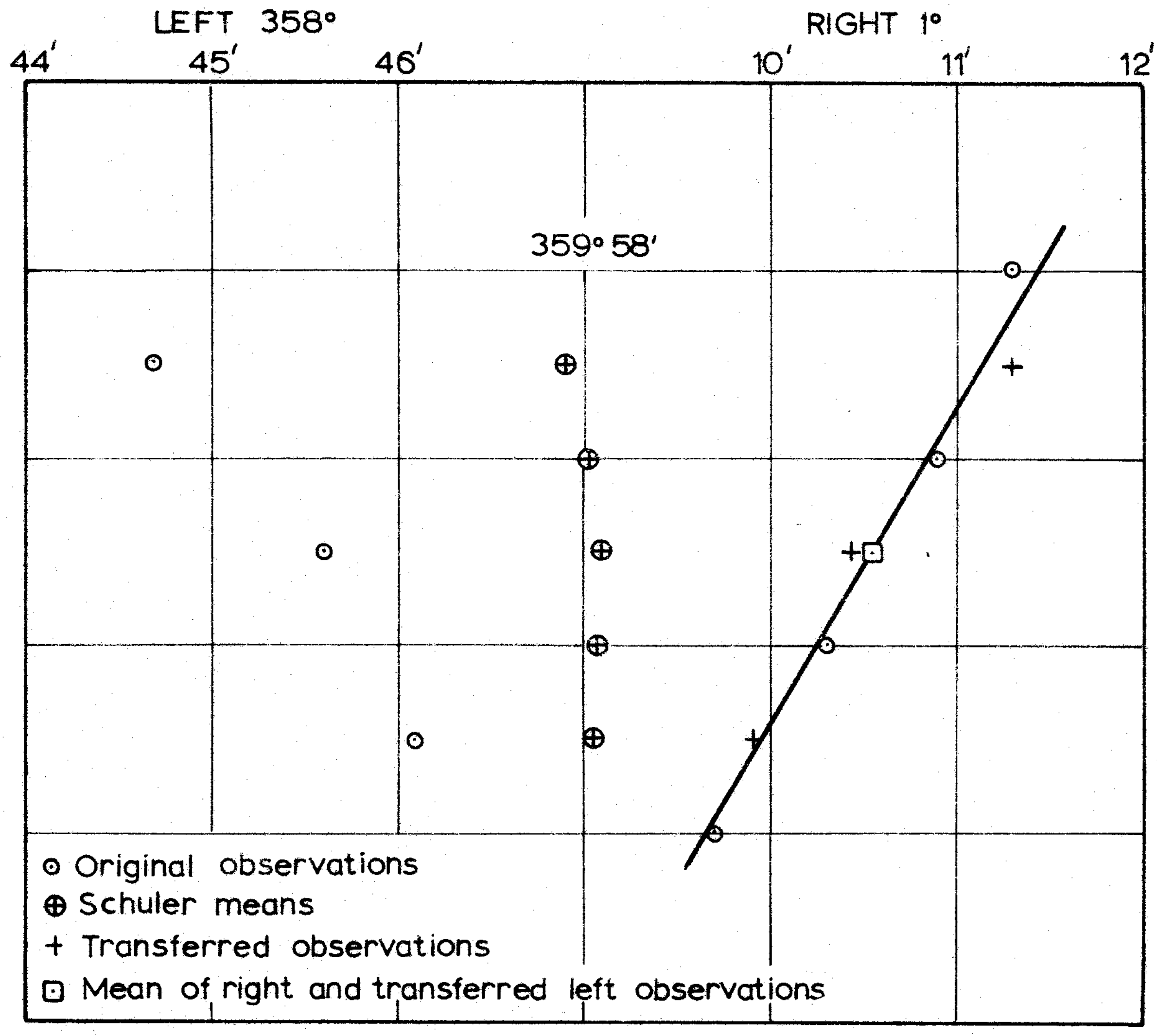
See previous calculations for Schuler Means and θ_o

Observations Left	Observations Right	Transferred Left ($2\theta_o$ - Left)	Graphical Values (See plot)	v
	1° 11' 18"		1° 11' 27"	+ 9
358° 44' 42"		1° 11' 19"	1 11 09	+ 10
	1 10 54		1 10 51	- 3
358 45 36		1 10 25	1 10 33	- 10
	1 10 18		1 10 15	- 3
358 46 06		1 09 55	1 09 57	2
	1 09 42		1 09 39	- 3

$2\theta_o$ = 359° 56" 01"

Mean of the right and transferred left observations	1° 10' 33"
Mean of the first two observations	1 11 18
Mean of the last two observations	1 09 48

$$\text{Slope} \quad a \quad = \quad \frac{1^\circ\ 11'\ 18'' \ - \ 1^\circ\ 09'\ 48''}{5} \quad = \quad 18'' \text{ per half period.}$$



OSCILLATION GRAPH

CONCLUSION.

Lauf (1967) has summarised the methods used by other writers in the solution of this problem and it is of interest to re-examine these in the light of this investigation. Kohlrausch (1955) advocates the use of an odd number of turning points and then combines the mean of the readings on one side with the mean of the readings on the other side to obtain a final mean. This approach is quite sound theoretically for a lightly damped oscillation as has been shown and it is hard to understand why Schuler (1932) criticises Kohlrausch. Schuler is also critical of Basch and Wilski (1917). This method is based on finding a line of best fit to observations on each side and then to combine them to give a final value. This technique is basically sound if we are considering the linear damped case. However, an extra condition must be added to the adjustment because the slope of the damping envelope must be the same on either side of the line of mean oscillation. The approach of Thomas (1965, 1967) is of interest because it attempts to take into account a heavier damping, which seldom prevails with gyro-theodolite observations. The Schuler and Thomas Means are generally quite sufficient for this work and there is no necessity to continue taking higher order Means.

Consideration must be given to the question of the number of observations required for the determination. If the internal precision of the observations is compatible with the external precision then the number of observations required will depend upon the accuracy required for the determination which can be estimated a priori from variance estimates based on past experience. From the calculation and adjustment standpoint it is immaterial whether we observe an odd or an even number although with an even number of observations the weighted Schuler Mean will give the least squares solution for quadratic damping.

In conclusion it can be stated that the derivations contained in this report are of a general nature and do not presume a limit or set values for the number of observations. A combination of the characteristics of each method gives a simple solution for numerical problems which is in sympathy with the observations. By application of the general law of propagation of variances it has been shown that we can still take advantage of simple relationships which are similar to those which have been proposed by Schuler whilst retaining the benefit of rigour in the least squares adjustment.

TABLE 1. SOLUTIONS OF NORMAL EQUATIONS FOR LINEAR DAMPING.

n	a									b								
		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
3	$\frac{1}{2}$ (	1		-1					)	$\frac{1}{2}$ (	3		-1					
4	$\frac{1}{4}$ (	1	-1	-1	1				)	$\frac{1}{2}$ (	2	-1		1				)
5	$\frac{1}{10}$ (	2	-1		1	-2			)	$\frac{1}{30}$ (	28	-9	10	9	-8			)
6	$\frac{1}{8}$ (	1	-1			-1	1		)	$\frac{1}{24}$ (	17	-9	8		-1	9		)
7	$\frac{1}{28}$ (	3	-2	1		-1	2	-3	)	$\frac{1}{28}$ (	19	-8	11		3	8	-5	)
8	$\frac{1}{40}$ (	3	-3	1	-1	-1	1	-3	3)	$\frac{1}{20}$ (	11	-6	7	-2	3	2	-1	6)

n	c									$\theta_o = \frac{b+c}{2}$								
		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
3	(	-1	1	1					)	$\frac{1}{4}$ (	1	2	1					)
4	$\frac{1}{4}$ (	-3	5	3	-1				)	$\frac{1}{8}$ (	1	3	3	1				)
5	$\frac{1}{5}$ (	-3	[illegible]		1	3			)	$\frac{1}{12}$ (	2	3	2	3	2			)
6	$\frac{1}{6}$ (	-3	5		2	3	-1		)	$\frac{1}{48}$ (	5	11	8	8	11	5		
7	$\frac{1}{21}$ (	-9	13	-3	7	3	1	9	)	$\frac{1}{4}$ (	3	[illegible]	3	4	3	4	3	)
8	$\frac{1}{8}$ (	-3	5	-1	3	1	1	3	-1)	$\frac{1}{80}$ (	7	13	9	11	11	9	13	7)

TABLE 1. (Contd.)

n	$B = \frac{b-c}{2}$								
		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
3	$\frac{1}{4}$ (	5	-2	-3					)
4	$\frac{1}{8}$ (	7	-7	-3	3				)
5	$\frac{1}{60}$ (	46	-33	10	3	-26			)
6	$\frac{1}{48}$ (	29	-29	8	-8	-13	13		)
7	$\frac{1}{168}$ (	93	-76	45	-28	-3	20	-51	)
8	$\frac{1}{80}$ (	37	-37	19	-19	1	-1	-17	17)

TABLE 2. THE PASCAL TRIANGLE.

n		Coefficients in Mean	Coefficients of Weight Matrix	Denominator 2^{n-1}
1	1			1
2	1 1	First		2
3	1 2 1	Schuler	First	4
4	1 3 3 1	Thomas		8
5	1 4 (6) 4 1		Schuler	16
6	1 5 10 10 5 1			32
7	1 6 15 (20) 15 6 1		Thomas	64
8	1 7 21 35 35 21 7 1			128
9	1 8 28 56 (70) 56 28 8 1			256
10	1 9 36 84 126 126 84 36 9 1			512
11	1 10 45 120 210 (252) 210 120 45 10 1			1024
12	1 11 55 165 330 462 462 330 165 55 11 1			2048
13	1 12 66 220 495 792 (924) 792 495 220 66 12 1			4096

The first term of the weight matrix circled.

TABLE 3. COEFFICIENTS OF THE MEANS.

n	First Mean									Schuler Mean								Thomas Mean						
		S^1_1	S^1_2	S^1_3	S^1_4	S^1_5	S^1_6	S^1_7			S^2_1	S^2_2	S^2_3	S^2_4	S^2_5	S^2_6			S^3_1	S^3_2	S^3_3	S^3_4	S^3_5	
3	$\frac{1}{2}$ (	1	1						)	(	1						)							
4	$\frac{1}{2}$ (	1		1					)	$\frac{1}{2}$ (	1	1					)	(	1					)
5	$\frac{1}{6}$ (	2	1	1	2				)	$\frac{1}{3}$ (	2	-1	2				)	$\frac{1}{2}$ (	1	1				)
6	$\frac{1}{3}$ (	1		1		1			)	$\frac{1}{12}$ (	5	1	1	5			)	$\frac{1}{6}$ (	5	-4	5			)
7	$\frac{1}{12}$ (	3	1	2	2	1	3		)	$\frac{1}{6}$ (	3	-2	4	-2	3		)	$\frac{1}{2}$ (	1			1		)
8	$\frac{1}{4}$ (	1		1		1		1	)	$\frac{1}{20}$ (	7	-1	4	4	-1	7	)	$\frac{1}{10}$ (	7	-8	12	-8	7	)

TABLE 4. INVERSE MATRICES $G^{1^{-1}}$, $G^{2^{-1}}$, $G^{3^{-1}}$, For $n \leqslant 8$.

$G^{1^{-1}}$

n = 3
$$\frac{4}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

n = 4
$$\frac{4}{4}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

n = 5
$$\frac{4}{5}\begin{bmatrix} 4 & -3 & 2 & -1 \\ -3 & 6 & -4 & 2 \\ 2 & -4 & 6 & -3 \\ -1 & 2 & -3 & 4 \end{bmatrix}$$

n = 6
$$\frac{4}{6}\begin{bmatrix} 5 & -4 & 3 & -2 & 1 \\ -4 & 8 & -6 & 4 & -2 \\ 3 & -6 & 9 & -6 & 3 \\ -2 & 4 & -6 & 8 & -4 \\ 1 & -2 & 3 & -4 & 5 \end{bmatrix}$$

n = 7
$$\frac{4}{7}\begin{bmatrix} 6 & -5 & 4 & -3 & 2 & -1 \\ -5 & 10 & -8 & 6 & -4 & 2 \\ 4 & -8 & 12 & -9 & 6 & -3 \\ -3 & 6 & -9 & 12 & -8 & 4 \\ 2 & -4 & 6 & -8 & 10 & -5 \\ -1 & 2 & -3 & 4 & -5 & 6 \end{bmatrix}$$

n = 8
$$\frac{4}{8}\begin{bmatrix} 7 & -6 & 5 & -4 & 3 & -2 & 1 \\ -6 & 12 & -10 & 8 & -6 & 4 & -2 \\ 5 & -10 & 15 & -12 & 9 & -6 & 3 \\ -4 & 8 & -12 & 16 & -12 & 8 & -4 \\ 3 & -6 & 9 & -12 & 15 & -10 & 5 \\ -2 & 4 & -6 & 8 & -10 & 12 & -6 \\ 1 & -2 & 3 & -4 & 5 & -6 & 7 \end{bmatrix}$$

TABLE 4. (Contd.)

$$G^{2^{-1}}$$

n = 3 $$\frac{16}{6}$$

n = 4 $$\frac{16}{10}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

n = 5 $$\frac{16}{10}\begin{bmatrix} 4 & -4 & 2 \\ -4 & 7 & -4 \\ 2 & -4 & 4 \end{bmatrix}$$

n = 6 $$\frac{16}{105}\begin{bmatrix} 50 & -60 & 45 & -20 \\ -60 & 114 & -96 & 45 \\ 45 & -96 & 114 & -60 \\ -20 & 45 & -60 & 50 \end{bmatrix}$$

n = 7 $$\frac{16}{84}\begin{bmatrix} 45 & -60 & 54 & -36 & 15 \\ -60 & 120 & -120 & 84 & -36 \\ 54 & -120 & 156 & -120 & 54 \\ -36 & 84 & -120 & 120 & -60 \\ 15 & -36 & 54 & -60 & 45 \end{bmatrix}$$

n = 8 $$\frac{16}{84}\begin{bmatrix} 49 & -70 & 70 & -56 & 35 & -14 \\ -70 & 145 & -160 & 134 & -86 & 35 \\ 70 & -160 & 220 & -200 & 134 & -56 \\ -56 & 134 & -200 & 220 & -160 & 70 \\ 35 & -86 & 134 & -160 & 145 & -7 \\ -14 & 5 & -56 & 70 & -70 & 49 \end{bmatrix}$$

TABLE 4. (Contd.)

$G^{3^{-1}}$

n = 4 $\frac{64}{20}$

n = 5 $\frac{64}{35}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$

n = 6 $\frac{64}{140}\begin{bmatrix} 25 & -30 & 15 \\ -30 & 52 & -30 \\ 15 & -30 & 25 \end{bmatrix}$

n = 7 $\frac{64}{42}\begin{bmatrix} 10 & -15 & 12 & -5 \\ -15 & 30 & -27 & 12 \\ 12 & -27 & 30 & -15 \\ -5 & 12 & -15 & 10 \end{bmatrix}$

n = 8 $\frac{64}{168}\begin{bmatrix} 49 & -84 & 84 & -56 & 21 \\ -84 & 184 & -204 & 144 & -56 \\ 84 & -204 & 264 & -204 & 84 \\ 56 & 144 & -204 & 184 & -84 \\ 21 & -56 & 84 & -84 & 49 \end{bmatrix}$

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G.G. BENNETT at present holds the appointment of Senior Lecturer in the Department of Surveying, University of New South Wales, to which he was appointed in 1962. He received a First Class Honours degree in Surveying from the University of Melbourne in 1954. After graduation, Mr. Bennett worked for the Snowy Mountains Hydro-Electric Authority from 1954 to 1959 where he specialised in Geodetic Astronomy. He joined the University of New South Wales in 1959 as a lecturer and completed his Master of Surveying degree at the University of Melbourne in 1962. In 1965 he spent a year as research officer with the Geodetic Surveys Section of the Department of Mines and Technical Surveys, Canada.

Mr. Bennett has published papers on both Geodetic Astronomy and the adjustment of control networks. His current research interests include in addition to the above topics, gyro-theodolites and their applications, as well as all aspects of error theory.

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