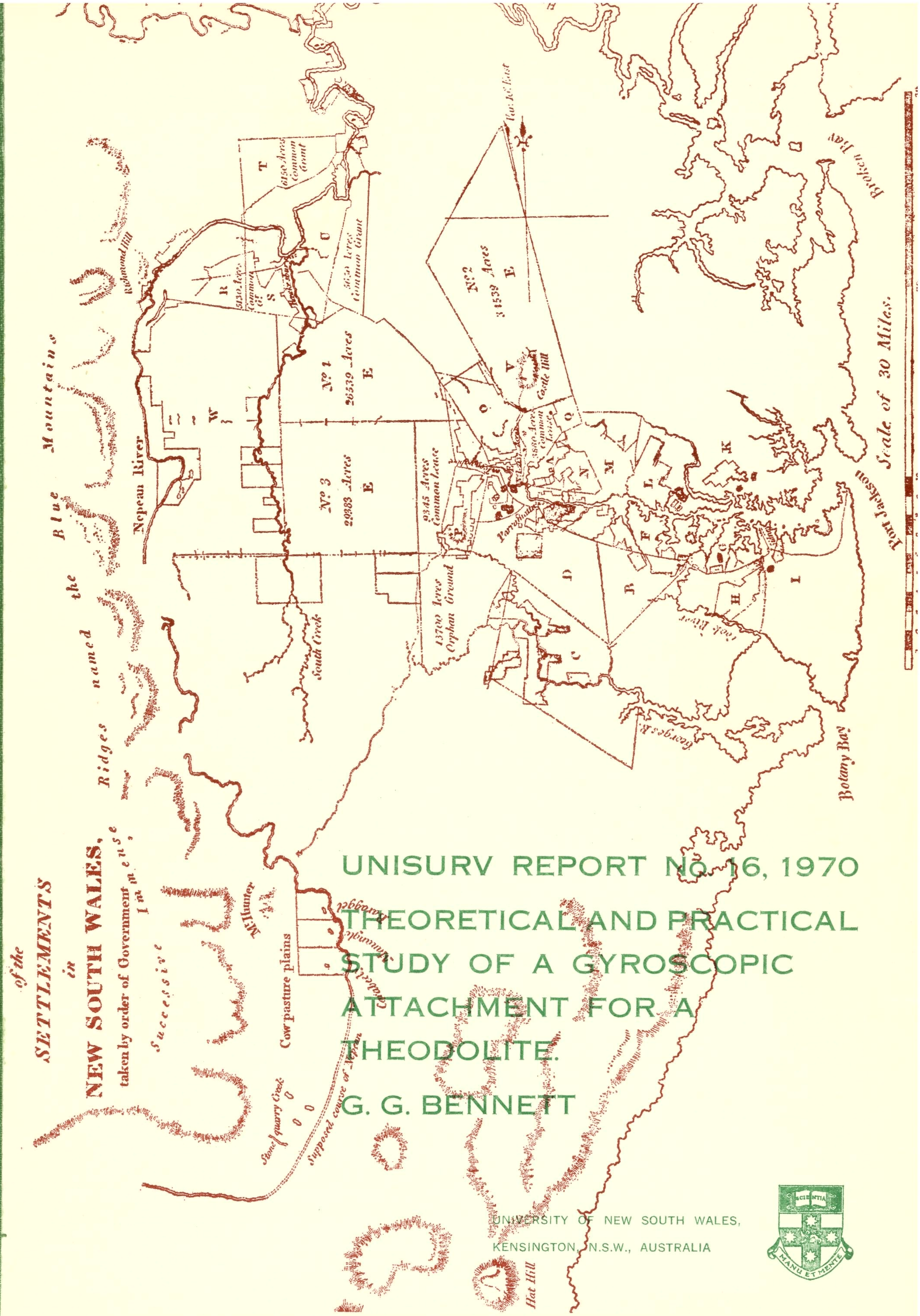

UNISURV REPORT No. 16, 1970

THEORETICAL AND PRACTICAL STUDY OF A GYROSCOPIC ATTACHMENT FOR A THEODOLITE.

G. G. BENNETT

UNIVERSITY OF NEW SOUTH WALES,
KENSINGTON, N.S.W., AUSTRALIA

Reference to Districts.

A Northern Boundaries
B Liberty Plains
C Banks Town
D Parramatta
EEEE Ground reserved for Govt. purposes
F Concord
G Petersham
H Bulanaming
I Sydney
K Hunters Hills
L Eastern Farms
M Field of Mars
N Ponds
O Toongabbey
P Prospect
Q
R Richmond Hill
S Green Hills
T Phillip
U Nelson
V Castle Hill
W Evan

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London: Published by John Booth, Duke Street, Portland Place, July 20th, 1810

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U N I S U R V   R E P O R T   NO. 16

THEORETICAL AND PRACTICAL STUDY OF A GYROSCOPIC ATTACHMENT FOR A THEODOLITE

G.G. Bennett

Received February, 1970

The Department of Surveying,
The University of New South Wales,
P.O. Box 1,
Kensington, N.S.W. 2033. Australia.

"Comme tous ces phénomènes dépendent du mouvement de la Terre et en sont des manifestations variées je propose de nominer 'gyroscope' l'instrument unique qui m'a servi a les constater." *

* From a memoir read by Léon Foucault before the Academy of Sciences in Paris, 1852.

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THEORETICAL AND PRACTICAL STUDY OF A GYROSCOPIC ATTACHMENT FOR A THEODOLITE.

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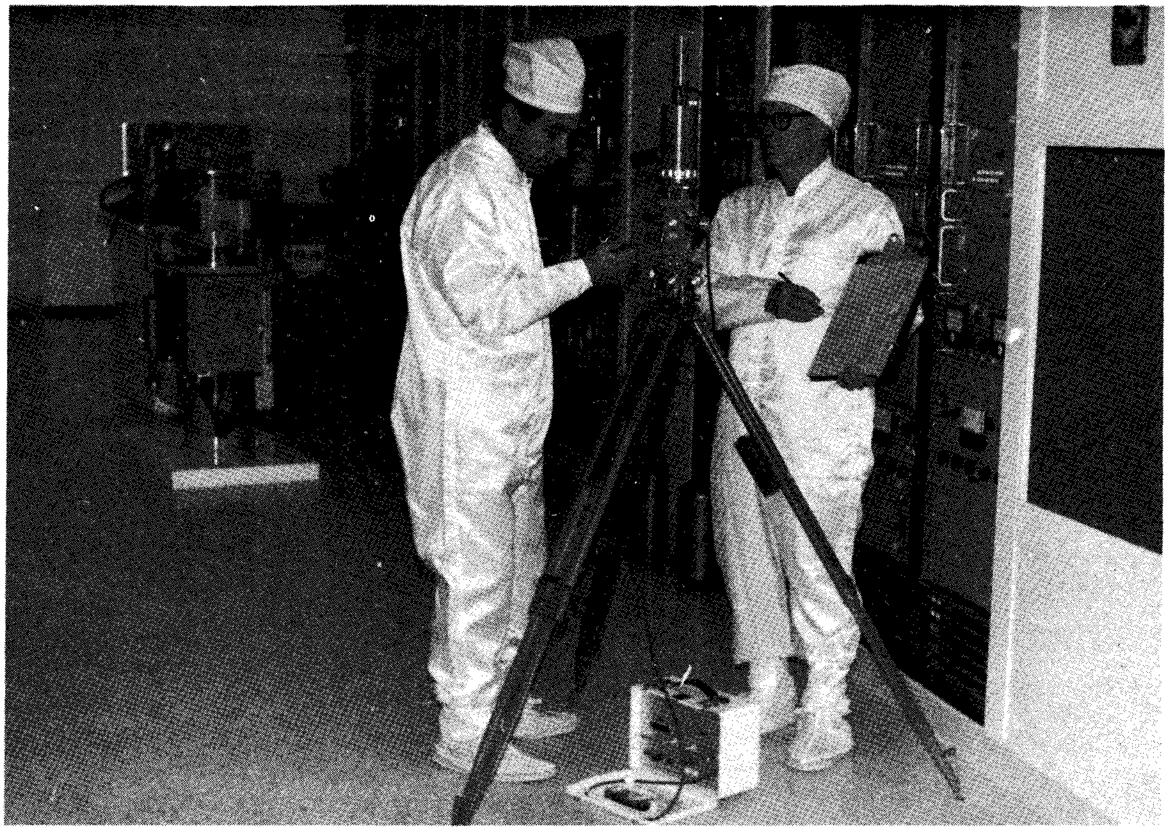
ERRATA.

Table of CONTENTS 7.4 For 'Instrumental Contents' read 'Instrumental Constants.'

Page	Line	
14	20	For 'Chausthal' read 'Clausthal.'
15	18	For 'sperical' read 'spherical.'
31	last	For α read α'.
95	20	For 13 59.72 read 12 59.72
95	last	For Mean = to 0.76 read Mean to = 0.76
114	20	For 'though' read 'through'
127	11	For 'place' read 'placed'
133	24	For 'Phillips' read 'Philips'
143	2	For $\cos(\pi+2\pi d\pi_{23})$ read $\cos(\pi-2\pi d_{23})$
143	4	For $\cos\pi-2\pi d\pi_{23}$ read $\cos(\pi-2\pi d\tau_{23})$
166	Lower left hand side of table	For -13'30" read -13'00"
170	7	For $-\frac{(i-1)}{2}$ read $-\alpha\frac{(i-1)}{2}$
170	12	For $\cos(i-1)$ read $\cos\pi(i-1)$
187	18	For 'dos' read 'does'
196	19	For 'reisdual' read 'residual'
207	13	For 'Anenometer' read 'Anemometer'
215	10	Insert) after 50.7
228	11	For 'bur' read 'but'
239	15	For 'Bewanger' read 'Berwanger'
262	7	For (a) read (b)
341	15	For h read η



CALIBRATION OF AIRCRAFT GYROS BY GYRO-THEODOLITE

IN THE "CLEAN ROOM" OF THE AVIONICS COMPLEX OF

QANTAS AIRWAYS, SYDNEY.

CONTENTS

SYNOPSIS.

In early 1967 the Mining Manager's Association of the Broken Hill group of mines donated a Wild GAK 1 gyro-attachment and T16 theodolite to the University of New South Wales. At this time Professor G.B. Lauf of the University of the Witwatersrand visited the University of New South Wales and gave a series of lectures on his experiences with this and other types of gyro-theodolite. It was apparent from Professor Lauf's lectures that the gyro-theodolite was an instrument of great potential value to the mining surveyor. The purpose of this work is to analyse and evaluate some theoretical and practical aspects of such an instrument.

The motion of a pendulous gyroscope about the meridian approximates closely to a lightly damped simple harmonic motion. The problem of finding the centre of oscillation as derived from a number of successive observations made in the region of minimum velocity of the motion has been investigated by many writers. A rigorous solution of the problem does not present any difficulty now that the high speed electronic computer is generally available. However it is of practical advantage to consider alternative approximate solutions when observations must be assessed under field conditions and sophisticated computational aids are not available. In particular the approach of Professor Max Schuler has been popular over many years, undoubtedly due to the simplicity of applying his formula. Writers such as Dr. F. Kohlrausch, Professor G.B.Lauf, Dr. T.L. Thomas and many others have contributed much to the understanding of this problem. Whilst investigating this problem, especially the linear damping case it was discovered that a rigorous solution of the direction of the line of mean oscillation could be obtained by a simple combination of Schuler Means. The work was later extended to include a solution of all of the unknown parameters by using a similar technique. A further extension of this work to the problem of finding the centre of oscillation from observations made in the region of maximum velocity was to reveal that these solutions were identical with those of the previous problem. What at first seemed to be two unrelated problems in fact were found to have a common solution. Examples and tables to assist in numerical reduction are given.

Since 1967 a great deal of experience has been gained in the handling of the instrument and in the interpretation of the results of observations. One aspect which has caused concern has been the results of observations made to the gyro in its non-spinning state. An explanation of the causes of some of the variations in these observations is given but anomalies still remain for which adequate answers cannot be given. A recent improvement in design of the braking system for the Wild GAK 1 should do much to overcome some of these latter problems.

With the standard methods of observation, namely, the turning point and transit methods, an interval of one period must elapse before a single determination of the meridian can be obtained. If redundant observations are to be made for the purpose of strengthening and providing checks on the solution an additional half period per observation is necessary. The theory and practice of two new methods of observation have been developed which allow multiple observations to be made in the region of the turning point or transit. Of these methods the modified transit method has been used extensively and with success in mines in Australia. In high latitudes where the period of oscillation is long these methods have proved to be valuable because of the time saved.

The gyro-theodolite can be used to advantage in high latitudes because there are many difficulties associated with the determination of azimuth by astronomical means especially at the time of the austral summer when perpetual daylight prevails. However, very little seems to be known of the performance characteristics of the instrument in high latitudes. In late 1968 a project was undertaken under the auspices of the United State National Science Foundation Antarctic Research Programme to test two GAK 1 instruments at stations in a range of latitudes between S34° and S80°. The results of these observations were encouraging but show that there are problems of instrument calibration which need to be overcome and further investigation is necessary.

Most of the work with the gyro-theodolites of the University of New South Wales has been made in areas where survey control is sparse and so to provide lines of known azimuth for calibration purposes astronomical observations for azimuth have been made at all stations where transfer of azimuth was required. This would appear to negate the usefulness of the instrument if astronomical observations are required each time the gyro-theodolite is used especially when only a relative transfer of azimuth is needed. The astronomical azimuth gives a very useful check on the calibration of the instrument and as will be shown will be essential if the instrument is to be used in a wide range of latitudes. For this reason an analysis has been made of the methods of determining astronomical azimuth in low, medium and high latitudes. In addition to this analysis a planisphere for predicting close circum-polar stars and a new time recording instrument have been developed for this work. Examples taken from practice illustrate these methods.

1. HISTORICAL REVIEW

In this section a chronological historical review of the significant developments of the gyroscope is made with particular reference to the gyro-compass, the forerunner of the gyro-theodolite.

Most accounts of the gyroscope commence with a brief description of the work of Léon (Jean Bernard) Foucault (1819 - 1868) and his celebrated experiments at Paris, Rheims and Amiens[1,2,*] but the origins of the gyroscope, although non-scientific, are to be found in the orient with the development by the Chinese of the spinning top[3]. Observations of the precession of the equinoxes, a fundamental gyroscopic phenomenon, were made as early as 120 B.C. by Hipparchus, but it was not until the 17th century A.D. at the time of Newton (1642 - 1727) that an appreciation of the phenomenon began to evolve[4]. The mathematical basis for describing gyroscopic behaviour is due to Euler (1707 - 1783), his name being associated with the dynamical equations relating the applied moment, inertia, angular velocity and accelerations, and the angles of rotation of a rigid body. By the end of the 19th century the mathematical theory of the gyroscope was virtually completed mainly by the significant contributions of men such as Jacobi, Clairaut, Poinsot and Poisson[4].

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* Where reference is made to a published work in this section, a raised numeral is shown against the author or at the end of the passage. These references are given in numerical order at the end of this Chapter. Any direct quotation is given in inverted commas. In all other Chapters, reference to published works is given by author and year, which are given in the main reference list in alphabetical order. If reference is made to more than one published work of an author in a given year then these references are distinguished by a letter following the year of publication e.g. Smith (1964a).

The earliest practical scientific application of gyroscopic principles is to be found in the work of the Englishman Serson in 1744 which was published in the "Gentleman's Magazine of 1754"[3,4,5]. An account of this invention is also given by Short[6]. The instrument was called the "Toupie" and consisted of a spinning top mounted on the deck of a ship and supported so as to be free of the ship's motion, with a mirror on the top horizontal surface which was used as an artificial horizon when the sea's horizon was obscured. Observations were carried out on H.M.S. Victory but the results of this work are not known because Serson lost his life when the ship foundered[5]. A similar device to the "Toupie" was used by Creighton, another Englishman, in 1819, which was later adapted by the French Admiral Fleuriais for his "Horizon Gyroscopique"[5], a description of which may be found in Deimel[7]. The artificial horizon which is used in all modern aircraft is the outcome of the development of this early device.

The first gyroscope* to be constructed is usually credited to Bohnenberger about 1810 and according to the Encyclopaedia Britannica (E.B.)[8] was described in Gilbert's Annalen for 1818. Bohnenberger called his apparatus "Maschinchen" or "baby machine"[3] and it was similar in many respects to that used by Foucault in his experiments. Richardson[3] and Schilovsky[10] also credit Bohnenberger with being the "inventor" of the instrument, now known as the gyroscope, but place the time of its use in 1832. Scarborough[9], however, although admitting that the "history of the instrument is rather obscure", also credits Bohnenberger with its construction and gives the date

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* A name coined by Foucault from the Greek words gyros (γύρος), circle or ring, and skopein (σκοπεῖν), to view[9].

as around 1810. It must be concluded from these sources that the construction of the first gyroscope must be accorded to Bohnenberger although there are inconsistencies in the dates. Professor Lauf[11,12] has shed some light on this latter aspect from an examination of Gilbert's Annalen for 1819, Vol 60 (note: not 1818, according to the E.B.) where Bohnenberger's article including a description of his apparatus is given. Lauf's extract of the description of the apparatus[12] agrees with the description in the E.B. and therefore there seems to be no doubt that Lauf and the E.B. are referring to the same work. According to Lauf[12], Bohnenberger's article was originally published in 1817 and in it reference is made to another work of Poisson in 1813, who in turn refers to Bohnenberger's instrument. The instrument's beginnings are thus documented back to 1813, although the earlier date of use in 1810 given by Scarborough and the E.B. may well be correct. It should be emphasised at this stage that Bohnenberger used the instrument to demonstrate some fundamental properties of a rigid spinning body inside a cardan suspension, such as directional stability and when a couple is applied to the rotational axis, precession. There is no evidence to suggest that it was intended then to use the instrument to demonstrate the earth's rotation, although the precession of the equinoxes was graphically demonstrated by analogy. Of interest etymologically is an improved version of the "Maschinchen" which was constructed by Johnson of Philadelphia in 1832 and was called a "rotascope".[9] Like Bohnenberger's instrument it was mainly used for demonstration purposes.

The credit for suggesting the use of a rotating device to prove the rotation of the earth directly, is usually given to Edward Sang, who in 1836 read a paper before the Royal Scottish Society of Arts[3,4,8,13], although this paper was not to be published until 20 years later, in the Transactions of the Society, 6 years after Foucault's epoch making experiments[3,4,8,13,14]. Sang suggested the use of such a device after working with an improved form of gyroscopic horizon produced by Troughton in 1819[4]. A quotation from Sang's paper given by the E.B. will dispel any doubts about Sang's intentions.

"'While using Troughton's top an idea occurred to me that a similar principle might be applied to the exhibition of the rotation of the earth. Conceive a large flat wheel, poised on several axes all passing directly through its center of gravity, and whose axis of motion is coincident with its principal axis of permanent rotation, to be put in very rapid motion. The direction of its axis would then remain unchanged. But the directions of all surrounding objects varying, on account of the motion of the earth, it would result that the axis of the rotating wheel would appear to move slowly.' This suggested experiment was actually carried out in 1852 by Foucault, probably without any knowledge of Lang's (sic) suggestion."

Arnold and Maunder[4] state that Sang did not have "the resources to construct a sufficiently accurate rotor and his experiments were inconclusive", and it remained for Foucault some years later to demonstrate the earth's rotation. Richardson[3] implies that Sang's suggestion was not the first by which the rotation of the earth could be shown and states "Even before this it is reported that a suggestion on these lines had been made and actually exhibited in Britain by Sir John Leslie", although

Richardson gives no reference source for this report. Sir John Leslie in fact died a few years before Sang's paper was read and so if Richardson's statement is correct then the ideas would have been Sir John Leslie's and not Sang's. In extracts from Sang's address to the Society quoted by Lauf[12] the following statement appears:-

"The only variety of the phenomena which he (Sir John Leslie) had not exhibited is that which refers to the rotation of the earth: and I took liberty of reminding the Society that, many years before, I had proposed that experiment, and illustrated the manner of carrying it out, by the very apparatus which had been used by Leslie, while at the same time, I had exhibited previously known experiments."

We must conclude from Sang's statement that the ideas were his and that his paper read to the Society in 1836 was the first public statement of a method by which the earth's rotation could be demonstrated.

Foucault's first experiment in 1850 convincingly demonstrated the rotation of the earth. The experiment was performed with a long wire pendulum with a lead filled brass case acting as the bob. The rotation of the vertical plane of oscillation was observed by noting the cuts in a ridge of sand placed near the extremities of the pendulum's movement. The theoretical period of oscillation of the plane of rotation is given by $1/\Omega \sin \phi$, where Ω is the sidereal angular velocity of the earth and ϕ is the latitude. For the latitude of Paris (ϕ = N49^{o}) the period is 31.8 hours which agreed well with the observed value of 32 hours.

Opinions are divided on the degree of success of Foucault's second experiment with his gyroscope. Scarborough[9] says that Foucault "constructed a very refined gyroscope to show the rotation of the earth." Hitchins and

May[14] state that he "first used a gyroscope to demonstrate the rotation of the earth." Arnold and Maunder[4] say that "he successfully demonstrated the earth's rotation". The E.B.[8] asserts that "the apparent effect of the earth's rotation on gyroscopes was first shown by Léon Foucault in 1852, the ability to construct sufficiently accurate instruments did not exist until the first decade of the 20th century." Richardson[3] says that Foucault, with reference to his gyroscope "was able to show, for example, that in latitude 50° north, the apparent angular movement of the horizon in space is about 1° in 5 minutes." Deimel[7], however, states that "The apparatus failed to give the expected results principally because he could not keep up the rotation for a long enough time." From these statements it would appear that the weight of opinion favours a successful demonstration of the earth's rotation but the quantitative results were not as convincing as those obtained with the pendulum experiment. Foucault's apparatus was a significant improvement on that of Bohnenberger's where the angular velocity was imparted by means of a string being unwound rapidly from the axle which passed through the diameter of the heavy spheroid. Foucault however had an auxiliary geared machine on which the gyro wheel could be brought to high speed and then lifted from the machine and placed in the gimbals of the horizontal suspension ring, the details of which can be seen from illustrations of a replica of the apparatus preserved in the Science Museum in London[3,8,9].

On the question of priorities in the development of the gyroscope it appears that Bohnenberger's "Maschinchen" was the first instrument to which we would now give the name gyroscope, although the idea of applying it to demonstrate the earth's rotation was Sang's. Foucault's contribution was to refine the instrument and apply it to demonstrate the earth's rotation and for this work was awarded the Copley Medal of the Royal Society. In conclusion, the words of Richardson[3] place the whole question in its correct perspective.

> "It is only fair to say, however, that there is no evidence of Foucault's having knowledge of these suggestions or demonstrations when he carried out his experiments in Paris in 1852 (news travelled slowly in those days) and the honours and fame accorded to him then are unlikely to be questioned now, a century later."

From the time of Foucault's demonstrations thoughts were turned to research into the practical application of gyroscopes, particularly for navigation. Smythies in 1856 took out a patent for a gyro-compass, but the means for driving it and making it North-seeking were not disclosed[14]. The first electrically driven gyro-compass is attributed to Trouvé in 1865 but no details beyond an illustration of the device appear to exist. From the illustration it can be seen that the instrument was fitted with a large pendulum[3,14]. Because of the absence of information about Trouvé's device, Hopkins is usually credited with being the first to apply the electric motor to the gyroscope in 1878. His apparatus was similar to Foucault's and it is reported to have shown the earth's rotation admirably but because of stiffness in the suspension threads and lack of precision in the methods of reducing friction, satisfactory quantitative values were not obtained[5,7,8,9]. Lord Kelvin in a British Association Report of 1884 suggested the use of a float to support the gyrostat* in order to reduce friction in the vertical suspension[13]. It is of interest to note that the first Anschütz gyro-compass hung from a steel float resting in an annular trough filled with mercury.

Of benefit to humanity, especially to those who are prostrated by 'mal de mer', was the application of the German Schlick in 1903, of the gyroscopic characteristic of directional stability to the problem of damping the pitch and roll of ships. A few years later in 1908, Sperry of the United States of America developed a more elaborate and efficient stabiliser[5,7]. At the time of Schlick's work a novel application of the gyroscopic characteristic was being applied to land vehicles by Brennan with his "monorail car" of 1903, Scherl with his "little car" (also a monorail) and Schilovsky with a motor car called an "auto car" in 1909[7,10].

The military significance of the gyroscope did not escape man's attention and so as with many of science's discoveries the first successful development of the gyroscope was not for peaceful purposes.

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* A term used by English writers following Lord Kelvin, to describe the gyroscope, according to Deimel[7].

In 1896 Orbry applied the gyroscope to the stabilisation of the course of the self-propelled torpedo. Subsequent developments by Whitehead a few years later enabled the torpedo to be maintained on a course which deviated by not more than a foot on either side of a straight line[5,15].

It was not until the turn of the century that the first reliable and accurate gyro-compasses were built. However, before this time there had been experiments by the French Navy in about 1884 with a gyroscopic device at sea to which we now give the name, gyro direction indicator or directional gyro. A three framed gyroscope with the rotor horizontal can be made to appear stationary by placing a weight at one end of the gyro casing which will compensate for the easterly movement of the gyro caused by the motion of the earth. The device was not used by the French for steering on a steady course, as in its modern form, but was used to check the magnetic compass by swinging the ship around at its moorings and at various headings comparing the readings of the compass with the gyro direction indicator. With regard to this instrument Hitchins and May[14] make an interesting observation concerning the beginnings of the Anschütz gyro-compass, viz.,

> "Curiously enough, Anschütz appears to have set out with the idea of making a gyro direction indicator which when once placed in the meridian would stay there, rather than a gyro-compass."

1908 marks the beginning of a new era in navigation and for that matter in surveying. In that year Dr. Hermann Anschütz-Kämfe patented the first sea worthy gyro-compass[3,10,14,16] from the principles which had been developed by Professor Max Schuler[8]. Trials were conducted on board the battleship Deutschland which proved so successful that from then on the gyro-compass became standard navigational equipment in the principal navies of the world[14]. Gyro-compasses were also constructed by Sperry in the United States, patent in 1911, and also in England by Brown in association with Perry in 1916. The first gyro-compasses were not without their faults, the chief one being the inter-cardinal rolling error caused by the rolling of the ship when on a course which was not in a cardinal direction. A solution to this problem was realised in the case of the Anschütz tri-gyro gyro-compass of 1911, by placing two additional gyros with the first in the form of an equilateral triangle[14].

An improved version of the tri-gyro gyro-compass, in which the East-West gyroscope was discarded leaving two gyroscopes placed at right anglés one above the other, was patented by Anschütz in 1922. Deimel[7] quoting Grammel[17] states that "Observations by Schuler on the Anschütz-Kämfe gyro-compass used at a fixed place ...... have shown a deviation from the meridian of as little as 20 seconds of arc." This statement probably refers to laboratory observations made with the 1922 gyro-compass. The gyro-compass afloat was an ingenious and very complicated mechanism which had to compensate for the rolling and pitching of the ship, conditions, fortunately, which land surveyors do not have to contend with. These instruments although complicated were very reliable and gave excellent service, in fact Hitchins and May state that the 1922 version is still (1952) in use although highly trained personnel are required for its servicing.

The first land-based forms of the gyro-compass which we now call gyro-theodolites were adapted from marine versions. The first of these instruments was constructed jointly by the firms of Anschütz and Breithaupt and according to Lauf[18] was designed specifically for the determination of directions in underground tunnels, although the instrument was never given a practical underground test. Modifications to the 1922 Anschütz instrument were carried out by Dr. Jungwirth, assistant to Professor Rellensmann of the Clausthal Mining Academy in 1948 and surface and underground test measurements were made at the Rammelsberg Mine in the same year[19]. A full description and illustrations of the instrument called the "Meridian Transferer" or "Meridian Indicator", may be found in the writings of Jungwirth[20] and Banks[21]. Subsequent observations at mines in the Ruhr showed maximum discrepancies of up to a minute between azimuths determined by the Meridian Indicator and precision surveys. South African experience beginning in 1951 substantially confirmed the quality of results obtained in Germany[18]. However the equipment was extremely heavy and several hours of patient work were necessary by a team of experts in order to complete a single azimuth determination.

Further research at Chausthal was to result in the design and construction of a greatly improved gyro-theodolite, the Fennel KT1 in 1959[22]. This instrument was but a fraction of the weight of its predecessors; power for driving the gyroscope was provided by a transistorised D.C./A.C. converter instead of a compressed air supply, the gyroscope instead of being immersed in a globe of conducting fluid was suspended underneath the theodolite by a small thin flat, steel ribbon so that any small errors in following the oscillation of the gyroscope would have little effect upon the measurements of the turning points (extreme positions of oscillation). In the later development of smaller instruments the suspended gyroscope in a case was mounted on a bridge attached to the standards of a theodolite. Examples of small attachment theodolites are the Fennel TK3, Wild GAK1, M.O.M. Gi C1, Gi C2 and Gi D1. Military versions usually called "aiming circles" which are chiefly used for the orientation of field artillery and missiles have the telescope, circles and gyroscope incorporated together in a case which is mounted on a tripod, examples of these are the Fennel KT2 and the Wild ARK 1.

An instrument which overcomes many of the problems of the suspension of the gyroscope in conventional gyro-theodolites is a "spinning-sphere gyro-compass" invented by Dr. Mueller of the Astro-Space Laboratories, Huntsville, Alabama in 1958[23]. The sensing element consists of a solid quartz sperical rotor suspended by means of a hydrostatic gas bearing inside a spherical outer housing. The rotor is brought up to a relatively low speed of 6,000 r.p.m. by contact with the housing which is driven by an electric motor. After reaching the required speed, air is pumped into the small space between the rotor and the housing; speed from then on being maintained by viscous drag. The rotor now attempts to maintain its axis fixed in space whilst the horizontal component of the earth rotation turns the gyro housing towards the East. The spin axis of the rotor and housing drift apart and a drag torque is exerted on the rotor which causes it to precess towards the East until its axis and the axis of the housing set up a constant angle between them which forms a vertical East-West plane. A small mirror mounted on the spin axis of the rotor provides a reference for the direction of this vertical plane which is observed by a Kern DKM-1 theodolite. For their production model the makers claim that the standard deviation of azimuth determination is ± 1', which is disappointing considering that tests with the proto-type gave a value of ± 24". One feature of the Astro-Space gyro-theodolite which is not found in other gyro-theodolites is the ability of the instrument to determine azimuth directly without external reference or calibration i.e. a reference line of known azimuth is not required for checking.

Over the last decade many of the improvements in the electronics and gyroscopes of gyro-theodolites can be attributed to research and development associated with ballistic missiles, nuclear submarines and weaponry. One result has been that in 1960 the British Aircraft Corporation in association with Hilger and Watts made a radical departure from current designs in a new instrument called the Precision Indicator of Meridian (PIM) using a floated integrating gyroscope developed originally by Dr. Draper of the Massachusetts Institute of Technology[24]. The gyroscope is attached rigidly to the vertical axis of the theodolite and is restrained from oscillating, unlike pendulous systems, by a torque motor. The current required to supply the restoring torque is displayed on a milliameter and from readings taken near the cardinal directions, the direction of gyro indicated North, i.e. zero meter reading, can be deduced. The instrument has been adopted by the British Army and has given good results in mines in many countries of the world[25,26].

In recent years two developments which will have great impact on the design of future gyro-theodolites are the gas squeeze bearing and the laser. In 1967 the Astronics Division of Lear Siegler made a developmental study of a miniature gyro-theodolite for military use. The heart of the instrument is a gas squeeze bearing which provides a frictionless support for the gyroscope, thus ridding the instrument of unstable torsion null positions of the suspension tape. The weight of the instrument including power pack is 15 lbs. A complete determination of the meridian, in which the horizontal circle is set automatically to North by critical damping occupies about 7 minutes including setting up the instrument. It is expected that the final precision of azimuth determination will be better than 2 minutes of arc (r.m.s.). With this development in conjunction with the latest small electro-magnetic distance measuring units, it now seems quite feasible that in the near future we will have a completely all purpose surveying instrument with which we will not only be able to measure accurate distances but orient the horizontal circle without reference to previously established stations or celestial bodies. The surveyor's dream come true!

Since the original discoveries of the maser and laser, scientists have been finding many diverse applications for these tools, one of which has been in the field of gyro-technology. In conventional gyroscopes the sensing element is a spinning mass and consequently there are problems in the design and construction of high precision bearings, asynchronous motors, etc. but most of all, problems in protecting the instrument from unwanted accelerations, especially those in inertial guidance systems. In essence the laser gyro or ring laser is a closed resonant cavity (triangular in the Honeywell gyro) in which contrarotating laser beams are travelling. If rotation takes place about an axis normal to the plane of the optical path then from considerations of general relativity there will be a path length change between the two beams. The change in path length causes a change in the resonant frequency of the cavity which is measured by counting the interference fringes formed when the beams are combined. Theoretically it is possible to measure the rate of angular rotation very precisely but at low angular rates, the circumstances which pertain with North-seeking gyros, the beams "lock in" and show no frequency shift. This problem can be overcome for example in two ways (1) by imparting a constant known angular velocity to the ring but which negates the advantage of having a static system or (2) by introducing Faraday bias cells in the optical path which simulates a rotation of the ring. Neither of these solutions have been entirely satisfactory but no doubt these problems will be overcome. The cost of development has been very high and many problems in high precision machining have had to be solved. For example the ring laser is enclosed in a fused quartz block which is required to be machined and polished to within tolerances of 0.1 micron. It will be some time before we see the laser gyro in production but research contracts have been awarded for feasibility studies as a North-seeking reference. Of the systems which are being evolved the laser gyro appears to be the most promising for future geodetic applications. The instrument will undoubtedly be costly but it has the inherent capability of giving a precise azimuth[27,28,29,30].

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2. INTRODUCTION

2.1 GENERAL

The gyro-theodolite, the land-based form of a gyro-compass, has been used mainly for the orientation of underground surveys and in military operations for the alignment of artillery and missile systems. Its application to topographical surveys and land navigation has yet to be fully exploited.

For the orientation of mining surveys the gyro-theodolite is first used to determine the gyro-azimuth of a surface survey line, then the equipment is taken underground and the gyro-azimuths of selected lines are determined. In this way the underground survey can be given the same orientation as the surface control survey. This method of azimuth transfer is purely relative because it matters little if the value of the gyro-azimuth and azimuth of a line are different; the chief concern is that this difference is invariant (within the limits of observation errors) and therefore the North seeking properties of the gyroscopic unit and the relationships between the gyroscopic unit and the mechanical and optical systems must be stable i.e. the difference, **azimuth - gyro-azimuth,** is constant. This constant is usually designated by E.

In military, topographical and navigational operations the value of the calibration constant E, must be known so that the azimuth of any line can be deduced. This calibration is usually made on a reference line whose azimuth has been previously determined from trigonometrical or astronomical observations.

If the gyro-theodolite is to be used for these latter operations on a world wide basis, then it will be necessary to evaluate the performance of the instrument over a wide latitude range so that estimates of accuracy and precision can be anticipated.

2.2 TOPOGRAPHICAL SURVEYS

For topographical mapping by photogrammetric methods some ground survey control points are essential for the scaling, orientation and heighting of the overlapping photography. In remote areas or where ground survey information is sparse the photogrammetrist makes considerable use of the technique of controlling blocks or strips of photographs from ground survey control points which may be widely spaced throughout the photography. The ground survey information is often supplemented with auxiliary data which has been obtained from such sources as the horizon camera, airborne profile recorder (A.P.R.), statoscope and solar periscope. Karara (1957, 1960) has described a method in spatial aerial triangulation which is especially suited for regions which do not have a geodetic survey. Brandenberger (1959, 1964), Colcord (1961) and Ghosh (1962) have also contributed to the understanding of this problem.

The position of control points established by astronomically determined latitudes and longitudes has been used extensively for the control of small scale mapping, the advantage lying in the fact that no ground or air survey connection is necessary between the control points (In Australia much of the control for the old 4 miles to 1 inch map series was surveyed by this method). Brandenberger (1964) makes the following pertinent statement concerning Antarctica:-

> "This means that the establishment of a relatively dense geodetic triangulation system (the Author is referring to cartographic exploration) would require the location of "fixed points" on the ice surface which cannot be considered as stable due to the ice flow. This inconvenience is further aggravated by the fact that many astronomic positioning methods are weak in the Antarctic, that no stable instrument positions can be found, and that the working conditions are extremely rough due to the cold and blizzards. All of these problems have to be overcome in the years to come since, at present, no continental geodetic triangulation network exists in the Antarctic, excepting a few scattered local surveying systems of variable accuracy."

For medium scale mapping the positions of control points established by astronomical methods is in doubt because of the unknown value of the deflection of the vertical. Karara quotes an error in position due to deflection of the vertical of the order of one mile (about one minute of arc). Deflections of this magnitude are probably rare and confined to isolated areas such as the African rift valley. In a private communication, J.G. Fryer of the University of New South Wales, states that the standard deviations of the deflection of the vertical for the continent of Australia are in the meridian $\xi = \pm 3.34''$, and in the prime vertical $\eta = \pm 3.91''$. These values were derived from 252 well spaced Laplace stations. But even errors of this magnitude may be objectionable e.g. for a strip 45 miles long at a photo scale of 1 : 43,000 containing 22 models with 60% overlap, Karara reports a mean square error in position of ±25 feet after adjustment, which is considerably smaller than what can be achieved economically by astronomical methods even if differential deflection of the vertical is ignored entirely. The method devised by Karara requires a "cross base" at each end of a strip of photographs. The length and azimuth of each base is to be measured together with height differences between the terminals of each base. Three absolute heights must be determined, one at each end and one in the middle of the strip. Two groups of three heights are also required at the beginning and at the end of the strip.

Over the last decade there has been considerable progress in the development of instrumentation and techniques which can be directed towards the types of measurement required for the "cross bases" method. The measurement of the length of the bases by any one of a number of electro-magnetic distance measuring instruments using either visible light (e.g. Geodimeter) or radio microwaves (e.g. Tellurometer) to an accuracy well within what is required for this method presents no difficulty. The instruments are light and portable and measurements can be completed in less than an hour even under extreme temperature conditions. Differential heights of the base terminals can be conveniently obtained by observing simultaneous reciprocal vertical angles at the time of distance measurement. The precaution of observing reciprocal vertical angles is necessary when making the observations because of possible abnormal refraction effects e.g. Angus-Leppan (1968) states that measurements taken over a frozen surface show markedly different refraction from those taken over normal surfaces;

the values of the coefficient of refraction, K, being remarkably high. Karara suggests the use of barometric methods for the determination of the heights at the ends of the strip. Allman (1968) has shown that using "Baromec" contact type barometers compared with "Wallace and Tiernan" and "Askania" barometers a considerable increase in reliability and accuracy is possible. In areas where the determination of azimuth by conventional astronomical methods is difficult, particularly when the sky is overcast for a considerable proportion of time, then the gyro-theodolite can be used with advantage. The azimuth accuracy require for the "cross bases" method, according to Karara, is of the order of 1 to 1½ minutes of arc and this accuracy should also be sufficient for orientation of lines in the slotted templet triangulation method, which is to be described later. This order of accuracy should be within the capabilities of most gyro-theodolites which are currently used in surveying, even in high latitudes.

Karara states that there is a further advantage in using the astronomical azimuth of "cross bases" because these azimuth determinations, and this also applies to gyro-azimuths, are but little affected by large deflections of the vertical. This is certainly correct because the difference between the astronomical and geodetic azimuth is governed by the Laplace equation, viz:-

$$\Delta A = \Delta\lambda \sin\phi$$

where ΔA is the difference between the astronomical and geodetic azimuth.

$\Delta\lambda$ is the difference between the astronomical and geodetic longitude.

ϕ is the latitude.

It will be seen that the azimuth difference cannot exceed the prime vertical deflection even in polar regions and that this difference will become progressively smaller as the latitude decreases. Besides being independent of cloud cover, gyro-theodolite azimuth determinations may be made without a precise knowledge of the latitude and longitude of the station, whereas with astronomical observations for azimuth especially in high South latitudes, observations to determine latitude and longitude are virtually essential for the reduction of the azimuth observations. An additional source of inconvenience in the Antarctic summer, the time of maximum field activity, besides the condition of perpetual daylight which makes astronomical observations difficult is the occasional period of fade-out of short wave radio communication caused by solar flares. Apart from disturbing normal radio communications and limiting aircraft flights, it is impossible to receive radio time signals in these periods and although the sky may be clear enough for astronomical observations, without suitable watch or chronometer checks the surveyor is severely handicapped.

Radial triangulation, a form of aerial triangulation, owes its popularity mainly to the fact that no expensive instruments are required for building up an assembly of overlapping near vertical aerial photographs. However, in contrast to spatial aerial triangulation, only maps showing planimetric information can be produced by this method. The use of astronomically oriented base lines in radial triangulation by the slotted templet method has been described by Visser (1958) and also by the International Training Centre for Aerial Survey (1963). A small base map of rigid transparent material is prepared for the area between the end points of the oriented base line and a local layout of slotted templets is prepared. When the local layout has been completed, the border templets are stuck to this base map which will act as a large templet in the overall layout. The grid of this small base map can now be viewed against the grid of the large base map and the grids oriented to one another. The accuracy of the azimuth determination is governed by the scale of the slotted templet assembly and the length of base lines employed, but a few minutes of arc will be sufficient for medium and small scale mapping.

2.3 NAVIGATION

The basic problems of navigating in unmapped or featureless areas are

(a) establishing the position of intermediate stations on the route between the terminals of the traverse and

(b) providing orientation for travelling between these stations.

Where weather conditons permit, position and orientation can be obtained from sun or star observations; reductions being made by any one of a number of quick calculation methods. When the weather inhibits astronomical observations the navigator resorts to dead reckoning methods, supplementing measured distances and angles by magnetic bearings. Unless an occasional check on position and orientation can be made then the errors in the traverse tend to accumulate. Relying entirely on magnetic orientation can be extremely hazardous in areas where there are local distortions in the earth's magnetic field. In polar regions these difficulties are further aggravated by the proximity of the magnetic poles because of the convergence of the isogonic lines and the weak horizontal magnetic intensity.

The gyro-theodolite would be a useful adjunct to the navigator because the system is self contained, requiring no external reference to stellar motions or the direction of the earth's magnetic field, only a firm base and a wind proof shelter are necessary for its operation. The equipment is robust and compact and requires little skill on the part of the operator to determine azimuth to within 0.1^{o}, which would satisfy most needs of exploratory surveys. If an approximate azimuth is required then one of the "quick methods" such as the ¼ period method or the amplitude method can be used. These methods have been described in detail by Schwendener (1966) and Halmos (1966). In high latitudes the gyro-theodolite may be used to determine latitude by measuring the period of oscillation of the turning point method, which will be referred to in Section 7.5.

3. THE LEAST SQUARES ADJUSTMENT OF GYRO-THEODOLITE OBSERVATIONS.

3.1 THE LINEAR DAMPED MODEL

Lauf (1963) has shown that the precession angle about the meridian of a pendulous gyroscope is of the form

$$\theta = Be^{-\alpha t} \cos (\beta t + \gamma) \qquad (1)$$

in which, according to Williams and Belling (1967a)

- B is the half-amplitude of the oscillation at some initial instant,
- e is the base of natural logarithms, and
- α and β are constants for a given instrument for a particular latitude when the angular momentum of the gyro-rotor is constant.
- γ is an arbitary phase-angle

It has also been shown by Lauf (1967a)that a good approximation to the precession angle at the "turning point" is when $(\beta t + \gamma) = 0, \pi, 2\pi, 3\pi$, etc. For convenience we may change the form of equation (1) to

$$\theta = -Be^{-2\alpha'\tau} \cos 2\pi\tau$$

where τ represents the fractional period elapsed.

"Turning points" will occur when $\tau = 0, \frac{1}{2}, 1 \ldots$ etc. For the gyro-theodolite the damping is usually slight and the exponential term may be expanded retaining only linear damping terms giving

$$\theta = -B\,(1 - 2\alpha'\tau) \cos 2\pi\tau$$

Now at the "turning points", $\tau = \frac{i}{2}$, where $i = 1, 2, \ldots n$, corresponding to the number of the "turning point". Therefore

$$\theta_i = B(1 - i\alpha')\,(-1)^{i-1}$$

The measurements of θ_i are made on a circle whose orientation with respect to the meridian is unknown and defined by θ_o. If these measurements are defined as y_i and their corresponding corrections by v_i then the previous equation takes the form

$$v_i = B(1 - i\alpha')(-1)^{i-1} + \theta_o - y_i \tag{2}$$

or, in extenso

$$\begin{aligned} v_1 &= B - \alpha' B + \theta_o - y_1 \\ v_2 &= -B + 2\alpha' B + \theta_o - y_2 \\ v_3 &= B - 3\alpha' B + \theta_o - y_3 \\ v_4 &= -B + 4\alpha' B + \theta_o - y_4 \\ &\text{etc.} \end{aligned}$$

which are the required correction equations.

With regard to the following adjustment procedure, the underlying assumptions are that the observations are stochastically independent and are made with equal precision. It will be noted that no hypothesis is made concerning the normality of the distribution, merely that a variance exists for this distribution. The least squares method seeks to satisfy the relationship (vv) = minimum, and requires no assumption regarding the distribution other than the existence of a variance. For elaboration of this point see Sunter (1966).

The correction equations in their present form are not suitable for a direct solution because they are not linear. Lauf (1967a)has used the familiar technique of introducing approximate values of the unknown parameters α and B and then expanding the non-linear term by Taylor's series to solve this problem. An alternative method of separating the unknown parameters is to substitute in the correction equations a new set of parameters such that the equations become linear in respect of these new values.

If we put $a = \alpha' B$, $b = \theta_o + B$, and $c = \theta_o - B$, then the general form of correction equations becomes

$$v_i = i(-1)^i a + \left(\frac{1 + (-1)^{i-1}}{2} \right) b + \left(\frac{1 + (-1)^i}{2} \right) c - y_i \qquad (3)$$

$$i = 1, 2, \ldots\ldots n$$

$$n \geqslant 3$$

or, in extenso

$$\begin{aligned} v_1 &= -a + b \qquad - y_1 \\ v_2 &= 2a \qquad + c - y_2 \\ v_3 &= -3a + b \qquad - y_3 \\ v_4 &= 4a \qquad + c - y_4 \end{aligned}$$

etc.

Our main interest will lie with the original parameters, which will be given by

$$\theta_o = \frac{b + c}{2}, \quad B = \frac{b - c}{2} \quad \text{and} \quad \alpha' = \frac{a}{B} \quad \text{or} \quad \alpha' = \frac{2a}{b - c} \qquad (4)$$

The geometrical interpretation of the adjustment is shown in Fig. 3.1 The mathematical model consists of a pair of straight lines; each line is inclined at the same angle to the line of mean oscillation and passes through its set of observed points.

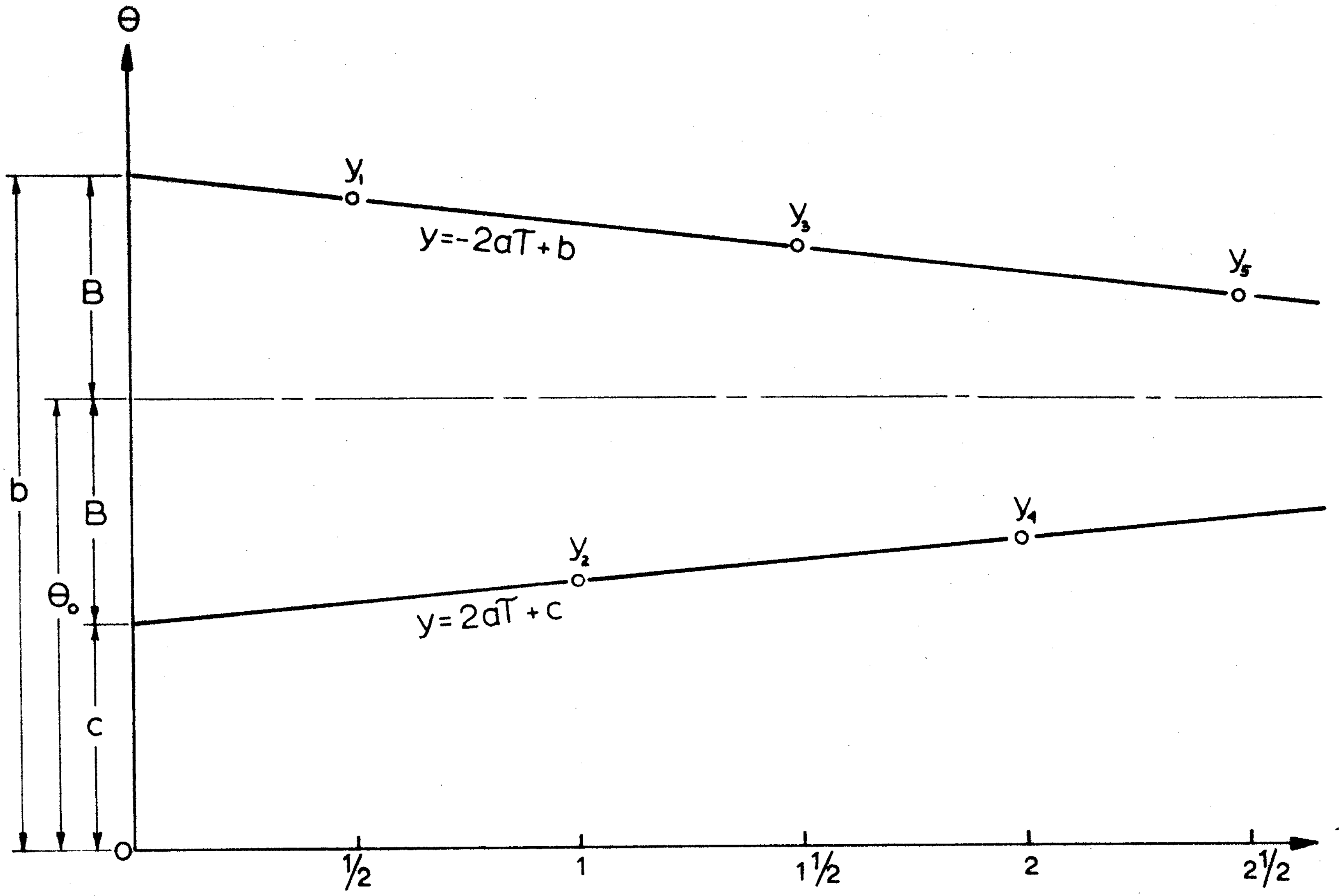

FIG. 3.1

Forming normal equations* we obtain

$$\begin{array}{cccc}
\underline{a} & \underline{b} & \underline{c} & \underline{\text{Absolute Term}} \\
\sum_{i=1}^{n} i^2 & \sum_{i=1}^{n} \frac{i(-1)^i\left(1+(-1)^{i-1}\right)}{2} & \sum_{i=1}^{n} \frac{i(-1)^i\left(1+(-1)^{i}\right)}{2} & -\sum_{i=1}^{n} i(-1)^i y_i = 0 \\
 & \sum_{i=1}^{n}\left(\frac{1+(-1)^{i-1}}{2}\right)^2 & 0 & -\sum_{i=1}^{n}\left(\frac{1+(-1)^{i-1}}{2}\right)y_i = 0 \\
 & & \sum_{i=1}^{n}\left(\frac{1+(-1)^{i}}{2}\right)^2 & -\sum_{i=1}^{n}\left(\frac{1+(-1)^{i}}{2}\right)y_i = 0
\end{array}$$

It has been found to be convenient in the following reduction process to treat two types of observation, one containing an even number and the other an odd number of observations.

The normal equations for n even are

$$\begin{array}{ccccc}
\underline{a} & \underline{b} & \underline{c} & \underline{\text{Absolute Term}} & \\
\frac{n(n+1)(2n+1)}{6} & -\frac{n^2}{4} & \frac{n}{4}(n+2) & \sum_{i=1}^{n} i^0 y^0_i - \sum_{i=1}^{n} i^E y^E_i & = 0 \\
 & \frac{n}{2} & 0 & -\sum_{i=1}^{n} y^0_i & = 0 \qquad (5) \\
 & & \frac{n}{2} & -\sum_{i=1}^{n} y^E_i & = 0
\end{array}$$

and for n odd are

$$\begin{array}{ccccc}
\underline{a} & \underline{b} & \underline{c} & \underline{\text{Absolute Term}} & \\
\frac{n(n+1)(2n+1)}{6} & -\left(\frac{n+1}{2}\right)^2 & \frac{n^2-1}{4} & \sum_{i=1}^{n} i^0 y^0_i - \sum_{i=1}^{n} i^E y^E_i & = 0 \\
 & \frac{n+1}{2} & 0 & -\sum_{i=1}^{n} y^0_i & = 0 \qquad (6) \\
 & & \frac{n-1}{2} & -\sum_{i=1}^{n} y^E_i & = 0
\end{array}$$

* For symmetrical matrices the lower triangular portion has been omitted.

where the superscript 0 and E denotes odd and even values only are to be taken, respectively.

From the practical viewpoint one will seldom be concerned with $n > 8$ and therefore equations (5) and (6) have been solved for $n = 3, 4 \ldots \ldots 8$. The results of these solutions are given in Table 3.1. The values of $\frac{b + c}{2}$ and $\frac{b - c}{2}$ have also been found, because these represent the principal unknowns θ_o and B. The values of θ_o for $n = 4$ and 8 agree precisely with those obtained by Lauf (1967a)who has been concerned mainly with those values of n which are a multiple of 4.

Two important relationships can be derived immediately from the sets of normal equations (5) and (6). The sum of the second and third normal equations of (5) is

$$\frac{n}{2} a + \frac{n}{2} b + \frac{n}{2} c = \sum_{i=1}^{n} y_i^0 + \sum_{i=1}^{n} y_i^E$$

therefore

$$\frac{b + c}{2} = \theta_o = \frac{1}{n} \sum_{i=1}^{n} y_i - \frac{a}{2} \qquad (7)$$

Also, from the set of normal equations (6), if we divide the second normal equation by $n + 1$ and the third normal equation by $n - 1$ and add, we obtain

$$\frac{b + c}{2} = \theta_o = \frac{1}{n^2 - 1} \left\{ (n - 1) \sum_{i=1}^{n} y_i^0 + (n + 1) \sum_{i=1}^{n} y_i^E \right\} \qquad (8)$$

which may be evaluated very simply by taking the mean of the odd numbered observations, the mean of the even numbered observations and then taking a grand mean.

It is worthwhile comparing the results of these derivations with those obtained by Williams and Belling (1967a). These authors state - "It must be recognised nevertheless that unless a better criterion can be found, [**vv**]* is as acceptable as any. Therefore, any simple reduction procedure which agrees very closely with the least squares result is as acceptable as this latter, and is indeed preferable to it if it offers a more general solution." They derive the following formulae on a basis of "symmetry" and "the principle of least displacement of random lines ....." :-

for n even

$$\theta_o = \frac{1}{n}\sum_{i=1}^{n} y_i - \frac{1}{4(n-2)} (y_1 - y_2 - y_{n-1} + y_n)$$

and for n odd

$$\theta_o = \frac{1}{n+1}\sum_{i=1}^{n} y_i^O + \frac{1}{n-1}\sum_{i=1}^{n} y_i^E$$

The agreement between these formulae and (7) and (8) is only partial. For n odd the agreement is exact but for n even $\frac{a}{2}$ is approximated by $\frac{1}{4(n-2)} (y_1 - y_2 - y_{n-1} + y_n)$. It is a fact that this approximation is reasonable because from the geometrical interpretation of the least squares solution it can be seen that $\frac{a}{2}$ can be derived relatively efficiently from the end observations only. These authors further state regarding their formulae that "The advantage of their application over the use of more complicated formulae derived via Gaussian theory is based on their simplicity of form and the consequent fact that θ_o can be rapidly obtained under field conditions." In fact, however, there is only a marginal difference in

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* This should probably be "[vv] is a minimum."

complexity between the rigorous least squares solution and that offered by these authors. The least squares solution is to be preferred. It will be shown further that an even simpler form of rigorous reduction is available from consideration of the Schuler means.*

Besides the solution of the estimates of the adjusted parameters, the calculation of their variances and co-variances is necessary in order to estimate the performance of both the mathematical model and the observations. A useful summary of the least squares adjustment techniques, using matrix algebra, has been prepared by Allman (1967) and has been used extensively in the following derivations.

The matrix of the weight coefficients of the adjusted parameters is given by the inverse matrix of the normal equations after removal of the absolute terms.

For n even the inverse matrix is

$$\begin{bmatrix} \dfrac{12}{n(n^2-4)} & \dfrac{6}{(n^2-4)} & \dfrac{-6}{n(n-2)} \\ & \dfrac{5n^2-8}{n(n^2-4)} & \dfrac{-3}{(n-2)} \\ & & \dfrac{5n+2}{n(n-2)} \end{bmatrix}$$

---

* In a recent communication from Professor Tárczy-Hornoch relating to the study of the Schuler Mean, he states ... "Allow me to mention in this connection that Schuler's formula given in the paper was not given first by Schuler in "Zeitschrift für angewandte Mathematik und Mechanik" Tom 12. 1932, but already earlier by Fox, who was professor at Clausthal. This was already stated by Professor Niemczyk on page 17 of "Mitteilungen aus dem Markscheidewesen" in 1930. I mentioned it already in my collected works "Markscheiderische Studien" edited in 1963, on page 904." The author would like to thank Professor Tárczy-Hornoch for correcting this error.

retaining the same order of parameters as in the normal equations,

and for n odd

$$\begin{bmatrix} \frac{12}{n(n^2-1)} & \frac{6}{n(n-1)} & \frac{-6}{n(n-1)} \\ & \frac{5n^2+4n+3}{n(n^2-1)} & \frac{-3(n+1)}{n(n-1)} \\ & & \frac{5n+3}{n(n-1)} \end{bmatrix}$$

If the estimate of the variance of the adjusted observations is $\hat{S}_y^2 = \frac{[vv]}{r}$ where r is the number of redundancies, then the estimates of the variance of the adjusted parameters a, b and c is given by the following -

$$\hat{S}_a^2 = Q_{aa}\,\hat{S}_y^2$$

$$\hat{S}_b^2 = Q_{bb}\,\hat{S}_y^2$$

$$\hat{S}_c^2 = Q_{cc}\,\hat{S}_y^2$$

As stated before, our interest lies with the original parameters

$$\theta_o = \frac{b+c}{2}, \quad B = \frac{b-c}{2} \quad \text{and} \quad \alpha' = \frac{a}{B}$$

Applying the general law of propagation of variances to these relationships we have

$$\hat{S}^2_{\theta_o} = \frac{1}{4}\left(Q_{bb} + 2Q_{bc} + Q_{cc}\right)\hat{S}^2_y$$

$$\hat{S}^2_B = \frac{1}{4}\left(Q_{bb} - 2Q_{bc} + Q_{cc}\right)\hat{S}^2_y$$

$$\hat{S}^2_{\alpha'} = \frac{4}{(b-c)^4}\left((b-c)^2Q_{aa} + a^2(Q_{bb} - 2Q_{bc} + Q_{cc}) - 2a(b-c)(Q_{ab} - Q_{ac})\right)\hat{S}^2_y$$

After substitution of the appropriate values from the inverse matrices we have for n even

$$\hat{S}^2_{\theta_o} = \frac{n^2-1}{n(n^2-4)} \cdot \hat{S}^2_y$$

$$\hat{S}^2_B = \frac{4n^2+6n-1}{n(n^2-4)} \cdot \hat{S}^2_y$$

$$\hat{S}^2_{\alpha'} = \frac{12}{n(n^2-4)B^2}\left(1 + \frac{\alpha'^2}{24}(5n+4)(n+1) - \alpha'(n+1)\right)\hat{S}^2_y$$

If n is not large as is the case in practice then this last expression may be approximated by

$$\hat{S}^2_{\alpha'} = \frac{12}{n(n^2-4)B^2}\hat{S}^2_y$$

and for n odd

$$\hat{S}^2_{\theta_o} = \frac{n}{n^2-1}\hat{S}^2_y$$

$$\hat{S}^2_B = \frac{4n^2+6n+3}{n(n^2-1)}\hat{S}^2_y$$

$$\hat{S}^2_{\alpha'} = \frac{12}{n(n^2-1)B^2}\left(1 + \frac{\alpha'^2}{12}(4n^2+6n+3) - \alpha'(n+1)\right)\hat{S}^2_y$$

which again may be approximated by

$$\hat{S}^2_{\alpha'} = \frac{12}{n(n^2-1)B^2} \cdot \hat{S}^2_y$$

If the origin of the τ axis is moved to the centre of the observations as Lauf (1967a)has done for his derivation, then a comparison is possible for the even values of n, thus if we substitute

$$b = b' + \frac{(n+1)}{2} a$$

and

$$c = c' - \frac{(n+1)}{2} a$$

in equations (5) and (6) we get

for n even

$$\begin{array}{ccccc} \underline{a} & \underline{b'} & \underline{c'} & \underline{\text{Absolute Term}} & \\ \frac{n(n^2-1)}{12} & \frac{-n^2}{4} & \frac{n}{4}(n+2) & \sum_{i=1}^{n} i^0 y^0_i - \sum_{i=1}^{n} i^E y^E_i & = 0 \\ \frac{n}{4} & \frac{n}{2} & 0 & -\sum_{i=1}^{n} y^0_i & = 0 \\ \frac{n}{4} & 0 & \frac{n}{2} & -\sum_{i=1}^{n} y^E_i & = 0 \end{array} \qquad (9)$$

and for n odd

$\underline{a}$	$\underline{b'}$	$\underline{c'}$	Absolute Term		
$\frac{n(n^2-1)}{12}$	$-\left(\frac{n+1}{2}\right)^2$	$\frac{(n^2-1)}{4}$	$\sum_{i=1}^{n} i^0 y_i^0 - \sum_{i=1}^{n} i^E y_i^E$	$= 0$	
0	$\frac{(n+1)}{2}$	0	$-\sum_{i=1}^{n} y_i^0$	$= 0$	(10)
0	0	$\frac{(n-1)}{2}$	$-\sum_{i=1}^{n} y_i^E$	$= 0$	

The solutions of these equations for "a" and $\frac{b+c}{2}$ will remain unchanged from those given for the solutions of equations (5) and (6). However, simpler expressions can be obtained for the "middle amplitude", B' where

$$B' = \frac{b'-c'}{2}$$

If we subtract the third from the second equation in (9) we obtain

for n even

$$B' = \frac{b'-c'}{2} = \frac{1}{n}\left(\sum_{i=1}^{n} y_i^0 - \sum_{i=1}^{n} y_i^E\right) \tag{11}$$

Also, if we do likewise in (10) we obtain

for n odd

$$B' = \frac{b'-c'}{2} = \frac{1}{n^2-1}\left((n-1)\sum_{i=1}^{n} y_i^0 - (n+1)\sum_{i=1}^{n} y_i^E\right) \tag{12}$$

which may be evaluated very simply by taking half the difference between the means of the odd and even numbered observations

The inverse matrices of (9) and (10) are

for n even

$$\begin{bmatrix} \frac{12}{n(n^2-4)} & \frac{6}{n^2-4} & \frac{-6}{n(n-2)} \\ \frac{-6}{n(n^2-4)} & \frac{2n^2-3n-8}{n(n^2-4)} & \frac{3}{n(n-2)} \\ \frac{-6}{n(n^2-4)} & \frac{-3}{n^2-4} & \frac{2n-1}{n(n-2)} \end{bmatrix}$$

and for n odd

$$\begin{bmatrix} \frac{12}{n(n^2-1)} & \frac{6}{n(n-1)} & \frac{-6}{n(n-1)} \\ 0 & \frac{2}{n+1} & 0 \\ 0 & 0 & \frac{2}{n-1} \end{bmatrix}$$

The estimate of the variance of the adjusted parameter θ_o will remain unchanged. Applying the general law of propagation of variances to B' and α'' where

$$B' = \frac{b'-c'}{2} \qquad \text{and} \qquad \alpha'' = \frac{2a}{b'-c'}$$

we obtain

$$\hat{S}^2_{B'} = \frac{1}{4}\left\{Q_{b'b'}-Q_{b'c'}-Q_{c'b'}+Q_{c'c'}\right\}\hat{S}^2_y$$

and

$$\hat{S}^2_{\alpha''} = \frac{4}{(b'-c')^4}\left\{(b'-c')^2Q_{aa}+a^2(Q_{b'b'}-Q_{b'c'}-Q_{c'b'}+Q_{c'c'}) +a(b'-c')(Q_{ac'}+Q_{c'a}-Q_{ab'}-Q_{b'a})\right\}\hat{S}^2_y$$

After substitution of the appropriate values from the inverse matrices we have

for n even

$$\hat{S}^2_{\theta_o} = \frac{n^2-1}{n(n^2-4)} \hat{S}^2_y$$

$$\hat{S}^2_{B'} = \frac{1}{n} \hat{S}^2_y$$

$$\hat{S}^2_{\alpha''} = \frac{12}{n(n^2-4)B'^2} \left\{ 1 + \frac{\alpha''^2}{12} (n^2-4) - \frac{\alpha''}{2} (n+1) \right\} \hat{S}^2_y$$

which again may be approximated by

$$\hat{S}^2_{\alpha''} = \frac{12}{n(n^2-4)B'^2} \hat{S}^2_y$$

and for n odd

$$\hat{S}^2_{\theta_o} = \frac{n}{n^2-1} \hat{S}^2_y$$

$$\hat{S}^2_{B'} = \frac{n}{n^2-1} \hat{S}^2_y$$

$$\hat{S}^2_{\alpha''} = \frac{12}{n(n^2-1)B'^2} \left\{ 1 + \frac{\alpha''^2 n^2}{12} - \frac{\alpha''}{2} (n+1) \right\} \hat{S}^2_y$$

which again may be approximated by

$$\hat{S}^2_{\alpha''} = \frac{12}{n(n^2-1)B'^2} \hat{S}^2_y \qquad (13)$$

The values for n even, agree precisely with those obtained by Lauf (1967a) after taking into account the difference in notation.

With the exception of a formula for the calculation of "a", simple general expressions for a least squares solution have been derived for any value of n, odd or even. Returning to equations (5) and

(6) or (9) and (10) we can eliminate the unknowns b and c or b' and c' and obtain

for n even

$$a = \frac{12}{n(n^2-4)} \left\{ \sum_{i=1}^{n} i^E y_i^E - \sum_{i=1}^{n} i^0 y_i^0 + \frac{n}{2} \sum_{i=1}^{n} y_i^0 - \frac{(n+2)}{2} \sum_{i=1}^{n} y_i^E \right\}$$

and for n odd (14)

$$a = \frac{12}{n(n^2-1)} \left\{ \sum_{i=1}^{n} i^E y_i^E - \sum_{i=1}^{n} i^0 y_i^0 + \frac{(n+1)}{2} \left[\sum_{i=1}^{n} y_i^0 - \sum_{i=1}^{n} y_i^E \right] \right\}$$

3.2 PROPERTIES OF THE SCHULER AND RELATED MEANS

For the turning or reversal point method with a pendulous gyroscope, it has been customary and convenient to derive the mean direction of the meridian from an approximate relationship given by Professor Max Schuler (1932), namely:

$$S_1 = \frac{y_1+2y_2+y_3}{4}, \quad S_2 = \frac{y_2+2y_3+y_4}{4}, \quad S_3 = \frac{y_3+2y_4+y_5}{4} \quad \ldots\ldots\ldots$$

$$\ldots\ldots\ldots\ldots\ldots \quad S_{n-2} = \frac{y_{n-2}+2y_{n-1}+y_n}{4}$$

and

$$\overline{S} = \frac{S_1 + S_2 + S_3 + \ldots\ldots\ldots\ldots S_{n-2}}{n-2}$$

The numerical calculation is simple and in the field the value of $\overline{S}$ can be obtained almost immediately after the completion of the observations. Convenience is not the only characteristic possessed by $\overline{S}$ which has led to its almost universal adoption but that the difference between $\overline{S}$ and the least squares estimate is seldom significantly big. Disadvantages of the technique are that :-

(a) It may not be known if the difference between the least squares estimate and $\overline{S}$ is significant.

(b) Because the original observations are combined, a poor observation or mistake may be hidden in the Schuler Means, thus biasing $\overline{S}$.

(c) A variance estimate based on the individual Schuler Means is invalid if these are treated as independent quantities.

See Lauf (1967a) for Schuler's original derivation.

(d) Neither the corrections to the original observations nor the remaining parameters and their variance estimates are disclosed.

The Schuler Mean is a special case of a series of Means which can be formed from the original observations. The necessary and sufficient number of observations required will be three, producing a minimum of two types of Mean in the series. For each additional observation a further Mean in the series may be found. Only the first three types of Mean will be considered here, which will be designated the First Mean, the Schuler Mean and the Thomas Mean. The third Mean has been named after Dr. T.L. Thomas who suggested its use and that of higher order Means in 1965. He named them the 3 point mean (Schuler Mean), 4 point mean, 5 point mean etc. The First Mean or 2 point mean was not considered. The process is summarised as follows:-

Observations	First Mean	Schuler Mean	Thomas Mean
y	S^1	S^2	S^3
y_1			
	$S^1_1 = \frac{y_1+y_2}{2}$		
y_2		$S^2_1 = \frac{S^1_1+S^1_2}{2} = \frac{y_1+2y_2+y_3}{4}$	
	$S^1_2 = \frac{y_2+y_3}{2}$		$S^3_1=\frac{S^2_1+S^2_2}{2} = \frac{y_1+3y_2+3y_3+y_4}{8}$
y_3		$S^2_2 = \frac{S^1_2+S^1_3}{2} = \frac{y_2+2y_3+y_4}{4}$	
	$S^1_3 = \frac{y_3+y_4}{2}$		$S^3_2=\frac{S^2_2+S^2_3}{2} = \frac{y_2+3y_3+3y_4+y_5}{8}$
y_4		$S^2_3 = \frac{S^1_3+S^1_4}{2} = \frac{y_3+2y_4+y_5}{4}$	
	$S^1_4 = \frac{y_4+y_5}{2}$		
y_5			

The coefficients of the original observations in each Mean may be obtained from the Binomial expansion $(a+b)^n$ and the denominator is the sum of these coefficients, according to Thomas (1965). Alternatively these values may be obtained more readily from the Pascal triangle, see Table 3.2. The denominator is 2^{n-1}.

It is instructive to examine the mathematical reasoning behind this process of taking means. Equation (2) is an approximation of the rigorous expression

$$y_i = B(-1)^{i-1} e^{-\alpha i} + \theta_o - v_i \qquad *$$

which in extenso gives :-

$$y_1 = B(1-\alpha + \frac{1}{2!}\alpha^2 - \frac{1}{3!}\alpha^3 + \frac{1}{4!}\alpha^4 \ldots\ldots\ldots\ldots) + \theta_o - v_1$$

$$\vdots$$

$$y_{j-2} = B(-1)^{j-3}\left\{1-(j-2)\alpha + \frac{1}{2!}\left[(j-2)\alpha\right]^2 - \frac{1}{3!}\left[(j-2)\alpha\right]^3 + \frac{1}{4!}\left[(j-2)\alpha\right]^4 \ldots\right\} + \theta_o - v_{j-2}$$

$$y_{j-1} = B(-1)^{j-2}\left\{1-(j-1)\alpha + \frac{1}{2!}\left[(j-1)\alpha\right]^2 - \frac{1}{3!}\left[(j-1)\alpha\right]^3 + \frac{1}{4!}\left[(j-1)\alpha\right]^4 \ldots\right\} + \theta_o - v_{j-1}$$

$$y_j = B(-1)^{j-1}\left\{1-j\alpha + \frac{1}{2!}(j\alpha)^2 - \frac{1}{3!}(j\alpha)^3 + \frac{1}{4!}(j\alpha)^4 \ldots\ldots\ldots\ldots\ldots\ldots\right\} + \theta_o - v_j$$

$$y_{j+1} = B(-1)^{j}\left\{1-(j+1)\alpha + \frac{1}{2!}\left[(j+1)\alpha\right]^2 - \frac{1}{3!}\left[(j+1)\alpha\right]^3 + \frac{1}{4!}\left[(j+1)\alpha\right]^4 \ldots\right\} + \theta_o - v_{j+1}$$

$$y_{j+2} = B(-1)^{j+1}\left\{1-(j+2)\alpha + \frac{1}{2!}\left[(j+2)\alpha\right]^2 - \frac{1}{3!}\left[(j+2)\alpha\right]^3 + \frac{1}{4!}\left[(j+2)\alpha\right]^4 \ldots\right\} + \theta_o - v_{j+2}$$

$$\vdots$$

---

* For convenience α instead of α' has been used in the following analysis

$$y_n = B(-1)^{n-1}\left(1-n\alpha + \frac{1}{2!}(n\alpha)^2 - \frac{1}{3!}(n\alpha)^3 + \frac{1}{4!}(n\alpha)^4 \ldots\ldots\ldots\ldots\right) + \theta_o - v_n$$

$$n \geq 3 \qquad j \geq 3$$

Taking means of consecutive equations gives :-

First Means.

$$S^1_{j-2} = \frac{y_{j-2}+y_{j-1}}{2} = \frac{B(-1)^{j-2}}{2}\left[-\alpha + \frac{(2j-3)\alpha^2}{2!} - \left((j-1)^3-(j-2)^3\right)\frac{\alpha^3}{3!}\right.$$
$$\left. + \left((j-1)^4-(j-2)^4\right)\frac{\alpha^4}{4!} \quad \ldots\right] + \theta_o - \frac{v_{j-2}+v_{j-1}}{2}$$

$$S^1_{j-1} = \frac{y_{j-1}+y_j}{2} = \frac{B(-1)^{j-1}}{2}\left(-\alpha + \frac{(2j-1)\alpha^2}{2!} - \left(j^3-(j-1)^3\right)\frac{\alpha^3}{3!} + \left(j^4-(j-1)^4\right)\frac{\alpha^4}{4!}\right.$$
$$\left. \ldots\ldots \right] + \theta_o - \frac{v_{j-1}+v_j}{2}$$

$$S^1_j = \frac{y_j+y_{j+1}}{2} = \frac{B(-1)^j}{2}\left(-\alpha + \frac{(2j+1)\alpha^2}{2!} - \left((j+1)^3-j^3\right)\frac{\alpha^3}{3!} + \left((j+1)^4-j^4\right)\frac{\alpha^4}{4!}\right.$$
$$\left. \ldots\ldots \right] + \theta_o - \frac{v_j+v_{j+1}}{2}$$

$$S^1_{j+1} = \frac{y_{j+1}+y_{j+2}}{2} = \frac{B(-1)^{j+1}}{2}\left[-\alpha + \frac{(2j+3)\alpha^2}{2!} - \left((j+2)^3+(j+1)^3\right)\frac{\alpha^3}{3!}\right.$$
$$\left. + \left((j+2)^4 - (j+1)^4\right)\frac{\alpha^4}{4!}\ldots\right] + \theta_o - \frac{v_{j+1}+v_{j+2}}{2}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Schuler Means

$$S^2_{j-2} = \frac{S^1_{j-2}+S^1_{j-1}}{2} = \frac{B(-1)^{j-1}}{4}\left[\alpha^2 - \frac{6(j-1)\alpha^3}{3!} + (12j^2-24j+14)\frac{\alpha^4}{4!} \;\ldots\ldots\right] + \theta_o - \frac{v_{j-2}+2v_{j-1}+v_j}{4}$$

$$S^2_{j-1} = \frac{S^1_{j-1}+S^1_{j}}{2} = \frac{B(-1)^{j}}{4}\left[\alpha^2 - \frac{6j\alpha^3}{3!} + (12j^2+2)\frac{\alpha^4}{4!} \;\ldots\ldots\right] + \theta_o - \frac{v_{j-1}+2v_{j}+v_{j+1}}{4}$$

$$S^2_{j} = \frac{S^1_{j}+S^1_{j+1}}{2} = \frac{B(-1)^{j+1}}{4}\left[\alpha^2 - \frac{6(j+1)\alpha^3}{3!} + (12j^2+24j+14)\frac{\alpha^4}{4!} \;\ldots\ldots\right] + \theta_o - \frac{v_{j}+2v_{j+1}+v_{j+2}}{4}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Thomas Means.

$$S^3_{j-2} = \frac{S^2_{j-2}+S^2_{j-1}}{2} = \frac{B(-1)^{j}}{8}\left[-\alpha^3+12(2j-1)\frac{\alpha^4}{4!} \;\ldots\ldots\right] + \theta_o - \frac{v_{j-2}+3v_{j-1}+3v_j+v_{j+1}}{8}$$

$$S^3_{j-1} = \frac{S^2_{j-1}+S^2_{j}}{2} = \frac{B(-1)^{j+1}}{8}\left[-\alpha^3+12(2j+1)\frac{\alpha^4}{4!} \;\ldots\right] + \theta_o - \frac{v_{j-1}+3v_{j}+3v_{j+1}+v_{j+2}}{8}$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Mr. M. Maughan of the Department of Surveying, University of New South Wales has shown that the series expressions in each Mean may be written concisely as follows :-

First Means	$B(-1)^{i-1}e^{-\alpha i}\left(\frac{1-e^{-\alpha}}{2}\right)$
Schuler Means	$B(-1)^{i-1}e^{-\alpha i}\left(\frac{1-e^{-\alpha}}{2}\right)^2$
Thomas Means	$B(-1)^{i-1}e^{-\alpha i}\left(\frac{1-e^{-\alpha}}{2}\right)^3$

It is apparent from the series expansions that the process of taking successive means eliminates terms containing powers of α i.e. for the First Means terms containing α^0 are removed, for the Schuler Means terms containing α are removed, for the Thomas Means terms containing α^2 are removed etc. Thus if we consider that a sufficient approximation to the mathematical model is

$$y_i \quad = \quad B(-1)^{i-1} + \theta_o - v_i$$

then we consider that $\alpha = 0$ and the First Means will contain only θ_o and combinations of the residuals v. For the Schuler Means the model is

$$y_i \quad = \quad B(-1)^{i-1}(1-i\alpha) + \theta_o - v_i \quad \text{etc.}$$

Thus if we accept the average (or some other combination) of the First Means or the average of the Schuler Means etc. as an estimate of θ_o then it is presumed that the quantities

$$B(-1)^{i-1}\,e^{-\alpha i}\left(\frac{1-e^{-\alpha}}{2}\right)$$

or

$$B(-1)^{i-1}\,e^{-\alpha i}\left(\frac{1-e^{-\alpha}}{2}\right)^2 \qquad \text{etc.}$$

are not numerically significant. This is demonstrated graphically in Fig. 3.2 for the First and Schuler Means.

An alternative explanation of this process is that if the First Mean is considered sufficient, then damping is ignored and the envelope of the turning points is composed of two parallel straight lines, for the Schuler Mean equally inclined straight lines, for the Thomas Mean quadratic curves and so on. Each of which, in their particular case are considered to be a sufficient approximation to an exponential envelope.

Whichever Mean is considered appropriate to the observations, there remains a serious objection to the second stage of taking an average. Taking the average is sound if we are considering original independent observations but not if the quantities to be adjusted are functions of these observations. Although the average in this case is inadmissable for a least squares adjustment, we may still use the derived quantities in an adjustment process provided that the mathematical correlation is taken into account.

Let the parametric equations under consideration be of the form

$$\begin{aligned}
\theta_o - S_1 &= V_1 \\
\theta_o - S_2 &= V_2 \\
\theta_o - S_3 &= V_3 \\
\vdots \quad\quad & \quad \vdots \\
\theta_o - S_m &= V_m
\end{aligned}$$

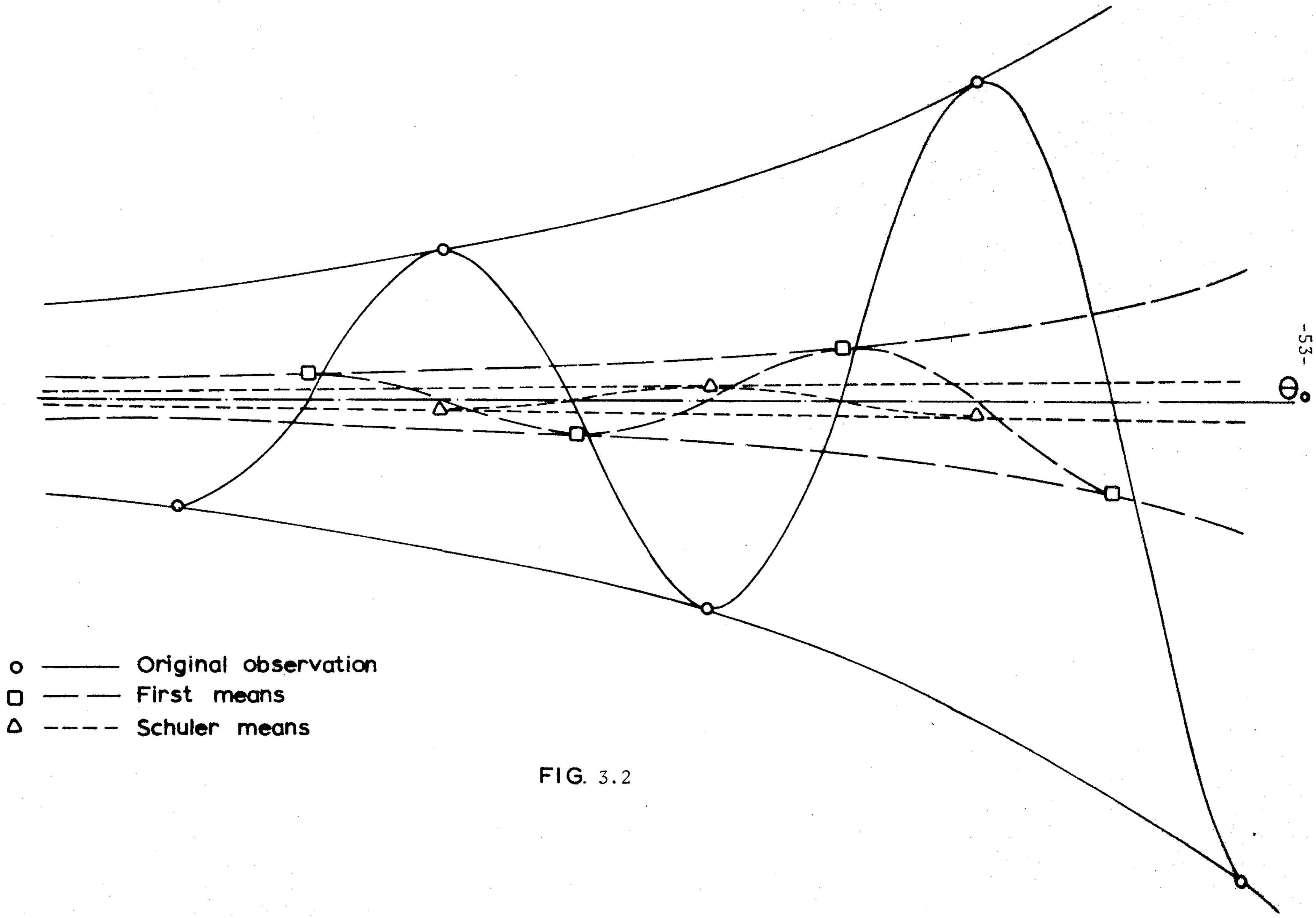


FIG. 3.2

where for the First Mean $m = n-1$

Schuler Mean $m = n-2$

Thomas Mean $m = n-3$ etc.

or in matrix notation

$$A\,\theta_o - S = V$$

where A is a unit column vector of dimension m, S is the column vector of the individual Means

$$\begin{bmatrix} S_1 \\ S_2 \\ S_3 \\ \cdot \\ \cdot \\ \cdot \\ S_m \end{bmatrix}$$

V is the column vector of the corrections

$$\begin{bmatrix} V_1 \\ V_2 \\ V_3 \\ \cdot \\ \cdot \\ \cdot \\ V_m \end{bmatrix}$$

and θ_o is 1 x 1 matrix

The coefficients in the weight coefficient matrix of the individual Means may be found by applying the general law of propagation of variances to

$$S_1^1 = \frac{y_1+y_2}{2}$$

$$S_2^1 = \frac{y_2+y_3}{2}$$

$$S_3^1 = \frac{y_3+y_4}{2} \qquad \text{etc. for the First Mean,}$$

and obtaining

$$g_{S_1^1} = \frac{1}{2} g_{y_1} + \frac{1}{2} g_{y_2}$$

$$g_{S_2^1} = \frac{1}{2} g_{y_2} + \frac{1}{2} g_{y_3}$$

$$g_{S_3^1} = \frac{1}{2} g_{y_3} + \frac{1}{2} g_{y_4} \qquad \text{etc.}$$

and

$$g_{S_1^1 S_1^1} = \frac{1}{4} g_{y_1 y_1} + \frac{1}{4} g_{y_2 y_2} = \frac{1}{2} g_{yy}$$

$$g_{S_1^1 S_2^1} = \frac{1}{4} g_{y_2 y_2} = \frac{1}{4} g_{yy} \qquad \text{etc.}$$

assuming that each observation is independent and of equal variance. Thus the required weight matrix is

$$G^1 = \frac{1}{4} \begin{bmatrix} 2 & 1 & & & & & \\ 1 & 2 & 1 & & & & \\ & 1 & 2 & 1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 2 \end{bmatrix}$$

Applying the same technique to the Schuler and Thomas Means gives

$$G^2 = \frac{1}{16}\begin{bmatrix} 6 & 4 & 1 & & & & & & & & \\ 4 & 6 & 4 & 1 & & & & & & & \\ 1 & 4 & 6 & 4 & 1 & & & & & & \\ & 1 & 4 & 6 & 4 & 1 & & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & & & & 1 & 4 & 6 & 4 & 1 & \\ & & & & & & 1 & 4 & 6 & 4 & 1 \\ & & & & & & & 1 & 4 & 6 & 4 \\ & & & & & & & & 1 & 4 & 6 \end{bmatrix}$$

and for G^3

$$\frac{1}{64}\begin{bmatrix} 20 & 15 & 6 & 1 & & & & & & & & & \\ 15 & 20 & 15 & 6 & 1 & & & & & & & & \\ 6 & 15 & 20 & 15 & 6 & 1 & & & & & & & \\ 1 & 6 & 15 & 20 & 15 & 6 & 1 & & & & & & \\ & 1 & 6 & 15 & 20 & 15 & 6 & 1 & & & & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & & \\ & & & & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & & & \\ & & & & & 1 & 6 & 15 & 20 & 15 & 6 & 1 & \\ & & & & & & 1 & 6 & 15 & 20 & 15 & 6 & 1 \\ & & & & & & & 1 & 6 & 15 & 20 & 15 & 6 \\ & & & & & & & & 1 & 6 & 15 & 20 & 15 \\ & & & & & & & & & 1 & 6 & 15 & 20 \end{bmatrix}$$

respectively.

The coefficients of these matrices may be conveniently obtained from the Pascal Triangle. The single normal equation is

$$A^T G^{-1} A \, \theta_o + A^T G^{-1} S = 0$$

and its solution

$$\theta_o = -(A^T G^{-1} A)^{-1} A^T G^{-1} S$$

This last equation is of little practical value unless general expressions can be found for the coefficients of the individual Means for any number of observations. An attempt was made to solve this problem by inverting the G matrices by pivotal condensation and bordering methods. This proved to be uneconomical by hand for the Schuler and Thomas Means when the number of observations was large. The writer is grateful for the assistance of Mr. M. Maughan who modified an existing IBM 360 structural engineering computer programme for matrix inversion to suit the requirements of these matrices. The inverse matrices for n up to 8 are shown in Table 3.4. From the computer output general expressions have been derived using the technique of divided differences.

<u>First Mean.</u>

n even

$$\theta_o = \frac{2}{n}(S^1_1 + S^1_3 + S^1_5 + S^1_7 \, \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots)$$

or

$$\theta_o = \frac{2}{n} \sum_{i=1}^{\frac{1}{2}n} S^1_{2i-1}$$

n odd

$$\theta_o = \frac{4}{n^2-1}\left[\frac{(n-1)}{2}S^1_1 + S^1_2 + \frac{(n-3)}{2}S^1_3 + 2S^1_4 + \frac{(n-5)}{2}S^1_5 + 3S^1_6 \ldots\ldots\right]$$

or

$$\theta_o = \frac{4}{n^2-1}\sum_{i=1}^{\frac{1}{2}(n-1)}\left[\frac{(n-2i+1)}{2}S^1_{2i-1} + iS^1_{2i}\right]$$

Schuler Mean.

n even

$$\theta_o = \frac{4}{n(n^2-4)}\left[\begin{array}{ll}(n-2)(n-1)\ S^2_1 & -\ (n-2)(n-7)\ S^2_2 \\ +2(n-4)(n-5)\ S^2_3 & -\ 2(n-4)(n-11)S^2_4 \\ +3(n-6)(n-9)\ S^2_5 & -\ 3(n-6)(n-15)S^2_6 \\ +4(n-8)(n-13)S^2_7 & -\ 4(n-8)(n-19)S^2_8 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}\right]$$

or

$$\theta_o = \frac{4}{n(n^2-4)}\sum_{i=1}^{\frac{1}{2}(n-2)} i(n-2i)\left[(n-4i+3)S^2_{2i-1} - (n-4i-3)S^2_{2i}\right]$$

n odd

$$\theta_o = \frac{4}{(n^2-1)}\left[\begin{array}{ll}(n-1)\ S^2_1 & -\ (n-3)\ S^2_2 \\ +2(n-3)\ S^2_3 & -\ 2(n-5)\ S^2_4 \\ +3(n-5)\ S^2_5 & -\ 3(n-7)\ S^2_6 \\ +4(n-7)\ S^2_7 & -\ 4(n-9)\ S^2_8 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \end{array}\right]$$

or

$$\theta_o = \frac{4}{(n^2-1)}\left[(n-1)S_1^2 + \sum_{i=1}^{\frac{1}{2}(n-3)} (n-2i-1)\left[-iS_{2i}^2 + (i+1)S_{2i+1}^2\right]\right]$$

Thomas Mean

n even

$$\theta_o = \frac{8}{n(n^2-4)}\Big[(n-2)(n-1)\,S_1^3 - 2(n-2)(n-4)\,S_2^3$$
$$+2(n-4)(2n-7)\,S_3^3 - 6(n-4)(n-6)\,S_4^3$$
$$+3(n-6)(3n-17)S_5^3 - 12(n-6)(n-8)\,S_6^3$$
$$+4(n-8)(4n-31)S_7^3 - 20(n-8)(n-10)S_8^3$$
$$\vdots \qquad \vdots \Big]$$

or

$$\theta_o = \frac{8}{n(n^2-4)}\Big[(n-2)(n-1)S_1^3 + \sum_{i=1}^{\frac{1}{2}(n-4)} \Big[-i(i+1)(n-2i)(n-2i-2)S_{2i}^3$$
$$+(i+1)(n-2i-2)\left[(i+1)n-2i^2-4i-1\right] S_{2i+1}^3\Big]\Big]$$

n odd

$$\theta_o = \frac{8}{(n^2-1)(n^2-9)}\Big[(n-3)(n^2-3n+2)\,S_1^3 - 2(n-3)(n^2-9n+14)\,S_2^3$$
$$+2(n-5)(2n^2-20n+42)\,S_3^3 - 6(n-5)(n^2-15n+46)S_4^3$$
$$+3(n-7)(3n^2-49n+170)S_5^3 - 12(n-7)(n^2-21n+94)S_6^3$$
$$+4(n-9)(4n^2-90n+434)S_7^3 - 20(n-9)(n^2-27n+158)S_8^3$$
$$+5(n-11)(5n^2-143n+882)S_9^3 - 30(n-11)(n^2-33n+238)S_{10}^3$$
$$\vdots \qquad \vdots \Big]$$

or

$$\theta_o = \frac{8}{(n^2-1)(n^2-9)} \sum_{i=1}^{\frac{1}{2}(n-3)} \Big\{ i(n-2i-1) \left\{ in^2-(3i-2)(2i+1)n+2(2i-1)(2i^2-1) \right\} S^3_{2i-1} - i(i+1)(n-2i-1) \left\{ n^2-3(2i+1)n+2(4i^2+4i-1) \right\} S^3_{2i} \Big\}.$$

To facilitate numerical calculation the values of the coefficients up to $n = 8$ are given in Table 3.3.

The estimate of the variance of the adjusted parameter θ_o is given by

$$\hat{S}^2_{\theta_o} = Q_{\theta_o \theta_o} \frac{V^T G^{-1} V}{r}$$

where $Q_{\theta_o \theta_o} = (A^T G^{-1} A)^{-1}$, which is tabulated below, and r is the number of redundancies.

$Q_{\theta_o \theta_o}$

	n odd	n even
First Mean	$\frac{n}{n^2-1}$	$\frac{1}{n}$
Schuler Mean	$\frac{n}{n^2-1}$	$\frac{n^2-1}{n(n^2-4)}$
Thomas Mean	$\frac{n(n^2-4)}{(n^2-1)(n^2-9)}$	$\frac{n^2-1}{n(n^2-4)}$

The process outlined does not overcome all the disadvantages of taking the average of the derived Means, as stated before, but it does remove the principal objection in that it gives the least squares estimate of θ_o. The method of obtaining this value is extremely simple and is preferred to any previous method. It may be noted that for an even number of observations, the solutions by the Schuler and Thomas Mean are identical and for an odd number of observations the First and Schuler Mean are identical. A suggested technique of calculation which combines the characteristics of the Linear Damped Model and the Schuler Mean has been used in the two numerical examples.

It was stated earlier that because original observations are combined, a poor observation or mistake may be hidden in the Schuler Means thus biasing $\bar{S}$. Jones (1970) has shown that the probability of detecting such a mistake from turning point observations up to n = 8 is very low. Kondrat'yev (1967) has described a method which attempts to guard against such a contingency, in the following way:-

(1) Determine the damping factor for a gyro-theodolite from a long series of observations before field work is commenced.

(2) Compile a table of the damping per period, with amplitude as argument.

(3) Take the difference between alternate observations in a set of field observations and compare these values with those given in the table.

(4) If a difference is lower or higher than the precalculated value by a set limit then one of the observations is unsatisfactory.

Here again the test requires a grouping of the observations into differences, and an objection could be lodged on similar grounds as before, although it should be noted that the number of unknown parameters has decreased from three to two, which improves the chances of detecting a blunder.

3.3 NUMERICAL CALCULATIONS

The two most important requirements for practical calculations are that :

(1) Mistakes may be readily detected at the time of observation. If any extra observations are then required they may be done with the minimum of inconvenience.

(2) Calculation methods should be compatible with the expected precision of the determination.

For both of these requirements, consideration must also be given to the conditions under which these calculations are performed. In the field, quick and simple methods are preferred and in most cases approximate methods will suffice. However in the office the second requirement above is of paramount importance. Furthermore the choice of method may be governed by the calculation aids. For the high speed digital computer it is immaterial which of the variety of rigorous expressions for the required unknowns is used. However, for a desk machine those expressions which offer speed and simplicity are preferred.

Suggested procedures are as follows:-

(a) Field Calculation.

(1) Calculate Schuler Means progressively as the observations are being made and plot the turning points and Schuler Means at a suitable scale on graph paper. This is a recommended procedure by the Wild Instrument Company in their handbook for the GAKI and is called the "oscillation graph."

(2) Calculate θ_o either approximately from the mean of the Schuler Means or rigorously from the weighted mean of the Schuler Means.

(3) Mistakes or large errors in the original observations may be concealed by this process of taking Schuler Means and may not be readily discernible in the oscillation graph. As stated before, the geometrical interpretation of the least squares adjustment is a pair of straight lines at the same inclination to the line of mean oscillation passing through the observations. As the position of the line of mean oscillation is known, it is a simple matter to transfer either the left or the right observations over this line and then construct a line of best fit through all of these points. This transfer may be done either by (i) folding the graph paper along the line of mean oscillation or (ii) subtracting either the left or right observations from $2\theta_o$ and replotting.

(4) Draw a line of best fit through these points. The deviations from this line will give a good estimate of the errors of the original observations. Instead of estimating the line of best fit by eye, a procedure proposed by Eddington given by Jeffreys (1948) is eminently suitable. Rainsford (1957) describes the method as follows:-

"Divide the data into 3 equal groups : the line joining the mean positions of the first and last groups gives the slope of the line, which is then fixed in position by making it pass through the mean position of all observations. Not only is this solution very simple but, provided that the observations are uniformly spaced, its efficiency is of the order of 8/9 of that of a rigorous least squares solution."

Office Calculation.

(1) Calculate θ_o from the weighted mean of the Schuler Means.

(2) Calculate, for n even, from equation (7)

$$a = 2\left(\frac{1}{n}\sum_{i=1}^{n} y_i - \theta_o\right)$$

and from equation (11)

$$B' = \frac{1}{n}\left(\sum_{i=1}^{n} y_i^O - \sum_{i=1}^{n} y_i^E\right)$$

For n odd, after re-arranging equation (14)

$$a = \frac{12}{n(n^2-1)}\left[\frac{n-1}{2}(y_1-y_n) - \frac{n-3}{2}(y_2-y_{n-1}) + \frac{n-5}{2}(y_3-y_{n-2}) \ldots\ldots \ldots\ldots(-1)^{\frac{1}{2}(n+1)}\left(y_{\frac{1}{2}(n-1)}-y_{\frac{1}{2}(n+3)}\right)\right]$$

and from equation (12)

$$B' = \frac{1}{2}\left[\frac{\sum_{i=1}^{n} y_i^O}{\frac{1}{2}(n+1)} - \frac{\sum_{i=1}^{n} y_i^E}{\frac{1}{2}(n-1)}\right]$$

(3) Calculate the variances according to equations (13)

Example A, n even.

Observations Left	Right	S.M. 359°59'	SM - $\overline{SM}$ V
358° 24' 18"			
(358 24 45)*	1° 33' 36"	10".5	-2".0
358 25 12	(1 33 15)	13.5	+1.0
(358 25 24)	1 32 54	9.0	-3.5
358 25 36	(1 32 48)	12.0	-0.5
(358 25 54)	1 32 42	18.0	+5.5
358 26 12	(1 32 12)	12.0	-0.5
	1 31 42		
	$\overline{SM}$	12.5	$\sum = 0$

$$\theta_o = 359°\ 59'\ 12''.5 + \frac{7(-2.0 - 0.5) - (1.0 + 5.5) + 4(-3.5 - 0.5)}{20}$$

$$\theta_o = 359°\ 59'\ 12''.5 - 2''.0 = \underline{\underline{359°\ 59'\ 10''.5}}$$

$$\sum y^O = 4(358°\ 20') + 21'\ 18'' \qquad \sum y^E = 4(1°\ 30') + 10'\ 54''$$

$$a = 2\left[\frac{1}{8}\left\{4(359°\ 50') + 32'\ 12''\right\} - 359°\ 59'\ 10''.5\right]$$

$$\underline{\underline{a = -18''}}$$

---

* The figures in brackets are the intermediate steps in the calculation of the Schuler Means.

$$B' = \frac{1}{8}\left\{4(358^\circ\ 20') + 21'\ 18'' - 4(1^\circ\ 30') - 10'\ 54''\right\}$$

$$B' = 358^\circ\ 26'\ 18''$$

$$\alpha'' = \frac{a}{B'} = \frac{-18}{-5622} = 3.20 \times 10^{-3}$$

$$B' + \theta_o = 358^\circ\ 25'\ 28''.5$$

$$-B' + \theta_o = 1^\circ\ 32'\ 52''.5$$

$$v_1 = B' + \theta_o - y_1 + \frac{7}{2}a = \begin{array}{rrr} 358^\circ & 25' & 28''.5 \\ -358 & 24 & 18.0 \\ & -\ 1 & 03.0 \\ \hline & v_1 & +\ 7.5 \end{array}$$

$$v_2 = -B' + \theta_o - y_2 - \frac{5}{2}a = \begin{array}{rrr} 1 & 32 & 52.5 \\ -1 & 33 & 36.0 \\ + & & 45.0 \\ \hline & v_2 & +\ 1.5 \end{array}$$

$$v_3 = B' + \theta_o - y_3 + \frac{3}{2}a = \begin{array}{rrr} 358 & 25 & 28.5 \\ -358 & 25 & 12.0 \\ & & -\ 27.0 \\ \hline & v_3 & -\ 10.5 \end{array}$$

$$v_4 = -B' + \theta_o - y_4 - \frac{1}{2}a = \begin{array}{rrr} 1 & 32 & 52.5 \\ -1 & 32 & 54.0 \\ & & +\ 9.0 \\ \hline & v_4 & +\ 7.5 \end{array}$$

$$v_5 = B' + \theta_o - y_5 - \frac{1}{2} a = \begin{array}{r} 358^\circ\ 25'\ 28''.5 \\ -358\ \ 25\ \ 36.0 \\ +\ \ 9.0 \\ \hline v_5\ +\ \ 1.5 \end{array}$$

$$v_6 = -B' + \theta_o - y_6 + \frac{3}{2} a = \begin{array}{r} 1\ \ 32\ \ 52.5 \\ -1\ \ 32\ \ 42.0 \\ -\ 27.0 \\ \hline v_6\ -\ 16.5 \end{array}$$

$$v_7 = B' + \theta_o - y_7 - \frac{5}{2} a = \begin{array}{r} 358\ \ 25\ \ 28.5 \\ -358\ \ 26\ \ 12.0 \\ +\ 45.0 \\ \hline v_7\ +\ \ 1.5 \end{array}$$

$$v_8 = -B' + \theta_o - y_8 + \frac{7}{2} a = \begin{array}{r} 1\ \ 32\ \ 52.5 \\ -1\ \ 31\ \ 42.0 \\ -\ 1\ \ 03.0 \\ \hline v_8\ +\ \ 7.5 \end{array}$$

Check $\Sigma v^O = \Sigma v^E = 0$

$$\Sigma v^2 = 558.00$$

$$\hat{S}^2_y = \frac{558}{5} = 111.6$$

$$\hat{S}^2_{\theta_o} = \frac{21}{160} \cdot \frac{558}{5} = 14.65$$

$$\hat{S}^2_{B'} = \frac{1}{8} \cdot \frac{558}{5} = 13.95$$

$$\hat{S}^2_{\alpha''} = \frac{1}{40(-5622)^2} \cdot \frac{558}{5} = 8.8272 \times 10^{8}$$

SUMMARY.

$n = 8,\quad r = 5$		$\hat{S}_y$	$= \pm\ 10''.6$
θ_o	$= 359^\circ\ 59'\ 10''.5$	$\hat{S}_{\theta_o}$	$= \pm\ 3.8$
B'	$= 358^\circ\ 26'\ 18''$	$\hat{S}_{B'}$	$= \pm\ 3.7$
α''	$= 3.20 \times 10^{-3}$	$\hat{S}_{\alpha''}$	$= \pm\ 0.30 \times 10^{-3}$

Graphical Solution.

See previous calculation for Schuler Means and θ_o

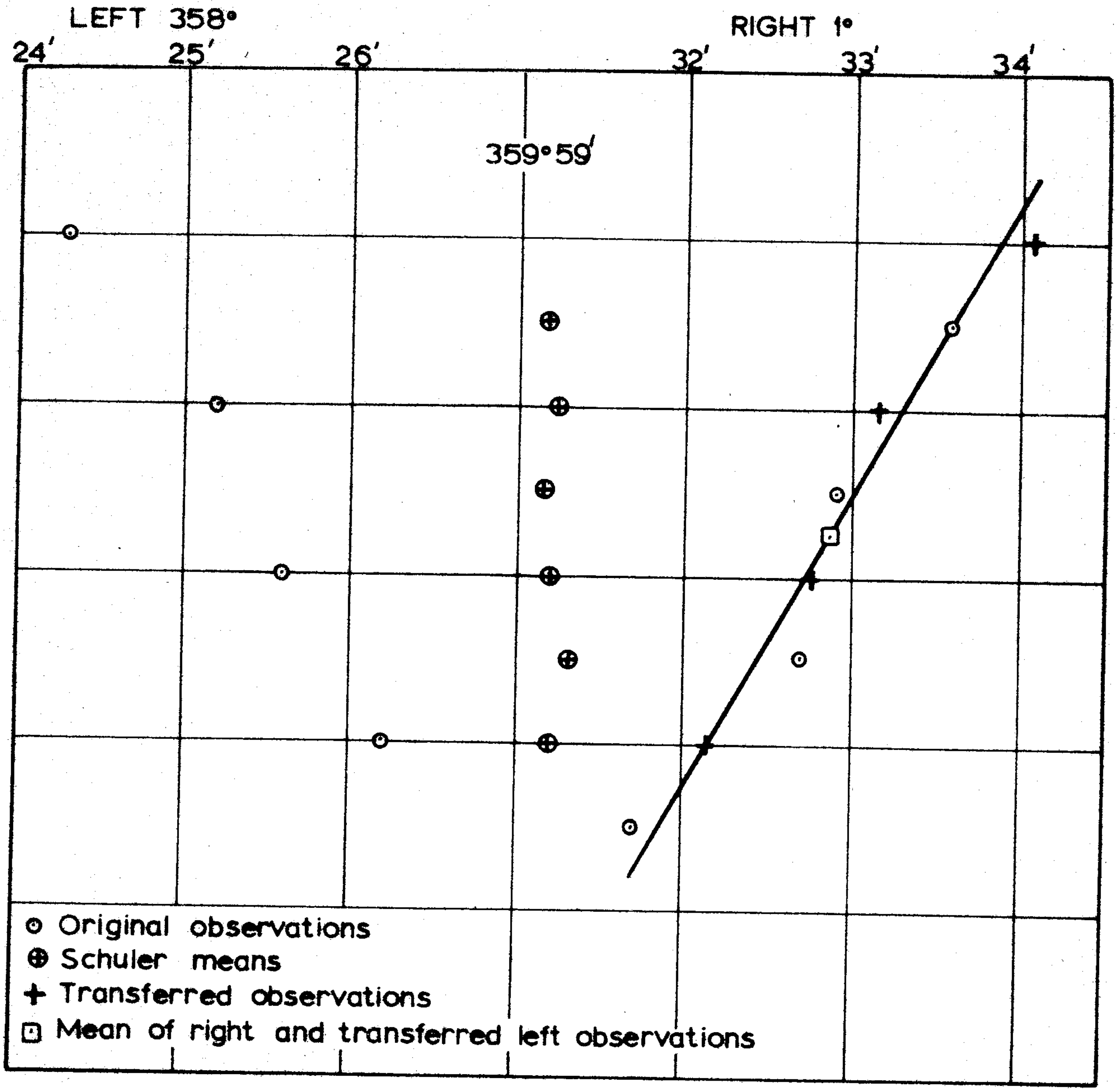
Observations

Left	Right	Transferred Left ($2\theta_o$ - Left)	Graphical Values (see plot)	v.
358°24'18"		1°34'03"	1°33'52"	+ 11"
	1°33'36"		1 33 35	- 1
358 25 12		1 33 09	1 33 18	- 9
	1 32 54		1 33 01	+ 7
358 25 36		1 32 45	1 32 44	+ 1
	1 32 42		1 32 27	- 15
358 26 12		1 32 09	1 32 10	- 1
	1 31 42		1 31 53	+ 11

$2\theta_o = 359^\circ\ 58'\ 21''$

Mean of the right and transferred left observations	1° 32' 52".5
Mean of the first three observations	1 33 36
Mean of the last three observations	1 32 11

$$\text{Slope } a = \frac{1^\circ\ 33'\ 36'' - 1^\circ\ 32'\ 11''}{5} = 17'' \text{ per half period.}$$



OSCILLATION GRAPH

Example B, n odd

Observations

Left	Right	SM $359°57'$	SM-$\overline{SM}$ V
	1° 11' 18"		
358° 44' 42"	(1 11 06)	54".0	- 7".8
(358 45 09)	1 10 54	61.5	- 0.3
358 45 36	(1 10 36)	66.0	+ 4.2
(358 45 51)	1 10 18	64.5	+ 2.7
358 46 06	(1 10 00)	63.0	+ 1.2
	1 09 42		
	$\overline{SM}$	61.8	$\sum = 0$

$$\theta_o = 359° \ 58' \ 01''.8 + \frac{3(-7.8 + 1.2) - 2(-0.3 + 2.7) + 4 \times 4.2}{6}$$

$$\theta_o = 359° \ 58' \ 01''.8 - 1''.3 = \underline{\underline{359° \ 58' \ 00''.5}}$$

$$\sum y^E = 3(358° \ 44') + 4' \ 24'' \qquad \sum y^0 = 4(1° \ 09') + 6' \ 12''$$

$$B' = \frac{1}{2}\left\{\frac{4(1° \ 09') + 6' \ 12''}{4} - \frac{3(358° \ 44') + 4' \ 24''}{3}\right\}$$

$$B' = \frac{1}{2}(1° \ 10' \ 33'' - 358° \ 45' \ 28'') = \underline{\underline{1° \ 12' \ 32''.5}}$$

$$\text{Note } \theta_o = \frac{358° \ 45' \ 28'' + 1° \ 10' \ 33''}{2} = \underline{\underline{359° \ 58' \ 00''.5}}$$

$$a = \frac{1}{28}(3 \times 1' \ 36'' + 2 \times 1' \ 24'' + 36'')$$

$$a = \underline{\underline{17''.57}}$$

$$\alpha'' = \frac{a}{B'} = \frac{17.57}{4352.5} = \underline{\underline{4.04 \times 10^{-3}}}$$

$$B' + \theta_o = 1° \ 10' \ 33''$$

$$-B' + \theta_o = 358° \ 45' \ 28''$$

$v_1 = B' + \theta_o + 3a - y_1 =$

1°	10'	33"
-1	11	18
	+	52.71
v_1	+	7.71

$v_2 = -B' + \theta_o - 2a - y_2 =$

358	45	28
-358	44	42
	-	35.14
v_2	+	10.86

$v_3 = B' + \theta_o + a - y_3 =$

1	10	33
-1	10	54
	+	17.57
v_3	-	3.43

$v_4 = -B' + \theta_o - y_4 =$

358	45	28
-358	45	36
v_4	-	8.00

$v_5 = B' + \theta_o - a - y_5 =$

1	10	33
-1	10	18
	-	17.57
v_5	-	2.57

$v_6 = -B' + \theta_o + 2a - y_6 =$

358	45	28
-358	46	06
	+	35.14
v_6	-	2.86

$v_7 = B' + \theta_o - 3a - y_7 =$

1	10	33
-1	09	42
	-	52.72
v_7	-	1.72

Check $\Sigma v^O = \Sigma v^E = 0$

$\Sigma v^2 = 270.89$

$$\hat{S}^2_y = \frac{270.89}{4} = 67.72$$

$$\hat{S}^2_{B'} = \hat{S}^2_{\theta_o} = \frac{7}{48} \cdot \frac{270.89}{4} = 9.88$$

$$\hat{S}^2_{\alpha''} = \frac{1}{28(4352.5)^2} \cdot \frac{270.89}{4} = 0.1277 \times 10^{-6}$$

SUMMARY.

n	=	7, r = 4	$\hat{S}_y$	=	± 8.″2
θ_o	=	359° 58' 00.″5	$\hat{S}_{\theta_o}$	=	± 3.1
B'	=	1° 12' 32.″5	$\hat{S}_{B'}$	=	± 3.1
α''	=	4.04×10^{-3}	$\hat{S}_{\alpha''}$	=	$\pm 0.36 \times 10^{-3}$

Graphical Solution.

See previous calculations for Schuler Means and θ_o

Observations

Left	Right	Transferred Left ($2\theta_o$ - Left)	Graphical Values (See plot)	v
	1° 11' 18"		1° 11' 27"	+ 9
358° 44' 42"		1° 11' 19"	1 11 09	+ 10
	1 10 54		1 10 51	- 3
358 45 36		1 10 25	1 10 33	- 10
	1 10 18		1 10 15	- 3
358 46 06		1 09 55	1 09 57	2
	1 09 42		1 09 39	- 3

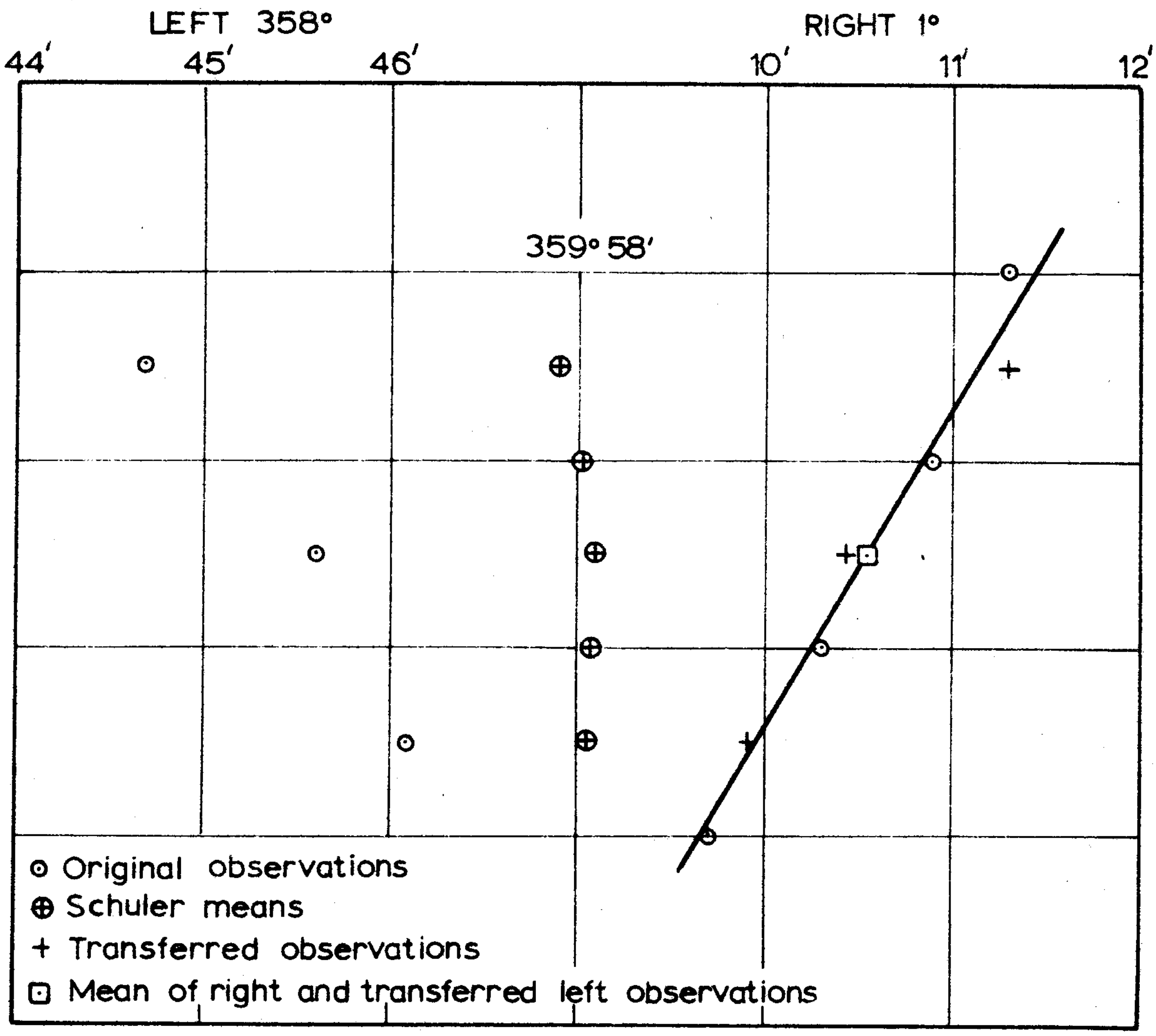
$2\theta_o$ = 359° 56" 01"

Mean of the right and transferred left observations 1° 10' 33"

Mean of the first two observations 1 11 18

Mean of the last two observations 1 09 48

$$\text{Slope } a = \frac{1^\circ\, 11'\, 18'' - 1^\circ\, 09'\, 48''}{5} = 18'' \text{ per half period.}$$



OSCILLATION GRAPH

3.4 CONCLUSION

Lauf (1967a) has summarised the methods used by other writers in the solution of this problem and it is of interest to re-examine these in the light of this investigation. Kohlrausch(1944) advocates the use of an odd number of turning points and then combines the mean of the readings on one side with the mean of the readings on the other side to obtain a final mean. This approach is quite sound theoretically for a lightly damped oscillation as has been shown and it is hard to understand why Schuler (1932) criticises Kohlrausch. Schuler is also critical of Basch and Wilski (1917). This method is based on finding a line of best fit to observations on each side and then to combine them to give a final value. This technique is basically sound if we are considering the linear damped case. However, an extra condition must be added to the adjustment because the slope of the damping envelope must be the same on either side of the line of mean oscillation. The approach of Thomas (1965, 1967) is of interest because it attempts to take into account a heavier damping, which seldom prevails with gyro-theodolite observations. The Schuler and Thomas Means are generally quite sufficient for this work and there is no necessity to continue taking higher order Means.

Consideration must be given to the question of the number of observations required for the determination. If the internal precision of the observations is compatible with the external precision then the number of observations required will depend upon the accuracy required for the determination which can be estimated a priori from variance estimates based on past experience. From the calculation and adjustment standpoint it is immaterial whether we observe an odd or an even number although with an even number of observations the weighted Schuler Mean will give the least squares solution for quadratic damping.

In conclusion it can be stated that the derivations contained in this Chapter are of a general nature and do not presume a limit or set values for the number of observations. A combination of the characteristics of each method gives a simple solution for numerical problems which is in sympathy with the observations. By application of the general law of propagation of variances it has been shown that we can still take advantage of simple relationships which are similar to those which have been proposed by Schuler whilst retaining the benefit of rigour in the least squares adjustment.

TABLE 3.1 SOLUTIONS OF NORMAL EQUATIONS FOR LINEAR D^MPING.

n	a									b								
		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
3	$\frac{1}{2}$	(1		-1					)	$\frac{1}{2}$	(3		-1					
4	$\frac{1}{4}$	(1	-1	-1	1				)	$\frac{1}{2}$	(2	-1		1				
5	$\frac{1}{10}$	(2	-1		1	-2			)	$\frac{1}{30}$	(28	-9	10	9	-8			)
6	$\frac{1}{8}$	(1	-1			-1	1		)	$\frac{1}{24}$	(17	-9	8		-1	9		)
7	$\frac{1}{28}$	(3	-2	1		-1	2	-3	)	$\frac{1}{28}$	(19	-8	11		3	8	-5	)
8	$\frac{1}{40}$	(3	-3	1	-1	-1	1	-3	3)	$\frac{1}{20}$	(11	-6	7	-2	3	2	-1	6)

n	c									$\theta_o = \frac{b+c}{2}$								
		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8
3		(-1	1	1					)	$\frac{1}{4}$	(1	2	1					)
4	$\frac{1}{4}$	(-3	5	3	-1				)	$\frac{1}{8}$	(1	3	3	1				)
5	$\frac{1}{5}$	(-3	4		1	3			)	$\frac{1}{12}$	(2	3	2	3	2			)
6	$\frac{1}{6}$	(-3	5		2	3	-1		)	$\frac{1}{48}$	(5	11	8	8	11	5		)
7	$\frac{1}{21}$	(-9	13	-3	7	3	1	9	)	$\frac{1}{4}$	(3	4	3	4	3	4	3	)
8	$\frac{1}{8}$	(-3	5	-1	3	1	1	3	-1)	$\frac{1}{80}$	(7	13	9	11	11	9	13	7)

TABLE 3.1 (Contd.)

n	$B = \frac{b-c}{2}$									
		y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	
3	$\frac{1}{4}$ (	5	-2	-3						)
4	$\frac{1}{8}$ (	7	-7	-3	3					)
5	$\frac{1}{60}$ (	46	-33	10	3	-26				)
6	$\frac{1}{48}$ (	29	-29	8	-8	-13	13			)
7	$\frac{1}{168}$ (	93	-76	45	-28	-3	20	-51		)
8	$\frac{1}{80}$ (	37	-37	19	-19	1	-1	-17	17	)

TABLE 3.2 THE PASCAL TRIANGLE.

n		Coefficients in Mean	Coefficients of Weight Matrix	Denominator 2^{n-1}
1	1			1
2	1 1	First		2
3	1 2 1	Schuler	First	4
4	1 3 3 1	Thomas		8
5	1 4 6 4 1		Schuler	16
6	1 5 10 10 5 1			32
7	1 6 15 20 15 6 1		Thomas	64
8	1 7 21 35 35 21 7 1			128
9	1 8 28 56 70 56 28 8 1			256
10	1 9 36 84 126 126 84 36 9 1			512
11	1 10 45 120 210 252 210 120 45 10 1			1024
12	1 11 55 165 330 462 462 330 165 55 11 1			2048
13	1 12 66 220 495 792 924 792 495 220 66 12 1			4096

The first term of the weight matrix circled.

TABLE 3.3 COEFFICIENTS OF THE MEANS.

n	First Mean								Schuler Mean							Thomas Mean					
		S^1_1	S^1_2	S^1_3	S^1_4	S^1_5	S^1_6	S^1_7		S^2_1	S^2_2	S^2_3	S^2_4	S^2_5	S^2_6		S^3_1	S^3_2	S^3_3	S^3_4	S^3_5
3	$\frac{1}{2}$ (	1	1					)	(	1					)						
4	$\frac{1}{2}$ (	1		1				)	$\frac{1}{2}$ (	1	1				)	(	1				)
5	$\frac{1}{6}$ (	2	1	1	2			)	$\frac{1}{3}$ (	2	-1	2			)	$\frac{1}{2}$ (	1	1			)
6	$\frac{1}{3}$ (	1		1		1		)	$\frac{1}{12}$ (	5	1	1	5		)	$\frac{1}{6}$ (	5	-4	5		)
7	$\frac{1}{12}$ (	3	1	2	2	1	3	)	$\frac{1}{6}$ (	3	-2	4	-2	3	)	$\frac{1}{2}$ (	1			1	)
8	$\frac{1}{4}$ (	1		1		1		1)	$\frac{1}{20}$ (	7	-1	4	4	-1	7)	$\frac{1}{10}$ (	7	-8	12	-8	7)

TABLE 3.4 INVERSE MATRICES $G^{1^{-1}}$, $G^{2^{-1}}$, $G^{3^{-1}}$, For $n \leqslant 8$.

$G^{1^{-1}}$

n = 3 $$\frac{4}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

n = 4 $$\frac{4}{4}\begin{bmatrix} 3 & -2 & 1 \\ -2 & 4 & -2 \\ 1 & -2 & 3 \end{bmatrix}$$

n = 5 $$\frac{4}{5}\begin{bmatrix} 4 & -3 & 2 & -1 \\ -3 & 6 & -4 & 2 \\ 2 & -4 & 6 & -3 \\ -1 & 2 & -3 & 4 \end{bmatrix}$$

n = 6 $$\frac{4}{6}\begin{bmatrix} 5 & -4 & 3 & -2 & 1 \\ -4 & 8 & -6 & 4 & -2 \\ 3 & -6 & 9 & -6 & 3 \\ -2 & 4 & -6 & 8 & -4 \\ 1 & -2 & 3 & -4 & 5 \end{bmatrix}$$

n = 7 $$\frac{4}{7}\begin{bmatrix} 6 & -5 & 4 & -3 & 2 & -1 \\ -5 & 10 & -8 & 6 & -4 & 2 \\ 4 & -8 & 12 & -9 & 6 & -3 \\ -3 & 6 & -9 & 12 & -8 & 4 \\ 2 & -4 & 6 & -8 & 10 & -5 \\ -1 & 2 & -3 & 4 & -5 & 6 \end{bmatrix}$$

n = 8 $$\frac{4}{8}\begin{bmatrix} 7 & -6 & 5 & -4 & 3 & -2 & 1 \\ -6 & 12 & -10 & 8 & -6 & 4 & -2 \\ 5 & -10 & 15 & -12 & 9 & -6 & 3 \\ -4 & 8 & -12 & 16 & -12 & 8 & -4 \\ 3 & -6 & 9 & -12 & 15 & -10 & 5 \\ -2 & 4 & -6 & 8 & -10 & 12 & -6 \\ 1 & -2 & 3 & -4 & 5 & -6 & 7 \end{bmatrix}$$

TABLE 3.4 (Contd.)

$$G^{2^{-1}}$$

n = 3 $\quad \frac{16}{6}$

n = 4 $$\frac{16}{10}\begin{bmatrix} 3 & -2 \\ -2 & 3 \end{bmatrix}$$

n = 5 $$\frac{16}{10}\begin{bmatrix} 4 & -4 & 2 \\ -4 & 7 & -4 \\ 2 & -4 & 4 \end{bmatrix}$$

n = 6 $$\frac{16}{105}\begin{bmatrix} 50 & -60 & 45 & -20 \\ -60 & 114 & -96 & 45 \\ 45 & -96 & 114 & -60 \\ -20 & 45 & -60 & 50 \end{bmatrix}$$

n = 7 $$\frac{16}{84}\begin{bmatrix} 45 & -60 & 54 & -36 & 15 \\ -60 & 120 & -120 & 84 & -36 \\ 54 & -120 & 156 & -120 & 54 \\ -36 & 84 & -120 & 120 & -60 \\ 15 & -36 & 54 & -60 & 45 \end{bmatrix}$$

n = 8 $$\frac{16}{84}\begin{bmatrix} 49 & -70 & 70 & -56 & 35 & -14 \\ -70 & 145 & -160 & 134 & -86 & 35 \\ 70 & -160 & 220 & -200 & 134 & -56 \\ -56 & 134 & -200 & 220 & -160 & 70 \\ 35 & -86 & 134 & -160 & 145 & -7 \\ -14 & 5 & -56 & 70 & -70 & 49 \end{bmatrix}$$

TABLE 3.4 (Contd.)

$G^{3^{-1}}$

n = 4 $\quad \frac{64}{20}$

n = 5 $$\frac{64}{35}\begin{bmatrix} 4 & -3 \\ -3 & 4 \end{bmatrix}$$

n = 6 $$\frac{64}{140}\begin{bmatrix} 25 & -30 & 15 \\ -30 & 52 & -30 \\ 15 & -30 & 25 \end{bmatrix}$$

n = 7 $$\frac{64}{42}\begin{bmatrix} 10 & -15 & 12 & -5 \\ -15 & 30 & -27 & 12 \\ 12 & -27 & 30 & -15 \\ -5 & 12 & -15 & 10 \end{bmatrix}$$

n = 8 $$\frac{64}{168}\begin{bmatrix} 49 & -84 & 84 & -56 & 21 \\ -84 & 184 & -204 & 144 & -56 \\ 84 & -204 & 264 & -204 & 84 \\ 56 & 144 & -204 & 184 & -84 \\ 21 & -56 & 84 & -84 & 49 \end{bmatrix}$$

4. THE GENERAL SCHULER MEAN SOLUTIONS

In Section 3.2 it was stated that $\bar{S}$ had been almost universally adopted because it is a very convenient way of reducing turning point observations and because of the simple arithmetic involved. The difference between $\bar{S}$ and the least squares estimate is seldom significantly big but in some cases it can be large as can be seen from Lauf (1967b) who states that "In particular cases, it has been found that the two expressions give results which differ by up to ten seconds of arc." It has been shown that with a slight increase in arithmetical manipulation the least squares solution of θ_o can be obtained, which could be of practical significance in the light of the magnitude of the above differences. In practice our interest often lies solely in the value of the unknown θ_o, an opinion which is shared by Janisch (1969) who states that "In production gyro-theodolite work, the values of B and f (damping factor) are not of interest". If we are interested in the solution of the estimates of the other parameters then it will be seen that a similar technique is also available.

4.1 THE TURNING POINT METHOD

We may write equation (2) of Section 3.1 in a slightly different form as follows:-

$$v_i = (-1)^{i-1} B + (i - 1) (-1)^{i-1} a + \theta_o - y_i \qquad (1)$$

This previous equation may be expressed in matrix notation as follows:-

$$V_1 = A_1X_1 - Y$$

where $V_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$ $A_1 = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 1 \\ 1 & -2 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ (-1)^{n-1} & (n-1)(-1)^{n-1} & 1 \end{bmatrix}$ $X_1 = \begin{bmatrix} B \\ a \\ \Theta_o \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \cdot \\ \cdot \\ \cdot \\ y_n \end{bmatrix}$

These expressions have been given before by Gregerson (1969a).

Schuler Means may be formed by taking successive sums of the original observations or by using a transformation matrix C_1, using the principles outlined by Allman (1969), where

$$C_1 = \tfrac{1}{4} \begin{bmatrix} 1 & 2 & 1 & & & & \\ & 1 & 2 & 1 & & & \\ & & \cdot & \cdot & \cdot & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & 1 & 2 & 1 \\ & & & & & 1 & 2 & 1 \end{bmatrix}$$

and the weight matrix

$$G_1 = C_1 C_1^T = \frac{1}{16} \begin{bmatrix} 6 & 4 & 1 & & & & & \\ 4 & 6 & 4 & 1 & & & & \\ 1 & 4 & 6 & 4 & 1 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & 1 & 4 & 6 & 4 & 1 \\ & & & & 1 & 4 & 6 & 4 \\ & & & & & 1 & 4 & 6 \end{bmatrix}$$

The resulting single normal equation is

$$B_1^T G_1^{-1} B_1 \Theta_o + B_1^T G_1^{-1} S = 0$$

and its solution is given by

$$\Theta_o = -(B_1^T G_1^{-1} A_1)^{-1} B_1^T G_1^{-1} S$$

where B_1 is a unit column vector of dimension n-2 and

S is the column vector of the individual Schuler Means .

General expressions for the coefficients of the derived values of Schuler Means have been obtained for any number of observations and are given in Section 3.2. To facilitate numerical calculation the values of these coefficients up to n = 8 are given in Table 3.3.

By a similar process we can find the adjusted value of "a" after removing Θ_o. This can be done conveniently by adding and subtracting $2\Theta_o$ from the first sum of the original observations. Then

$$V_1 = A_2 X_2 - Y$$

where $A_2 = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 1 & -2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ (-1)^{n-1} & (n-1)(-1)^{n-1} \end{bmatrix}$ $\qquad X_2 = \begin{bmatrix} B \\ a \end{bmatrix}$

The transformation matrix $C_2 = \begin{bmatrix} 1 & 1 & & & & \\ & -1 & -1 & & & \\ & & 1 & 1 & & \\ & & & \cdot & \cdot & \\ & & & & \cdot & \cdot \\ & & & & & \cdot \quad \cdot \end{bmatrix}$

and the weight matrix

$$G_2 = C_2 C_2^T = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}$$

and similarly as in the previous derivation, general expressions for the coefficients of the derived values of "a" have been found and are given in Table 4.1 on Page 109 together with numerical values of these coefficients up to $n = 8$.

The estimate of B can now be found by eliminating Θ_o and "a" from the original observations and taking a simple average. The differences from this mean are the corrections to the original observations. The variances of the adjusted parameters are calculated from:-

$$\hat{s}_y^2 = \frac{\sum v^2}{r} \qquad \hat{s}_{\Theta_o}^2 = Q_{\Theta_o\Theta_o} \; \hat{s}_y^2$$

$$\hat{s}_a^2 = Q_{aa} \; \hat{s}_y^2$$

$$\hat{s}_B^2 = Q_{BB} \; \hat{s}_y^2$$

where $r = n-3$ and for

n even

$$Q_{\Theta_o\Theta_o} = \frac{n^2-1}{n(n^2-4)}$$

$$Q_{aa} = \frac{12}{n(n^2-4)}$$

$$Q_{BB} = \frac{4n^2-6n-1}{n(n^2-4)}$$

n odd

$$Q_{\Theta_o\Theta_o} = \frac{n}{n^2-1}$$

$$Q_{aa} = \frac{12}{n(n^2-1)}$$

$$Q_{BB} = \frac{4n^2-6n+3}{n(n^2-1)}$$

For convenience the values of the square root of the above expressions (weight coefficients) have been calculated up to $n = 8$ and are given in Table 4.2 on page 110. An example to illustrate this technique is given below.

EXAMPLE OF THE TURNING POINT METHOD (n=8)

Observations (1)	$\sum_1$	$\sum_2$	$\pm(\sum_1 - 2\Theta_o)$	(2)	(1) - (2)	v	v
2°36'24"				y_o = 44.4"	2°35'39.6"	+0.6"	
	1'30"		+1.2"				
357 25 06		2'54"		y_o+a = 48.6	357 24 17.4		+2.4'
	1 24		+4.8				
2 36 18		3 00		y_o-2a = 36.0	2 35 42.0	-1.8	
	1 36		+7.2				
357 25 18		3 00		y_o+3a = 57.0	357 24 21.0		-1.2
	1 24		+4.8				
2 36 06		3 00		y_o-4a = 27.6	2 35 38.4	+1.8	
	1 36		+7.2				
357 25 30		3 06		y_o+5a = 65.4	357 24 24.6		-4.8
	1 30		-1.2				
2 36 00		3 00		y_o-6a = 19.2	2 35 40.8	-0.6	
	1 30		+1.2				
357 25 30				y_o+7a = 73.8	357 24 16.2		+3.6
					Mean = B 2 35 40.2	$\sum$=0	$\sum$=0

$\sum v^2 = 50.4$

$$\Theta_o = \tfrac{1}{4}\left\{2'\ 54'' + \frac{7(0+6)-(6+12)+4(6+6)''}{20}\right\}$$

$\Theta_o = 44''.4$ $\qquad 2\Theta_o = 1'\ 28.8''$

$$a = \frac{1}{84}\{7(1.2+1.2)+12(4.8-1.2)+15(7.2+7.2)+16\text{x}4.8\}''$$

$a = 4.2''$

$\hat{S}_y = \left(\frac{50.4}{8-3}\right)^{\frac{1}{2}} = \pm 3.17''$

$\hat{S}_{\Theta_o} = \pm 0.362 \times 3.17 = \pm 1.15$

$\hat{S}_a = \pm 0.158 \times 3.17 = \pm 0.50$

$\hat{S}_B = \pm 0.657 \times 3.17 = \pm 2.08$

The solutions for the parameters Θ_o and B will be found to agree precisely with those derived in Chapter 3. In the work of Rack (1968) a summary of the work of Basch (1914) is given of which this author was not aware at the time of his derivation. General solutions were given by Basch for all the parameters and their variances using a linear relationship similar to equation (1). Expressions for the solutions of the parameters were given in terms of the original observations will be found to be the same as that obtained by combining observations etc. according to the aforegoing theory. Also it was previously supposed that the least squares solution for an odd number of observations was first given by F. Kohlrausch in 1944 but Rack credits B.F. Kohlrausch with having used the formula in 1880.

4.2 THE TRANSIT METHOD

In this method, timing observations are made on an index mark placed near the centre of the oscillation and the motion may be expressed in the form of

$$y = B\, e^{-\frac{2\alpha}{T}(t_i+v_i-t_o)} \sin\left\{\frac{2\pi}{T}(t_i+v_i-t_o)\right\}$$

where T is the period of oscillation.

t_i is the observed time and v_i its correction.

Then because α and the sine term are small

$$\Delta y \simeq \frac{2\pi}{T} B(-1)^{-1} \{t_i+v_i-\frac{(i-1)}{2}T-t_o\}$$

and $\Delta y = -\frac{2\pi}{T} B\ \delta t$

where $\delta t = (-1)^{-1}\ \{t_i + v_i - \frac{(i-1)}{2} T - t_o\}$

or $v_i = t_o + \frac{(i-1)}{2} T + (-1)^i\ \delta t - t_i$ (3)

which is shown in graphical form in Fig. 4.1.

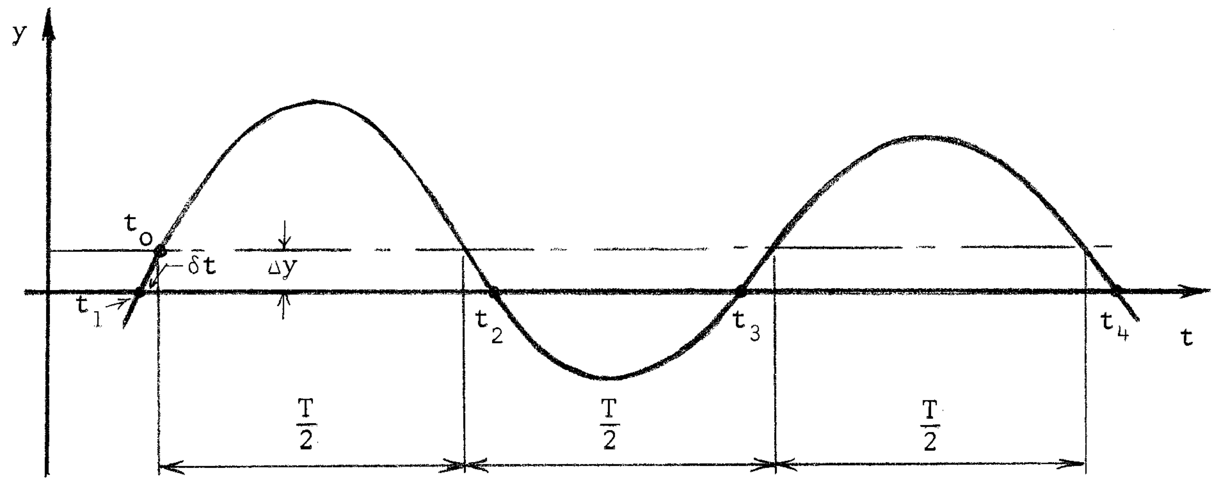


FIG. 4.1 TRANSIT METHOD

The dominant error in Δy from the above approximation is $\Delta y \frac{(i-1)}{2}\alpha$, if B is taken as the amplitude in the middle of the observations. E.g. for $B = 3^o$, $T = 500^s$, $\alpha = 0.002$, $\delta t = 1^s$ and for 8 observations this error is about 1''.

Equation (3) can be expressed in matrix notation as follows:-

$$V_2 = A_3 X_3 - T_1$$

where
$$V_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ \cdot \\ \cdot \\ \cdot \\ v_n \end{bmatrix} \quad A_3 = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & -1 \\ 1 & 2 & 1 \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot \\ 1 & (n-1) & (-1)^n \end{bmatrix} \quad X_3 = \begin{bmatrix} t_o \\ \frac{T}{2} \\ \delta t \end{bmatrix} \quad T_1 = \begin{bmatrix} t_1 \\ t_2 \\ t_3 \\ \cdot \\ \cdot \\ \cdot \\ t_n \end{bmatrix}$$

These expressions have also been given by Gregerson (1969a) .

If we take successive differences instead of taking successive sums as with the turning point method then we will isolate δt. Taking a simple average of these derived values of δt can be done in the same way as the turning point method, i.e.

$$\overline{T} = \frac{T_1 + T_2 + T_3 \ldots\ldots\ldots T_{n-2}}{n-2}$$

but the disadvantages as stated before also apply to this average. The transformation matrix in this case will be

$$C_3 = \frac{1}{4} \begin{bmatrix} 1 & -2 & 1 & & & & \\ & -1 & 2 & -1 & & & \\ & & 1 & -2 & 1 & & \\ & & & \cdot & \cdot & \cdot & \\ & & & & \cdot & \cdot & \cdot \\ & & & & & \cdot & \cdot & \cdot \end{bmatrix}$$

and the weight matrix

$$G_3 = C_3\, C_3^T = \frac{1}{16} \begin{bmatrix} 6 & 4 & 1 & & & & & \\ 4 & 6 & 4 & 1 & & & & \\ 1 & 4 & 6 & 4 & 1 & & & \\ & \cdot & \cdot & \cdot & \cdot & \cdot & & \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ & & & 1 & 4 & 6 & 4 & 1 \\ & & & & 1 & 4 & 6 & 4 \\ & & & & & 1 & 4 & 6 \end{bmatrix}$$

which is identical with G_1. Thus the solution for δt will be the same as for θ_o after changing the signs of the even numbers of second differences.

In a similar way we eliminate δt by correcting the first differences by $\pm\, 2\delta t$ and then

$$V_2 = A_4\, X_4 - T_1$$

where $A_4 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 2 \\ \cdot & \cdot \\ \cdot & \cdot \\ \cdot & \cdot \\ 1 & (n-1) \end{bmatrix}$ $\quad X_4 = \begin{bmatrix} t_o \\ \frac{T}{2} \end{bmatrix}$

The transformation matrix $C_4 = \begin{bmatrix} -1 & 1 & & & & & & \\ & -1 & 1 & & & & & \\ & & -1 & 1 & & & & \\ & & & \cdot & \cdot & & & \\ & & & & \cdot & \cdot & & \\ & & & & & -1 & 1 & \\ & & & & & & -1 & 1 \\ & & & & & & & -1 \quad 1 \end{bmatrix}$

and the weight matrix

$$G_4 = C_4 C_4^T = \begin{bmatrix} 2 & -1 & & & & & & \\ -1 & 2 & -1 & & & & & \\ & -1 & 2 & -1 & & & & \\ & & \cdot & \cdot & \cdot & & & \\ & & & \cdot & \cdot & \cdot & & \\ & & & & -1 & 2 & -1 & \\ & & & & & -1 & 2 & -1 \\ & & & & & & -1 & 2 \end{bmatrix}$$

which is identical with G_2 and thus the solution for $\frac{T}{2}$ will be the same as that for "a".

The estimate of t_o can now be found by eliminating δt and $\frac{T}{2}$ from the original observations and taking a simple average as with the turning point method. The differences from this mean are the corrections to the original observations and the variances of the adjusted parameters are calculated from expressions which are identical with those for the turning point method. An example to illustrate this technique is given below.

EXAMPLE OF THE TRANSIT METHOD (n=7)

Observations (1)	Δ_1	$\pm\Delta_2$	$\Delta_1 \pm 2\delta t$	(2)	(1)-(2)	v
$0^m 00.0^s$				$\delta t = -0.62^s$	0.62^s	$+0.14^s$
	$3^m\ 16.5^s$		$3^m\ 15.25^s$			
3 16.5		-2.7^s		$\frac{T}{2} - \delta t = 3^m 15.71$	0.79	-0.03
	3 13.8		3 15.05			
6 30.3		-2.6		$T + \delta t =$ 6 29.55	0.75	+0.01
	3 16.4		3 15.15			
9 46.7		-2.4		$\frac{3T}{2} - \delta t =$ 9 45.88	0.82	-0.06
	3 14.0		3 15.25			
13 00.7		-2.2		$2T + \delta t =$ 13 59.72	0.98	-0.22
	3 16.2		3 14.95			
16 16.9		-2.7		$\frac{5T}{2} - \delta t =$ 16 16.06	0.84	-0.08
	3 13.5		3 14.75			
19 30.4				$3T + \delta t =$ 19 29.89	0.51	+0.25
				Mean = t_o	0.76	$\sum = +0.01$
						$\sum v^2 = 0.14$

$$\delta t = -\tfrac{1}{4} \{ 2.0^s + \frac{3(0.7+0.7)-2(0.6+0.2)+ 4x\ 0.4^s}{6} \}$$

$\underline{\delta t = -0.62^s}$ $\quad$ $\underline{2\delta t = -1.25^s}$ $\qquad$ $\hat{S}_t = (\frac{0.14}{7-3})^{\frac{1}{2}} = \pm 0.19^s$

$$\frac{T'}{2} = 3^m 14.75^s + \frac{3(0.50+0)+5(0.30+0.20)+6(0.40+0.50)^s}{28} \qquad \hat{S}_{\delta t} = 0.382 \times 0.19 = \pm 0.07$$

$$\frac{T}{2} = 3^m 15.08^s_6 \qquad \hat{S}_{\frac{T}{2}} = 0.189 \times 0.19 = \pm 0.04$$

$$\hat{S}_{t_o} = 0.684 \times 0.19 = \pm 0.13$$

4.3 CONCLUSION

It is apparent from the preceeding theory and the illustrative examples that there are further advantages than those stated before

(1) If our interest lies solely in the principal unknowns θ_o and δt, then the algorithm enables us to solve for these quantities immediately without recourse to calculating the remaining parameters in the problem.

(2) The calculation techniques are remarkably similar. What appears at first glance to be two unrelated computational problems have in fact a common basic algorithm. Thus one set of precomputed auxiliary values serves for both methods.

(3) The formulae given for these solutions are of a general nature and do not presume a limit or a particular value for the number of observations. However the algorithm does not take into account the unlikely circumstance of an omitted or rejected observation.

(4) The models chosen for this adjustment process are good approximations to a lightly damped simple harmonic motion and in many cases further calculations by means of an electronic computer with the correct model may not be warranted.

4.4 THE DIFFERENCE BETWEEN THE LEAST SQUARES ESTIMATE AND THE SCHULER MEAN

It is instructive to examine the size of the difference between the least squares estimate and the average of the Schuler Means (the average of the derived values of θ_o or δt). This difference L.S. - $\bar{S}$ = x say, can be expressed in terms of the original observations and therefore an estimate of the variance of x may be derived if the observations are considered to be of equal precision and correlation free. The following derivation is common to both the turning point and transit methods. It can be proved that for

n even

$$x = \frac{1}{4n(n^2-4)} \Big\{ (3n-2)(n-4)(y_1+y_n) + (n+10)(n-4)(y_2+y_{n-1})$$

$$-4(5n-14)(y_3+y_{n-2}) + 4(n-22)(y_4+y_{n-3})$$

$$-4(5n-26)(y_5+y_{n-4}) + 4(n-34)(y_6+y_{n-5})$$

$$-4(5n-38)(y_7+y_{n-6}) + 4(n-46)(y_8+y_{n-7})$$

$$\vdots \qquad\qquad \vdots \Big\}$$

and for n odd

$$x = \frac{1}{4(n-2)(n^2-1)} \Big\{ 3(n-1)(n-3)(y_1+y_n) + (n+1)(n-5)(y_2+y_{n-1}) - 12(n-1)(y_3+y_5+\ldots\ldots+y_{n-4}+y_{n-2}) - 4(n+1)(y_4+y_6+\ldots\ldots+y_{n-5}+y_{n-3}) \Big\}$$

$n > 3$

and therefore after applying the law of propagation of variances to these expressions we obtain for

n even

$$\hat{S}_x^2 = \frac{1}{8\{n(n^2-4)\}^2} \Big\{ \{(3n-2)(n-4)\}^2 + \{(n+10)(n-4)\}^2 + \{4(5n-14)\}^2 + \{4(n-22)\}^2 + \{4(5n-26)\}^2 + \{4(n-34)\}^2 \; \vdots \qquad \vdots \Big\} \hat{S}_y^2 \; *$$

* Note: The number of terms used to evaluate this expression should be the same as that used to evaluate the previous expression for x (n even).

and for n odd

$$\hat{S}_x^2 = \frac{1}{\{4(n-2)(n^2-1)\}^2} \left\{ 2\{3(n-1)(n-3)\}^2 + 2\{(n+1)(n-5)\}^2 + \frac{n-3}{2}\{12(n-1)\}^2 + \frac{n-5}{2}\{4(n+1)\}^2 \right\} \hat{S}_y^2$$

From these results we may estimate 'a priori' the difference between calculating θ_o from $\bar{S}$ instead of from least squares. It will be noted that the quantity $\hat{S}_x$ depends upon the value of $\hat{S}_y$ and thus we can go a stage further and estimate the effect of neglecting x in terms of $\hat{S}_{\theta_o}$, which also depends upon $\hat{S}_y$. We had previously that for n even

$$\hat{S}_{\theta_o}^2 = \frac{n^2-1}{n(n^2-4)} \hat{S}_y^2 \quad \text{and for n odd} \quad \hat{S}_{\theta_o}^2 = \frac{n}{n^2-1} \hat{S}_y^2 .$$

If we assume that the value of x is confined within the bounds of $\pm 3\hat{S}_x$ (which will embrace 99.7% of our observations) then the effect of neglecting x will virtually never exceed the following values:-

n	$\pm 3\ \hat{S}_x$
5	$\pm 1.34\ .\ \hat{S}_{\theta_o}$
6	$\pm 1.01\ .\ \hat{S}_{\theta_o}$
7	$\pm 1.22\ .\ \hat{S}_{\theta_o}$
8	$\pm 1.00\ .\ \hat{S}_{\theta_o}$

Thus if differences of the above magnitude can be tolerated then a simple average will suffice. However in view of the slight extra work which is required to find the least squares estimate then there seems to be no real barrier to calculating the weighted mean of the derived values of θ_o or δt.

A verification of the aforegoing theory was attempted using the results of 80 sets of observations made with three different Wild GAK 1 gyro-theodolites using the turning point method, each set having 8 consecutive turning points. Each set of 8 observations was analysed in groups 3, 4, 5, 6, 7 and 8 consecutive observations. Thus for each set of 8 there were 6 groups of 3, 5 of 4, 4 of 5, 3 of 6, 2 of 7 and 1 of 8. Each group was treated as though it were independent of all other groups, an assumption which should have little adverse effect on the results, considering that the sample sizes are large, i.e. 320 groups of 5, 240 of 6, 160 of 7 and 80 of 8. The model parameters, their estimated standard deviations based on the least squares solution and $\bar{S}$ were calculated on a digital computer. Histograms of the difference, x, were prepared for n = 5, 6, 7 and 8 and are shown in Figures 4.2, 4.3, 4.4 and 4.5. The structure of these distribution histograms resembles a normal distribution and the probability density functions of the normal distribution has been superimposed over each histogram.

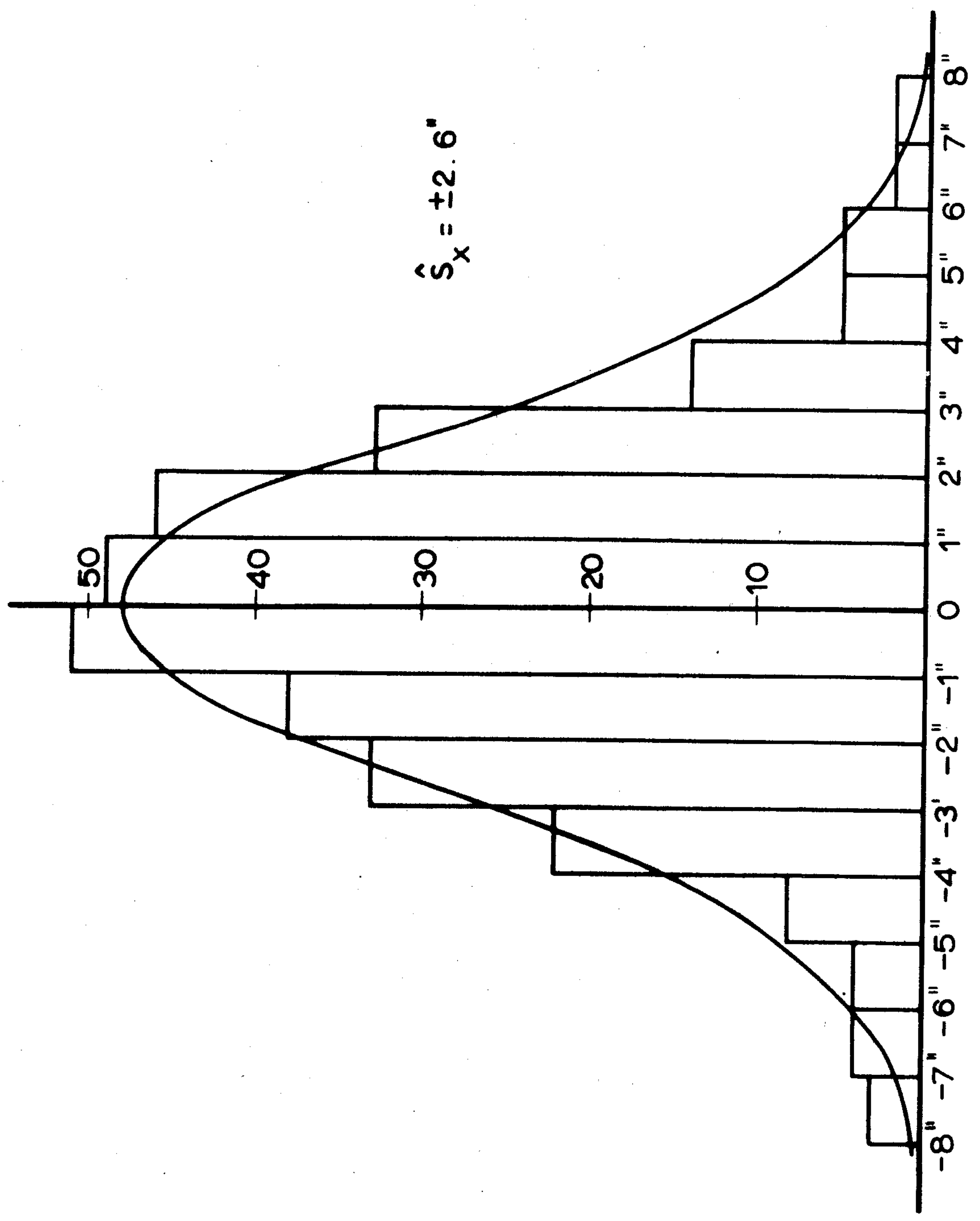


FIG. 4.2: L.S. − $\overline{\text{S.M}}$ n = 5

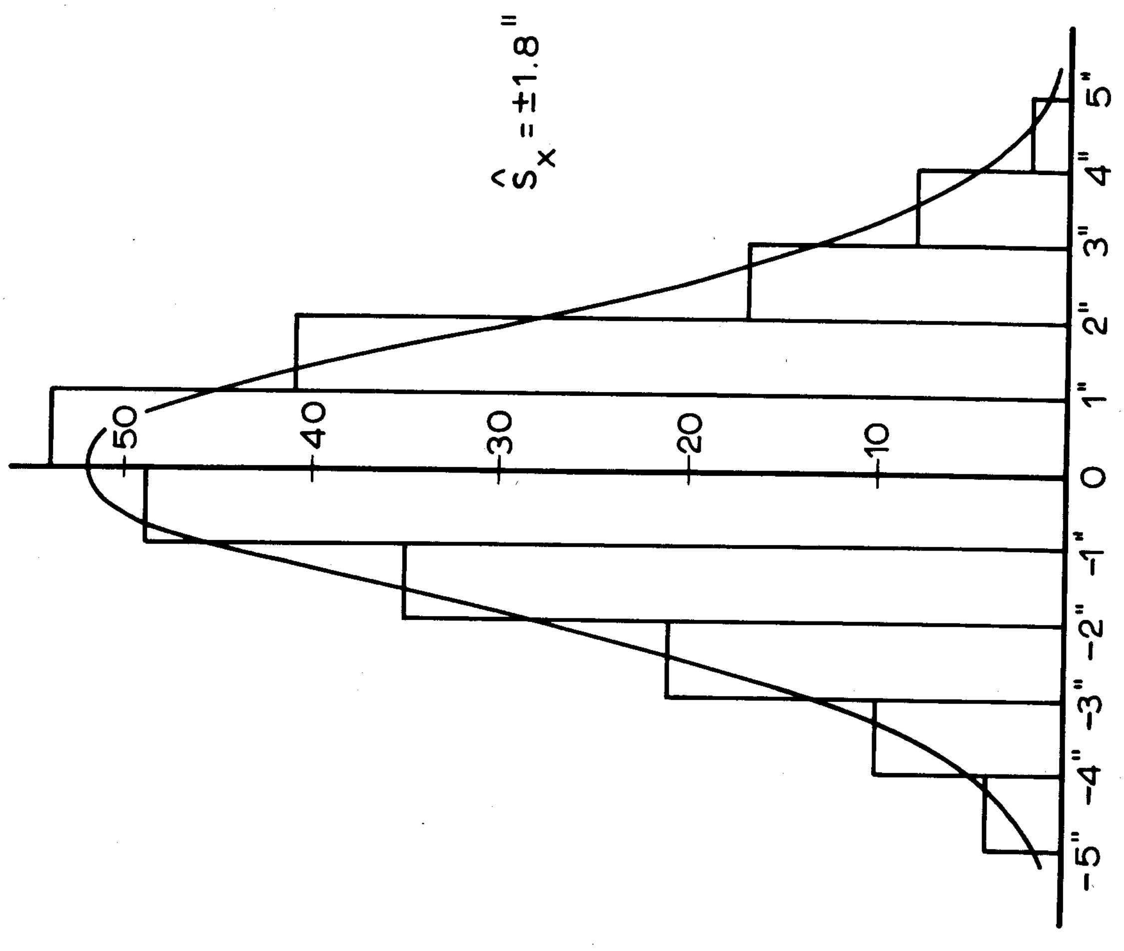


FIG. 4.3: L.S. − $\overline{\text{S.M}}$ n = 6

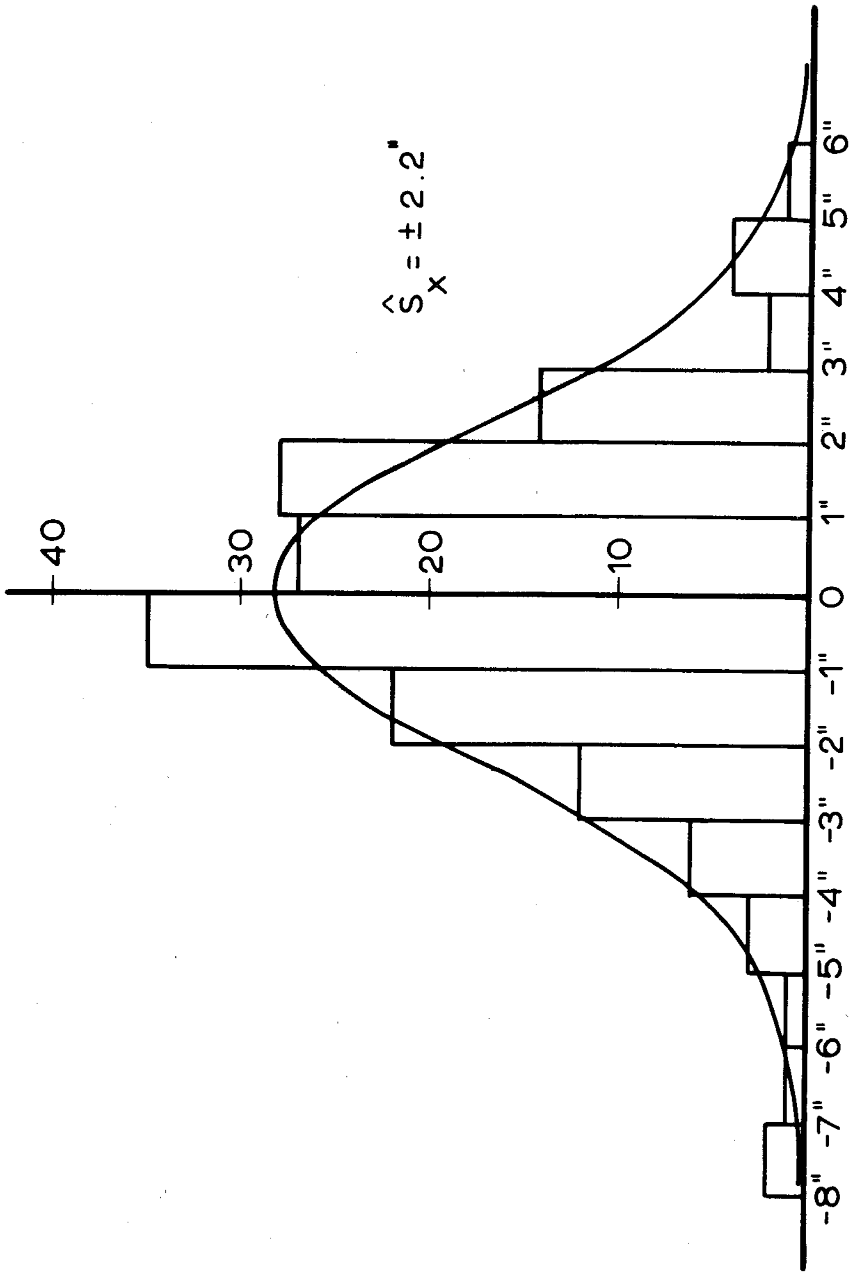


FIG. 4.4: L.S. − $\overline{\text{S.M.}}$ n = 7

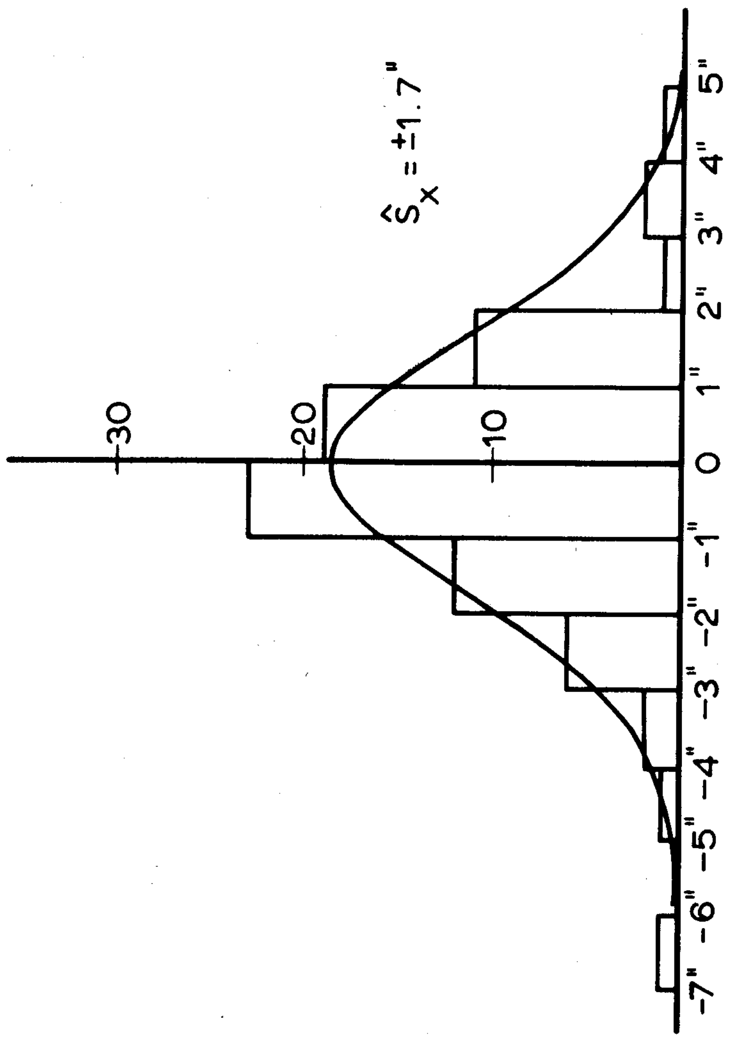


FIG.4.5: L.S. − $\overline{S.M.}$ n = 8

For each of these distributions the estimates $\hat{S}_y$ can be deduced as follows:-

n	$\hat{S}_x$	$\hat{S}_y$	
5	±2.6"	$\sqrt{24}.\hat{S}_x$	= ±12.7"
6	±1.8	$\sqrt{48}.\hat{S}_x$	= ±12.4
7	±2.2	$\sqrt{\frac{1200}{29}}.\hat{S}_x$	= ±14.1
8	±1.7	$\sqrt{\frac{480}{7}}.\hat{S}_x$	= ±14.1

These results are in reasonable agreement with the value of $\hat{S}_y$ = ±15" which has been derived from all observations. The computer programme and an example of the above analysis are given in Appendix II.

4.5 THE MODEL

A convenient form of equation (1) in Section 3.1 for n turning points may be written as follows:-

$$v_i \;=\; (-1)^{i-1}Be^{(\frac{n+1-2i}{2})\alpha} + \theta_o - y_i \qquad (1)$$

where the time origin lies in the centre of the observations. Expanding the exponential function gives

$$v_i \;=\; (-1)^{i-1}B\{1+(\frac{n+1-2i}{2})\alpha + \frac{1}{2}(\frac{n+1-2i}{2})^2\alpha^2 \ldots\ldots\} + \theta_o - y_i \qquad (2)$$

In Section 3.1 two approximations were made before entering into a least squares adjustment:-

(1) terms containing powers of α greater than one were neglected, and

(2) substitutions of $\theta_o = \frac{b+c}{2}$, $B = \frac{b-c}{2}$ and $\alpha = \frac{a}{B}$ were made in order to make the equation linear with respect to the new parameters, a, b and c.

The error arising from the first approximation will be a maximum for the first and last equation, e.g. for $B = 2^{o}$, $\alpha = 0.002$ and $n = 8$, the term containing α^2 is 0.18". Errors of this magnitude are of little practical significance.

Although the substitution of new parameters in equation (2) creates a linear equation with respect to a, b and c, these new parameters are not all independent. The error in the adjustment process caused by the inter-dependence of these parameters is not easy to evaluate because the true linear form is complex and an iterative calculation procedure is required in order to obtain precise values for the unknown parameters unless good approximate values are known. If we substitute from

$$\theta_o = \theta'_o + \Delta\theta'_o$$

$$B = B' + \Delta B'$$

$$\alpha = \alpha' + \Delta\alpha'$$

in equation (2) we obtain after some reduction the following linear forms:-

Linear damping

$$v_i = (-1)^{i-1}\Delta B'[1+(\frac{n+1-2i}{2})\alpha'] + (-1)^{i-1}\Delta\alpha' B'(\frac{n+1-2i}{2})$$
$$+ \Delta\theta'_o + \theta'_o + (-1)^{i-1}B'[1+(\frac{n+1-2i}{2})\alpha'] - y_i \qquad (3)$$

Quadratic damping.

$$v_i = (-1)^{i-1}\Delta B'[1+(\frac{n+1-2i}{2})\alpha' + \tfrac{1}{2}(\frac{n+1-2i}{2})^2\alpha'^2]$$
$$+(-1)^{i-1}\Delta\alpha' B'[(\frac{n+1-2i}{2}) + (\frac{n+1-2i}{2})^2\alpha']$$
$$+ \Delta\theta'_o + \theta'_o + (-1)^{i-1}[1+(\frac{n+1-2i}{2})\alpha' + \tfrac{1}{2}(\frac{n+1-2i}{2})^2\alpha'^2] - y_i \qquad (4)$$

The differences between the solutions of the parameters from the weighted Schuler Means and equation (3), and the weighted Thomas Means and equation (4), were evaluated on the computer for 78 sets of 8 consecutive turning point observations made with the GAK 1. The solutions using equation (3) and (4) were found using the preliminary values obtained from the Schuler Mean and Thomas Mean solutions respectively. All calculations were made with double precision, the programme iterating until $\Delta\theta' < 0.000001''$. Below are given the maximum differences between the solutions of the parameters designated by $d\theta_o$, dB and $d\alpha$.

Linear damping

$d\theta_o = 0$ $\qquad \frac{dB}{B} = 0$ $\qquad \frac{d\alpha}{\alpha} = \frac{0.0000007}{0.00200}$

Quadratic damping

$d\theta_o = 0.02''$ $\qquad \frac{dB}{B} = \frac{0.42''}{8,962''}$, $\qquad \frac{d\alpha}{\alpha} = \frac{0.000007}{0.004748}$

From these results it may be concluded that the weighted Schuler Means and Thomas Means give solutions which are completely adequate for practical purposes.

4.6 MATRIX TRANSFORMATION.

In Section 3.2 transformation matrices were used to eliminate all but the principal parameter θ_o. This process was extended in Section 4.1 and 4.2, so that after the solution of the principal parameter the remaining parameters could be solved in turn in a similar way. The success of this technique of reduction depends upon the validity of the matrix expressions which have been derived using the transformation matrices as follows:-

If we have n observations denoted by y_i (i = 1,...,n) and each observation is made with the same precision and is independent of other observations then

$$AX = Y + V$$

where A is the matrix of coefficients

X is a column vector of parameters

Y is a column vector of observations

V is the column vector of corrections

Then the least squares estimate of X will be given by

$$X = -(A^TA)^{-1} A^TY \qquad (1)$$

Consider a transformation matrix C which will be used to combine the original observations in some desired manner thus

$$CAX = CY + CV$$

which will have a weight of CC^T. Then the least squares estimate of X will be given by

$$X = - \{(CA)^T(CC^T)^{-1}CA\}^{-1} (CA)^T(CC^T)^{-1}CY$$

$$= - \{A^TC^T(CC^T)^{-1}CA\}^{-1} A^TC^T(CC^T)CY$$

$$= - \{A^TC_oA\}^{-1}A^TC_oY \qquad (2)$$

where $C_o = C^T(CC^T)^{-1}C$

If we wish to preserve identical solutions for the least squares estimate of X then expressions (1) and (2) should be equal. Two obvious conditions will preserve this identity (a) if C is the identity matrix I or (b) if C is square and non-singular because

$$C_o = C^T(C^T)^{-1}C^{-1}C$$

$$= I . I$$

$$= I$$

The transformation matrices which have been used for the adjustment of gyro-theodolite observations do not satisfy either of the above conditions although in all cases identical solutions are obtained from equations (1) and (2)

TABLE 4.1. COEFFICIENTS OF THE DERIVED VALUES OF a AND $\frac{T}{2}$ (S^2 or T^2).

$$a = \frac{T}{2} = \frac{6}{n(n^2-1)} \Big[(n-1)S_1^2 + 2(n-2)S_2^2 + 3(n-3)S_3^2 + 4(n-4)S_4^2 + 5(n-5)S_5^2 + 6(n-6)S_6^2 \; \cdots \Big]$$

or for n even

$$a = \frac{T}{2} = \frac{6}{n(n^2-1)} \left[(n-1)S^2 + \sum_{i-1}^{\frac{1}{2}(n-2)} \{2i(n-2i)S_{2i}^2 + (2i+1)(n-2i-1)S_{2i+1}^2\} \right]$$

or for n odd

$$a = \frac{T}{2} = \frac{6}{n(n^2-1)} \sum_{i-1}^{\frac{1}{2}(n-1)} \{(2i-1)(n-2i+1)S_{2i-1}^2 + 2i(n-2i)S_{2i}^2\}$$

n		S_1^2	S_2^2	S_3^2	S_4^2	S_5^2	S_6^2	S_7^2
3	$\frac{1}{2}$ (	1	1					)
4	$\frac{1}{10}$ (	3	4	3				)
5	$\frac{1}{10}$ (	2	3	3	2			)
6	$\frac{1}{35}$ (	5	8	9	8	5		)
7	$\frac{1}{28}$ (	3	5	6	6	5	3	)
8	$\frac{1}{84}$ (	7	12	15	16	15	12	7)

TABLE 4.2 WEIGHT COEFFICIENTS.

n	$Q_{\theta_o\theta_o}^{1/2}$ or $Q_{\delta t\delta t}^{1/2}$	$Q_{aa}^{1/2}$ or $Q_{\frac{T}{2}\frac{T}{2}}^{1/2}$	$Q_{BB}^{1/2}$ or $Q_{t_o t_o}^{1/2}$
4	0.559	0.500	0.901
5	0.456	0.316	0.780
6	0.427	0.250	0.747
7	0.382	0.189	0.684
8	0.362	0.158	0.657

5. NEW METHODS OF OBSERVATION WITH THE WILD GAK 1 GYRO-THEODOLITE.

5.1 INTRODUCTION.

After an initial trial and training period a gyro-theodolite survey was made at the Huntley Colliery, Dapto, on the Sydney coalfield. Access to this mine is by way of two adits into the face of a steep escarpment; a mode of access which seldom presents problems of azimuth transfer. The mine has been worked for many years so that the working faces are remote from the adits, resulting in difficulties in maintaining adequate ventilation. To overcome this latter problem it has been proposed to sink a vertical ventilation shaft from the surface to near the present working area. The present survey connections are in the form of a long closed traverse (about 50 lines) between the adits with a short unclosed spur traverse leading off the closed traverse to the proposed shaft site. The circumstances are further complicated by the fact that many of the traverse lines are short (some are about 50 ft. long). A gyro-theodolite survey was requested for the purpose of strengthening the bearings in the closed traverse and checking the bearings in the spur traverse.

For a relative transfer of azimuth from the surface to underground workings by means of the gyro-theodolite the following sequence of observations is usually made:-

Surface line AB, Underground line PQ.

Sequence A-B, P-Q, Q-P, B-A.

This bracketing procedure provides a check on the stability of the instrument. In addition a regular practice has been made of calibrating the instrument at the University on a reference line of known azimuth before and after each gyro-theodolite survey. If A_{AB} is the azimuth of the line AB etc. and G_{AB} is the measured gyro-azimuth of the same line then

$$A_{PQ} = A_{AB} + \frac{G_{PQ} + G_{QP} \pm 180^{o}}{2} - \frac{G_{AB} + G_{BA} \pm 180^{o}}{2}$$

Applying the law of propagation of variances to this expression we find that the standard deviation of the transferred azimuth A_{PQ} is equal to the standard deviation of a single measured gyro-azimuth provided that the azimuth of the surface line A_{AB} is considered error free and all observations have equal variance and are correlation free.

The results of the gyro-theodolite survey were satisfactory except for one disturbing feature. The difference between the forward and reverse observation on the reference line was large when compared with the differences between observations made at both ends of three other lines, namely 39" as compared with 5", 21", 23". The oscillation graphs of all observations appeared to be normal and no external causes were encountered which could explain this singularly large discrepancy. The reference line was reobserved the next day and the results of the four determinations are as follows:-

Date	Line	Azimuth		v
3/3/67	A-B	109^{o} 50' 16"	Δ=39"	-25"
	B-A	289^{o} 49' 37"		+14"
4/3/67	A-B	109^{o} 49' 44"	Δ=4"	+ 7"
	B-A	289^{o} 49' 48"		+ 3"
Mean	A-B	$\underline{109^{o}\ 49'\ 51''}$		

On later examination it was found that 6" of this 39" difference could be accounted for by applying a more rigorous reduction formula to the observations i.e. instead of taking the simple average of the Schuler Means a weighted average was used as described in Chapters 3 and 4. The reason for considering the difference of 39" large was that with our experience before this time, differences of this magnitude had never occurred. If the standard deviation of a single gyro determination is $\pm$ s then the standard deviation of the difference between forward and backward azimuths will be $\pm \sqrt{2}s$ and the range of differences at the 5% significance level will be $\pm 1.960 \sqrt{2}s = \pm 2.77s$. If the difference of 33" is attributed to an extreme value at the 5% significance level then the population standard deviation would be ± 12" which is corroborated by the results from experiments and tests which were carried out later and which are to be described. If an absolute azimuth is required then the uncertainty in the determination of the E value must be included. If the uncertainty in the value of E is of the same magnitude as that of a single gyro determination then the standard deviation of an absolute azimuth would be ±17" which agrees with Schwendener (1966) who reports that "tests by various civilian and military authorities resulted in an absolute mean square error of a measured azimuth of ± 15" to ±30"."

The method chosen for all of these previous determinations was the turning point method using eight reversal points - a procedure recommended by Professor Lauf. There is little to choose between the turning point and transit methods and their main characteristics are tabulated below.

	Turning Point	Transit
Accuracy of preliminary orientation	± 1^o	± 10'
Instrument constants	E	E,c.
Calculations	(Un)weighted Schuler Means	Small Calculation of ΔN
Extra equipment	Extended tangent screw	Split hand timer

For the turning point method a number of independent circle readings are obtained, and it is usual to calculate progressive Schuler Means immediately to check the work. However, with the transit method the observation requires the horizontal circle to be set close to the reading which corresponds to gyro-indicated North and a mistake in this setting may go unnoticed. There is no doubt that the turning point method is more tiring for the observer, more because the observer must concentrate continuously rather than for the reason that the turning point period is longer than that of the transit method. The transit method has more appeal for those surveyors who have an infrequent need to use the gyro-theodolite, because less skill is required in timing the gyro mark though the vee slot than in keeping the gyro mark centrally placed by means of the tangent screw in the vee slot in the turning point method. The constant manipulation of the instrument in the latter method may also impart small irregularities to the gyroscope's motion: a fact which some manufacturers have recognised by installing automatic following up devices.

From a practical view point we wish to choose an observation technique which is certain, economical, convenient and which will produce acceptable accuracy. Economy of time is certainly not a characteristic possessed by either the turning point or transit methods because a time interval of at least one period must elapse before a single determination of the direction of the meridian can be obtained. It was this aspect which prompted an investigation into the development of techniques which would give a greater number of observations per period and thus a reduction in observation time. Also to guard against uncertainty in the standards by which future observations would be judged it was thought necessary to make a series of observations using different methods on a reference line.

5.2 THE MODIFIED TRANSIT METHOD

The transit method as originally devised by Schwendener (1964) and further described by Strasser and Schwendener (1966) consists of timing the gyro-mark across a central index mark (vee slot) on the auxiliary scale of the autocollimator. The correction to the approximate North setting of the theodolite is given by

$$\Delta N = c.\ a.\ \Delta t$$

Where ΔN is the correction to the North setting to give the direction of gyro indicated North

c is an instrument constant

a is the amplitude of oscillation as determined from readings made on the auxiliary scale.

Δt is the second difference of times of transit of the gyro-mark through the vee slot.

The constant c may be evaluated from the relationship

$$c = m \frac{\pi}{2} \frac{T_u^{\,2}}{T_D^{\,3}}$$

where m is the arc value of each division of the auxiliary scale,

T_u is the period for the turning point method,

T_D is the period for the transit method,

or c may be determined empirically from a number of observations made at settings on either side of the meridian.

A gyro-theodolite method which uses multiple observations has been described by Grafarend (1967, 1969). Grafarend's technique which has been used for the reduction of automatically timed observations is of interest because the solution is independent of the amplitude of the oscillation. The passage of the gyro-mark across slits is detected by means of photo-electric cells and the times are recorded in either digital form using either a counter and printer or precision watches or in analogue form using either an oscilloscope or magnetic tape. For three slits the reduction formula, according to Grafarend, is as follows:

$$\alpha = \frac{2(\Delta t_{12} - \Delta t_{23})}{W^2 \Delta t_{12} \Delta t_{23} \Delta t_{13}} + \alpha_2$$

where Δt_{12} etc. are the time intervals between observations made to slits 1 and 2 etc.

α The required North direction

α_2 The horizontal circle setting

$W = \frac{2\pi}{T}$, where T is the period of oscillation, which can be pre-determined by timing a number of transits of the gyro-mark across the slits.

The success of the method depends upon the accurate determination of the time difference ($\Delta t_{12} - \Delta t_{23}$), which requires highly precise timing apparatus. Theoretically one transit across the slits would be sufficient for a single determination of the direction of gyro indicated North.

Instead of making one timing observation as the gyro-mark passes through the central vee slot, multiple observations can be made by timing the gyro-mark on graduation lines on either side of zero on the auxiliary scale. It is not possible to make a sensitive time estimation when the gyro-mark is centred over one of the graduation lines but if the coincidence of the leading edge with a scale line is observed, a good time estimate of this event can be made. Figure 5.12 (see Section 5.7) shows the scheme of observation. It can be proved (see Section 5.7) that the first term is dominant in a series expression for ΔN i.e.

$$\Delta N = c\ a\ (1 - \frac{n^2}{a^2})^{\frac{1}{2}}\ \Delta\ t \qquad (1)$$

Where n is the number of the graduation line

$$\Delta t = (t_4 - t_1) - (t_6 - t_3)$$

$$\Delta t = (t_3 - t_2) - (t_5 - t_4) \text{ etc.}$$

$$\Delta N = \left(\frac{T_u}{T_D}\right)^2 \Delta y$$

and c, a, have the same meaning as before.

It may be noted that equation (1) is independent of the width of the gyro-mark and is a general equation in that it embraces the normal transit method i.e. $n = 0$, $t_1 = t_2$, $t_3 = t_4$, $t_5 = t_6$ etc.

If each timing observation is recorded as though it were observed on the previous graduation line, then for a gyro-mark of normal width (for the GAK 1, approximately $2\frac{1}{2}$ divisions) there would be a reduction in the size of the neglected terms in the series. Moreover if the gyro-mark was an integral number of divisions wide, then observations could be simulated as though the mark had zero width. The error caused by the neglect of terms other than the first may be reduced by

(1) Observing only on those scale lines which are close to the zero of the auxiliary scale.

(2) Using a large amplitude.

(3) Keeping ΔN small.

(4) Using a modified auxiliary scale with multiple vee slots. A suggested arrangement is shown in Fig. 5.1 on Page 119.

It should be noted that the above error can be eliminated by arranging ΔN to be of opposite sign when making reciprocal observations on a survey line.

In Section 5.7.1 are given the results of calculation of examples for some selected values of Δy and "a". The proportionality factor "c" is not sensitive to latitude changes (see Strasser and Schwendener (1966)) and therefore these examples illustrate in general the magnitude of errors caused by using the approximate relationship for ΔN. In practice the quantity $K = c\ (a^2 - n^2)^{\frac{1}{2}}$ has been precalculated and tabulated on a digital computer so that field calculations can be kept to a minimum. Then $\Delta N = K.\ \Delta t.$ A sample table is given in Appendix I.

The most significant feature of the modified transit method from the user's viewpoint is the increase in the number of timing observations. No difficulty has been found in observing and recording with the GAK 1 where the observations are spaced at intervals of about 5^s in Sydney (ϕ = S34^o).

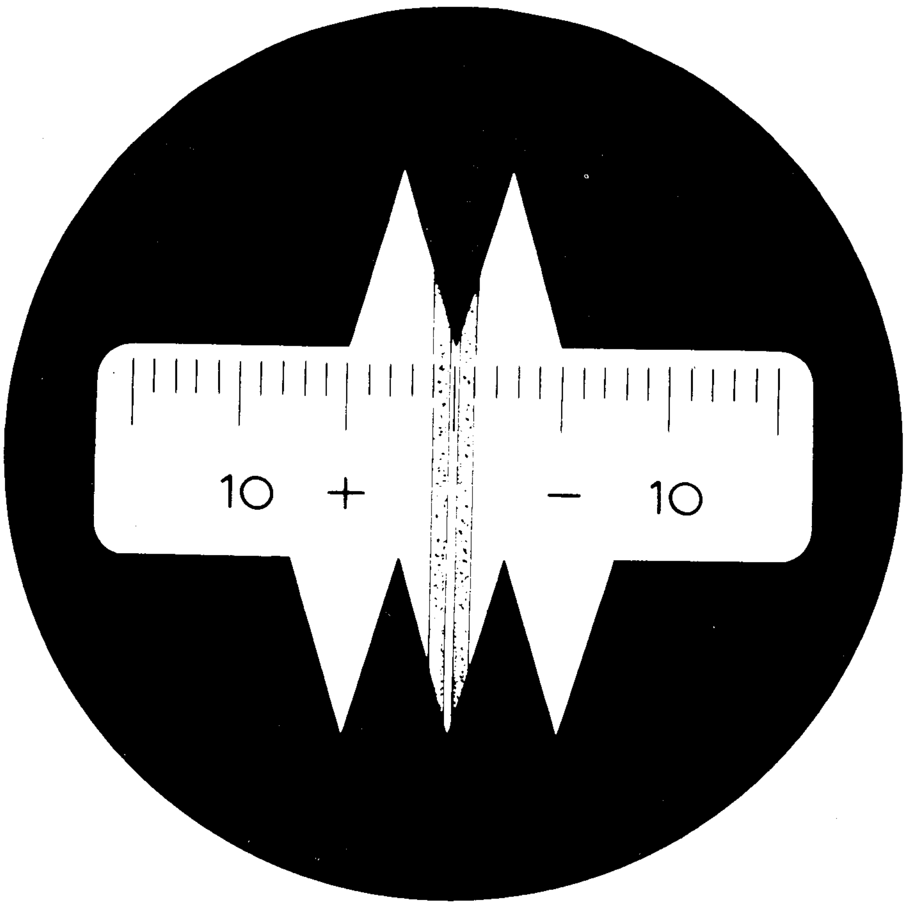


FIG. 5.1 MODIFIED AUXILIARY SCALE

The amplitude readings made at the start and finish of the observations provide a useful preliminary check on the calculation of ΔN. It has been found that the value of ΔN calculated purely from amplitude readings is correct to within 1 to 2' (Schwendener (1966) refers to this technique as the "Amplitude Method" under "Quick Methods"). Before moving from a station it is useful to check the consistency of the observations. If the time differences (Column Δ in Section 5.7.1) are calculated then a consistency check can be made by examining the values of Δ in alternate columns to see if they are in reasonable agreement. An example taken from practice is given in Section 5.7.2.

5.3 THE MODIFIED TURNING POINT METHOD.

If multiple observations are to be made with this method then the optimum circumstance will occur in the vicinity of the turning point. The observation consists of recording timed horizontal circle readings: a technique which is more suited to optical scale rather than micrometer theodolites. It may be proved (see Section 5.8) that the correction to the observed horizontal circle reading required to bring it to the turning point is given by

$$\Delta y = \frac{B(-1)^{i-1}(2\pi)^2 \, \Delta\tau^2}{2!} - \frac{B(-1)^{i-1}(2\pi)^4 \, \Delta\tau^4}{4!} \ldots \quad (2)$$

Where B and $\Delta\tau$ are the amplitude and fractional period respectively. This equation may be expressed more simply as

$$\Delta y = \pm \frac{B' \, \pi^2 \, \Delta t^2}{T^2} \mp \frac{B' \, \pi^4 \, \Delta t^4}{3T^4} \ldots\ldots\ldots \quad (2a)$$

Where B', T and Δt are the double amplitude, period, and time interval before or after the turning point respectively.

Equations (2) or (2a) give the first terms in a rapidly converging series for Δy. In Section 5.8 the effect of neglecting the remaining terms in the series is discussed. The technique of observation and reduction is analogous to the circum-elongation observation for azimuth in field astronomy.

The method requires two observers, one to track the gyro-mark and the other to observe the horizontal circle and record the times. Three techniques of observing have been investigated by trial observation in the following ways

(1) Observing times when the horizontal circle index corresponds with a circle graduation line.

(2) As in (1) with the exception that the same main circle graduations are observed on either side of the turning point.

(3) Making observations at random times and estimating the horizontal circle reading.

Technique (1) is satisfactory with the exception that in the immediate vicinity of the turning point the motion of the gyro-mark is extremely slow and therefore there is a long time interval between readings when the horizontal circle index corresponds with a circle graduation line. Thus we are prevented from taking readings during the optimum observation period.

The second technique suffers from the same defect as the first but the second has the advantage that the quantity Δt can be deduced simply from half the difference of the times of observation made to the same circle graduation line on either side of the turning point. It is also possible to deduce the period of oscillation from these observations made by this technique because the instant of the turning point is the average of these times, and the period will be twice the time intervals between consecutive turning points. Again there is an analogy between this technique and the equal altitude method of meridian latitude determination in field astronomy. The disadvantage of the limitation in the observing period in techniques (1) and (2) can be avoided by adopting the procedure outlined for the third technique. The presence of the other observer does not distract the observer who is tracking, in fact it is more relaxing for him because he is relieved of the task of taking the circle readings. This procedure could well be adopted in the usual turning point method. It has been found in practice that taking three observations before and three after the turning point and one observation without time at the turning point occupies from between 1 minute and 1 minute 40 seconds. The third technique was found to be satisfactory and was adopted for all subsequent observations with this method.

It is significant that the first and second terms in equations (2) or (2a) are dominant and dependent upon the amplitude, period and time difference between the observation and the turning point and not upon the damping factor, α. These quantities are readily available from the observations. The amplitude may be deduced from the horizontal circle readings taken at the instant of the turning point. Simple general expressions for the least squares estimate of the "middle amplitude" have been given in Section 3.1. The period of oscillation can be predetermined in a particular locality or it may be deduced from recording times when the horizontal circle reads zero (the circle is usually oriented to North approximately). The period will be the difference between the alternate values of these time instants. The average of consecutive values of these recorded times will also give the instant of the turning point, and the time interval Δt will be the difference between the instant of observation and the turning point. These aspects are readily seen from an example given in Section 5.8.1.

The analysis of the results of the trial observations, apart from gauging the merits or demerits of the three methods of observation referred to previously, disclosed the presence of a systematic error in the observations. On the assumption that the mean of the observations, after reduction to the turning point, represented the most probable value of the turning point, the residuals on one side of the turning point showed a preponderance of the same sign. It was concluded that the observer had a tendency to lag behind in his following of the gyro-mark in the vee slot on both sides of the turning point. This tendency may be natural on the part of the observer, because near the turning point he does not wish to "over run" with the tangent screw and miss the turning point. After the observer was made aware of this effect he paid greater attention to centering the gyro-mark in the vee slot and then the residuals were found to assume a random distribution in sign.

5.4 EXPERIMENTAL SERIES.

As indicated previously there were two reasons for conducting an experimental series of observations, (1) to evaluate new observation methods and (2) to establish standards by which future observations could be judged. The experiments were made in the period between 3rd April and 15th May 1967. The results of these tests were used as a basis for establishing an observational procedure for gyro-theodolite surveys at the Broken Hill Group of mines, which were made in the period between 17th May and 13th July 1967 (including post-calibration observations).

The observations were made with the gyro-theodolite mounted on a bracket fixed to the inside of the East-West wall of the Civil Engineering Building on the University campus. The advantages of this arrangement were that observations could be made in comfort in nearly all weather conditions and errors of centering could be virtually eliminated. The referring mark was a fixed target placed on the Biological Sciences Building at a distance of about 1,000 feet. The azimuth of this line was derived from a small triangulation scheme which had been oriented by astronomical observations. To minimise the correlation which exists between successive observations made in a short period of time, no more than one set of observations was made on any day. Each set consisted of the following observations:-

<u>First Day</u>

(1) 5 auxiliary scale readings of the non-spinning gyro.

(2) 3 pointings in circle left and right on the referring mark.

(3) Turning point method with 8 reversal points.

(4) Repeat (2) after arresting the gyro.

(5) Transit method with 8 transits, observing scale lines 2 and 3 as well as through the vee slot. Scale lines 2 to 6 were observed for the first and last three of the eight transits.

(6) Repeat (2) after arresting the gyro.

(7) Repeat (1).

Second Day

The same sequence of observations as on the first day with (3) and (5) interchanged.

The third day's observations were made in the same sequence as the first day and so on. In all, 27 sets of observations were made before the gyro-theodolite surveys were undertaken at Broken Hill.

The experiments were designed to give information about the following aspects:-

(1) The performance of the turning point and transit method and in particular whether there was a significant difference between the means of 4 and 8 observations. Halmos (1967) after extensive tests with the gyro-theodolite MOM-Gi-B1 considers that four reversal points are sufficient because of the likelihood of systematic errors. Halmos also considers that it is advisable to "idle" the motor before beginning the first measurements in order to bring the instrument to an even temperature, the temperature changes being brought about by bearing friction and current supply. Thus the sequence of observations in each set in these experiments were alternated so that half of the observations were made in a "cold" state and the other half in a "warm" state.

(2) The performance of the modified transit method and in particular whether the increase in the number of observations over the normal method would give a greater precision. An increase in the number of observations can be achieved in two ways (a) over a long period by observing scale lines 2 and 3

for 8 transits or (b) over a short period by observing scale lines 2 to 6 for 3 transits.

The modified turning point method was not developed until after this experimental series.

The results of this experimental series are shown in graphical form in Figures 5.2 to 5.7 on Pages 164 to 169. The information from this series was limited when after the fifth set of observations the attachment was lifted from the bridge with the gyro unclamped. The effect of this can be seen in all graphs where there is a sudden change of about 1' in the value of E. In the next nine sets of observations the graphs of E show an upward trend towards, but not quite reaching the original value of E. It is remarkable that the strain on the suspension tape was not removed until after a long set of observations indicating that the stress strain relationship was complex. Probably some fibres were stressed beyond the elastic limit. It was considered that the last 13 sets were free from this effect. To avoid a recurrence of this accident a procedure sheet was drawn up, setting out in detail the sequence of observations required of the observer. The recorder then assumed the responsibility of guiding the observer through the observations. All subsequent observations have been made in this manner and this accident has not been repeated. The manufacturer could give some thought to this aspect which would require a device to prevent the attachment from being lifted from the bridge when the gyro is in a lowered position.

Theoretically it should be possible to account for these changes in E by analysing the values of the mean position of oscillation of the non-spinning gyro readings. A change of 0.1 div. of the non-spinning gyro results in a change of 16" in the value of E for this instrument in this latitude ($S34^o$). The results of all the non-spinning gyro readings are shown in Figure 5.8 on Page 128. Mean values for the periods 3rd - 7th April, 10th - 20th April, 24th April - 15th May are shown, which give changes of 35" and 27" for the value of E. These changes agree well with the changes in the average values of E determined by all observation methods. However it will be noted that the non-spinning readings show a large scatter and thus in practice little reliance can be place on corrections to E deduced from such readings.

Graphs of the individual observations for the turning point and transit method are shown in Figures 5.9 and 5.10. For the turning point method there is a marked tendency for the first observations in the cold state to lie on the left side of the mean line. When the instrument is warm the observations adopt a random pattern. In the transit method this tendency is not so marked. The range of observations in the turning point method is considerably smaller than that of the transit method and if these ranges are indicative of the external precision then it would be anticipated that the turning point method would give a higher precision. This was not borne out by a later analysis when the reverse appeared to be the case.

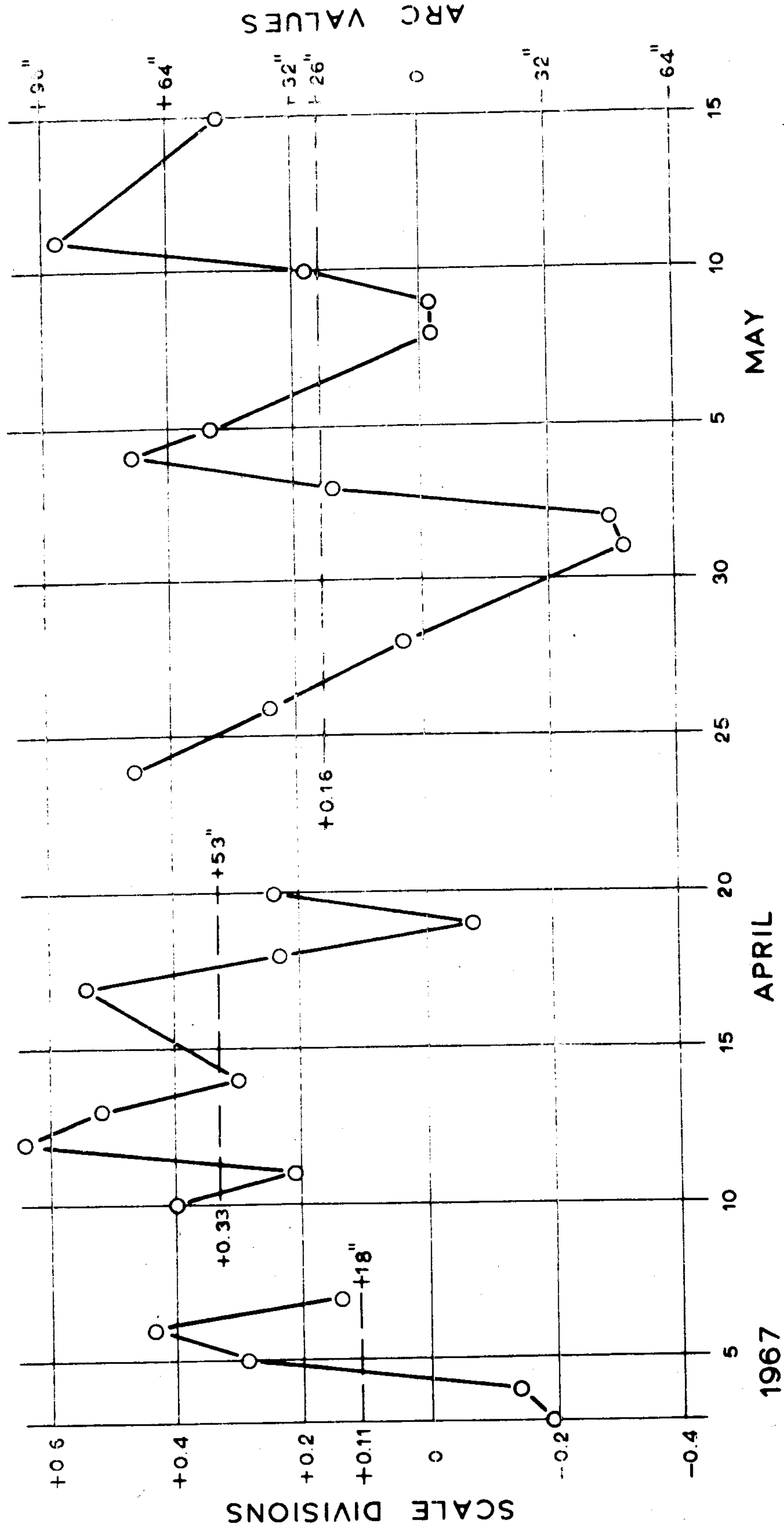


FIG. 5.8 NON-SPINNING GYRO READINGS

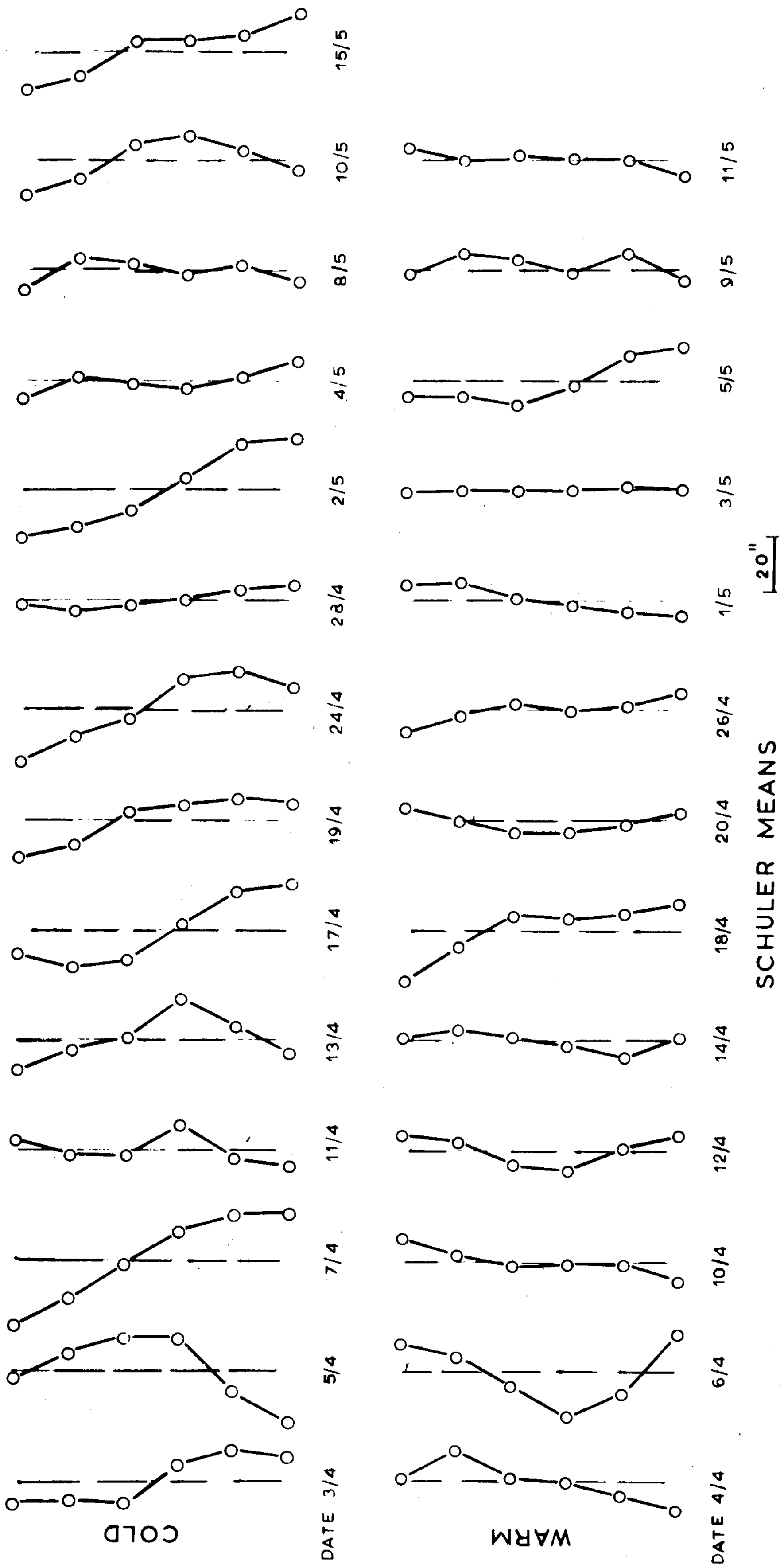


FIG. 5.9 TURNING POINT METHOD

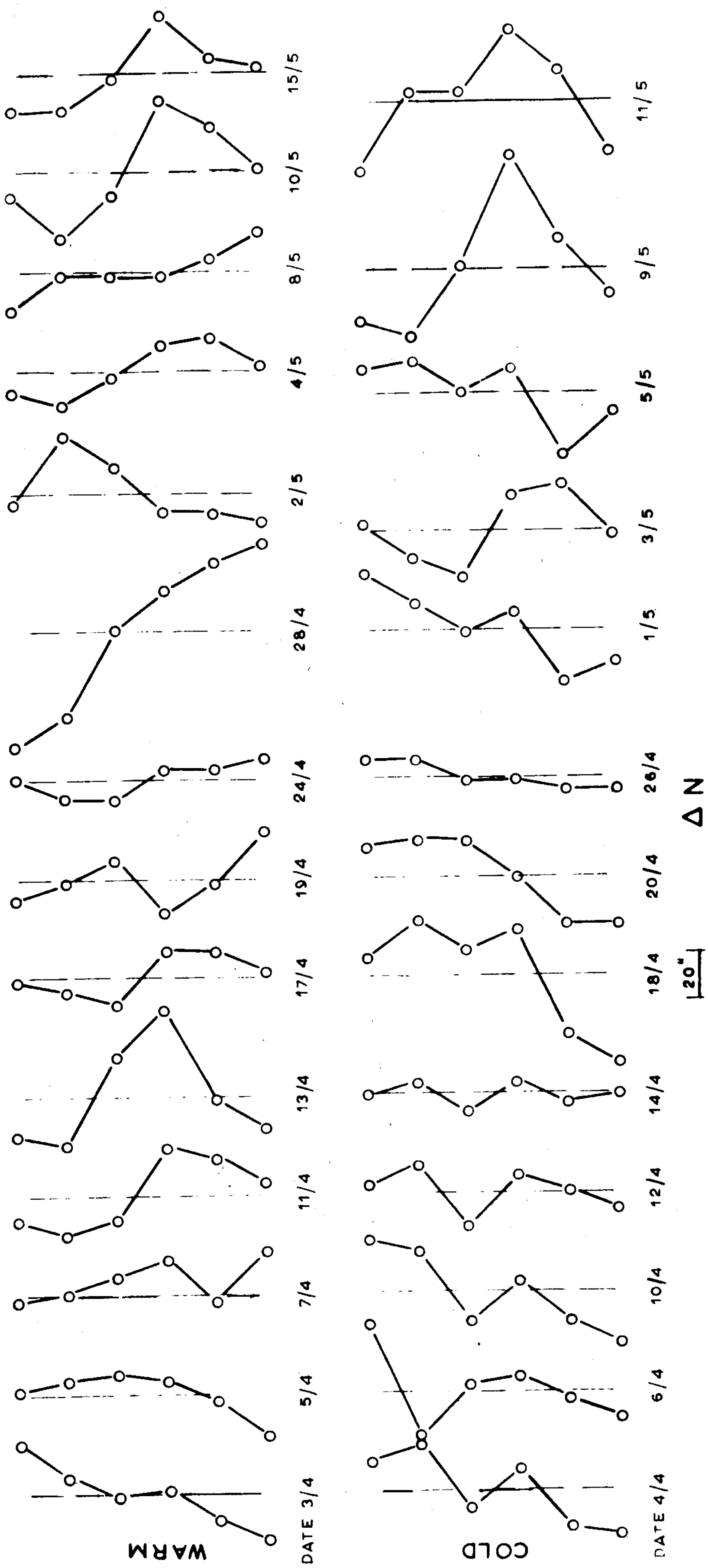


FIG. 5.10 TRANSIT METHOD

The analysis of the results from the last 13 sets of observations are given in Table 5.1. The sample size is somewhat reduced because of the need to reject those observations taken at the beginning and those affected by the tape strain. Conclusions drawn from such a sample cannot be definitive but will give some indication of the precision of the various methods and of the presence of systematic effects. It was concluded that:-

(a) The transit method gave better results than the turning point method when 8 observations were taken. (Methods A and B, see Figures 5.2 and 5.3).

(b) There appeared to be some improvement in precision for both methods when 8 observations were taken rather than 4.

(c) The difference between mean values in the cold and warm states was more pronounced with the turning point method.

(d) Increasing the number of observations by observing scale lines 2 and 3 did not lead to a significant improvement when compared with observations made with the vee slot. This may not be the case if a modified auxiliary scale (see Figure 5.1) is used where one may expect an improvement in the precision of timing. (Method C, see Figure 5.4)

(e) Observations made on scale lines 2 to 6 for 3 transits gave results which were significantly better than taking 4 transits through the vee slot. The results from the last 3 transits gave better results than the first three which may be due to the fact that temperature effects at the end of 8 transits have diminished, although there is little difference in the

TABLE 5.1

Analysis of results of the observations made in the period 24th April to 15th May 1967.

	Method	State	n	Range	S.D.	Mean	Difference Cold - Warm
	Turning Point	All	13	35"	±13"	47"	
A	(8 T.P.'s)	Cold	7	23	10	57	+20"*
		Warm	6	18	7	37	
	Turning Point	All	13	49	17	52	
	(4 T.P.'s)	Cold	7	32	12	65	+27*
		Warm	6	17	6	38	
	Transit (Vee Slot)	All	13	23	7	46	
B	(8 Transits)	Cold	6	23	8	46	0
		Warm	7	18	6	46	
	Transit (Vee Slot)	All	13	36	12	44	
	(4 Transits)	Cold	6	28	11	44	0
		Warm	7	36	13	44	
	Transit	All	13	23	8	37	
C	(8 Transits)	Cold	6	23	8	40	+6
	(Scale lines 2 & 3)	Warm	7	21	8	34	
	Transit	All	13	28	10	33	
D1	(Scale lines 2 to 6)	Cold	6	28	10	40	+13*
	(First 3 transits)	Warm	7	22	7	27	
	Transit	All	13	24	7	32	
D2	(Scale lines 2 to 6)	Cold	6	16	7	36	+6
	(Last 3 Transits)	Warm	7	16	6	30	
	Transit	All	13	20	6	33	
D3	(Scale lines 2 to 6)	Cold	6	10	4	38	+10
	(Mean of First & Last 3 Transits)	Warm	7	11	4	28	
E	Mean of all		13	14	4	39	

* Those differences marked with an asterisk are significant at the 5% significance level with the "t" test.

values of the means. (Methods D1 and D2, see Figure 5.5)

(f) The mean of scale lines 2 to 6 combined for the first and last 3 transits gave excellent results (Standard Deviations of ± 6", 4", 4") indicating that it would be well worthwhile making further tests with this method.
(Method D3, see Figure 5.6).

(g) The mean of all results also gave excellent results.
(Method E, see Figure 5.7)

The small standard deviations achieved with some of the methods are remarkable considering that the least count of the Wild T16 theodolite is 1' and readings can only be estimated to 0.1'.

A second experimental series was conducted in the period between the 19th and 27th July, 1969 with a Wild GAK 1 gyro attachment S/N 3243 and T16 theodolite S/N 112615. Each set of observations consisted of

(1) Auxiliary scale readings of the non-spinning gyro in azimuths 90^{o} and 270^{o}.

(2) 3 pointings in circle left and circle right to the referring mark.

(3) Timing observations to all scale lines from -5 to +5 for 4 transits and amplitude readings on the auxiliary scale.

(4) Repeat (2) after arresting the gyro.

(5) Repeat (1).

When a recorder was not available it was found that the observations could be conveniently recorded on a small dictaphone (Phillips 85 pocket memo) and transcribed afterwards.

The results of these 30 sets of observations are shown in the sequence in which they were observed in graphical form in Figure 5.11 on Page 135. These results do not show any obvious periodicity or indicate the presence of other systematic effects and it has been assumed that the scatter of the observations is caused by random errors. As with previous work with gyro-theodolite GAK 1 S/N 2871 the first observation exhibited a large difference from the mean, although in this case this is not an isolated large residual. Halmos (1967) considers that after long transportation the tape may have minor deformations and recommends "to charge the band with the weight of the oscillation system before the first measurement." The instrument was transported before the experimental series and the effect as described by Halmos could have been present. The estimate of the standard deviation of a single observation was ± 10" compared with ± 7" and ± 10" obtained with GAK 1 S/N 2871 using a similar observing technique.

5.5 BROKEN HILL GYRO-THEODOLITE SURVEY

For the observations at the Broken Hill group of mines it was decided to adopt the modified transit method, observing scale lines 1 - 5 (excluding the vee slot) for 4 transits with no warm up period. The observing time of about 10 minutes is about one third of that required for the turning point method with 8 reversal points. It was expected that observations made under field conditions would give a lower precision than those taken under laboratory conditions. In the laboratory the observer is working in comfort with a stable instrument support and the centering of instrument and target are under

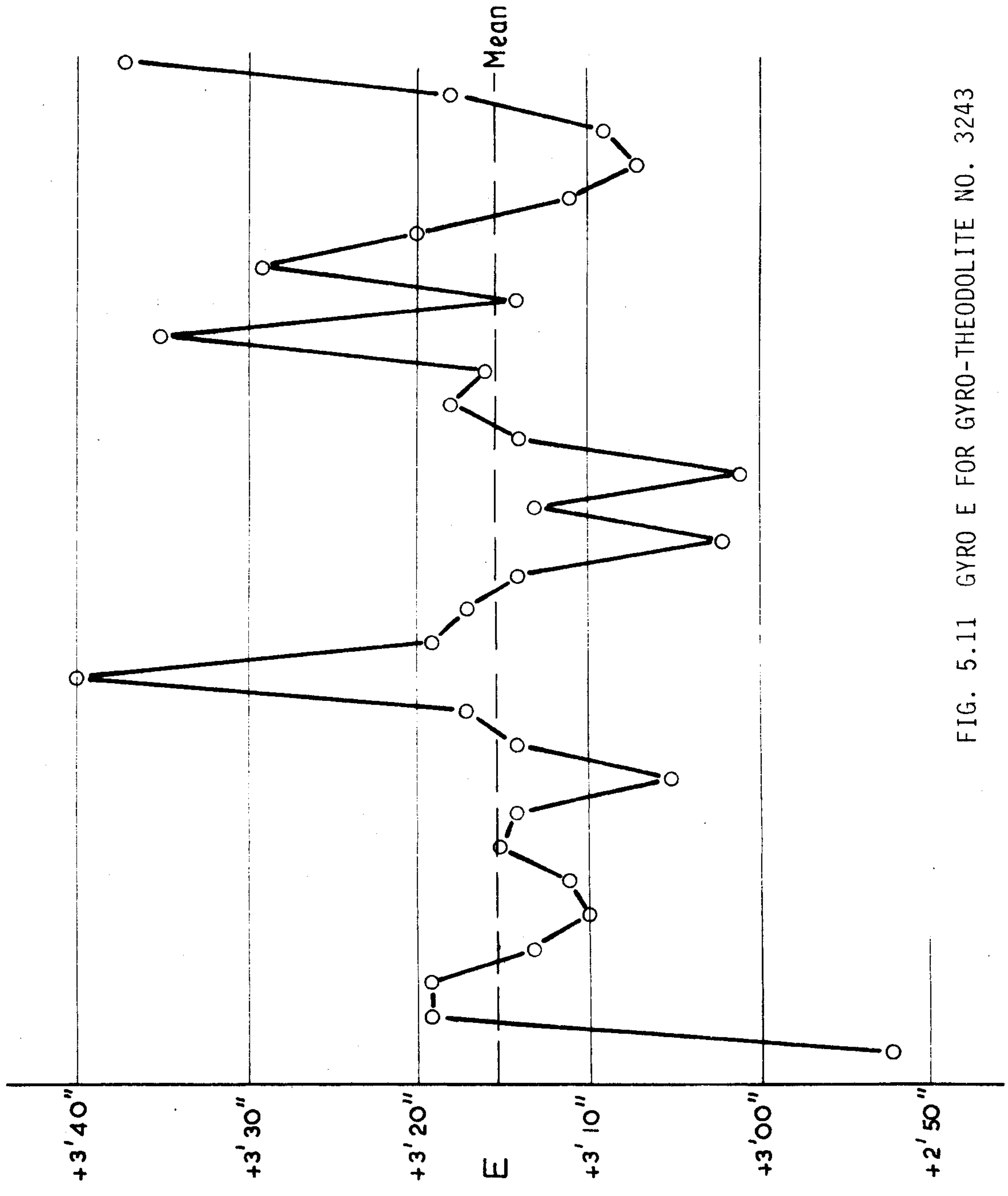


FIG. 5.11 GYRO E FOR GYRO-THEODOLITE NO. 3243

rigid control. It was estimated that the precision of an azimuth determination under laboratory conditions was about ± 10" and under field conditions this would be extended to about ± 15". A precision of this magnitude is quite acceptable for normal mining survey work.

Observations for the transfer of azimuth between surface and underground workings were made at the following mines -

The New Broken Hill Consolidated Mine

The Broken Hill South Mine

The Broken Hill North Mine

These mines are located in North Western New South Wales and their principal products are lead, silver and zinc. Astronomical observations for azimuth were made at each of these mines to provide reference lines of known azimuth for future gyro-theodolite surveys. Each mine uses an independent arbitarily chosen co-ordinate origin for its survey work with co-ordinate axes oriented in the direction of the initial mining development. With the rapid expansion of mining activities in the post war years the mine workingshave become quite extensive and a knowledge of the interrelation of the various mines is of prime importance not only from the point of view of economical development but for safe operation.

The results of the 27 gyro-theodolite azimuths made at these mines are given in Table 5.2 on Page 137.

On the whole the results were quite satisfactory, the estimate of the standard deviation of a single determination is ± 15". The observations were made with minimum of delay to mining production. This was partly due to the short observation period of the method. This last aspect is

TABLE 5.2 SUMMARY OF BROKEN HILL GYRO-THEODOLITE RESULTS

Date	Line	Gyro Azimuth			Convergence	Reduced Gyro Azimuth			v
27/5	127-L	323°	48'	52"	-1"	323°	48'	51"	Reject
27/5	L-127	143	47	37	-3	143	47	34	+16"
28/5	127-L	323	48	20	-1	323	48	19	-29
29/5	L-127	143	47	41	-3	143	47	38	+12
27/5	145G-152G	207	15	52	-14	207	15	38	- 7
27/5	152G-145G	27	15	38	-14	27	15	24	+ 7
28/5	52-107	226	43	42	+ 8	226	43	50	- 2
28/5	107-52	46	44	09	-22	46	43	47	+ 1
28/5	F-E	319	03	30	+ 1	319	03	31	+15
28/5	E-F	139	04	02	0	139	04	02	-16
29/5	122-CS32	319	24	31	- 2	319	24	29	-20
29/5	CS32-122	139	23	52	- 3	139	23	49	+20
29/5	523G-530G	228	05	38	-22	228	05	16	-13
29/5	530G-523G	48	05	11	-20	48	04	51	+12

N.B.H.C. Mine

Date	Line	Gyro Azimuth			Convergence	Reduced Gyro Azimuth			v
30/5	A-B	48°	59'	03"	-5"	48°	58'	58"	-5"
30/5	B-A	228	58	51	-3	228	58	48	+5
31/5	B-A	228	58	55	-3	228	58	52	+1
30/5	19-28	34	02	30	-8	34	02	22	-9
30/5	28-19	214	02	11	-7	214	02	04	+9
30/5	22-10	116	04	11	-9	116	04	02	-10
30/5	10-22	296	03	51	-8	296	03	43	+ 9
31/5	39-38	143	07	34	-8	143	07	26	- 7
31/5	38-39	323	07	22	-9	323	07	13	+ 6

South Mine

Date	Line	Gyro Azimuth			Convergence	Reduced Gyro Azimuth			v
1/6	T16-T20	299°	48'	04"	+25"	299°	48'	29"	-5"
1/6	T20-T16	119	48	02	+18	119	48	20	+4
1/6	4-5	70	24	48	+22	70	25	10	-6
1/6	5-4	250	24	35	+23	250	24	58	+6

North Mine

of some importance because delays due to survey operations can inconvenience production personnel and are unpopular with them. No difficulties were found in using the method in the mines. The recorder was able to check for internal consistency quickly (see Section 5.2) before moving from the station. The final calculations were made later with a slide rule and could be completed in a few minutes.

As with the gyro-theodolite survey at the Huntley Colliery the first observation at Broken Hill gave a value which was considerably different from later determinations made on the same line. In this case the instrument was used directly after transportation without preliminary observations and again the effect as described previously by Halmos could have been present.

5.6 CONCLUSION

The modified methods which have been described have been used successfully in practice, especially the modified transit method. Their advantage lies in the fact that a greater number of observations per period can be made than with the normal methods. In high latitudes where the period of oscillation is long these methods are of particular value and if the observation errors are of a purely random nature then it may be possible to extend the latitude range of the gyro-theodolite because of the increase in precision of the determination.

The transit method has also been extended by Halmos (1969b) to include observations to scale lines on either side of the zero of the auxiliary scale. If $\Delta t = \frac{1}{2}\{(t_3 - t_2) + (t_4 - t_1) - (t_6 - t_3) + (t_5 - t_4)\}$ etc., see Figure 5.12, Section 5.7, then the correction to the approximate North setting of the theodolite is given by

$$\Delta N = c\ a\ \Delta\ t$$

This method differs from the previous approach in that the formula does not include a correcting term of $(1 - \frac{n^2}{a^2})^{\frac{1}{2}}$.

Halmos has also described a combined transit and turning point method for a gyro-theodolite fitted with an automatic following device. Times are recorded when the zeros of the autocollimator scale are in coincidence for preset value(s) of the horizontal circle. Halmos (1969b) has reported that these methods have given improved results for the M.O.M. gyro-theodolite series Gi - B1, Gi - B2, Gi - C2, Gi - D1 where these methods are appropriate.

5.7 PROOF OF THE MODIFIED TRANSIT METHOD

Lauf (1963) has shown that the oscillation of a pendulous gyroscope is of the form -

$$y = m\,a\,e^{\frac{-\alpha(t-t_o)}{T}} \cos\left\{\frac{2\pi(t-t_o)}{T} + \gamma\right\}$$

where y is the displacement in arc

m is the arc value of each scale division

a is the amplitude in scale division at time $t = t_o$

T is the period of oscillation

t_o is the initial time instant

t is any time instant

α is the damping constant

γ is an arbitrary phase angle

Consider observations to be made when the leading edge of the gyro mark is coincident with a graduation line on the auxiliary scale. Then for the first half period

$$\Delta y = y_{t_i} - \frac{mw}{2} - nm = m\,a\,e^{\frac{-\alpha(t_i-t_o)}{T}} \cos\left\{\frac{2\pi(t_i-t_o)}{T} + \gamma\right\} - \frac{mw}{2} - nm \qquad (1)$$

and for the second half period

$$\Delta y = y_{t_i} + \frac{mw}{2} - nm = m\,a\,e^{\frac{-\alpha(t_i-t_o)}{T}} \cos\left\{\frac{2\pi(t_i-t_o)}{T} + \gamma\right\} + \frac{mw}{2} - nm \qquad (2)$$

where Δy is the displacement of the zero graduation line from the axis of mean oscillation.

y_{t_i} is the displacement of the centre of the gyro mark at time t_i.

w is the width of the gyro mark in scale divisions.

n is the number of the scale graduation on either side of the zero graduation line.

For convenience put $t_o = 0$ and $\gamma = 0$. For simplicity in the following derivation we will consider observations made to a single scale division on either side of the zero graduation line with the timing sequence renumbered accordingly, see Figure 5.12.

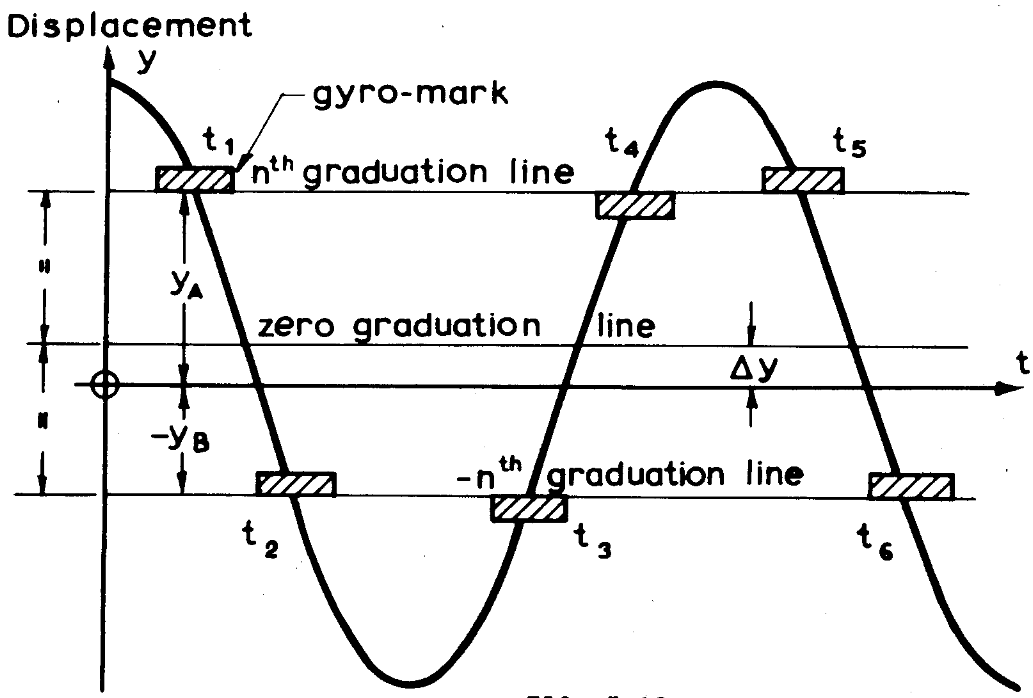


FIG. 5.12

From (1) we may write after expanding the exponential coefficient

$$y_A = ma\ (1-\alpha\tau_1 + \frac{\alpha^2\tau_1^2}{2!}\ ....)\ \cos 2\pi\ \tau_1 - \frac{mw}{2} \qquad (3)$$

$$y_B = ma\ (1-\alpha\tau_2 + \frac{\alpha^2\tau_2^2}{2!}\ ....)\ \cos 2\pi\ \tau_2 - \frac{mw}{2} \qquad (4)$$

$$y_A = ma\ (1-\alpha\tau_5 + \frac{\alpha^2\tau_5^2}{2!}\ ....)\ \cos 2\pi\ \tau_5 - \frac{mw}{2} \qquad (5)$$

$$y_B = ma\ (1-\alpha\tau_6 + \frac{\alpha^2\tau_6^2}{2!}\ ....)\ \cos 2\pi\ \tau_6 - \frac{mw}{2} \qquad (6)$$

and from (2)

$$y_B = ma\ (1-\alpha\tau_3 + \frac{\alpha^2\tau_3^2}{2!}\ ....)\ \cos 2\pi\ \tau_3 + \frac{mw}{2} \qquad (7)$$

$$y_A = ma\ (1-\alpha\tau_4 + \frac{\alpha^2\tau_4^2}{2!}\ ....)\ \cos 2\pi\ \tau_4 + \frac{mw}{2} \qquad (8)$$

after replacing $\frac{t_1}{T}$, $\frac{t_2}{T}$ .... by $\tau_1\ \tau_2$....

The mean of (4) and (7) gives

$$y_B = \frac{ma}{2}\ (\cos 2\pi\tau_2 + \cos 2\pi\tau_3) - \frac{ma\alpha}{2}\ (\tau_2 \cos 2\pi\tau_2 + \tau_3 \cos 2\pi\tau_3)$$

$$+ \frac{ma\alpha^2}{4}(\tau_2^2 \cos 2\pi\tau_2 + \tau_3^2 \cos 2\pi\tau_3)\ .... \qquad (9)$$

$$y_B = ma \cos 2\pi\ (\frac{\tau_2+\tau_3}{2})\ \cos 2\ \pi\,(\frac{\tau_2-\tau_3}{2}) - \frac{ma\alpha}{2}\ (\tau_2 \cos 2\pi\tau_2 + \tau_3 \cos 2\pi\tau_3)$$

$$...... \qquad (10)$$

put $\frac{\tau_2 + \tau_3}{2} = \frac{1}{2} + d\tau_{23}$ (10a)

$\therefore$ $\cos 2\pi(\frac{\tau_2 + \tau_3}{2}) = \cos(\pi + 2\pi d\pi_{23})$ and a Taylor expansion of the right hand side gives

$$\cos \pi - 2\pi d\pi_{23} \sin \pi - \frac{4\pi^2 d\tau_{23}^2}{2!} \cos\pi \;\ldots\ldots\ldots$$

$$= -1 + \frac{4\pi^2 d\tau_{23}^2}{2} \;\ldots\ldots\ldots$$

$\therefore$ (10) becomes

$$y_B = -ma \cos 2\pi \left(\frac{\tau_2 - \tau_3}{2}\right) + \frac{ma\ 4\pi^2 d\tau_{23}^2}{2} \cos 2\pi \left(\frac{\tau_2 - \tau_3}{2}\right)$$

$$- \frac{ma\alpha}{2}(\tau_2 \cos 2\pi\tau_2 + \tau_3 \cos 2\pi\tau_3) \quad \ldots(12)$$

The mean of (5) and (8) gives

$$y_A = \frac{ma}{2}(\cos 2\pi\tau_4 + \cos 2\pi\tau_5) - \frac{ma\alpha}{2}(\tau_4 \cos 2\pi\tau_4 + \tau_5 \cos 2\pi\tau_5)$$

$$+ \frac{ma\alpha^2}{4}(\tau_4^2 \cos 2\pi\tau_4 + \tau_5^2 \cos 2\pi\tau_5) \ldots \quad (13)$$

$$y_A = ma \cos 2\pi\left(\frac{\tau_4 + \tau_5}{2}\right)\cos 2\pi\left(\frac{\tau_4 - \tau_5}{2}\right) - \frac{ma\alpha}{2}(\tau_4 \cos 2\pi\tau_4 + \tau_5 \cos 2\pi\tau_5)$$

$$\ldots \quad (14)$$

put $\frac{\tau_4 + \tau_5}{2} = 1 + d\tau_{45}$ (14a)

$\therefore$ $\cos 2\pi\left(\frac{\tau_4 + \tau_5}{2}\right) = \cos(2\pi + 2\pi d\tau_{45})$ and a Taylor expansion of the right hand side gives

$$\cos 2\pi - 2\pi d\tau_{45} \sin 2\pi - \frac{4\pi^2}{2!} d\tau_{45}^2 \cos 2\pi \ldots\ldots$$

$$= 1 - \frac{4\pi^2 d\tau_{45}^2}{2} \ldots\ldots$$

$\therefore$ (14) becomes

$$y_A = ma \cos 2\pi \left(\frac{\tau_4 - \tau_5}{2}\right) - \frac{ma\ 4\pi^2\ \tau_{45}^2}{2} \cos 2\pi \left(\frac{\tau_4 - \tau_5}{2}\right)$$

$$- \frac{ma\alpha}{2} (\tau_4 \cos 2\pi\tau_4 + \tau_5 \cos 2\pi\tau_5) .. \qquad (15)$$

From Fig. 5.12 $y_A - \Delta y = - y_B + \Delta y$

$$\therefore \quad \Delta y = \frac{y_A + y_B}{2}$$

and substituting (12) and (15) in this equation gives

$$\Delta y = \frac{ma}{2} \{\cos 2\Pi \left(\frac{\tau_4 - \tau_5}{2}\right) - \cos 2\Pi \left(\frac{\tau_2 - \tau_3}{2}\right) \}$$

$$+ ma\ \Pi^2 \{d\tau_{23}^2 \cos 2\Pi \left(\frac{\tau_2 - \tau_3}{2}\right. - d\tau_{45}^2 \cos 2\Pi \left(\frac{\tau_4 - \tau_5}{2}\right) \}$$

$$- \frac{ma\alpha}{4} (\tau_2 \cos 2\Pi\tau_2 + \tau_3 \cos 2\Pi\tau_3 + \tau_4 \cos 2\Pi\tau_4 + \tau_5 \cos 2\Pi\tau_5) \ ... \qquad (16)$$

The first term of (16) $= -ma \sin 2\Pi \left(\frac{\tau_2 - \tau_3 + \tau_4 - \tau_5}{4}\right) \sin 2\Pi \left(\frac{-\tau_2 + \tau_3 + \tau_4 - \tau_5}{4}\right)$

$$(17)$$

Taking the difference between (10a) and (14a) gives

$$\tau_2 + \tau_3 - \tau_4 - \tau_5 \quad = \quad -1 + 2(d\tau_{23} - d\tau_{45})$$

$$\tau_2 - \tau_3 + \tau_4 - \tau_5 \quad = \quad -1 + 2(\tau_4 - \tau_3) + 2(d\tau_{23} - d\tau_{45})$$

and substituting in (17) gives

$$-ma \sin 2\Pi \left\{\frac{-1 + 2(\tau_4 - \tau_3) + 2(d\tau_{23} - d\tau_{45})}{4}\right\} \sin 2\Pi \ \frac{(-\tau_2 + \tau_3 + \tau_4 - \tau_5)}{4}$$

$$= - ma \sin \{\frac{2\Pi(\tau_4-\tau_3)}{2} - \frac{\Pi}{2} + 2\Pi \frac{(d\tau_{23}-d\tau_{45})}{2}\}\sin 2\Pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4}$$

A Taylor expansion of the first term gives

$$-ma \sin \{2\Pi \frac{(\tau_4-\tau_3)}{2} - \frac{\Pi}{2}\}\sin 2\pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4}$$

$$-ma\ 2\Pi \frac{(d\tau_{23}-d\tau_{45})}{2}\cos\{2\Pi \frac{(\tau_4-\tau_3)}{2} - \frac{\Pi}{2}\} \sin 2\Pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4}$$

$$+\frac{ma4\Pi^2}{2} \frac{(d\tau_{23}-d\tau_{45})^2}{4} \sin \{ 2\Pi \frac{(\tau_4-\tau_3)}{2} - \frac{\Pi}{2}\} \sin 2\Pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4} \quad \ldots.$$

$$= ma \cos 2\Pi \frac{(\tau_4-\tau_3)}{2} \sin 2\Pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4}$$

$$-ma\ 2\Pi \frac{(d\tau_{23}-d\tau_{45})}{2} \sin 2\Pi \frac{(\tau_4-\tau_3)}{2} \sin 2\Pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4}$$

$$- \frac{ma\Pi^2}{2} (d\tau_{23}-d\tau_{45})^2\cos 2\Pi \frac{(\tau_4-\tau_3)}{2} \sin 2\Pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4} \quad \ldots\ldots$$

Resubstituting in (16)

$$\Delta y = ma \cos 2\pi \frac{(\tau_4-\tau_3)}{2} \sin 2\Pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4}$$

$$-ma\ 2\Pi \frac{(d\tau_{23}-d\tau_{45})}{2}\sin 2\Pi \frac{(\tau_4-\tau_3)}{2}\sin 2\Pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4}$$

$$- \frac{ma\Pi^2}{2} (d\tau_{23}-d\tau_{45})^2 \cos 2\Pi \frac{(\tau_4-\tau_3)}{2} \sin 2\Pi \frac{(-\tau_2+\tau_3+\tau_4-\tau_5)}{4}$$

$$+ ma\Pi^2\{d\tau_{23}^2 \cos 2\Pi \frac{(\tau_2-\tau_3)}{2} - d\tau_{45}^2 \cos 2\Pi \frac{(\tau_4-\tau_5)}{2}\}$$

$$- \frac{ma\alpha}{4} (\tau_2 \cos 2\Pi\tau_2 + \tau_3 \cos 2\Pi\tau_3 + \tau_4 \cos 2\Pi\tau_4 + \tau_5 \cos 2\Pi\tau_5) \quad \ldots\ldots.$$

put $(t_3-t_2) - (t_5- t_4) = \Delta t$

then $\frac{2\Pi(-\tau_2+\tau_3+\tau_4-\tau_5)}{4} = \frac{\Pi\Delta t}{2T}$

$$\therefore \Delta y = ma \cos 2\Pi \frac{(\tau_4-\tau_3)}{2} \sin \frac{\Pi\Delta t}{2T}$$

$$- ma\Pi (d\tau_{23}-d\tau_{45}) \sin 2\Pi \frac{(\tau_4-\tau_3)}{2} \sin \frac{\Pi\Delta t}{2T}$$

$$- \frac{ma\Pi^2}{2} (d\tau_{23}-d\tau_{45})^2 \cos 2\Pi \frac{(\tau_4-\tau_3)}{2} \sin \frac{\Pi\Delta t}{2T}$$

$$+ ma\Pi^2\{ d\tau_{23}^2 \cos 2\Pi \frac{(\tau_2-\tau_3)}{2} - d\tau_{45}^2 \cos 2\Pi \frac{(\tau_4-\tau_5)}{2} \}$$

$$- \frac{ma\alpha}{4} (\tau_2 \cos 2\Pi\tau_2 + \tau_3 \cos 2\Pi\tau_3 + \tau_4 \cos 2\Pi\tau_4 + \tau_5 \cos 2\Pi\tau_5) \quad \ldots\ldots \quad (18)$$

From Fig. 5 12 $\frac{y_A - y_B}{2} = mn$

and substituting from (7) and (8)

$$\frac{y_A-y_B}{2} = \frac{ma}{2} (\cos 2\Pi\tau_4 - \cos 2\Pi\tau_3) - \frac{ma\alpha}{2} (\tau_4 \cos 2\Pi\tau_4 - \tau_3 \cos 2\Pi\tau_3)\ldots\ldots$$

$$= - ma \sin 2\Pi \frac{(\tau_4+\tau_3)}{2} \sin 2\Pi \frac{(\tau_4-\tau_3)}{2} - \frac{ma\alpha}{2} (\tau_4 \cos 2\Pi\tau_4 - \tau_3 \cos 2\Pi\tau_3)\ldots$$

put $\frac{\tau_3+\tau_4}{2} = \frac{3}{4} + d\tau_{34}$ and substitute in the first term

$$\frac{y_A-y_B}{2} = - ma \sin 2\Pi(\frac{3}{4} + d\tau_{34})\sin 2\Pi \frac{(\tau_4-\tau_3)}{2} - \frac{ma\alpha}{2}(\tau_4 \cos 2\Pi\tau_4 - \tau_3 \cos 2\Pi\tau_3) \ldots\ldots$$

but $\sin 2\Pi (\frac{3}{4} + d\tau_{34}) = \sin (\frac{3\Pi}{2} + 2\Pi d\tau_{34})$

$$= \sin\frac{3\Pi}{2} + 2\pi d\tau_{34}\cos\frac{3\Pi}{2} - 4\Pi^2\frac{d\tau^2_{34}}{}\sin\frac{3\Pi}{2} \quad \ldots\ldots$$

$$= -1 + 4\Pi^2\frac{d\tau^2_{34}}{2} \quad \ldots\ldots\ldots\ldots$$

and $\therefore$ $mn = \frac{y_A+y_B}{2} = ma\sin 2\Pi\frac{(\tau_4-\tau_3)}{2} - 4\Pi^2 ma\frac{d\tau^2_{34}}{2}\sin 2\Pi\frac{(\tau_4-\tau_3)}{2}$

$$-\frac{ma\alpha}{2}(\tau_4\cos 2\Pi\tau_4 - \tau_3\cos 2\Pi\tau_3) \ldots\ldots$$

$$\sin 2\Pi\frac{(\tau_4-\tau_3)}{2} = (1-2\Pi^2 d\tau^2_{34}\ \ldots)^{-1}\left\{\frac{n}{a} + \frac{\alpha}{2}(\tau_4\cos 2\Pi\tau_4 - \tau_3\cos 2\Pi\tau_3)\ \ldots\right\}$$

$$= \frac{n}{a} + \frac{\alpha}{2}(\tau_4\cos 2\Pi\tau_4 - \tau_3\cos 2\Pi\tau_3) + \frac{2n\Pi^2 d\tau^2_{34}}{a} \quad \ldots\ldots$$

and $\cos 2\Pi\frac{(\tau_4-\tau_3)}{2} = \left\{1 - \frac{n^2}{a^2} - \frac{n\alpha}{a}(\tau_4\cos 2\Pi\tau_4 - \tau_3\cos 2\Pi\tau_3)\right.$

$$\left. - \frac{4n^2\Pi^2 d\tau^2_{34}}{a^2}\ \ldots\right\}^{\frac{1}{2}}$$

$$= \left(1-\frac{n^2}{a^2}\right)^{\frac{1}{2}} - \tfrac{1}{2}\left(1-\frac{n^2}{a^2}\right)^{-\frac{1}{2}}\left\{\frac{n\alpha}{a}(\tau_4\cos 2\Pi\tau_4\right.$$

$$\left. - \tau_3\cos 2\Pi\tau_3) + \frac{4n^2\Pi^2 d\tau^2_{34}}{a^2}\ \ldots\right\} \ldots$$

Substituting in (18) and also putting $\sin\frac{\Pi\Delta t}{2T}$,

which is a very small quantity $= \frac{\Pi\Delta t}{2T}$ we have

$$\Delta y = \frac{ma\,\Pi\Delta t}{2T}\left(1-\frac{n^2}{a^2}\right)^{\frac{1}{2}} \qquad \text{(i)}$$

$$-\frac{ma\,\Pi\,\Delta\,t}{4T}\left(1-\frac{n^2}{a^2}\right)^{-\frac{1}{2}}\left\{\frac{n\alpha}{a}(\tau_4\cos 2\Pi\tau_4 - \tau_3\cos 2\Pi\tau_3) + 4n^2\Pi^2 d\tau^2_{34}\ \ldots\right\} \qquad \text{(ii)}$$

$$- \frac{ma\Pi^2\Delta t}{2T} (d\tau_{23} - d\tau_{45})\{\frac{n}{a} + \frac{\alpha}{2}(\tau_4 \cos 2\Pi\tau_4 - \tau_3 \cos 2\Pi\tau_3) + \frac{2n\Pi^2 d\tau_{34}^2}{a} \ldots\}$$ (iii)

$$- \frac{ma\Pi^3\Delta t}{4T} (d\tau_{23} - d\tau_{45})^2 (1 - \frac{n^2}{a^2})^{\frac{1}{2}}$$ (iv)

$$+ ma\,\Pi^2\{d\tau_{23}^2 \cos 2\Pi \frac{(\tau_2 - \tau_3)}{2} - d\tau_{45}^2 \cos 2\Pi \frac{(\tau_4 - \tau_5)}{2}\}$$ (v)

$$- \frac{ma\ \alpha}{4} (\tau_2 \cos 2\Pi\tau_2 + \tau_3 \cos 2\Pi\tau_3 + \tau_4 \cos 2\Pi\tau_4 + \tau_5 \cos 2\Pi\tau_5) \ldots$$ (vi) (19)

where $\Delta t = (t_3 - t_2) - (t_5 - t_4)$

$$d\tau_{23} = \frac{\tau_2 + \tau_3}{2} - \frac{1}{2}$$

$$d\tau_{34} = \frac{\tau_3 + \tau_4}{2} - \frac{3}{4}$$

$$d\tau_{45} = \frac{\tau_4 + \tau_5}{2} - 1$$

It may also be proved that

$$\Delta y = \frac{ma\Pi\Delta t'}{2T} (1 - \frac{n^2}{a^2})^{\frac{1}{2}}$$ (i)

$$- \frac{ma\Pi\Delta t'}{4T} (1 - \frac{n^2}{a^2})^{-\frac{1}{2}}\{\frac{n\alpha}{a}(\tau_4 \cos 2\Pi\tau_4 - \tau_3 \cos 2\Pi\tau_3) + \frac{4n^2\Pi^2 d\tau_{34}^2}{a^2} \ldots\}$$ (ii)

$$+ \frac{ma\Pi^2\Delta t'}{2T} (d\tau_{14} - d\tau_{36})\{\frac{n}{a} + \frac{\alpha}{2} (\tau_4 \cos 2\Pi\tau_4 - \tau_3 \cos 2\Pi\tau_3) + \frac{2n\Pi^2 d\tau_{34}^2}{a} \ldots\}$$ (iii)

$$- \frac{ma\Pi^3\Delta t'}{4T} (d\tau_{14} - d\tau_{36})^2 (1 - \frac{n^2}{a^2})^{\frac{1}{2}}$$ (iv)

$$+ ma\Pi^2\{d\tau_{14}^2 \cos 2\Pi \frac{(\tau_1 - \tau_4)}{2} - d\tau_{36}^2 \cos 2\Pi \frac{(\tau_3 - \tau_6)}{2}\}$$ (v)

$$- \frac{ma\alpha}{4} (\tau_1 \cos 2\Pi\tau_1 + \tau_3 \cos 2\Pi\tau_3 + \tau_4 \cos 2\Pi\tau_4 + \tau_6 \cos 2\Pi\tau_6) \ldots (vi)$$

(20)

where $\Delta t' = (t_4 - t_1) - (t_6 - t_3)$

$$d\tau_{14} = \frac{\tau_1 + \tau_4}{2} - \tfrac{1}{2}$$

$$d\tau_{36} = \frac{\tau_3 + \tau_6}{2} - 1$$

Example to assess the magnitude of each term in (19) and (20)

$\Delta y = +10'$ $\qquad a = 10$ div. $\qquad T = 390^s$ $\qquad m = 12'/\text{div}$

$\alpha = \frac{1}{150}$ $\qquad w = 2\frac{1}{2}$ div. $\qquad \Delta t = 24.92^s$ $\qquad \Delta t' = 24.37^s$

$t_1 = 48.59^s$	$\tau_1 = 0.1246$	$\tau_1 \cos 2\Pi\tau_1 = 0.0883$
$t_2 = 115.92$	$\tau_2 = 0.2972$	$\tau_2 \cos 2\Pi\tau_2 = -0.0867$
$t_3 = 256.79$	$\tau_3 = 0.6584$	$\tau_3 \cos 2\Pi\tau_3 = -0.3567$
$t_4 = 322.22$	$\tau_4 = 0.8262$	$\tau_4 \cos 2\Pi\tau_4 = 0.3786$
$t_5 = 438.17$	$\tau_5 = 1.1235$	$\tau_5 \cos 2\Pi\tau_5 = 0.7083$
$t_6 = 506.05$	$\tau_6 = 1.2976$	$\tau_6 \cos 2\Pi\tau_6 = -0.3785$

$d\tau_{23} = -0.0222$ $\qquad \cos 2\Pi \frac{(\tau_2 - \tau_3)}{2} = 0.4223$

$d\tau_{34} = -0.0077$ $\qquad \cos 2\Pi \frac{(\tau_4 - \tau_5)}{2} = 0.5946$

$d\tau_{45} = -0.0251$ $\qquad \cos 2\Pi \frac{(\tau_1 - \tau_4)}{2} = -0.5918$

$d\tau_{14} = -0.0246$ $\qquad \cos 2\Pi \frac{(\tau_3 - \tau_6)}{2} = -0.4237$

$d\tau_{36} = -0.0220$

Equation (19)

Term (i)	+10.43'
(ii)	- 0.02
(iii)	- 0.06
(iv)	0.00
(v)	- 0.20
(vi)	- 0.15
	$\sum$ +10.00

Equation (20)

Term	(i)	+10.19'
	(ii)	- 0.01
	(iii)	- 0.05
	(iv)	0.00
	(v)	- 0.18
	(vi)	+ 0.05
	$\sum$	+10.00

5.7.1. SAMPLE CALCULATIONS

The following calculations are based on the formula

$$y = m\,a\,e^{\frac{-\alpha t}{T}} \cos \frac{2\Pi t}{T} \pm \frac{mw}{2}$$

with $m = 12'/\text{div.}$ $\quad T = 6^m\, 30^s$

$\alpha = \frac{1}{150}$ $\quad w = 2\frac{1}{2}$ div.

and $\Delta y = K \Delta t$

where $K = \frac{ma\Pi}{2T} \left(1 - \frac{n^2}{a^2}\right)^{\frac{1}{2}} = c\,(a^2 - n^2)^{\frac{1}{2}}$

Two methods of carrying out these sample calculations have been made in order to assess the effect of having a gyro-mark of smaller width than that of the GAKI, which is 2½ divisions wide. In method A the observations are recorded against the observed scale graduation number. In method B the observations are recorded as though they were observed against the previous graduation line. The effect of this latter variation is to simulate observations with a gyro-mark reduced in width by two scale divisions.

Method A

a = 15 div   Δy = + 10'

Scale	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time
5	66.ˢ 93	244.ˢ 96	311.ˢ 89	144.ˢ 83	456.ˢ 72	245.ˢ 30	702.ˢ 02	144.ˢ 49	846.ˢ 51	245.ˢ 64	1092.ˢ 15
4	71. 55	236. 00	307. 55	153. 82	461. 37	236. 28	697. 65	153. 54	851. 19	236. 56	1087. 75
3	76. 00	227. 31	303. 31	162. 54	465. 85	227. 53	693. 38	162. 32	855. 70	227. 75	1083. 45
2	80. 36	218. 75	299. 11	171. 13	470. 24	218. 91	689. 15	170. 97	860. 12	219. 07	1079. 19
1	84. 62	210. 31	294. 93	179. 60	474. 53	210. 42	684. 95	179. 49	864. 44	210. 53	1074. 97
0	88. 84	201. 92	290. 76	188. 02	478. 78	201. 97	680. 75	187. 97	868. 72	202. 02	1070. 74
-1	93. 02	193. 58	286. 60	196. 39	482. 99	193. 57	676. 56	196. 40	872. 96	193. 56	1066. 52
-2	97. 15	185. 26	282. 41	204. 74	487. 15	185. 19	672. 34	204. 81	877. 15	185. 12	1062. 27
-3	101. 30	176. 86	278. 16	213. 17	491. 33	176. 73	668. 06	213. 30	881. 36	176. 60	1057. 96
-4	105. 47	168. 39	273. 86	221. 66	495. 52	168. 21	663. 73	221. 84	885. 57	168. 03	1053. 60
-5	109. 67	159. 79	269. 46	230. 29	499. 75	159. 55	659. 30	230. 53	889. 83	159. 31	1049. 14
	K		Δt	Δy	Δt	Δy	Δt	Δy	Δt	Δy	
5	0.6835		+14.ˢ 67	+10'.03	+14.ˢ 72	+10'.06	+14.ˢ 77	+10'.10	+14.ˢ 82	+10'.13	
4	0.6987		14. 34	10 .02	14. 39	10 .05	14. 44	10 .09	14. 49	10 .12	
3	0.7103		14. 14	10 .04	14. 19	10 .08	14. 23	10 .11	14. 28	10 .14	
2	0.7185		14. 01	10 .07	14. 06	10 .10	14. 10	10 .13	14. 15	10 .17	
1	0.7233		13. 92	10 .07	13. 97	10 .10	14. 02	10 .14	14. 07	10 .18	
0	0.7250		13. 90	10 .08	13. 95	10 .11	14. 00	10 .15	14. 05	10 .19	
-1	0.7233		13. 98	10 .11	14. 03	10 .15	14. 08	10 .18	14. 13	10 .22	
-2	0.7185		14. 13	10 .15	14. 17	10 .18	14. 22	10 .22	14. 26	10 .25	
-3	0.7103		14. 32	10 .17	14. 36	10 .20	14. 41	10 .24	14. 45	10 .26	
-4	0.6987		14. 57	10 .18	14. 62	10 .21	14. 67	10 .25	14. 72	10 .28	
-5	0.6835		14. 96	10 .23	15. 02	10 .26	15. 06	10 .29	15. 11	10 .33	
		Mean		+10 .10		+10 .14		+10 .17		+10 .21	
		Error		0 .10		0 .14		0 .17		0 .21	

Method B

a = 15 div Δy = +10'

Scale	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time
6	66^s.93				456^s.72				846^s.51		
5	71.55				461.37				851.19		
4	76.00	235^s.89	311^s.89	153^s.96	465.85	236^s.17	702^s.02	153^s.68	855.70	236^s.45	1092^s.15
3	80.36	227.19	307.55	162.69	470.24	227.41	697.65	162.47	860.12	227.63	1087.75
2	84.62	218.69	303.31	171.22	474.53	218.85	693.38	171.06	864.44	219.01	1083.45
1	88.84	210.27	299.11	179.67	478.78	210.37	689.15	179.57	868.72	210.47	1079.19
0	93.02	201.91	294.93	188.06	482.99	201.96	684.95	188.01	872.96	202.01	1074.97
-1	97.15	193.61	290.76	196.39	487.15	193.60	680.75	196.40	877.15	193.59	1070.74
-2	101.30	185.30	286.60	204.73	491.33	185.23	676.56	204.80	881.36	185.16	1066.52
-3	105.47	176.94	282.41	213.11	495.52	176.82	672.34	213.23	885.57	176.70	1062.27
-4	109.67	168.49	278.16	221.59	499.75	168.31	668.06	221.77	889.83	168.13	1057.96
-5			273.86				663.73				1053.60
-6			269.46				659.30				1049.14

	K		Δt	Δy	Δt	Δy	Δt	Δy	Δt	Δy
4	0.6987		+14^s.30	+ 9'.99	+14^s.35	+10'.03	+14^s.40	+10'.06	+14^s.45	+10'.10
3	0.7103		14.08	10.00	14.13	10.04	14.18	10.07	14.23	10.11
2	0.7185		13.96	10.03	14.01	10.07	14.05	10.09	14.10	10.13
1	0.7233		13.88	10.04	13.93	10.08	13.97	10.10	14.02	10.14
0	0.7250		13.85	10.04	13.90	10.08	13.95	10.11	14.00	10.15
-1	0.7233		13.94	10.08	13.98	10.11	14.03	10.15	14.07	10.18
-2	0.7185		14.08	10.12	14.12	10.15	14.17	10.18	14.21	10.21
-3	0.7103		14.25	10.12	14.30	10.16	14.35	10.19	14.40	10.23
-4	0.6987		14.53	10.15	14.58	10.19	14.63	10.22	14.68	10.23
		Mean		+10.06		+10.10		+10.13		+10.17
		Error		0.06		0.10		0.13		0.17

Method A

a = 10 div

Δy = +10'

Scale	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time
5	48^{s}59	273^{s}63	322^{s}22	115^{s}95	438^{s}17	274^{s}26	712^{s}43	115^{s}32	827^{s}75	274^{s}89	1102^{s}64
4	56.86	258.52	315.38	131.16	446.54	259.00	705.54	130.68	836.22	259.48	1095.70
3	64.36	244.45	308.81	145.31	454.12	244.80	698.92	144.96	843.88	245.15	1089.03
2	71.36	231.05	302.41	158.77	461.18	231.30	692.48	158.52	851.00	231.55	1082.55
1	78.02	218.13	296.15	171.74	467.89	218.28	686.17	171.59	857.76	218.43	1076.19
0	84.45	205.46	289.91	184.45	474.36	205.53	679.89	184.38	864.27	205.60	1069.87
-1	90.75	192.88	283.63	197.07	480.70	192.87	673.57	197.08	870.65	192.86	1063.51
-2	96.98	180.30	277.28	209.70	486.98	180.20	667.18	209.80	876.98	180.10	1057.08
-3	103.22	167.53	270.75	222.51	493.26	167.34	660.60	222.70	883.30	167.15	1050.45
-4	109.49	154.46	263.95	235.62	499.57	154.18	653.75	235.90	889.65	153.90	1043.55
-5	115.92	140.87	256.79	249.26	506.05	140.47	646.52	249.66	896.18	140.07	1036.25
	K		Δt	Δy	Δt	Δy	Δt	Δy	Δt	Δy	
5	0.4185		+24^{s}37	+10$'$19	+24^{s}52	+10$'$26	+24^{s}60	+10$'$30	+24^{s}75	+10$'$36	
4	0.4430		22.90	10.14	23.02	10.20	23.10	10.23	23.22	10.29	
3	0.4610		21.94	10.11	22.03	10.16	22.10	10.19	22.19	10.23	
2	0.4735		21.35	10.11	21.43	10.15	21.50	10.18	21.58	10.22	
1	0.4809		21.06	10.13	21.13	10.16	21.20	10.20	21.27	10.23	
0	0.4833		21.01	10.15	21.08	10.19	21.15	10.22	21.22	10.26	
-1	0.4809		21.14	10.17	21.21	10.20	21.28	10.23	21.35	10.27	
-2	0.4735		21.53	10.19	21.60	10.23	21.68	10.27	21.75	10.30	
-3	0.4610		22.22	10.24	22.29	10.28	22.38	10.32	22.45	10.35	
-4	0.4430		23.30	10.32	23.38	10.36	23.50	10.41	23.58	10.45	
-5	0.4185		24.92	10.43	25.00	10.46	25.15	10.53	25.23	10.56	
		Mean		+10.20		+10.24		+10.28		+10.32	
		Error		0.20		0.24		0.28		0.32	

Method B

a = 10 div Δy = 10'

Scale	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time
6	48^{s}59				438^{s}17				827^{s}75		
5	56.86				446.54				836.22		
4	64.36	257^{s}86	322^{s}22	131^{s}90	454.12	258^{s}31	712^{s}43	131^{s}45	843.88	258^{s}76	1102^{s}64
3	71.36	244.02	315 38	145.80	461.18	244.36	705.54	145.46	851.00	244.70	1095.70
2	78.02	230.79	308.81	159.08	467.89	231.03	698.92	158.84	857.76	231.27	1089.03
1	84.45	217.96	302.41	171.95	474.36	218.12	692.48	171.79	864.27	218.28	1082.55
0	90.75	205.40	296.15	184.55	480.70	205.47	686.17	184.48	870.65	205.54	1076.19
-1	96.98	192.93	289.91	197.07	486.98	192.91	679.89	197.09	876.98	192.89	1069.87
-2	103.22	180.41	283.63	209.63	493.26	180.31	673.57	209.73	883.30	180.21	1063.51
-3	109.49	167.79	277.28	222.29	499.57	167.61	667.18	222.47	889.65	167.43	1057.08
-4	115.92	154.83	270.75	235.30	506.05	154.55	660.60	235.58	896.18	154.27	1050.45
-5			263.95				653.75				1043.55
-6			256.79				646.52				1036.25

	K		Δt	Δy	Δt	Δy	Δt	Δy	Δt	Δy
4	0.4430		+22^{s}56	+ 9'.99	+22^{s}65	+10'.03	+22^{s}73	+10'.07	+22^{s}82	+10'.11
3	0.4610		21.73	10.02	21.81	10.05	21.89	10.09	21.97	10.13
2	0.4735		21.16	10.02	21.23	10.05	21.30	10.09	21.37	10.12
1	0.4809		20.89	10.05	20.96	10.08	21.03	10.11	21.10	10.15
0	0.4833		20.85	10.08	20.92	10.11	20.99	10.14	21.06	10.18
-1	0.4809		20.98	10.09	21.05	10.12	21.12	10.16	21.19	10.19
-2	0.4735		21.33	10.10	21.45	10.16	21.47	10.17	21.54	10.20
-3	0.4610		21.99	10.14	22.07	10.17	22.15	10.21	22.23	10.25
-4	0.4430		22.93	10.16	23.01	10.19	23.10	10.23	23.18	10.27
		Mean		+10.08		+10.11		+10.14		+10.18
		Error		0.08		0.11		0.14		0.18

Method A

a = 10 div

Δy = 0'

Scale	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time
5	55^{s}.55	260^{s}.94	316^{s}.49	128^{s}.73	445^{s}.22	261^{s}.44	706^{s}.66	128^{s}.23	834^{s}.89	261^{s}.94	1096^{s}.83
4	63.15	246.74	309.89	143.00	452.89	247.12	700.01	142.62	842.63	247.50	1090.13
3	70.21	233.27	303.48	156.49	459.97	233.58	693.55	156.18	849.73	233.89	1083.62
2	76.92	220.26	297.18	169.60	466.78	220.43	687.21	169.43	856.64	220.60	1077.24
1	84.40	207.54	290.94	182.36	473.30	207.63	680.93	182.27	863.20	207.72	1070.92
0	89.71	194.97	284.68	194.98	479.66	194.97	674.63	194.98	869.61	194.97	1064.58
-1	95.95	182.40	278.35	207.59	485.94	182.31	668.25	207.68	875.93	182.22	1058.15
-2	102.17	169.68	271.85	220.35	492.20	169.51	661.71	220.52	882.23	169.34	1051.57
-3	108.44	156.65	265.09	233.42	498.51	156.34	654.85	233.73	888.58	156.03	1044.61
-4	114.83	143.19	258.02	246.93	504.95	142.81	647.76	247.31	895.07	142.43	1037.50
-5	121.40	128.99	250.39	261.18	511.57	128.49	640.06	261.68	901.74	127.99	1029.73

	K		Δt	Δy	Δt	Δy	Δt	Δy	Δt	Δy
5	0.4185		-0^{s}.24	-0'.10	-0^{s}.24	-0'.10	-0^{s}.24	-0'.10	-0^{s}.24	-0'.10
4	0.4430		-0.19	-0.08	-0.19	-0.08	-0.19	-0.08	-0.19	-0.08
3	0.4610		-0.15	-0.07	-0.15	-0.07	-0.15	-0.07	-0.15	-0.07
2	0.4735		-0.09	-0.04	-0.09	-0.04	-0.09	-0.04	-0.09	-0.04
1	0.4809		-0.05	-0.02	-0.05	-0.02	-0.05	-0.02	-0.05	-0.02
0	0.4833		-0.01	0	-0.01	0	-0.01	0	-0.01	0
-1	0.4809		+0.04	+0.02	+0.04	+0.02	+0.04	+0.02	+0.04	+0.02
-2	0.4735		+0.08	+0.04	+0.08	+0.04	+0.08	+0.04	+0.08	+0.04
-3	0.4610		+0.16	+0.07	+0.16	+0.07	+0.16	+0.07	+0.16	+0.07
-4	0.4430		+0.19	+0.08	+0.19	+0.08	+0.19	+0.08	+0.19	+0.08
-5	0.4185		+0.26	+0.11	+0.26	+0.11	+0.26	+0.11	+0.26	+0.11
		Mean		0.00		0.00		0.00		0.00
		Error		0		0		0		0

Method B

a = 10 div   Δy = 0'

Scale	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time
6	55^{s}55				445^{s}22				834^{s}89		
5	63.15				452.89				842.63		
4	70.21	246^{s}28	316^{s}49	143^{s}48	459.97	246^{s}69	706^{s}66	143^{s}07	849.73	247^{s}10	1096^{s}83
3	76.92	232.97	309.89	156.89	466.78	233.23	700.01	156.63	856.64	233.49	1090.13
2	83.40	220.08	303.48	169.82	473.30	220.25	693.55	169.65	863.20	220.42	1083.62
1	89.71	207.47	297.18	182.48	479.66	207.55	687.21	182.40	869.61	207.63	1077.24
0	95.95	194.99	290.94	195.00	485.94	194.99	680.93	195.00	875.93	194.99	1070.92
-1	102.17	182.51	284.68	207.52	492.20	182.43	674.63	207.60	882.23	182.35	1064.58
-2	108.44	169.91	278.35	220.16	498.51	169.74	668.25	220.33	888.58	169.57	1058.15
-3	114.83	157.02	271.85	233.10	504.95	156.76	661.71	233.36	895.07	156.50	1051.57
-4	121.40	143.69	265.09	246.48	511.57	143.28	654.85	246.89	901.74	142.87	1044.61
-5			258.02				647.76				1037.50
-6			250.39				640.06				1029.73

	K		Δt	Δy	Δt	Δy	Δt	Δy	Δt	Δy
4	0.4430		-0^{s}20	-0'09	-0^{s}20	-0'09	-0^{s}20	-0'09	-0^{s}20	-0'09
3	0.4610		-0.13	-0.06	-0.13	-0.06	-0.13	-0.06	-0.13	-0.06
2	0.4735		-0.08	-0.04	-0.08	-0.04	-0.08	-0.04	-0.08	-0.04
1	0.4809		-0.05	-0.02	-0.05	-0.02	-0.05	-0.02	-0.05	-0.02
0	0.4833		-0.01	0	-0.01	0	-0.01	0	-0.01	0
-1	0.4809		+0.03	+0.01	+0.03	+0.01	+0.03	+0.01	+0.03	+0.01
-2	0.4735		+0.09	+0.04	+0.09	+0.04	+0.09	+0.04	+0.09	+0.04
-3	0.4610		+0.13	+0.06	+0.13	+0.06	+0.13	+0.06	+0.13	+0.06
-4	0.4430		+0.21	+0.09	+0.21	+0.09	+0.21	+0.09	+0.21	+0.09
		Mean		0.00		0.00		0.00		0.00
		Error		0		0		0		0

Method A

a = 15 div $\Delta y = 0'$

Scale	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time
5	$70^{s}79$	$237^{s}48$	$308^{s}27$	$152^{s}33$	$460^{s}60$	$237^{s}78$	$698^{s}38$	$152^{s}03$	$850^{s}41$	$238^{s}08$	$1088^{s}49$
4	75.28	228.73	304.01	161.12	465.13	228.96	694.09	160.89	854.98	229.19	1084.17
3	79.63	220.17	299.80	169.71	469.51	220.34	689.85	169.54	859.39	220.51	1079.90
2	83.92	211.71	295.63	178.20	473.83	211.82	685.65	178.09	863.74	211.93	1075.67
1	88.14	203.32	291.46	186.62	478.08	203.37	681.45	186.57	868.02	203.42	1071.44
0	92.31	194.98	287.29	194.99	482.28	194.98	677.26	194.99	872.25	194.98	1067.23
-1	96.47	186.63	283.10	203.36	486.46	186.58	673.04	203.41	876.45	186.53	1062.98
-2	100.62	178.25	278.87	211.77	490.64	178.14	668.78	211.88	880.66	178.03	1058.69
-3	104.77	169.81	274.58	220.24	494.82	169.64	664.46	220.41	884.87	169.47	1054.34
-4	108.97	161.23	270.20	228.85	499.05	161.00	660.05	229.08	889.13	160.77	1049.90
-5	113.22	152.47	265.69	237.64	503.33	152.17	655.50	237.94	893.44	151.87	1045.31
	K		Δt	Δy	Δt	Δy	Δt	Δy	Δt	Δy	
5	0.6835		$-0^{s}16$	$-0'.11$	$-0^{s}16$	$-0'.11$	$-0^{s}16$	$-0'.11$	$-0^{s}16$	$-0'.11$	
4	0.6987		-0.12	-0.08	-0.12	-0.08	-0.12	-0.08	-0.12	-0.08	
3	0.7103		-0.07	-0.05	-0.07	-0.05	-0.07	-0.05	-0.07	-0.05	
2	0.7185		-0.06	-0.04	-0.06	-0.04	-0.06	-0.04	-0.06	-0.04	
1	0.7233		-0.04	-0.03	-0.04	-0.03	-0.04	-0.03	-0.04	-0.03	
0	0.7250		-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	
-1	0.7233		+0.01	+0.01	+0.01	+0.01	+0.01	+0.01	+0.01	+0.01	
-2	0.7185		+0.05	+0.04	+0.05	+0.04	+0.05	+0.04	+0.05	+0.04	
-3	0.7103		+0.10	+0.07	+0.10	+0.07	+0.10	+0.07	+0.10	+0.07	
-4	0.6987		+0.11	+0.08	+0.11	+0.08	+0.11	+0.08	+0.11	+0.08	
-5	0.6835		+0.14	+0.10	+0.14	+0.10	+0.14	+0.10	+0.14	+0.10	
		Mean		0.00		0.00		0.00		0.00	
		Error		0		0		0		0	

Method B

a = 15 div

$\Delta y = 0'$

Scale	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time	Δ	Time
6	70^{s}.79				460^{s}.60				850^{s}.41		
5	75.28				465.13				854.98		
4	79.63	228^{s}.64	308^{s}.27	161^{s}.24	469.51	228^{s}.87	698^{s}.38	161^{s}.01	859.39	229.10	1088^{s}.49
3	83.92	220.09	304.01	169.82	473.83	220.26	694.09	169.65	863.74	220.43	1084.17
2	88.14	211.66	299.80	178.28	478.08	211.77	689.85	178.17	868.02	211.88	1079.90
1	92.31	203.32	295.63	186.65	482.28	203.37	685.65	186.60	872.25	203.42	1075.67
0	96.47	194.99	291.46	195.00	486.46	194.99	681.45	195.00	876.45	194.99	1071.44
-1	100.62	186.67	287.29	203.35	490.64	186.62	677.26	203.40	880.66	186.57	1067.23
-2	104.77	178.33	283.10	211.72	494.82	178.22	673.04	211.83	884.87	178.11	1062.98
-3	108.97	169.90	278.87	220.18	499.05	169.73	668.78	220.35	889.13	169.56	1058.69
-4	113.22	161.36	274.58	228.75	503.33	161.13	664.46	228.98	893.44	160.90	1054.34
-5			270.20				660.05				1049.90
-6			265.69				655.50				1045.31

	K		Δt	Δy	Δt	Δy	Δt	Δy	Δt	Δy
4	0.6987		-0^{s}.11	-0$'$.08	-0^{s}.11	-0$'$.08	-0^{s}.11	-0$'$.08	-0^{s}.11	-0$'$.08
3	0.7103		-0.09	-0.06	-0.09	-0.06	-0.09	-0.06	-0.09	-0.06
2	0.7185		-0.06	-0.04	-0.06	-0.04	-0.06	-0.04	-0.06	-0.04
1	0.7233		-0.03	-0.02	-0.03	-0.02	-0.03	-0.02	-0.03	-0.02
0	0.7250		-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01	-0.01
-1	0.7233		+0.02	+0.01	+0.02	+0.01	+0.02	+0.01	+0.02	+0.01
-2	0.7185		+0.05	+0.04	+0.05	+0.04	+0.05	+0.04	+0.05	+0.04
-3	0.7103		+0.08	+0.06	+0.08	+0.06	+0.08	+0.06	+0.08	+0.06
-4	0.6987		+0.12	+0.08	+0.12	+0.08	+0.12	+0.08	+0.12	+0.08
		Mean		0.00		0.00		0.00		0.00
		Error		0		0		0		0

The error in the means derived from each of the previous calculations are shown below in the form of a summary in which the columns headed 1,2,3,4 refer to successive values of the errors as the observations progress.

Summary of errors (unit 0.01')

	Method A								Method B							
	a=10				a=15				a=10				a=15			
	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
Δy=0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
Δy=10'	20	24	28	32	10	14	17	21	8	11	14	18	6	10	13	17

It should be noted that the mathematical model assumes "a" to be the amplitude at t=0. In practice amplitude readings are made on the auxiliary scale at the beginning and end of each determination and the mean value is used in the calculation. The effect of this approximation would be to smooth the errors slightly.

5.7.2 Example of the Modified Transit Method

Broken Hill South Mine, 2,400 ft. level. Line 28 to 19.

Date: 30th May 1967

Scale	t	Δ	t	Δ	t	Δ	t
+5	0^m 00.0^s	$-4^m00.4^s$	$4^m00.4^2$	$+2^m26.8^s$	$6^m27.2^s$	$4^m01.0^s$	$10^m28.2$
+4	0 05.7	-3 49.6	3 55.3	+2 37.5	6 32.8	3 49.9	10 22.7
+3	0 11.0	-3 39.3	3 50.3	+2 47.7	6 38.0	3 39.8	10 17.8
+2	0 15.7	-3 30.2	3 45.9	+2 56.8	6 42.7	3 30.5	10 13.2
+1	0 20.6	-3 19.9	3 40.5	+3 07.2	6 47.7	3 20.4	10 08.1
0	0 25.4	-3 10.7	3 36.1	+3 16.9	6 53.0	3 10.3	10 03.3
-1	0 30.2	-3 01.0	3 31.2	+3 26.1	6 57.3	3 01.0	9 58.3
-2	0 35.4	-2 50.6	3 26.0	+3 36.6	7 02.6	2 50.4	9 53.0
-3	0 40.1	-2 40.7	3 20.8	+3 46.6	7 07.4	2 40.6	9 48.0
-4	0 45.0	-2 30.6	3 15.6	+3 56.8	7 12.4	2 30.3	9 42.7
-5	0 50.3	-2 19.5	3 09.8	+4 07.8	7 17.6	2 19.4	9 37.0

Scale	K*	Δt	ΔN	Δt	ΔN
5	0.590	$+7.4^s$	+4.36'	$+7.4^s$	+4.36'
4	0.609	+7.2	+4.38	+7.2	+4.38
3	0.624	+7.3	+4.55	+7.1	+4.42
2	0.634	+6.4	+4.06	+6.4	+4.06
1	0.641	+6.2	+3.97	+6.2	+3.97
0	0.643	+6.2	+3.98	+6.6	+4.24
1	0.641	+6.2	+3.97	+5.7	+3.65
2	0.634	+6.2	+3.93	+6.1	+3.87
3	0.624	+7.0	+4.37	+6.8	+4.24
4	0.609	+6.9	+4.20	+6.9	+4.20
5	0.590	+7.3	+4.30	+6.8	+4.01

$*K = c(a^2-n^2)^{\frac{1}{2}}$

c = 0.051 amplitude a = 12.6 div.

Mean ΔN		+4.16'
Circle Setting	0^o	16.00
Circle Reading of G.I.N.	0	20.16
Mean R.O. Circle Reading	214	36.93
Gyro Azimuth	214	16.77
E		-13.59
Azimuth of R.O.	214	03.18

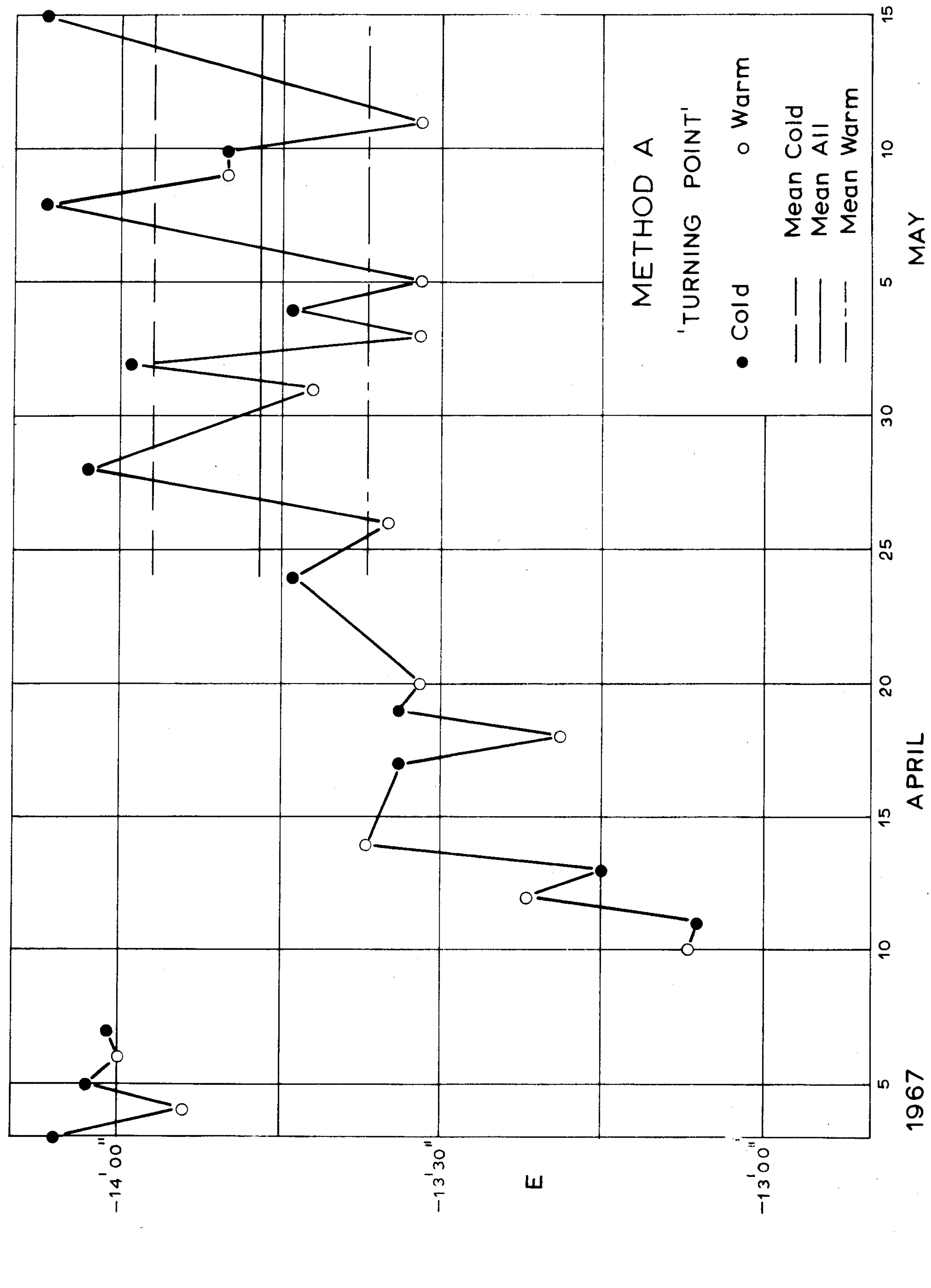
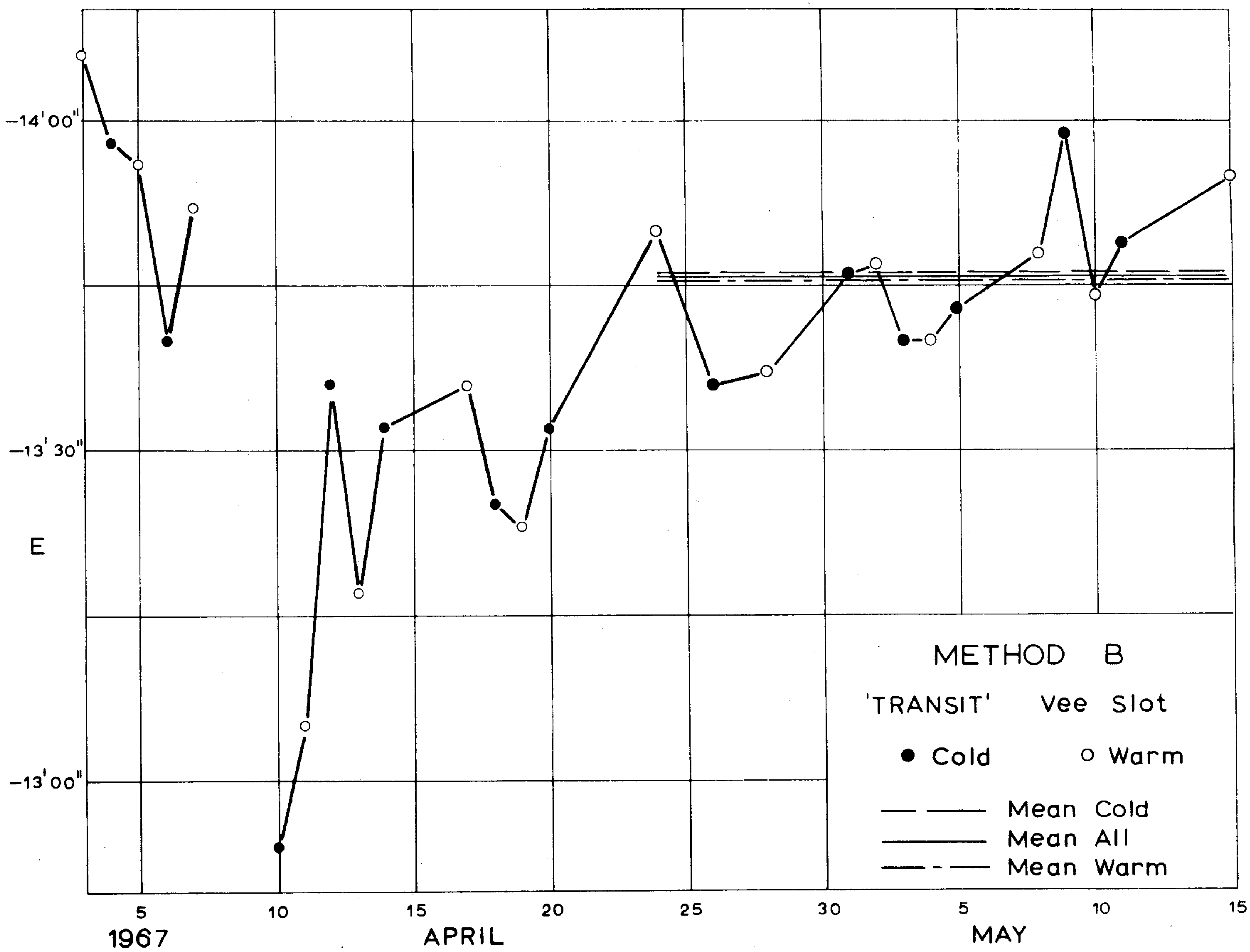

METHOD A
'TURNING POINT'
● Cold
○ Warm
Mean Cold
Mean All
Mean Warm
1967
APRIL
MAY
5
10
15
20
25
30
5
10
15
E
−14'00"
−13'30"
−13'00"

FIG. 5.2



-14'00"
-13'30"
-13'00"
E
METHOD B
'TRANSIT' Vee Slot
Cold
Warm
Mean Cold
Mean All
Mean Warm
5
10
15
20
25
30
5
10
15
1967
APRIL
MAY

FIG. 5.3

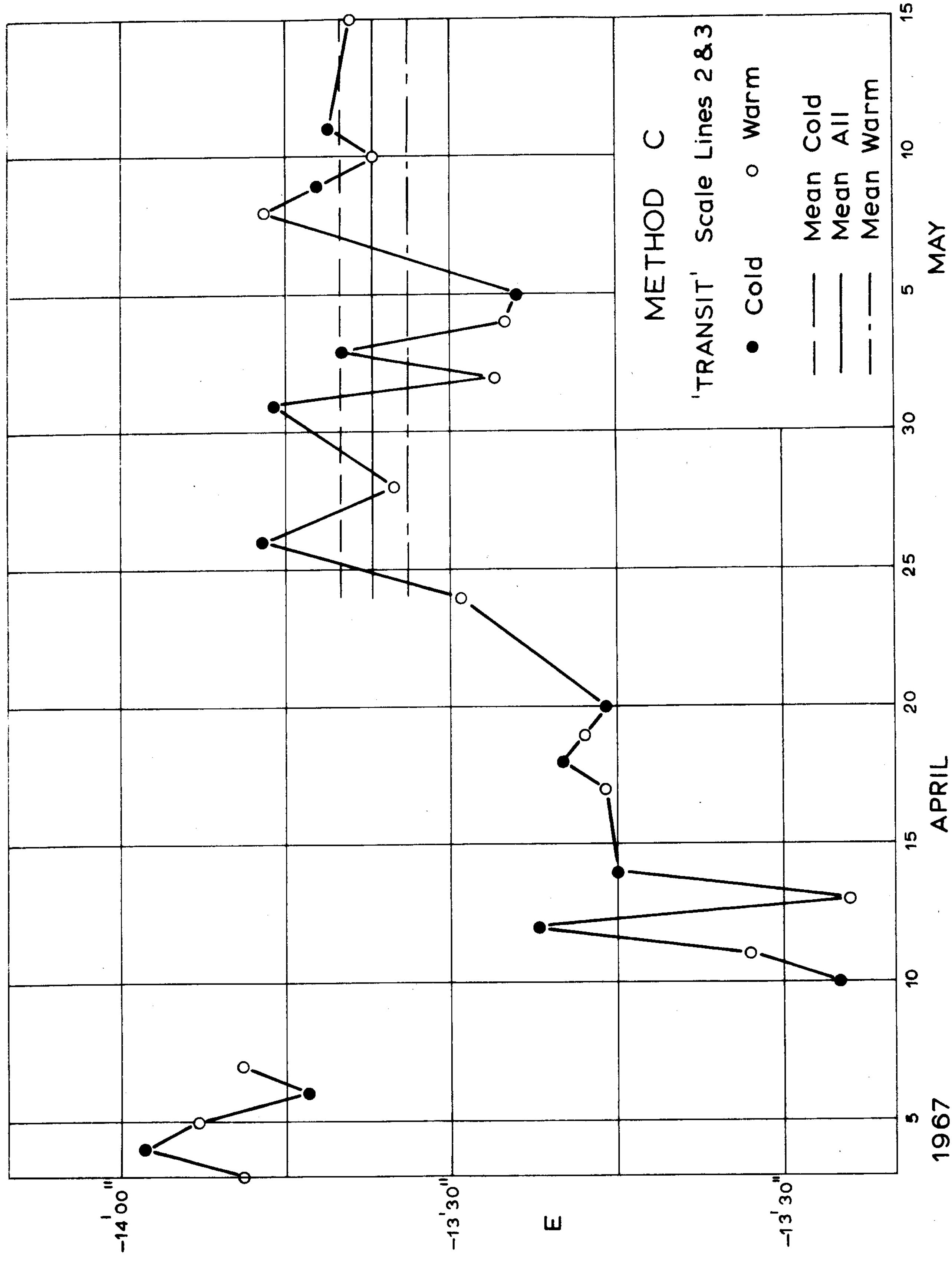

METHOD C
'TRANSIT' Scale Lines 2&3
Cold
Warm
Mean Cold
Mean All
Mean Warm
1967
5
10
15
20
25
30
5
10
15
APRIL
MAY
-14'00"
-13'30"
-13'30"
E

FIG. 5.4

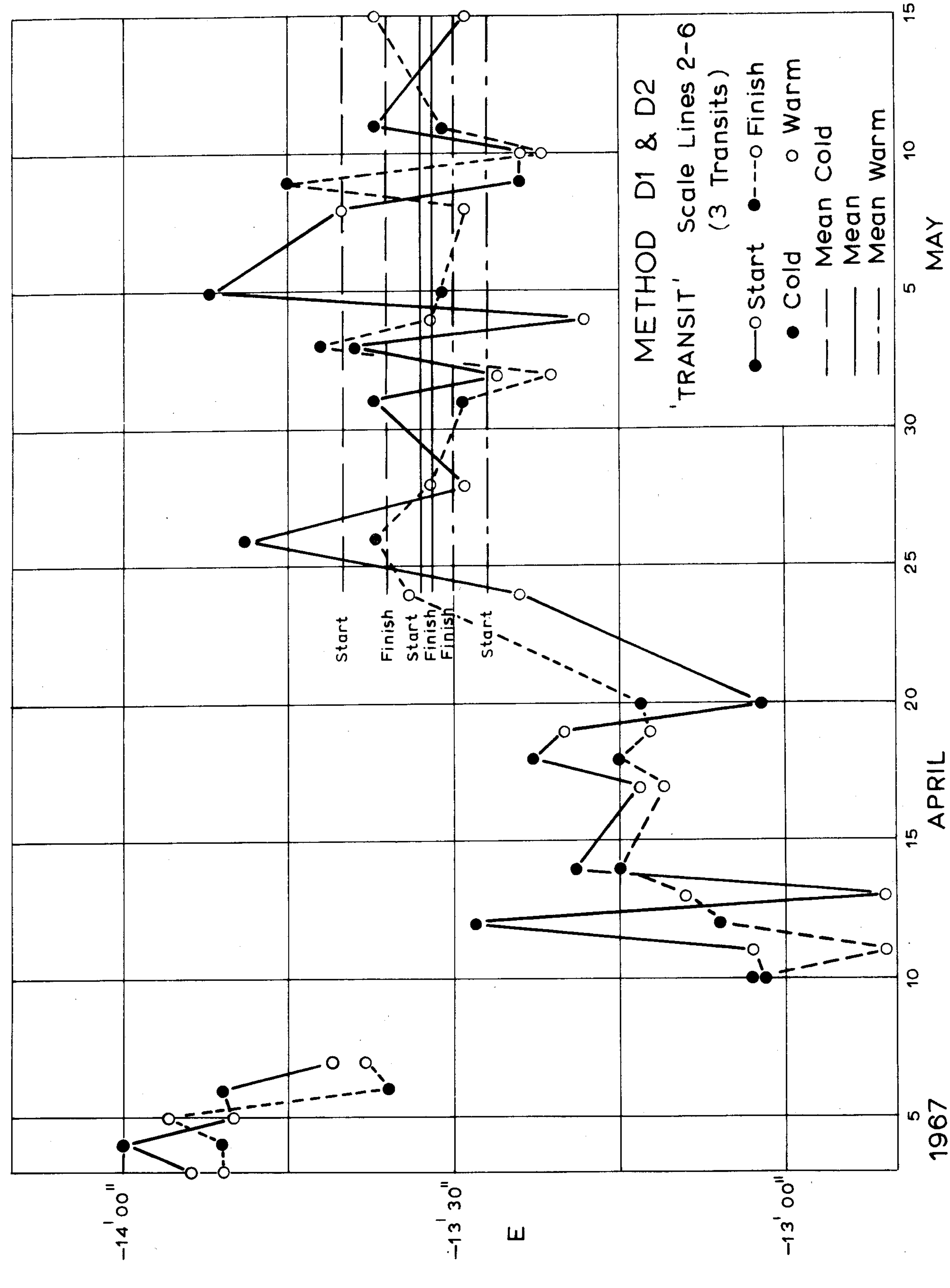

METHOD D1 & D2
'TRANSIT' Scale Lines 2-6
(3 Transits)
Start
Finish
Cold
Warm
Mean Cold
Mean
Mean Warm
Start
Finish
Start
Finish
Finish
Start
−14' 00"
−13' 30"
−13' 00"
E
5
10
15
20
25
30
5
10
15
1967
APRIL
MAY

FIG. 5.5

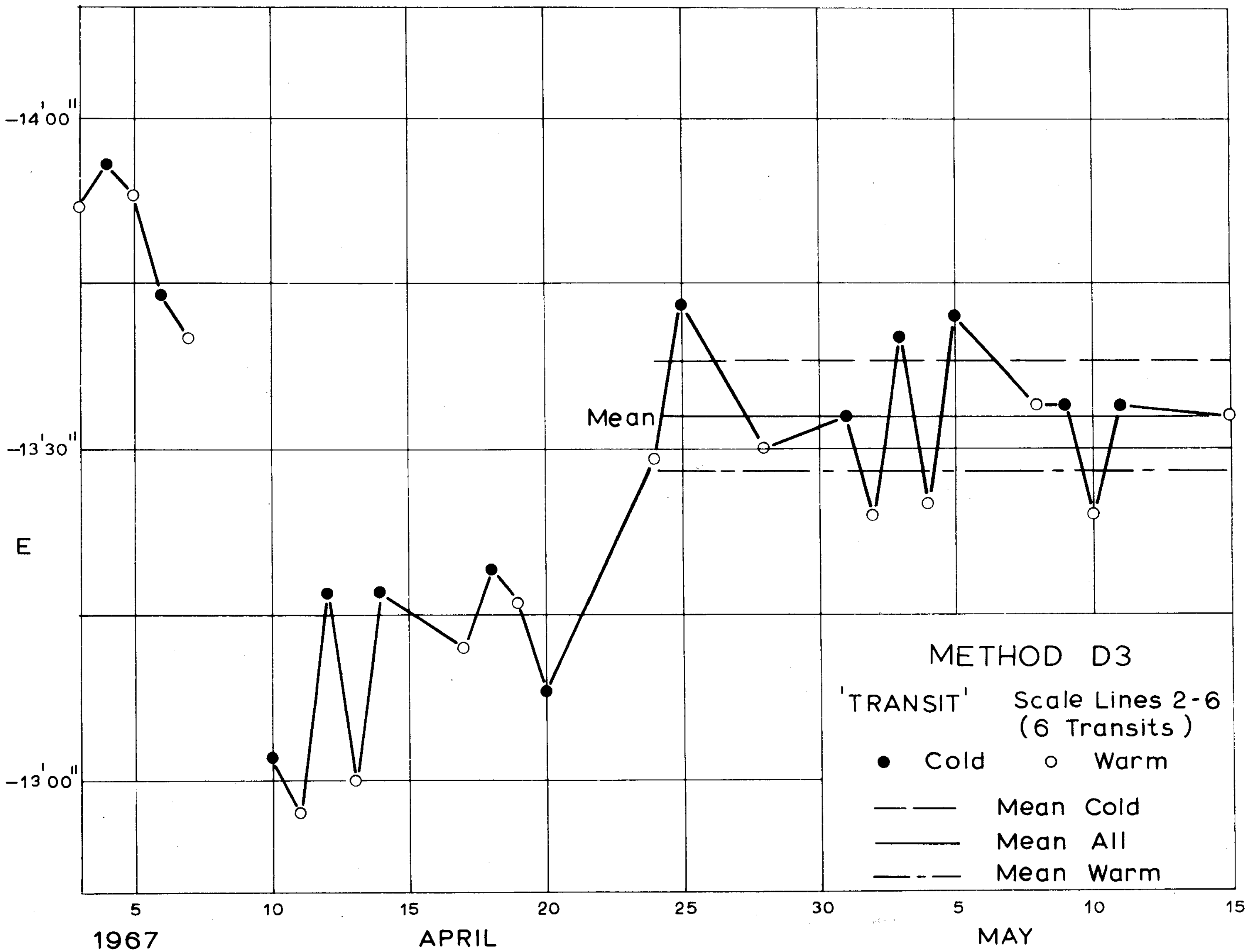

-14'00"
-13'30"
-13'00"
E
Mean
METHOD D3
'TRANSIT' Scale Lines 2-6
(6 Transits)
Cold
Warm
Mean Cold
Mean All
Mean Warm
5
10
15
20
25
30
5
10
15
1967
APRIL
MAY

FIG. 5.6

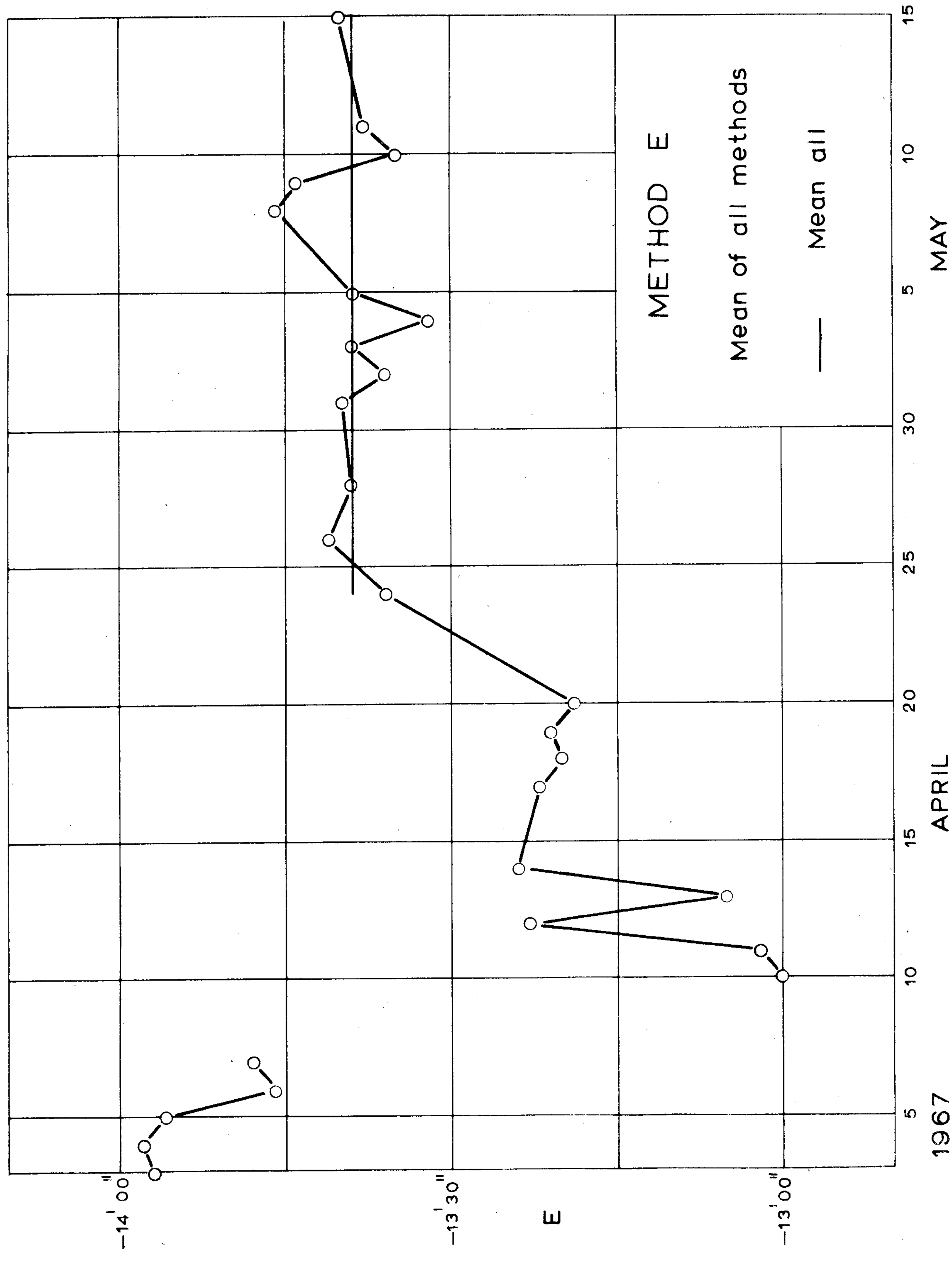

METHOD E
Mean of all methods
— Mean all
−14°00″
−13°30″
−13°00″
E
1967
5
10
15
20
25
30
5
10
15
APRIL
MAY

FIG 5.7

5.8 PROOF OF THE MODIFIED TURNING POINT METHOD

Adopting a similar mathematical model to that used for the proof of the modified transit method, i.e.

$$y = Be^{-\alpha\tau} \cos 2\pi\tau \qquad (1)$$

where B = ma, then "turning points" y_i will occur when $\tau = 0, \frac{1}{2}, 1\ldots$ etc.

At the "turning point"

$$y_i = Be^{-\frac{(i-1)}{2}} (-1)^{i-1} \qquad (2)$$

where i is a positive integer and corresponds to the number of the "turning point".

Let $\tau = \frac{i-1}{2} + \Delta\tau$, and substituting this in (1) gives

$$y = B(-1)^{i-1} e^{-\alpha\frac{(i-1)}{2}} e^{-\alpha\Delta\tau} \cos 2\pi\Delta\tau$$

since $\sin \pi(i-1) = 0$ and $\cos (i-1) = (-1)^{i-1}$.

A series expansion of the exponential and cosine quantities gives

$$y = B(-1)^{i-1} \left\{ 1 - \frac{\alpha(i-1)}{2} + \frac{\alpha^2(i-1)^2}{2\,!\,4} \ldots\ldots \right\} \left\{ 1 - \alpha\Delta\tau + \frac{\alpha^2\Delta\tau^2}{2!} \ldots\ldots \right\}$$
$$\left\{ 1 - \frac{(2\pi)^2\Delta\tau^2}{2\,!} + \frac{(2\pi)^4\Delta\tau^4}{4\,!} \ldots \right\} \qquad (3)$$

Put $\Delta y = y_i - y$, and from (2) and (3),

$$\Delta y = \frac{B(-1)^{i-1}\,(2\pi)^2\Delta\tau^2}{2\,!} - \frac{B(-1)^{i-1}\,(2\pi)^4\Delta\tau^4}{4\,!} \ldots\ldots$$

$$+ B(-1)^{i-1}\,\alpha\Delta\tau \left\{ 1 - \frac{(2\pi)^2\Delta\tau^2}{2\,!} + \frac{(2\pi)^4\Delta\tau^4}{4\,!} \ldots\ldots \right\}$$

$$- \frac{B(-1)^{i-1}\alpha(i-1)}{2} \left[\frac{(2\pi)^2\Delta\tau^2}{2!} - \frac{(2\pi)^4\Delta\tau^4}{4!} \cdots\cdots \right]$$
$$- \frac{B(-1)^{i-1}\alpha^2(i-1)\Delta\tau}{2} \left[1 - \frac{(2\pi)^2\Delta\tau^2}{2!} + \frac{(2\pi)^4\Delta\tau^4}{4!} \cdots \right] \qquad (4)$$

In order to estimate the number of terms required for a practical solution, each term in (4) has been evaluated using conservative values of the parameters B, α and $\Delta\tau$,

i.e. $B = 180'$, $\alpha = \frac{1}{150}$, $\Delta\tau = 0.1$.

Term	i=1	i=2	i=3	i=4
1	+35.53'	-35.53'	+35.53'	-35.53'
2	- 1.17	+ 1.17	- 1.17	+ 1.17
3	+ 0.10	- 0.10	+ 0.10	- 0.10
4	0	+ 0.11	- 0.23	+ 0.34
5	0	0	0	0
$\Sigma = \Delta y$	+34.46	-34.35	+34.23	-34.12

In practice it would be convenient to use term 1 only. The remaining terms could be reduced in magnitude by using a small amplitude and/or a small time difference. The neglect of terms other than the first may be significant, leading to a distortion of the value of the turning point which is derived from the mean of the corrected observations. This distortion will be such as to displace each mean turning point away from the line of mean oscillation by a constant amount if the observations are distributed in time in the same manner about each turning point. However, there will be negligible effect on the resulting Schuler Means. The previous numerical example will serve to illustrate this aspect where each observation is made at a time instant $\Delta\tau = 0.1$ before or after the turning point.

	Observation	Correction (1st term)	Corrected observ. L	Corrected observ. R	Schuler Mean
y_1	+145.54'	+35.53'	+181.07'		
y_2	-145.05	-35.53	(+180.58)	-180.58'	0.00'
y_3	+144.57	+35.53	+180.10	(-180.10)	0.00
y_4	-144.09	-35.53		-179.62	

Even though the effect on the final values of the Schuler Means of neglecting all terms other than the first will be minute, it must be recognised that each observation after reduction will suffer from a systematic effect and thus the error distribution will be distorted.

5.8.1 EXAMPLE OF THE MODIFIED TURNING POINT METHOD

Byrd Station, Antarctica. Line Jamesway to New Byrd Astro (Ecce).

Date: 2nd December, 1968.

Hor. Circle		t		Δt	AΔt^2	(AΔt^2)^2/3B'	T.P.
0°	00'	0^m	00^s	t_o			
356	52.0	3	11.9	39.3^s	-6.98'	+0.04'	45.06'
	48.0	3	24.8	26.4	3.15	0.01	44.86
	45.0	3	41.8	9.4	0.40	0	44.60
	44.8	(3	51.2)				44.80
	47.5	4	15.7	24.5	2.71	0.01	44.80
	53.0	4.	33.5	42.3	8.09	0.06	44.97
356	60.0	4	49.7	58.5	-15.47	+0.21	44.74
0	00	7	42.4	t_1		Mean	(44.83)
3	04.0	10	48.0	43.6	+ 8.59	-0.06	12.53
	10.0	11	07.3	24.3	2.67	0.01	12.66
	13.0	11	26.5	5.1	0.12	0	13.12
	13.3	(11	31.6)				13.30
	09.5	11	59.9	28.3	3.62	0.01	13.11
3	03.0	12	18.3	46.7	9.86	0.08	12.78
2	56.0	12	33.7	62.1	+17.43	-0.26	13.17
0	00	15	20.7	t_2		Mean	(12.95)
356	55.0	18	28.8	42.2	- 8.05	+0.06	47.01
	49.0	18	49.7	21.3	2.05	0	46.95
	47.0	19	07.3	03.7	0.06	0	46.94
	46.8	(19	11.0)				46.80
	50.0	19	37.7	26.7	3.22	0.01	46.79
	54.3	19	51.3	40.3	7.34	0.05	47.01
356	64.0	20	13.5	62.5	-17.66	+0.27	46.61
0	00	23	01.3	t_3		Mean	(46.87)

Period $T = t_2 - t_o$ =	15^m	20.7^s	
$t_3 - t_1$ =	15	18.9	
Mean	15	19.8	(919.8^s)

L		R		
356^o	44.8'			
(356	45.8)	3^o	13.3'	B' = difference = $6^o27.5'$
356	46.8			= 387.5'
				3B' = 1162'

$$A = \frac{B'\pi^2}{T^2} = 0.00452$$

L		R		Schuler Mean	
356^o	44.83'				
(356	45.85)	3^o	12.95'	359^o	59.40'
356	46.87				

Mean R.O. Circle Reading	26^o	50.03'
Circle Reading of G.I.N.	359	59.40
Gyro Azimuth	26	50.63
E		-12.93
Azimuth R.O.	26	37.70

5.9 THE LEAST SQUARES ADJUSTMENT OF THE MODIFIED TRANSIT METHOD

Equation (1) of 5.2 is an approximation and it is of interest to examine the difference between this approximation and a least squares solution. Because the observations are made over a short time interval, namely 2 periods, the damping factor can be ignored without causing serious error and the model modified to give the following correction equation:

$$a \sin \{\frac{2\pi}{T}(t_i+v_i-t_o) + \frac{Ka2\pi\Delta t}{T} - \{(2n+1)P - n - i + w\} (-1)^P = 0 \qquad (2)$$

where t_i is the instant of observation with associated correction v_i.

i varies from	1	to	$2n+1$	then	$P = 1$
	$2n+2$	to	$4n+2$	then	$P = 2$
	$4n+3$	to	$6n+3$	then	$P = 3$
	$6n+4$	to	$8n+4$	then	$P = 4$
		etc.			etc.

n is the number of scale lines observed on either side of the zero scale line.

$K=1$ if the first transit is from right to left.

$K=-1$ if the first transit is from left to right.

t_o is the instant of time corresponding to the transit of the gyro-mark across the zero scale line.

$\Delta t = t_{n+1} - t_o$

w is the half width of the gyro-mark.

Equation (2), however, is non-linear with respect to the unknown parameters T and t_o.

Substituting

$$t_o = t_o' + \Delta t_o \qquad w = w' + \Delta w \quad \text{and} \quad T = T' + \Delta T$$

in equation (2), expanding by Taylor's series and ignoring second and higher order terms gives

$$\frac{a2\pi}{T'^2} (t_i - t_o') \cos \{ \frac{2\pi}{T'} (t_i - t_o')\} \Delta T + \frac{Ka2\pi\Delta t}{T'}$$

$$+ \quad \frac{a2\pi}{T'} \cos \{ \frac{2\pi}{T'} (t_i - t_o)\} \Delta t_o + (-1)^P \Delta w - a \sin \frac{2\pi}{T'} (t_i - t_o')$$

$$+ \quad \{(2n+1)P - n - i + w'\} (-1)^P = \frac{2\pi a}{T'} \cos \{ \frac{2\pi}{T'} (t_i - t_o')\} v_i \qquad (3)$$

If we apply the law of propagation of variances to the right hand side of the correction equation (3) we obtain the variance of this equation, viz.,

$$\left[\frac{2\pi a}{T'} \cos \{ \frac{2\pi}{T'} (t_i - t_o')\} \right]^2 \sigma_{t_i}^2 \qquad (4)$$

where $\sigma_{t_i}^2$ is the variance of the timing of the instant of observation which will depend upon the velocity of the gyro-mark across the auxiliary scale. We may also write that the approximate displacement of the gyro-mark from the centre of the oscillation is given by

$$\Theta = a \sin \{ \frac{2\pi}{T'} (t_i - t_o')\}$$

Applying the law of propagation of variances to this expression gives

$$\sigma_\theta^2 = \left[\frac{2\pi a}{T'} \cos \{ \frac{2\pi}{T'} (t_i - t_o')\}\right]^2 \sigma_{t_i}^2 \qquad (5)$$

According to Roelofs (1950) it has been shown that for astronomical observations the error in pointing is independent of the speed of a star at transit. It would be reasonable to assume that there is a direct analogy between timing a star across fixed wires and timing a gyro-mark across the fixed lines of an auxiliary scale. Therefore it can be seen from equations (4) and (5) that the variance of the correction equation is σ_θ^2 which may be taken as constant and therefore equation (3) can be used directly in a least squares adjustment with unit weight.

In the experimental series previously described, the observations in a single set to 5 scale lines on either side of zero give rise to 44 equations typified by equation (3) in which there are 4 unknown parameters ΔT, Δt, Δt_o and Δw. A least squares solution of the 30 sets of observations in the second experimental series has been made on an IBM 360 computer. The 30 sets of observations were also reduced according to equation (1) using the method which simulates a narrower gyro-mark where the observations are recorded as though they were observed on the previous graduation line. This latter technique gives 36 equations for the least squares solution which was also processed on the computer. The results of these four methods of reduction are shown in the Table 5.3. It will be noticed that there is no significant difference between the results of any of the methods of reduction and therefore the simple technique which uses the original observations in equation (1) (Method A) will be sufficiently accurate in practice. However, this simple technique of reduction does not disclose the corrections to the original observations and a poor observation may pass unnoticed. If one or two poor

TABLE 5.3. GYRO CONSTANT E

No.	A	B	C	D
1	+2'52"	+2'50"	+2'49"	+2'50"
2	3 19	3 21	3 19	3 19
3	3 19	3 20	3 20	3 20
4	3 13	3 12	3 13	3 12
5	3 10	3 10	3 10	3 10
6	3 11	3 10	3 12	3 13
7	3 15	3 14	3 15	3 14
8	3 14	3 17	3 14	3 16
9	3 05	3 07	3 06	3 07
10	3 14	3 16	3 15	3 16
11	3 17	3 17	3 17	3 17
12	3 40	3 41	3 41	3 41
13	3 19	3 18	3 19	3 19
14	3 17	3 14	3 17	3 15
15	3 14	3 14	3 13	3 13
16	3 02	3 06	3 06	3 06
17	3 13	3 15	3 13	3 15
18	3 01	3 02	3 01	3 01
19	3 14	3 14	3 14	3 14
20	3 18	3 18	3 19	3 18
21	3 16	3 17	3 16	3 14
22	3 35	3 35	3 36	3 36
23	3 14	3 14	3 14	3 15
24	3 29	3 29	3 28	3 29
25	3 20	3 18	3 21	3 19
26	3 11	3 11	3 11	3 11
27	3 07	3 08	3 07	3 08
28	3 09	3 09	3 10	3 08
29	3 18	3 18	3 17	3 18
30	3 37	3 42	3 38	3 42

A 5 scale lines, reduction according to equation (1)
B 5 scale lines, reduction according to equation (3)
} gyro-mark 2½ divisions wide

C 4 scale lines, reduction according to equation (1)
D 4 scale lines, reduction according to equation (3)
} gyro-mark ½ division wide

observations are included in a series of 44 observations then little effect will be felt on the final value of ΔN. The computer programme and an example of two of the above solutions are given in Appendix III.

Times were recorded with a split hand timer with a least count of 0.2 sec. estimating to 0.1 sec. From the computer output for the least squares adjustment the estimate of the standard deviation of a single observation varied from ±0.11 to ±0.22 sec., with an average value of about ±0.15 sec.

6. THE NON-SPINNING GYROSCOPE

Of major concern with gyro-theodolites, which have the gyroscope suspended by a thin steel ribbon tape, is the rest or zero position when the gyroscope occupies a stationary position when freely hung in the non-spinning state. Ideally the rest position should coincide with the zero of the auxiliary scale, or vee slot, so that the observations of the spinning gyro, which are also referred to the zero of the scale are unaffected by tape torsion in that position.

The cross section of the tape is chosen to be extremely thin (for the GAK 1 approximately 0.4 x 0.02 mm) so that a minimum twisting moment is produced from a given angle of twist commensurate with the shear strength of the material. For a detailed account of this analysis see Lauf (1963). In addition to the suspension tape which takes the weight of the gyroscope, a number of fine power leads transfer the current between the external housing of the instrument and the gyro motor. The size, cross section and shape of these leads are also chosen so that they will have the least torsional effect on the suspended gyroscope. The rest position is usually determined by reading the position of turning points of the gyro-mark on the auxiliary scale, and combining the readings in the form of Schuler Means. Theoretically the zero position should remain constant provided the instrument is handled carefully, but if changes do occur then a correction (see Appendix IV) can be applied to the direction of gyro indicated North.

Soon after taking delivery of the instrument a series of test observations were made in order to assess the precision of measuring the tape zero position. In the first test the gyro was lowered in the non-spinning state and the extreme positions of swing were read continuously over a period of $1\frac{1}{2}$ hours. The average of each three consecutive Schuler Means was taken as an estimate of the tape zero position. The range of these averages was 0.09 divisions* with an estimated standard deviation of ± 0.023 divisions. A second set of observations was made, also over a protracted period of time, but in this test after every 5 scale readings the gyro was clamped and lowered again. The range of averages was 0.46 divisions with an estimated standard deviation of ± 0.10 divisions. The results of the first test showed that observations of the gyro-mark against the auxiliary scale could be made with high precision. However, the results of the second test showed that the precision of observation was offset by the variability of the tape zero position due to clamping and unclamping. A further set of observations made in the same way as the first test gave a similar result as before. It is of interest to note that once the gyro is lowered in the non-spinning state the tape zero position remains constant without appreciable change, but on clamping and lowering again the tape zero position occupies a slightly different, although constant, position. Halmos (1968) has found that stresses are introduced into the suspension tape caused by clamping. Small stresses will be introduced into the suspension tape because the movement of the case enclosing the gyro when clamped and unclamped will not be entirely vertical.

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* 1 scale division for the GAK 1 corresponds to about ten minutes of arc.

Some further experience with the GAK 1 gyro-attachment has been at variance with the manufacturer's description of the expected performance in the non-spinning state. For example the following statement appears in the handbook for the instrument:-

> "The zero position is determined before measurement as a check on the state of adjustment, because it influences the calibration of the instrument. The zero position, therefore, does not have to lie exactly in the middle of the index. On the other hand its consistency is important. If the value determined before measurement differs from the calibrated value by no more than 0.3 units the actual measurement can begin at once. With bigger deviations, possibly caused by heavy shaking or by dropping, the gyro is left hanging freely for about 20 - 30 minutes. After this the zero position normally returns to its previous value. With careful treatment and increasing experience in the handling of the instrument it is not always necessary to determine the zero position before every measurement. Only occasionally should it be necessary to check it e.g. after long journeys."

In the period 3rd April to 15th May 1967, observations of the non-spinning gyro for the GAK 1 Number 2871 were taken before and after 27 gyro azimuth determinations. Each tape zero position was based on 5 readings of the extreme position of swing, giving rise to 3 Schuler Means. A summary of the ranges of these results is given below.

	Range	Mean
Before	-0.35 to + 0.91 div.	+ 0.26 div.
After	-0.43 to + 1.05 "	+ 0.18 "

The results of the observations appeared to have a random distribution about the mean values. It should be noted that these results include observations taken in the period when the tape was known to have been strained, see Section 5.4. The ranges and means are very little affected if the observations when the strain was present, are excluded. In all 50% of all non-spinning observations fell outside of the limits of ± 0.3 divisions which could not be attributed entirely to poor handling of the instrument.

If corrections to the gyro azimuth determinations were made on a basis of the above results then the range of the values of E would have been increased considerably. Furthermore it will be noticed from the results of the observations given in Section 5.4 that when the tape was strained a consistent value of E was not regained until after 9 further observations were made, each observation lasting nearly an hour. It would appear that if tape strain is present then a longer period than 20 - 30 minutes may be necessary before the tape is restored to its normal position. All non-spinning observations were made with the instrument in the same orientation i.e. approximately North-South, and therefore the results should have been little affected by the local magnetic field. This effect will be discussed later.

In the period between 16th May and 13th July 1967 a further 38 gyro azimuth determinations were made including non-spinning observations at the beginning and end of the observations. Once again the values of the zero position showed the same characteristics of range and scatter. If the gyro azimuths were corrected on a basis of these results then the agreement between forward and back azimuths on each line would have been considerably worse than if the correction were ignored. From the results of this work it was concluded that the tape zero position as deduced from non-spinning gyro readings was uncertain and would only be of value in detecting the presence of a large external magnetic field or if the instrument had been damaged.

In late October 1968 delivery was taken of a new GAK 1 gyro-attachment Number 3243. Testing and calibration of this instrument and also GAK 1 Number 2888 of the University of New South Wales School of Mining Engineering were undertaken immediately because the instruments were to be taken to the Antarctic for high latitude observations. The first test on Number 3243 showed that the tape zero position was approximately + 6 divisions. It was assumed that the instrument had been damaged while being transported to the University because the manufacturer's representative had checked the instrument before delivery. The instrument was found to be undamaged after being returned to the representative but further tests at the University showed that the tape zero position was still far removed from the centre of the scale. It was suspected that the instrument was being influenced by an external magnetic field and so the tape zero position was observed at intervals of 10^o azimuth. The results are shown in graphical form in Figure 6.1 on Page 185. A smoothed curve through the points is sinusoidal with a period of 2π and has a pronounced amplitude. Because the observations were made in an instrument laboratory it was suspected that the cause of the effect was the proximity of

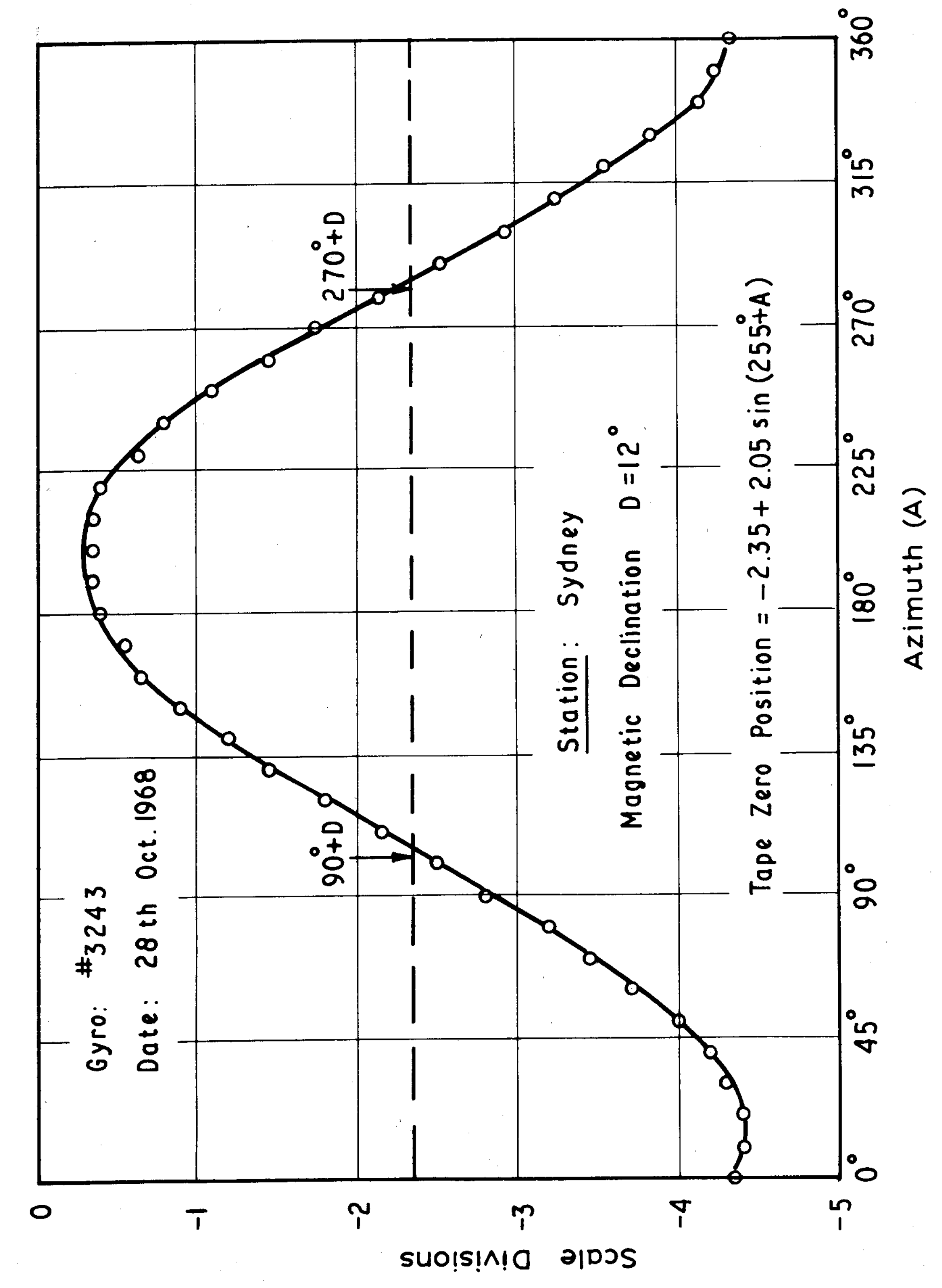


Fig.6.1: OBSERVED TAPE ZERO POSITION

a strong local magnetic field. The experiment was repeated in the open, away from likely sources of local magnetic disturbance. The form of the curve resulting from this latter experiment was the same as with the laboratory experiment, suggesting that the gyro was being influenced by the earth's magnetic field.

The direction of the residual magnetic field within the gyro motor is at right angles to the spin axis and therefore minimum deflection of the tape zero position should occur when the gyro is oriented at right angles to the local magnetic meridian i.e. when the direction of the two fields is parallel. Thus zero deflection should occur at azimuths of (90° + D) and (270° + D) and maximum deflection at azimuths of (D) and (180° + D) where (D) is the magnetic declination. It will be seen from Figure 6.1 that in fact the maximum and minimum points are close to the above azimuths thus confirming that the earth's magnetic field was influencing the tape zero position. Because of the constantly changing polarity of the magnetic field within the gyro motor the gyro in its spinning state is unaffected by external magnetic fields.

Further tests on gyro Number 2888 gave a similar result but with a greatly reduced amplitude. A deflection of the tape zero position would depend upon the following factors:-

(1) Horizontal intensity of the local magnetic field.

(2) Effectiveness of the mu metal shield surrounding the suspended gyro.

(3) Horizontal intensity of the magnetic field within the motor.

(4) Angle between the directions of the local and motor magnetic fields.

(5) Torsion characteristics of the suspension tape and power transfer leads.

On the assumption that the mu metal shield considerably reduces the effect of the earth's magnetic field in the vicinity of the gyro motor then the cause of the pronounced amplitude of the tape zero deflection for gyro Number 3243 would probably be due to a strong horizontal field within the gyro motor. Gregerson (1969a) with the assistance of the Wild Company of Canada has modified the converter by placing a variable potentiometer in the circuit of the power supply and then the amplitude of the current braking the 3 phase motor was gradually decreased to zero as the motor came to a standstill. Gregerson reported that after application of the potentiometer the residual magnetism was a small fraction of that found when the potentiometer was not used.

Earlier in 1968 the Wild Company was approached concerning some irregularities which were observed with gyro Number 2871. Similar irregularities had been observed by Baker (1967) who also reported to the Company that "It has been noticed, however, that tape-zero, when checked immediately after an observation, can vary by 1 or 2 divisions and this is probably due to residual magnetism with the gyro motor". The Company in reply proposed that the sentence be changed to ...."and this is due to a residual magnetism induced only after the gyro motor is braked to a standstill. This magnetism dos not exist in the spinning gyro and therefore does not affect the determination of North. As the induced magnetism disappears again after a time, a tape zero adjustment, which is rarely necessary, should never be made shortly after a gyro operation". After the irregularities experienced with gyro Number 2871 and Number 3243 the Melbourne Metropolitan Board of Works and Mt. Isa Mines were requested to make observations of the tape zero position of their GAK 1 instruments at azimuth settings of 0^{o}, 45^{o}, 90^{o} etc. around the circle in order to see if these instruments were also affected by the earth's magnetic field. In addition, observations for tape zero position were made at Christchurch, Hallett, McMurdo and Byrd Stations with gyro Number 2888 and Number 3243.

A convenient algorithm for curve fitting to these observations was derived as follows:-

Let y be the measured value of the tape zero position at azimuth intervals x, where $x = \frac{21\pi}{n}$ where n is the total number of measurements and 1 = o, 1 .... , n-1. The curve is assumed to be sinusoidal of period 2π which can be represented by

$$y = A + B \sin (C + x) \quad (1)$$

where A, B and C are unknown constants.

Expanding (1) gives

$$y = A + B \sin C \text{ cox } x + B \cos C \sin x \quad (2)$$

Put B sin C = X and B cos C = Y and substitute in (2) giving

$$y = A + X \cos x + Y \sin x$$

Therefore correction equations will be of the form

$$v_i = A + X \cos x_1 + Y \sin x_1 - y_i$$

where i = 1, 2, ... , n.

Forming normal equations

A		X		Y	abs.	= 0
n	+	[Cos x]	+	[sin x]	-[y]	
		[cos^2 x]	+	[sin x cos x]	+[y cos x]	
				[sin^2 x]	-[y sin x]	

It may be proved that $[\cos x] = [\sin x] = [\sin x \cos x] = 0$

and $[\cos^2 x] = [\sin^2 x] = \frac{n}{2}$, according to Whittaker and Robinson (1944).

$$\therefore \quad A = \frac{[y]}{n} \qquad X = \frac{[y \cos x]}{[\cos^2 x]} \qquad Y = \frac{[y \sin x]}{[\sin^2 x]}$$

then
$$\tan C = \frac{X}{Y} = \frac{[y \cos x]}{[y \sin x]}$$

and
$$B = \frac{X}{\sin C} = \frac{Y}{\cos C}$$

E.g. for $n = 8$ (i.e. $x = 0^\circ, 45^\circ \ldots\ldots 315^\circ$)

$$[\cos^2 x] = [\sin^2 x] = 4$$

$$[y \sin x] = y_3 - y_7 + \frac{\sqrt{2}}{2} (y_2 + y_4 - y_6 - y_8)$$

$$[y \cos x] = y_1 - y_5 + \frac{\sqrt{2}}{2} (y_2 - y_4 - y_6 + y_8)$$

Example

Sydney Gyro No. 3243

i	y	x	C+x	sin(C+x)	Bsin(C+x)	A+Bsin(C+x)	v
1	-0.35	0°	283°	-0.97	-1.40	-0.32	+0.03
2	+0.45	45	328	-0.53	-0.76	+0.32	-0.13
3	+1.40	90	13	+0.23	+0.33	+1.14	+0.01
4	+2.20	135	58	+0.85	+1.22	+2.30	+0.10
5	+2.50	180	103	+0.97	+1.40	+2.48	-0.02
6	+1.95	225	148	+0.53	+0.76	+1.84	-0.11
7	+0.75	270	193	-0.23	-0.33	+0.75	0
8	-0.25	315	238	-0.85	-1.22	-0.14	+0.11

$$A = \frac{\Sigma y_i}{n} = \frac{+\ 8.65}{8} = +\ 1.08$$

$$X = \frac{[y \cos x]}{[\cos^2 x]} = \tfrac{1}{4}\{-0.35 - 2.50 + \frac{\sqrt{2}}{2}(0.45 - 2.20 - 1.95 - 0.25)\}$$
$$= -1.41$$

$$Y = \frac{[y \sin x]}{[\sin^2 x]} = \tfrac{1}{4}\{+1.40 - 0.75 + \frac{\sqrt{2}}{2}(0.45 + 2.20 - 1.95 + 0.25)\}$$
$$= +0.33$$

$$\tan C = \frac{X}{Y} = \frac{-1.41}{+0.33} = -4.27, \quad C = 283^{\circ}$$

$$B = \frac{X}{\sin C} = \frac{-1.41}{+0.97} = +1.45$$

$$B = +1.44$$

$$B = \frac{Y}{\cos C} = \frac{+0.33}{+0.23} = +1.43$$

$$\underline{y = +1.08 + 1.44 \sin (283^{\circ} + x)}$$

The results of these curve fittings are shown in Figures 6.2 to 6.6 on Pages 191 to 195. It will be seen that all of the GAK 1 gyro-attachments tested were influenced in varying degrees by the earth's magnetic field which is confirmed by the phase relationship of the curves with respect to the magnetic meridian. It is important to note that gyro Number 2875 was not used for several months before the tests and also gyro Number 2887 was not used for 4 weeks before testing. Thus contrary to the manufacturer's statement it would appear that for some instruments, residual magnetism in the gyro motor remains for a considerable time after operation in the spinning state. An examination of the tape zero results for gyro Number 2888 and Number 3243 show

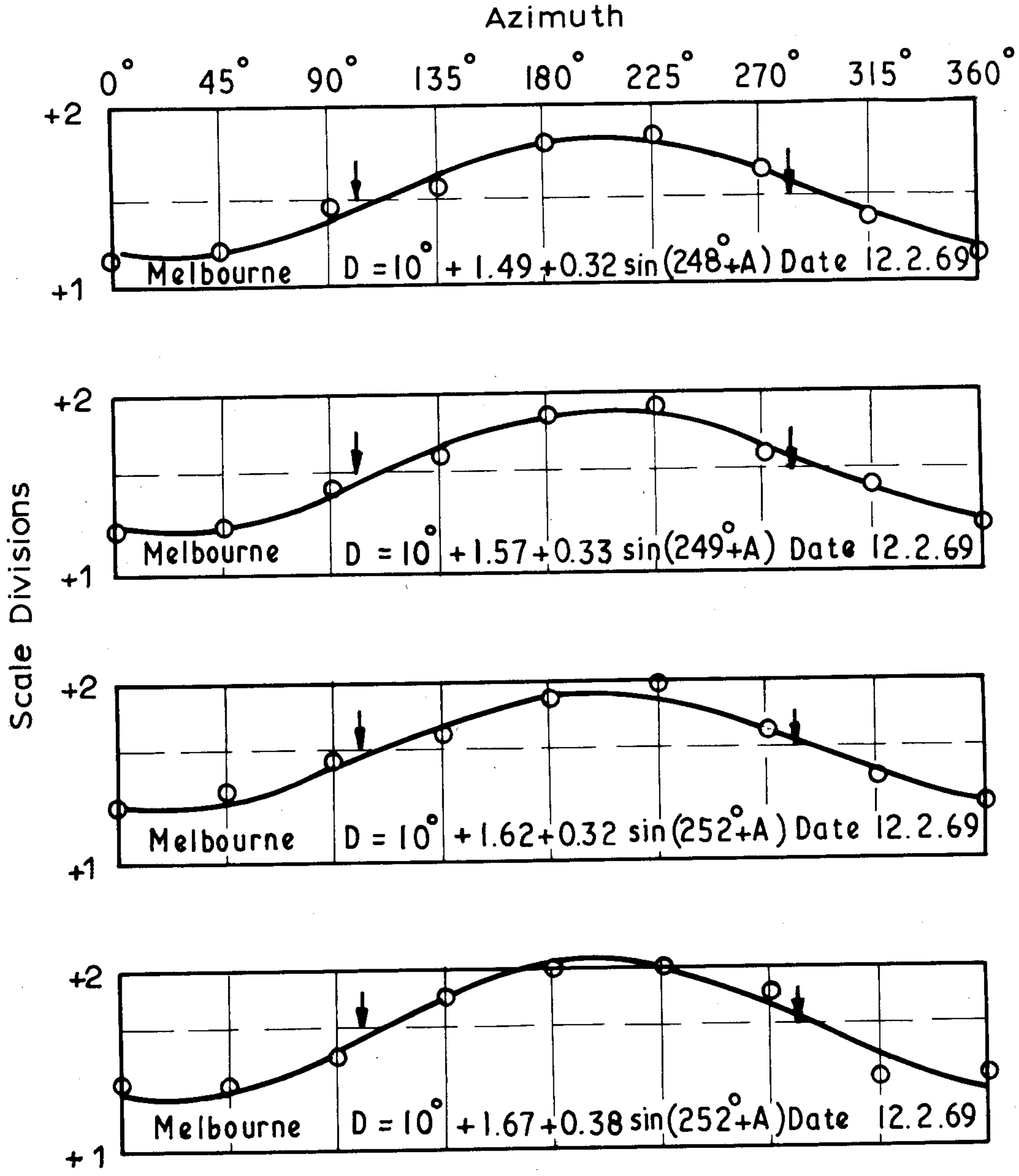


FIG. 6.2: TAPE ZERO POSITION GYRO # 2875

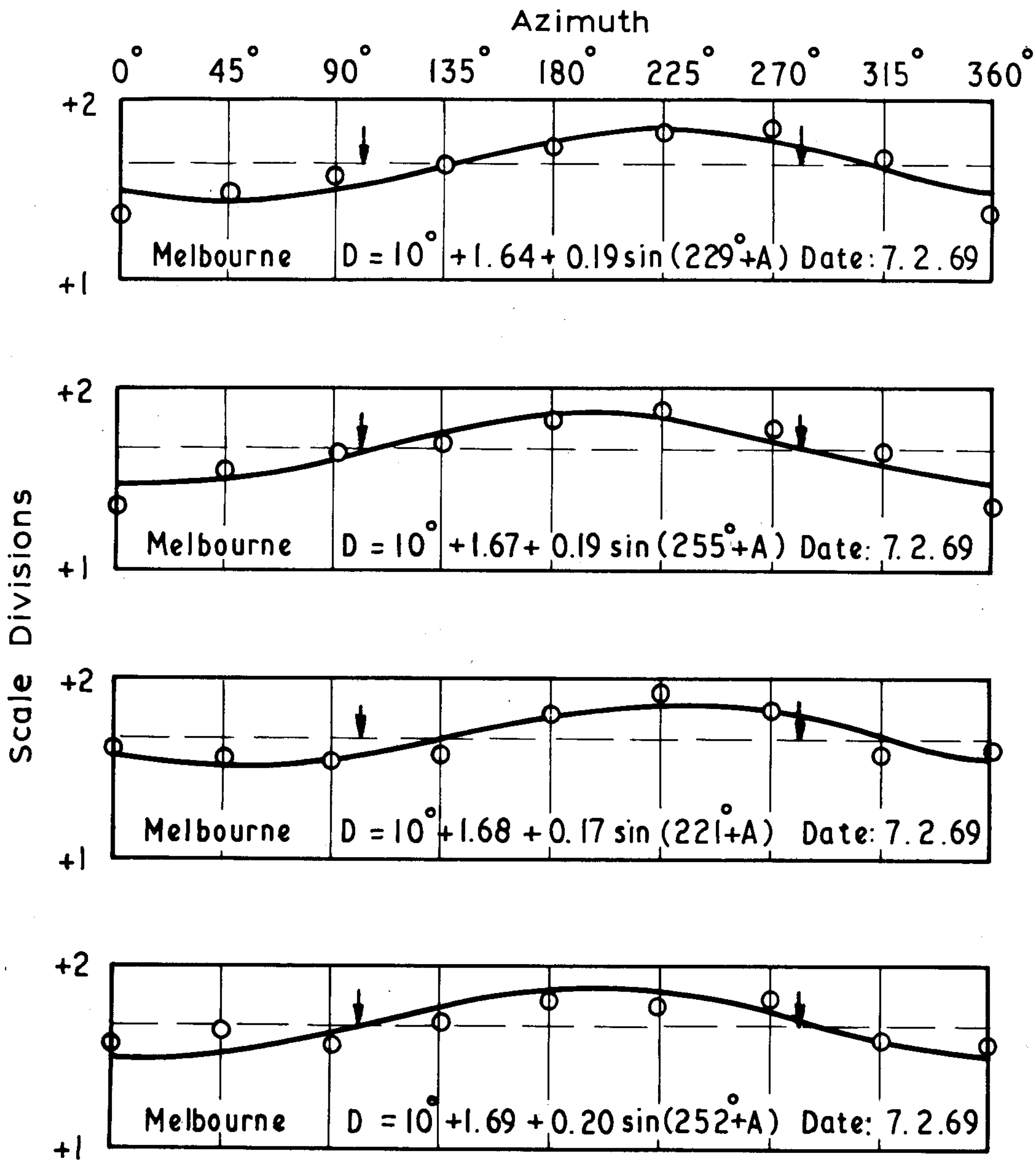


Fig. 6.3: TAPE ZERO POSITION GYRO # 2875

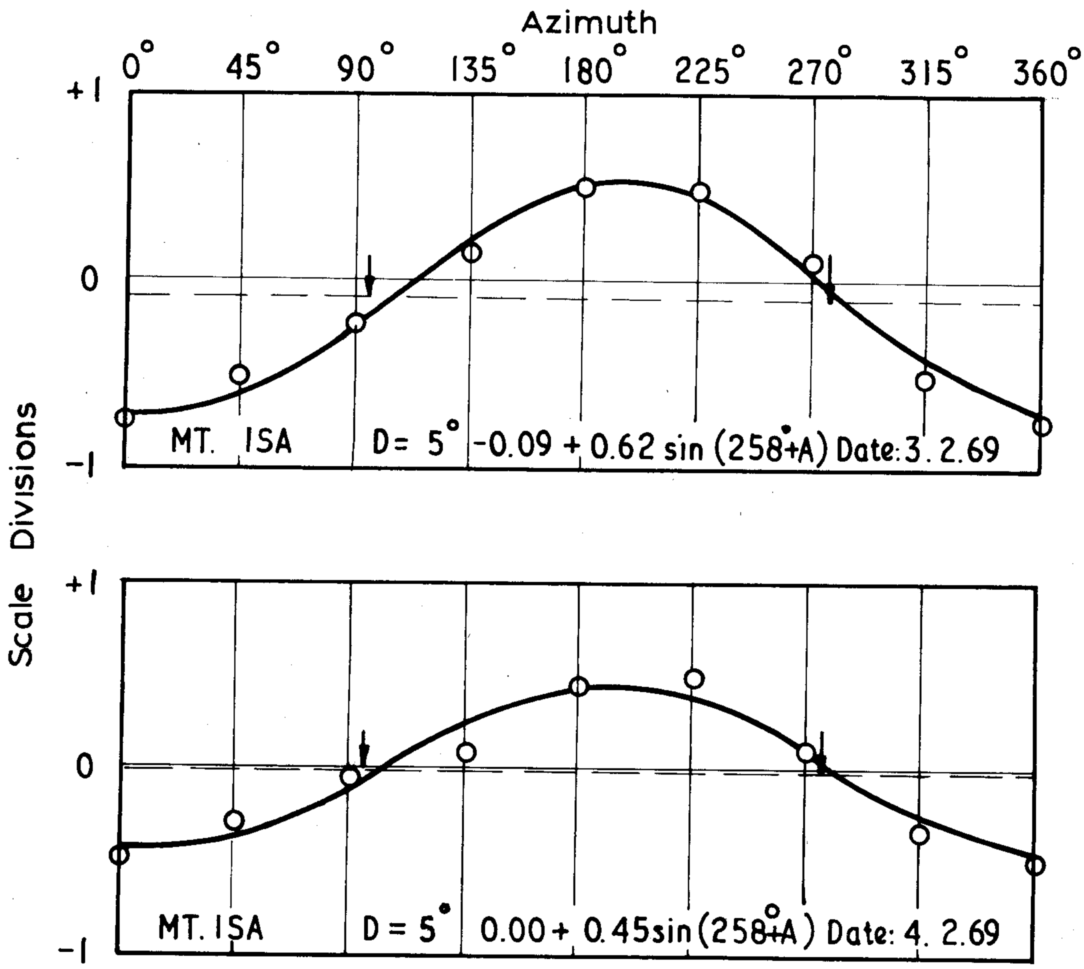


Fig. 6.4: TAPE ZERO POSITION GYRO #2887

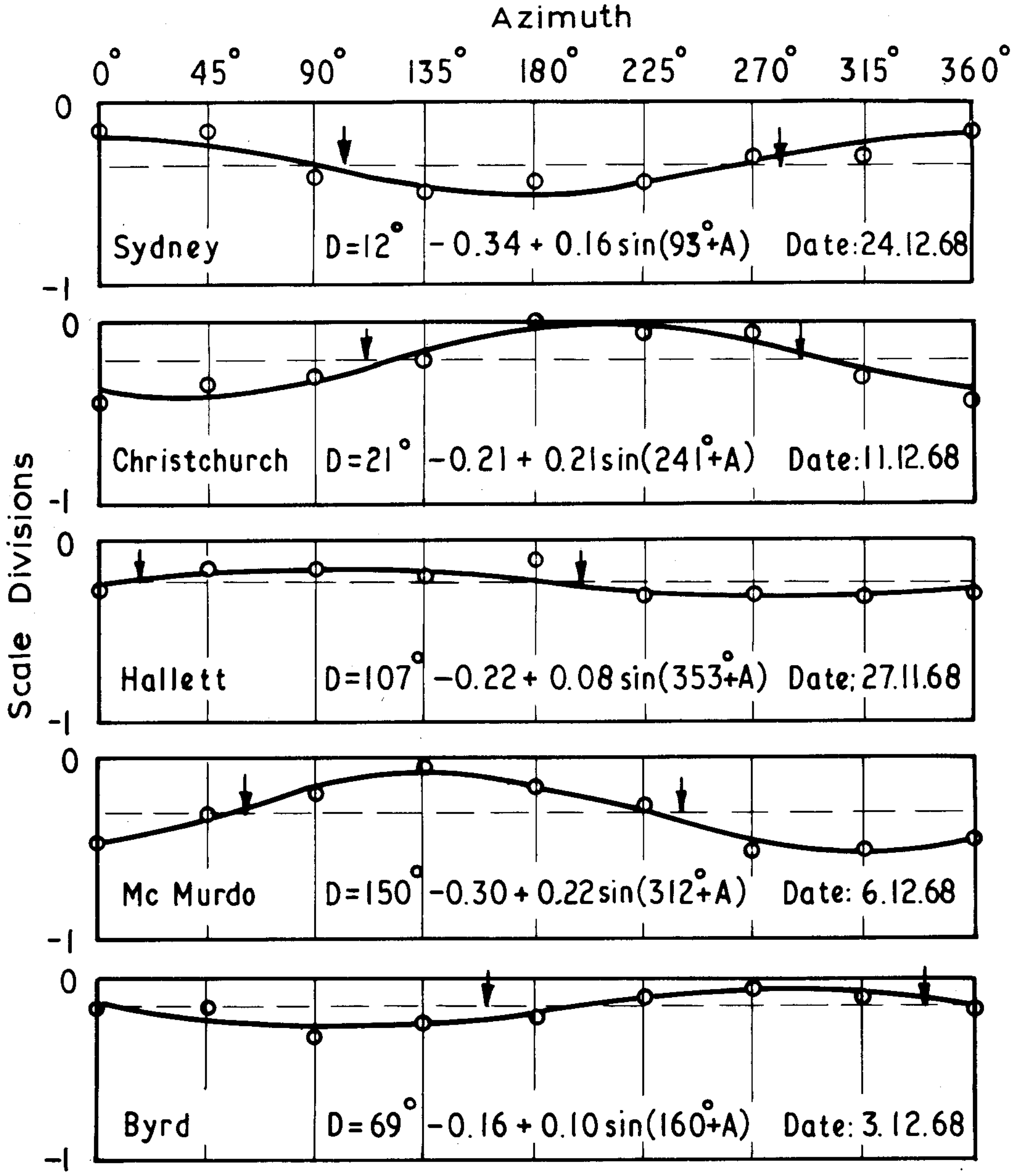


Fig. 6.5: TAPE ZERO POSITION GYRO # 2888

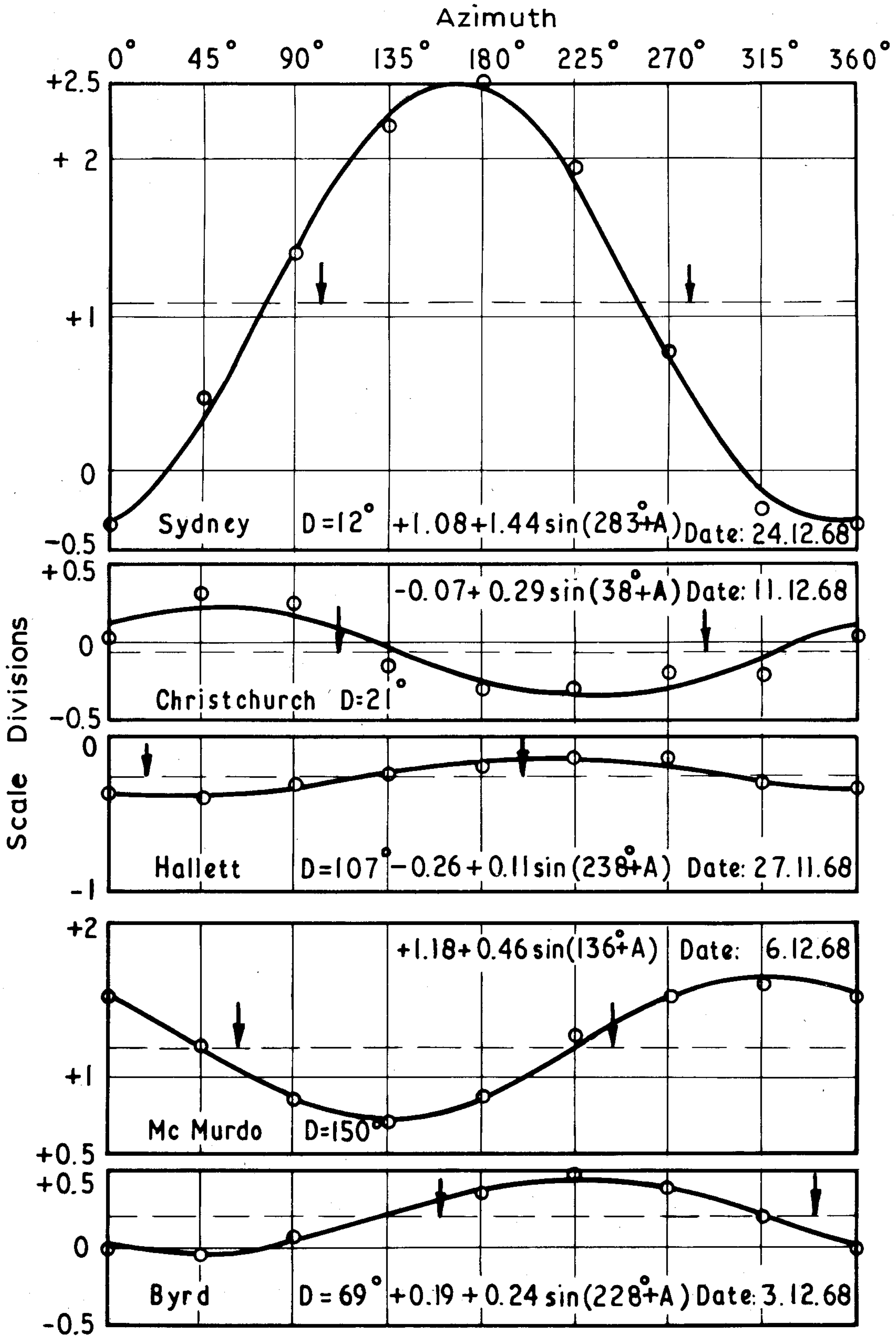


Fig.6.6: TAPE ZERO POSITION, GYRO #3243

that Number 2888 appears to be little affected by residual magnetism judging by the amplitude of the curves at all stations, and also the tape zero values do not depart by more than .09 divisions from a mean value of -0.25 divisions. However, with gyro Number 3243 curve amplitudes vary from 0.11 to 1.44 divisions and tape zero values have a range between -0.26 and +1.08 divisions.

It has been seen that there is considerable variability in the results of observations made with GAK 1 gyro attachments. It has been found that the greatest variability in tape zero position seems to occur when the amplitude of the curve is large which would also indicate the presence of residual magnetism. The effect of the earth's magnetic field can be eliminated by taking the mean of non-spinning readings made at an even number of equally spaced settings of the horizontal circle. If the form of the curve is not required then the effect may be eliminated simply by taking the mean of two sets of observations in azimuths 180^{o} apart.

If the braking of the spinning gyro is performed with the 3 phase A.C. current then the stator and its associated magnetic field will come to rest in a random position which may cause the gyro in its case to be deflected towards the mu metal cylindrical shield surrounding the gyro. If this is so then the tape zero position will depend upon the magnitude of the reisdual field and the stationary position of the stator. In the M.O.M. Gi.B and Gi C gyro-theodolite series the position of the tape zero can be determined accurately by means of the auxiliary D.C. braking system. With D.C. braking the stator is brought into a final stationary position such that the direction of the residual magnetic field always lies parallel to the axis of the shield, thus there is no attractive component towards the shield. A similar system is now used in the Wild ARK Aiming Circle series. Braking to a standstill is performed automatically with the 12 volt D.C. system. If the tape zero position is required to be checked after transportation then the converter can be actuated for a few tenths of a second to impart a small angular velocity (to the gyro) and then the D.C. braking system automatically brings the stator to the desired stationary position in a 3 second period. The converter GKK 3 which is used with this instrument is compatible with the GAK 1 gyro-theodolite and should be used in preference to the GKK 1 converter.

In the Historical Review reference was made to the development of the gas squeeze bearing as a supporting device for the Lear Siegler miniature gyro-theodolite. The bearing consists of a hybrid type of piezo-electric transducer, resonant at about 15,000 Hz. The upper surface of the transducer is a spherically concave glass plate which matches the shape of another glass plate on the underside of the case housing the gyro. When the gyro is in operation the spherical surfaces are kept about 100 to 150 microinches apart thus providing a frictionless support for the gyro. By supporting the case on its underside the height of the instrument is reduced considerably. The only physical connection between the gyro case and the instrument housing will be the power transfer leads. With conventional tape supported gyroscopes we have seen that there are problems with unstable tape torsion and variability in the tape zero position, problems which have been virtually overcome by this novel form of support.

7. THE HIGH LATITUDE PERFORMANCE OF THE GAK 1 GYRO-THEODOLITE

7.1 GENERAL CONSIDERATIONS

Under the influence of a directional couple given by $J\omega\cos\phi \sin A$, the horizontal spin axis of a freely suspended gyro-motor will precess about the vertical, where:-

- J is the angular momentum of the gyro-motor
- ω is the angular velocity of the earth
- ϕ is the latitude of the place of observation, and
- A is the azimuth of the spin axis.

It will be seen from the previous expression that when $A = 0$ the directional couple disappears and the spin axis will lie in the meridian plane. However, the gyroscopic unit of the gyro-theodolite is very lightly damped and will oscillate about the meridian with a well defined amplitude in contrast to the gyro-compass which is heavily damped and brought to rest in the meridian plane. The previous expression also shows that the directional couple is a maximum at the equator and will decrease towards the poles. At the poles themselves the couple disappears and the gyro rotates in a random fashion. Thus we would expect that the best results should be obtained in equatorial latitudes where the directional couple has its greatest value.

Strasser and Schwendener (1964) and the Wild Instrument Company in their technical manual of the gyro-theodolite GAK 1 state that the operational range of the equipment is from latitude 0° to 75°. These authors justify this limit by considering the tape and earth torque for the GAK 1 Number 3, where these torques are of approximately equal value at latitude 75°. However, there have been significant changes in the torsion characteristics of GAK 1 suspension tapes of later models which would indicate that a higher latitude limit is possible. The following table gives values of the ratio of the tape and earth torque expressed as a percentage, for GAK 1 gyro-theodolites which have been used by this author in addition to the value for GAK 1 Number 3. This ratio is called the useful torque ratio by Strasser and Schwendener.

Useful Torque Ratio

GAK 1 No.	Latitude	
	0°	75°
3	25%	98%
2871	18	69
2888	17	67
3243	14	52

For GAK 1 Number 3243 the useful torque ratio is 100% at latitude 82°. Ideally, therefore we should seek an instrument with the lowest useful torque ratio if observations are to be made in high latitudes.

The useful torque ratio is not the only characteristic which is of importance but also the drift must be considered, which is present in varying degrees with gyro-theodolites. Halmos (1967) states that there will be small changes in the torsion free position of the suspension tape due to changes in temperature, mechanical stress and torque of the current carrying spirals. Abrupt mechanical stresses should be avoided which may occur during transportation and releasing and braking the gyroscopic unit. Pusztai (1966) is also of the same opinion, citing temperature change as a major cause of drift and also molecular stresses introduced in the rolling of the sheet metal from which the tapes are cut. Gregerson (1969b) confirms Pusztai's findings with regard to temperature change and he has found that a long warm up period of about an hour is necessary before stability is reached. These previous remarks apply to the MOM GiB series of gyro-theodolites but significant differences in the gyro constant E for the Wild GAK 1 in the cold and warm states have been noted previously in Section 5.4. Thermal equilibrium in this latter instrument is not reached until the instrument has been running for about two hours which can be seen from Figure 7.1 on Page 201. In the experiment to determine the temperature behaviour, the temperature sensitive element, a chromium aluminium thermocouple was strapped to the external case of the instrument, thus at the maximum point on the curve the thermal gradients within the instrument should be stable.

In protracted observations with the turning point and transit methods, Williams and Belling (1967b)have found secondary harmonic (SHAR) effects which are not explained by gyroscopic theory so far developed. Chrzanowski (1969) has also found harmonic changes including both long and short period terms. The pattern of these cyclic changes were found with repeated observation series which indicated the presence of a diurnal effect. This diurnal effect has also been reported by Gregerson (1969a) who offers a luni-solar gravity theory to explain the phenomena but states that so far there is not sufficient evidence to either confirm or deny its presence.

In equatorial and medium latitudes we can tolerate the errors arising from many of these sources provided we are not seeking geodetic precision because the earth torque will dominate, but in high latitudes the aggregation of these small irregularities may dominate the earth torque and then the drift will become intolerably large.

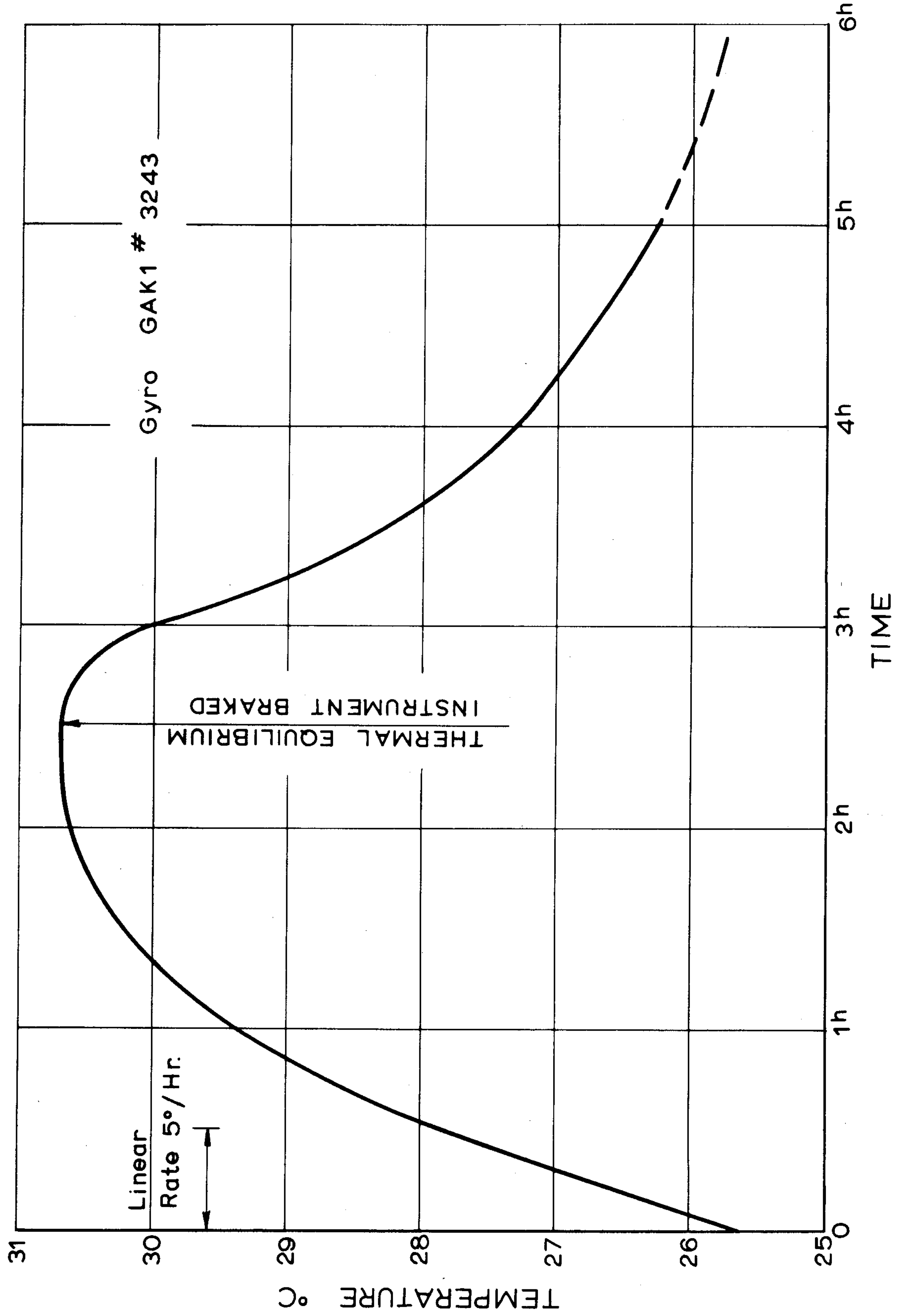


FIG. 7.1: TEMPERATURE CHARACTERISTICS

A better insight into the causes of drift of gyro-theodolites would be possible if high latitude observations could be simulated in the laboratory. It was stated before that the directional couple causing the spin axis of the gyro to precess is directly proportional to the angular momentum of the gyro-motor and therefore high latitude observations could be simulated if the angular velocity were reduced. Ing. L. Strassburg from the Institute of Mining Surveying, Bochum has stated in private discussion that the results of experiments over a range of angular velocities indicate that changes occur in the bearing of the gyro-motor which mask the effect of a pure angular momentum change. Therefore at the present stage of gyro-theodolite development it will be necessary to take these instruments into polar regions to assess their high latitude performance.

7.2 OBSERVING METHODS

The minimum observation time required for the turning point and transit methods is one oscillation period, but this of course does not provide any check on the readings and a further half a period must elapse before a check can be obtained. Halmos (1966) has devised a method of reduction which allows the total observation time to be reduced to half a period. This method requires a knowledge of the gyro-theodolite damping constant. With the aid of a small auxiliary table or diagram the calculations can be performed quickly. However, Halmos advises that the method gives results which on the average are about 15-20% less precise than the turning point method using 4 reversion points.

From Figure 7.2 on Page 203 it is seen that the period for the turning point method is longer than the transit method and that this difference becomes larger as the pole is approached. Both methods require the instrument

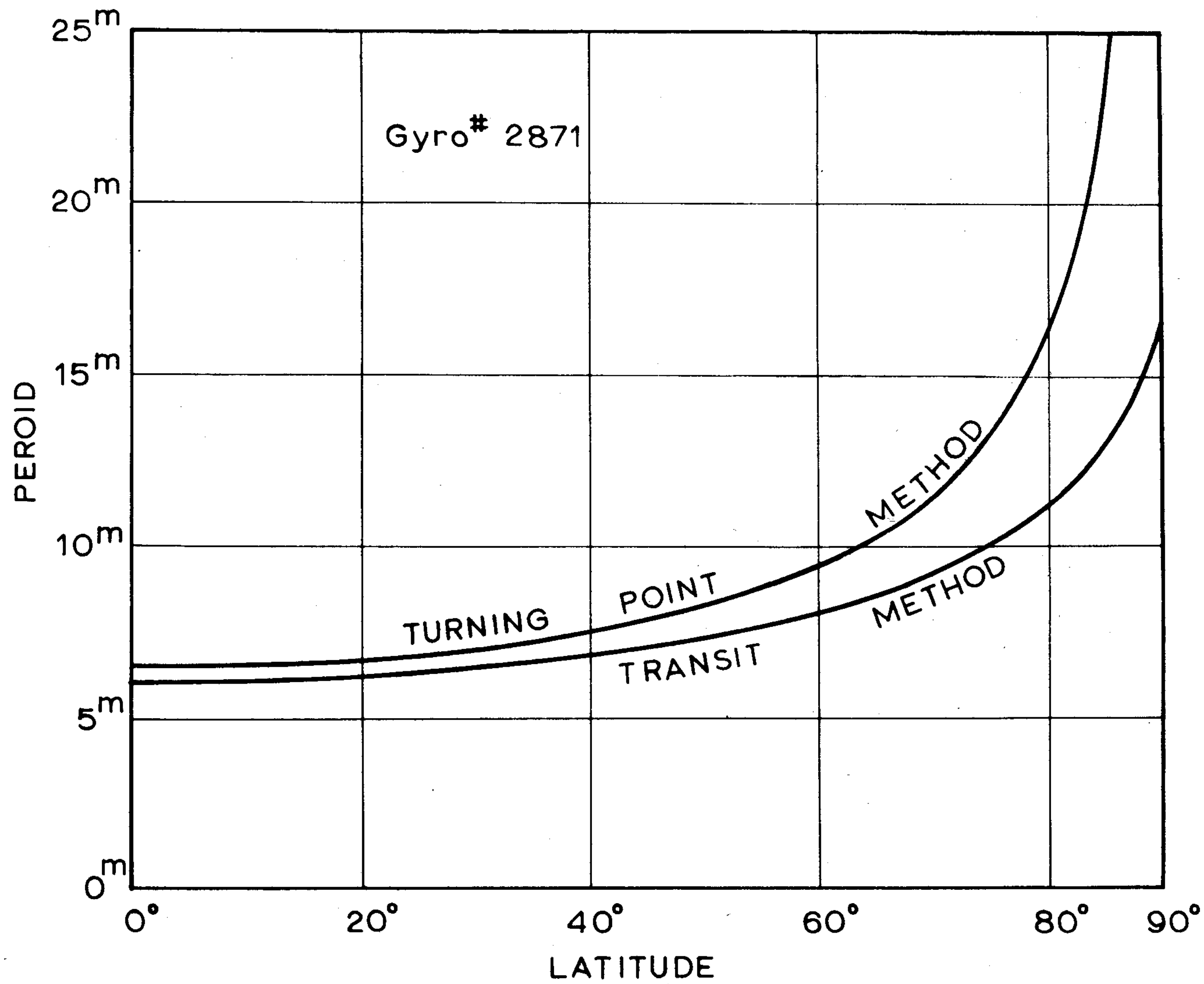


FIG. 7.2: OSCILLATION PERIODS

support to be stable in azimuth during the observation period. There is about a 50% increase in observing time for the turning point method in latitude 80° and thus there would be a distinct preference for the transit method on this count alone. In addition the transit method is more relaxing for the observer since only a timing observation is necessary when the gyro-mark passes through the central vee slot, whereas for the turning point method the observer must concentrate continuously on keeping the gyro-mark centred in the vee slot. The constant manipulation of the instrument in this latter method may also impart small irregularities to the gyroscope's motion and thus introduce further errors. These errors can be minimised if a gyro-theodolite with an automatic following device is used such as in the M.O.M. Gi B2 instrument, but the extra weight of this type of instrument would be an additional factor to consider. It has been found that turning point observations are strenuous for the observer when working in low temperatures; if the observer wears gloves the following of the gyro-mark is erratic but with the fingers exposed at temperatures of -20°F the observer is uncomfortable and also if care is not taken the observer's breath condenses on the observation tube thus obscuring the auxiliary scale. In Chapter 5 detailed proofs etc. are given of observing methods which have been developed in order to reduce the total observation time for the transit and turning point methods. In high latitudes where the period of oscillation is long these methods are of particular value because a determination including redundant observations can be completed in a relatively short time, e.g. at Byrd Station (ϕ = S 80°) 44 timing observations were made in 17 minutes for the transit method and 21 timed horizontal circle readings were obtained for the turning point method in 23 minutes, with gyro-theodolite Number 3243.

7.3 TEST OBSERVATIONS.

In 1968 the National Science Foundation approved of a request for assistance in the testing of gyro-theodolites in high latitudes under the United States Antarctic Research Programme. The experiments were designed to give information about the internal and external accuracy of a small pendulous gyroscopic attachment for a theodolite. Two Wild gyro-theodolites GAK 1/T16 configuration were tested at stations which would give a wide latitude range. The stations selected were as follows:-

	Latitude (ϕ)	Longitude (λ)
Sydney	S 33° 55'	E 151° 14'
Christchurch	S 43 32	E 173 35
Hallett	S 72 19	E 170 13
McMurdo	S 77 51	E 166 40
Byrd	S 80 01	W 119 32

Figure 7.3 illustrates the disposition of these stations and shows the routes which were taken.

At each station it was anticipated that 2 to 4 gyro-theodolite determinations could be made in the available time with each instrument, using the modified transit method at all stations and the turning point method at the low latitude stations and the modified turning point method at the high latitude stations. Observations were also made at Sydney and Christchurch on the return journey so that the stability of the instruments could be checked. In addition astronomical observations for latitude, longitude and azimuth were to be made if the weather permitted. The latitude and longitude results were then to be used

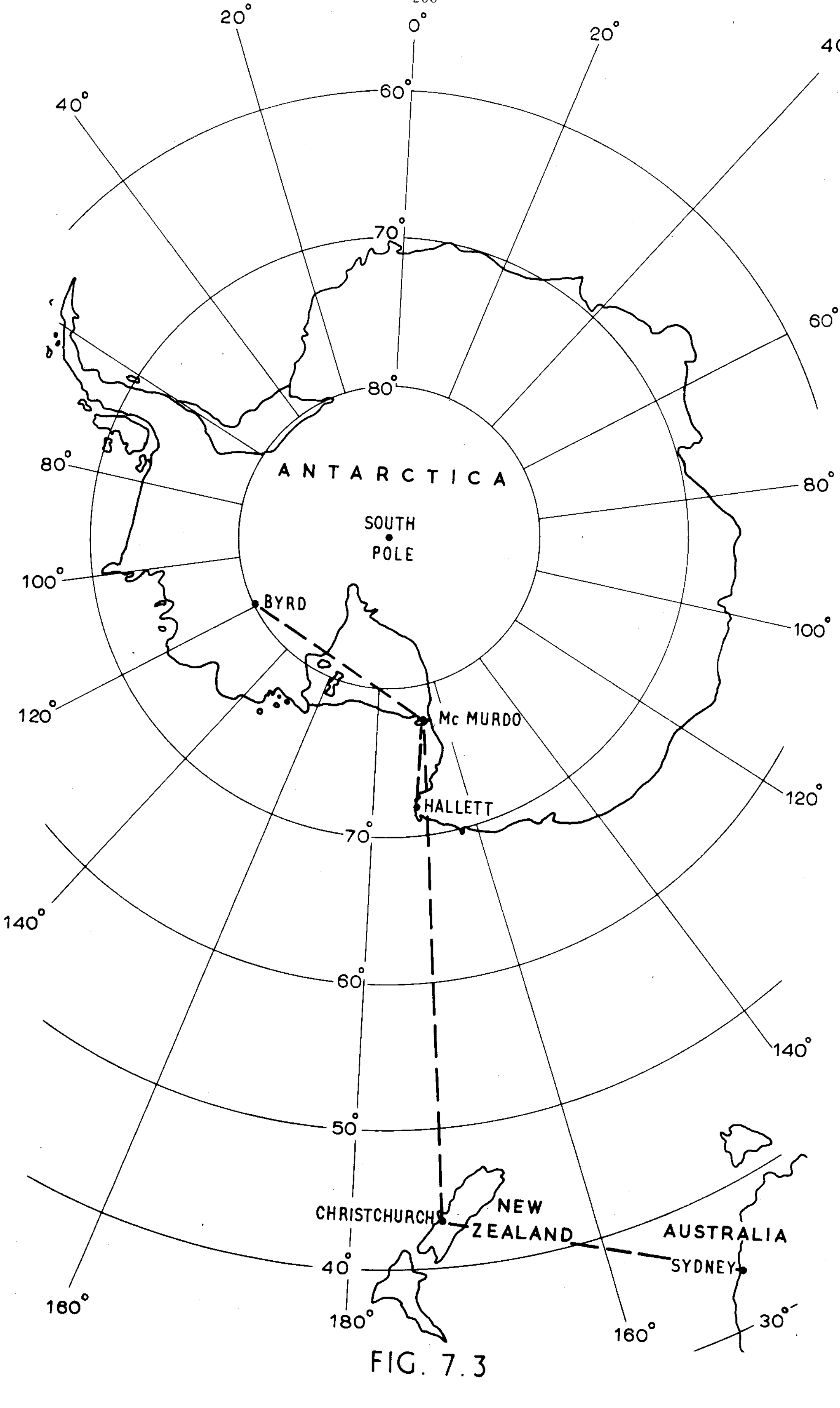

ANTARCTICA
SOUTH POLE
BYRD
Mc MURDO
HALLETT
CHRISTCHURCH
NEW ZEALAND
AUSTRALIA
SYDNEY
0°
20°
40°
60°
80°
100°
120°
140°
160°
180°
70°
50°
30°

FIG. 7.3

for the reduction of the azimuth observations. The final astronomical azimuth results were then used for checking the external accuracy of the gyro azimuth determinations. All observations, including calibration were completed in the period 6th November 1968 to 24th December 1968, which included 17 days spent in Antarctica. As well as gyro-theodolite and astronomical observations, Professor P.V. Angus-Leppan made observations of vertical refraction over ice and snow surfaces for two 24 hour periods. The equipment required for all of this work was quite extensive weighing about 700 lbs. and consisted of:-

1 Wild T2 theodolite	3 tripods
1 Wild T16 theodolite	6 Aspirated thermistor thermometers plus associated circuitry
2 Wild GAK gyro-attachments	
2 Wild GKK converters	1 observing tent
1 Wild GEL charger	1 Anenometer
3 Wild tripods	1 Chronometer and stop watches
2 Zeiss Ni2 levels	1 Short wave radio

Observing sheets, reduction and astronomical tables, tools etc.

With all this equipment the party of two was self supporting technically. Transportation between stations was by Australian and New Zealand commercial airlines, United States military aircraft LC-130F "Hercules" and C-121J "Super Constellation" involving a round trip of 9,700 miles.

A brief description of the reference lines and the method of azimuth determination is as follows:-

<u>Sydney</u> See Section 5.4 for a description of this line.

<u>Christchurch</u> Instrument station, survey mark Number 8418 on the campus of the University of Canterbury at Ilam to T.S. Sugarloaf.

Azimuth by observations on close circum-polar star σ Octantis. Latitude and longitude by position lines from star observations.

Hallett — Reference line across the Adelie penguin rookery with the disused magnetic hut at one terminal. Azimuth by star hour angles. Latitude and longitude by position lines. All observations to daylight stars.

McMurdo — Reference line from a point near the USARP administration building to the small rock cairn of the Shrine of Our Lady of Sorrows which is situated about 330 yards North East of Scott's hut. Azimuth and position observations as at Hallett.

Byrd — Reference line from instrument station established in the emergency Jamesway hut to U.S. Geological Survey Station "M Pier Astro" which is situated about 1400 ft. to the North East of the Jamesway hut. Azimuth by sun hour angles.

7.4 INSTRUMENTAL CONSTANTS.

It is convenient when operating the gyro-theodolite in a wide range of latitudes to evaluate beforehand certain constants which are to be used for correcting and reducing observations. It is also convenient to have a fore-knowledge of the lengths of the periods of oscillation which can then be used for preliminary orientation by the ¼ period method and for anticipating the occurrence of turning points. In this experimental series the constants for each station were evaluated from observations made at Sydney, using the theoretical relationships given in Appendix IV.

7.4.1 INSTRUMENTAL CONSTANT K.

If the rest position of the non-spinning gyro does not coincide with the zero of the auxiliary scale then a correction is to be applied to the gyro indicated North for all methods of observations, which is given by:-

$$K \, \Delta \alpha \; = \; m \left\{ \left(\frac{T_U}{T_D} \right)^2 - 1 \right\} \Delta \alpha \; = \; \frac{m \, \mu \, \Delta \alpha}{J \, \omega \cos \phi}$$

where $\Delta \alpha$ is the deviation from the zero position.

Schwendener (1966) has shown from an examination of the differential relationship

$$d \, K \; = \; \frac{m \, \mu \tan \phi \, d \phi}{2 \, J \, \omega \cos \phi}$$

that in mid-latitudes the variation in K is unimportant over a latitude range of about $\pm \, 2.4^{\circ}$ for the amplitude method. In high latitudes K becomes large and the variation in K important for all methods. The final accuracy of azimuth will also depend upon the size and accuracy of the deviation $\Delta \alpha$ which has been discussed in Chapter 6. As with the proportionality constant c, the sensitivity to latitude and the size of K can be reduced by selecting a gyro with a low useful torque ratio as will be seen from Table 7.1 on Page 210 where gyro Number 3243 has a lower useful torque ratio than Number 2888.

Table 7.1 $K = m\left(\frac{T_U^2}{T_D^2} - 1\right)$ min./div.

Station	Gyro			
	No. 3243		No. 2888	
	Calculated	Observed	Calculated	Observed
Sydney		1.74		2.16
Christchurch	1.99	2.01	2.48	2.48
Hallett	4.74	4.84	5.92	6.04
McMurdo	6.84	7.02	8.54	8.73
Byrd	8.32	8.46	10.37	10.61

The table shows the values of K which have been calculated from observations made at the individual stations and those which have been calculated from observations made at Sydney. The differences between the observed and calculated values are relatively small with a maximum variation of about 2% at latitude 80°. Thus using either the observed or calculated value of K should not introduce a serious error provided $\Delta\alpha$ is small.

7.4.2 THE PROPORTIONALITY FACTOR c.

The proportionality factor c is required for the reduction of transit and modified transit observations in the following equations:-

$$\Delta N = c\, a\, \Delta t \qquad \text{Transit Method}$$

$$\Delta N = c\, a\, \left(1 - \frac{n^2}{a^2}\right)^{\frac{1}{2}} \Delta t \qquad \text{Modified Transit Method}$$

where $$c = m\,\frac{\Pi}{2}\,\frac{T_U^2}{T_D^3} = \frac{m\Pi}{2T_{U_o}} \cos \phi^{\frac{1}{2}} (1+\psi)^{\frac{3}{2}} \; *$$

where ψ is the useful torque ratio $= \dfrac{\mu}{J\,\omega\,\cos\phi} = \left(\dfrac{T_U}{T_D}\right)^2 - 1$

Schwendener (1966) has noted that the factor c is not sensitive to latitude change in low to medium latitudes and shows that c may be taken as an instrumental constant in medium latitudes (without serious error) over a range of about $\pm\, 2.3^{\circ}$ from consideration of the differential relationship:-

$$d\,c = \frac{m\,\Pi \sin\phi}{2\,T_{U_o}\cos^2\phi}\left\{\left(\cos\phi + \frac{\mu}{J\,\omega\,\cos\phi}\right)^{\frac{3}{2}} - \frac{3}{2}\cos\phi\left(\cos\phi + \frac{\mu}{J\,\omega\,\cos\phi}\right)^{\frac{1}{2}}\right\} d\,\phi$$

which may be expressed in a simpler form as follows:-

$$d\,c = c\,\tan\phi\,\left\{1 - \frac{3}{2}\left(\frac{T_D}{T_U}\right)^2\right\} d\,\phi$$

with the notation used in Appendix IV.

In high latitudes c is sensitive to latitude change which can be seen from Figure 7.4 on Page 212. From an examination of the previous expression for c marked * before, it will be seen that this sensitivity can be reduced if a gyro is chosen which has a low useful torque ratio although by doing this the period of oscillation will increase.

In Table 7.2 are shown the values of c which were calculated from observations made at each station and those which were calculated from observations made at Sydney.

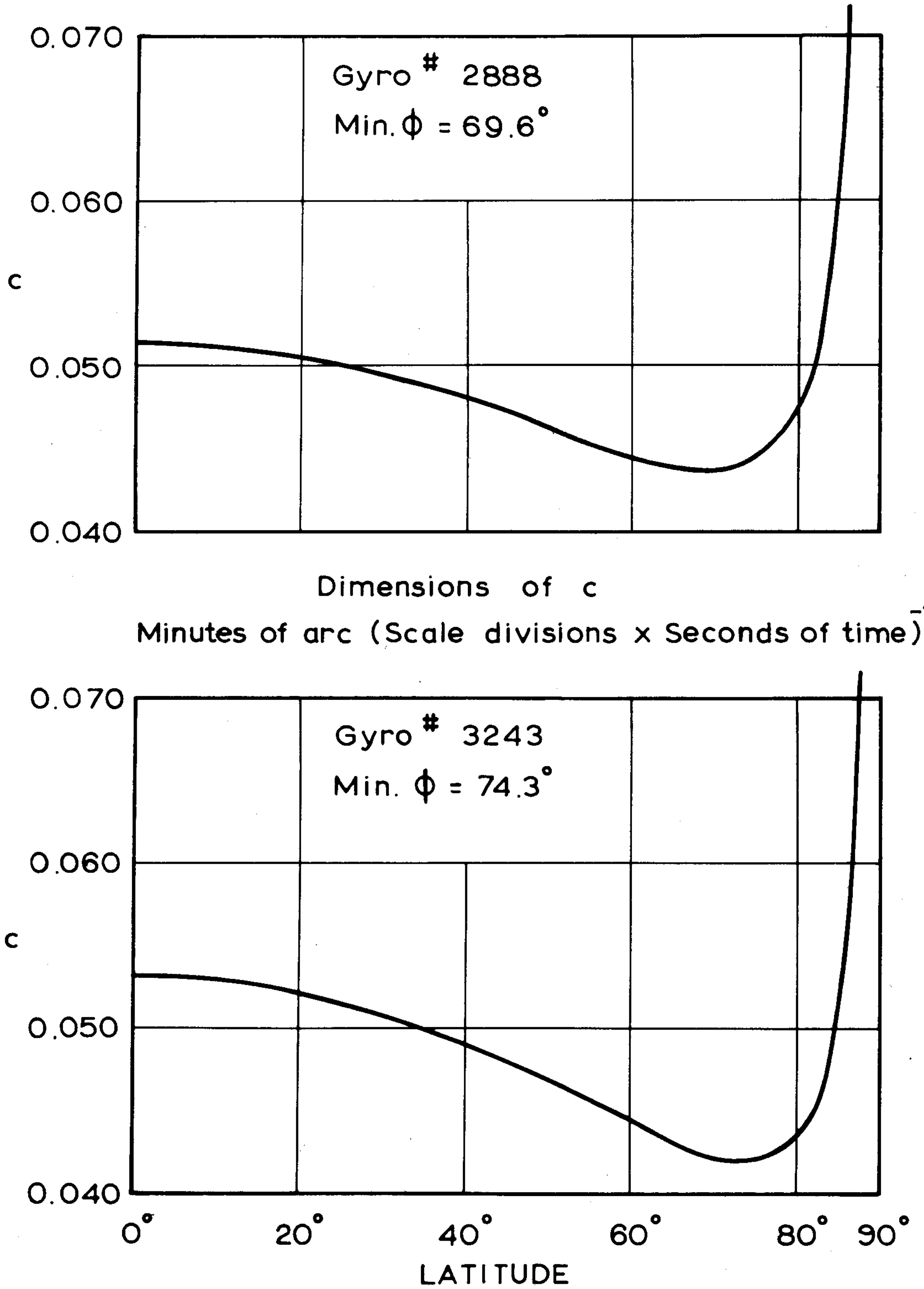


FIG. 7.4: Gyro Constant $c = m \cdot \frac{\pi}{2} \cdot \frac{T_U^2}{T_D^3}$

Table 7.2 $c = m \frac{\Pi}{2} \frac{{T_U}^2}{{T_D}^3}$ min./div. sec.

Station	Gyro			
	No. 3243		No. 2888	
	Calculated	Observed	Calculated	Observed
Sydney		0.0502		0.0490
Christchurch	0.0484	0.0484	0.0475	0.0475
Hallett	0.0421	0.0423	0.0439	0.0445
McMurdo	0.0425	0.0429	0.0458	0.0466
Byrd	0.0435	0.0437	0.0478	0.0487

In general there is good agreement between the observed and computed values. The maximum variation is about 2% which should not be serious provided ΔN is not allowed to become too large.

7.5 DETERMINATION OF LATITUDE FROM THE TURNING POINT METHOD.

The oscillation period for the turning point method is given by $T_{U_\phi} = T_{U_o} \cos \phi^{-\frac{1}{2}}$. It will be seen from this expression that if T_{U_o} is known then the latitude ϕ can be deduced by measuring T_{U_ϕ}. T_{U_ϕ} is insentitive to latitude change in low latitudes but in high latitudes Strasser and Schwendener (1966) have shown that from an examination of the differential relationship $\frac{d\,T_U}{d\,\phi}$ that the latitude should be able to be determined with an accuracy better than one minute of arc, e.g. these authors show that the change in period for GAK 1 Number 3 for 1' of latitude in latitude 75^o is 0.43 seconds and thus their conclusions are theoretically justified because timing errors are of the order of about ± 0.2 seconds.

In Table 7.3 on Page 215 a summary is given of the observed periods and the corresponding values of the calculated periods based on the observations made at Sydney. It will be seen from the tabulated values that the variation in observed periods at each station is considerably greater than what would be expected if timing errors alone were present. Variations in observed period were greater at the high latitude than at the low latitude stations, amounting to 2.6 seconds at Hallett. Variations of the same order of magnitude were also noted during a single gyro-azimuth determination. The reason for these variations is not known but in the light of these results the accuracy of latitude determination as proposed by Strasser and Schwendener must be modified. These authors suggest that latitude could be determined in high latitudes if the gyro attachment were calibrated, which would mean that calibration must be made at one of the high latitude stations. Latitude would then be deduced from the relationship

$$\cos \phi = \left(\frac{T_{U_1}}{T_U} \right)^2 \cos \phi_1$$

where T_{U_1} is the observed period at the calibration station in latitude ϕ_1. Using the observed value of the period at Hallett, i.e. treating Hallett as the calibration station, we can calculate the latitude of McMurdo and Byrd. The results of which are as follows:-

Station	Gyro		Astro
	No. 3243	No. 2888	
McMurdo	77° 51'	77° 50'	77° 51'
Byrd	80 03	80 01	80 01

TABLE 7.3 SUMMARY OF CALCULATED AND OBSERVED PERIODS

Station	Gyro No. 3243		Gyro No. 2888	
	Turning Point	Transit	Turning Point	Transit
Sydney	$6^m57.7^s$ [2]	$6^m27.3^s$	$7^m20.6^s$	$6^m40.6^s$
Christchurch	7 27.8 [1] 27.8 28.2 27.4 28.2 28.0 7 27.9 [2] (27.2)	6 50.5 50.2 51.2 51.0 50.8 50.8 6 50.8 (50.7	7 51.7 50.3 51.5 52.0 50.8 7 51.2 (51.7)	7 02.6 03.0 03.0 03.1 03.8 7 03.1 (03.6)
Hallett	11 33.3 35.9 34.0 33.4 11 34.2 (30.3)	9 35.2 36.5 36.0 34.6 9 35.6 (34.3)	12 07.0 06.9 07.2 06.4 12 06.9 (08.2)	9 37.8 36.8 37.6 37.0 9 37.3 (40.6)
McMurdo	13 53.0 55.2 53.4 53.6 13 53.8 (49.3)	10 47.5 48.0 47.0 46.7 10 47.3 (47.1)	14 34.0 31.8 32.3 32.0 14 32.5 (34.9)	10 44.7 42.1 41.1 41.1 10 42.2 (47.2)
Byrd	15 21.1 20.3 19.8 20.8 15 20.5 (13.8)	11 27.4 28.3 26.5 26.6 11 27.2 (24.8)	16 01.0 02.9 02.6 01.4 16 02.0 (04.0)	11 15.0 14.6 17.0 16.0 11 15.6 (20.8)

1 Observed values.

2 Mean observed values.
Figures in brackets are calculated periods from the observations made at Sydney.

In the light of these results it would appear that latitude can be determined to an accuracy of about one to two minutes of arc which is a somewhat lower accuracy than that claimed by Strasser and Schwendener. A possible improvement in accuracy could be obtained if calibrations were made at more than one station.

7.6 DISCUSSION OF RESULTS OF TEST OBSERVATIONS.

At each station, observations were made with the turning point and transit methods in succession using each gyro-attachment alternately. The reasons for adopting this sequence of observations were twofold -

(a) the results of the experimental series in Section 5.4 indicated that combining the results of both methods improved the accuracy of the determination, and

(b) alternating each gyro-attachment would decrease the correlation which exists between successive observations made with the same instrument.

The results of the observations are given in Tables 7.4 to 7.9 on Pages 217 to 222. A summary table of variances is given below.

Variance of a single determination (square seconds).

Station	Gyro No. 3243			Gyro No. 2888		
	Turning Point	Transit	Combined	Turning Point	Transit	Combined
Sydney	142	202	150	91	106	86
Christchurch	510	831	659	114	140	51
Hallett	2,481	1,537	1,978	697	286	390
McMurdo	2,414	2,859	2,622	422	1,198	682
Byrd	7,628	1,285	1,792	514	294	195

TABLE 7.6 SUMMARY OF RESULTS

Gyro: S/N 3243

Method: Turning Point

Station	Date (1967)	Gyro Azimuth	v	Mean, Range, Var.	Astro. Azimuth	$Astro^E$-Gyro	Calculated E	V
Sydney	9 Nov.	67° 41' 56"	-18"	67° 41' 38"	67° 41' 45"	+7"	+3"	-14"
	9 Nov.	67 41 40	- 2	29				+ 2
	10 Nov.	67 41 52	-14	142				-10
	20 Dec.	67 41 27	+11					+15
	20 Dec.	67 41 31	+ 7					+11
	24 Dec.	67 41 27	+11					+15
	24 Dec.	67 41 32	+ 6					+10
Christchurch	17 Nov.	148 04 48	-22	148 04 26	148 03 56	-30	-27	-25
	17 Nov.	148 04 43	-17	56				-20
	11 Dec.	148 03 52	+34	510				+31
	12 Dec.	148 04 42	-16					-19
	12 Dec.	148 04 24	+ 2					- 1
	12 Dec.	148 04 07	+19					+16
Hallett	26 Nov.	52 34 48	+23	52 35 11	52 28 20	- 6' 51	- 5' 57	-31
	26 Nov.	52 35 34	-23	1 53				-77
	27 Nov.	52 36 07	-56	2,481				-110
	27 Nov.	52 34 14	+57					+ 3
McMurdo	24 Nov.	316 15 54	-56	316 14 58	316 05 28	- 9 30	-10 08	-18
	24 Nov.	316 15 21	-23	1 52				+15
	25 Nov.	316 14 37	+21	2,414				+59
	25 Nov.	316 14 02	+56					+94
Byrd	2 Dec.	26 51 19	+18	26 51 37	26 38 45	-12 52	-12 52	+18
	2 Dec.	26 50 38	+59	1 58				+59
	3 Dec.	26 52 36	-59	7,628				-59
	3 Dec.	26 51 55	-18					-18

TABLE 7.5 SUMMARY OF RESULTS

Gyro: S/N 3243

Method: Transit

Station	Date (1967)	Gyro Azimuth	v	Mean, Range, Var.	Astro. Azimuth	$Astro^E$-Gyro	Calculated E	V
Sydney	9 Nov.	67°41'13"	- 1"	67°41' 12"	67°41'45"	+33"	+27"	+ 5"
	9 Nov.	67 41 19	- 7	41				- 1
	10 Nov.	67 41 39	-27	202				- 21
	20 Dec.	67 40 58	+14					+ 20
	20 Dec.	67 41 07	+ 5					+ 11
	24 Dec.	67 41 08	+ 4					+ 10
	24 Dec.	67 40 58	+14					+ 20
Christchurch	17 Nov.	148 04 32	-29	148 04 03	148 03 56	- 7	+ 2	- 38
	17 Nov.	148 04 20	-17	1 18				- 26
	11 Dec.	148 03 14	+49	831				+ 40
	12 Dec.	148 04 20	-17					- 26
	12 Dec.	148 04 06	- 3					- 12
	12 Dec.	148 03 45	+18					+ 10
Hallett	26 Nov.	52 33 06	+13	52 33 19	52 28 20	-4' 59	-4' 33	- 13
	26 Nov.	52 33 36	-17	1 31				- 43
	27 Nov.	52 34 02	-43	1,537				- 69
	27 Nov.	52 32 31	+48					+ 22
McMurdo	24 Nov.	316 13 52	-58	316 12 54	316 05 28	-7 26	-8 04	- 20
	24 Nov.	316 13 20	-26	2 02				+ 12
	25 Nov.	316 12 33	+21	2,859				+ 59
	25 Nov.	316 11 50	+64					+102
Byrd	2 Dec.	26 49 31	- 5	26 49 26	26 38 45	-10 41	-10 31	- 15
	2 Dec.	26 48 34	+52	1 22				+ 42
	3 Dec.	26 49 56	-30	1,285				- 40
	3 Dec.	26 49 41	-15					- 25

TABLE 7.6 SUMMARY OF RESULTS

Gyro: S/N 3243

Method: Turning Point & Transit

Station	Date (1967)	Gyro Azimuth	v	Mean, Range, Var.	Astro. Azimuth	$Astro^E$-Gyro	Calculated E	V
Sydney	9 Nov.	67°41'34"	- 9"	67°41' 25"	67°41'45"	+20"	+15"	- 4"
	9 Nov.	67 41 30	- 5	34				0
	10 Nov.	67 41 46	-21	150				-16
	20 Dec.	67 41 12	+13					+18
	20 Dec.	67 41 19	+ 6					+11
	24 Dec.	67 41 18	+ 7					+12
	24 Dec.	67 41 15	+10					+15
Christchurch	17 Nov.	148 04 40	-26	148 04 14	148 03 56	-18	-12	-32
	17 Nov.	148 04 32	-18	1 07				-24
	11 Dec.	148 03 33	+41	659				+35
	12 Dec.	148 04 31	-17					-23
	12 Dec.	148 04 15	- 1					- 7
	12 Dec.	148 03 56	+18					+12
Hallett	26 Nov.	52 33 57	+18	52 34 15	52 28 20	-5' 55	-5' 15	-22
	26 Nov.	52 34 35	-20	1 43				-60
	27 Nov.	52 35 04	-49	1,978				-89
	27 Nov.	52 33 22	+53					+13
McMurdo	24 Nov.	316 14 53	-57	316 13 56	316 05 28	-8 28	-9 06	-19
	24 Nov.	316 14 20	-24	1 57				-14
	25 Nov.	316 13 35	+21	2,622				+59
	25 Nov.	316 12 56	+60					+98
Byrd	2 Dec.	26 50 25	+ 6	26 50 31	26 38 45	-11 46	-11 48	+ 8
	2 Dec.	26 49 36	+55	1 40				+57
	3 Dec.	26 51 16	-45	1,792				-43
	3 Dec.	26 50 48	-17					-15

TABLE 7.7 SUMMARY OF RESULTS

Gyro: S/N 2888

Method: Turning Point

Station	Date (1967)	Gyro Azimuth	v	Mean, Range, Var.	Astro. Azimuth	$Astro^E$-Gyro	Calculated E	V
Sydney	9 Nov.	66°28'23"	+ 4"	66°28' 27"	67°41'45"	+1°13'18"	+1°13'17"	+ 5"
	10 Nov.	66 28 19	+ 8	27				+ 9
	10 Nov.	66 28 17	+10	91				+ 11
	20 Dec.	66 28 34	- 7					- 6
	20 Dec.	66 28 44	-17					- 16
	24 Dec.	66 28 29	- 2					- 1
	24 Dec.	66 28 22	+ 5					+ 6
Christchurch	17 Nov.	146 49 53	+ 9	146 50 02	148 03 56	+1 13 54	+1 13 55	+ 8
	17 Nov.	146 49 51	+11	24				+ 10
	11 Dec.	146 50 11	- 9	114				- 10
	12 Dec.	146 50 15	-13					- 14
	12 Dec.	146 50 00	+ 2					+ 1
Hallett	26 Nov.	51 07 53	-16	51 07 37	52 28 20	+1 20 43	+1 20 42	- 15
	26 Nov.	51 08 04	-27	58				- 26
	27 Nov.	51 07 06	+31	697				+ 32
	27 Nov.	51 07 25	+12					+ 13
McMurdo	24 Nov.	314 39 59	-22	314 39 37	316 05 28	+1 25 51	+1 25 52	- 23
	24 Nov.	314 39 35	+ 2	49				+ 1
	25 Nov.	314 39 10	+27	422				+ 26
	6 Dec.	314 39 44	- 7					- 8
Byrd	2 Dec.	25 14 09	+14	25 14 23	26 38 45	+1 24 22	+1 29 28	-292
	2 Dec.	25 13 59	+24	49				-282
	3 Dec.	25 14 48	-25	514				-331
	3 Dec.	25 14 35	-12					-318

TABLE 7.8 SUMMARY OF RESULTS

Gyro: S/N 2888

Method: Transit

Station	Date (1967)	Gyro Azimuth	v	Mean, Range, Var.	Astro. Azimuth	$Astro^E$-Gyro	Calculated E	V
Sydney	9 Nov.	66°29'43"	- 7"	66°29' 36"	67°41'45"	+1°12'09"	+1°12'08"	- 6"
	10 Nov.	66 29 32	+ 4	33				+ 5
	10 Nov.	66 29 21	+15	106				+ 16
	20 Dec.	66 29 34	+ 2					+ 3
	20 Dec.	66 29 54	-18					- 17
	24 Dec.	66 29 32	+ 4					+ 5
	24 Dec.	66 29 35	+ 1					+ 2
Christchurch	17 Nov.	146 51 27	-12	146 51 15	148 03 56	+1 12 41	+1 12 43	- 14
	17 Nov.	146 51 07	+ 8	29				+ 6
	11 Dec.	146 50 58	+17	140				+ 15
	12 Dec.	146 51 21	- 6					- 8
	12 Dec.	146 51 20	- 5					- 7
Hallett	26 Nov.	51 09 17	+ 1	51 09 18	52 28 20	+1 19 02	+1 19 00	+ 3
	26 Nov.	51 09 41	-23	41				- 21
	27 Nov.	51 09 16	+ 2	286				+ 4
	27 Nov.	51 09 00	+18					+ 20
McMurdo	24 Nov.	314 41 58	-16	314 41 42	316 05 28	+1 23 46	+1 23 47	- 17
	24 Nov.	314 41 23	+19	1 17				+ 18
	25 Nov.	314 41 06	+36	1,198				+ 35
	6 Dec.	314 42 23	-41					- 42
Byrd	2 Dec.	25 16 26	-17	25 16 09	26 38 45	+1 22 36	+1 27 10	-291
	2 Dec.	25 15 53	+16	33				-258
	3 Dec.	25 15 56	+13	294				-261
	3 Dec.	25 16 22	-13					-287

TABLE 7.9 SUMMARY OF RESULTS

Gyro: S/N 2888

Method: Turning Point & Transit

Station	Date (1967)	Gyro Azimuth	v	Mean, Range, Var.	Astro. Azimuth	$Astro^{E}$-Gyro	Calculated E	V
Sydney	9 Nov.	66°29'03"	- 1"	66°29' 02"	67°41'45"	+1°12'43"	+1°12'43"	- 1"
	10 Nov.	66 28 56	+ 6	30				+ 6
	10 Nov.	66 28 49	+13	86				+13
	20 Dec.	66 29 04	- 2					- 2
	20 Dec.	66 29 19	-17					-17
	24 Dec.	66 29 00	+ 2					+ 2
	24 Dec.	66 28 58	+ 4					+ 4
Christchurch	17 Nov.	146 50 40	- 2	146 50 38	148 03 56	+1 13 18	+1 13 19	- 3
	17 Nov.	146 50 29	+ 9	19				+ 8
	11 Dec.	146 50 34	+ 4	51				+ 3
	12 Dec.	146 50 48	-10					-11
	12 Dec.	146 50 40	- 2					- 3
Hallett	26 Nov.	51 08 35	- 7	51 08 28	52 28 20	+1 19 52	+1 19 51	- 6
	26 Nov.	51 08 52	-24	41				-23
	27 Nov.	51 08 11	+17	390				+18
	27 Nov.	51 08 12	+16					+17
McMurdo	24 Nov.	314 40 58	-18	314 40 40	316 05 28	+1 24 48	+1 24 49	-19
	24 Nov.	314 40 29	+11	56				+10
	25 Nov.	314 40 08	+32	682				+31
	6 Dec.	314 41 04	-24					-25
Byrd	2 Dec.	25 15 18	- 2	25 15 16	26 38 45	+1 23 29	+1 28 19	-292
	2 Dec.	25 14 56	+20	32				-270
	3 Dec.	25 15 22	- 6	195				-296
	3 Dec.	25 15 28	-12					-302

It will be noticed that in general the variance of an individual determination increases as the latitude increases (the stations are arranged in order of increasing latitude). If the variances for the turning point and transit method for one instrument are compared then it will be seen that there is little to choose between these methods. For gyro-attachment Number 3243, the combined method does not give any improvement but for Number 2888 there is a decided improvement for this method. It will also be seen that the variance for gyro-attachment Number 2888 is lower than for Number 3243.

The results in the tables also show that the value of E is not constant for each instrument but changes for both method and station. The reason for the difference in E between each method is due to the fact that the vee slot is used as the observing mark in the turning point method, whilst in the modified transit method the observations are made entirely on the auxiliary scale lines. For both instruments which were used there is a slight slope of the gyro-mark, so that the vee slot and zero of the scale do not give the same azimuth index even though they may lie vertically above one another. The causes of the change in E with changes of the station position are discussed later.

It can be assumed 'a priori' that the variance should exhibit little change in low latitudes but change rapidly in high latitudes and approach infinity near the poles, because the useful torque ratio is a function of the secant of the latitude. A simple relationship which approximates the observed variances and satisfies the 'a Priori' characteristics is of the form

$$\text{Variance} = A \sec \phi - B$$

where A and B are constants. The results of fitting such a curve of the above form to the observed variances is shown in Figure 7.5 on Page 224. The mean of

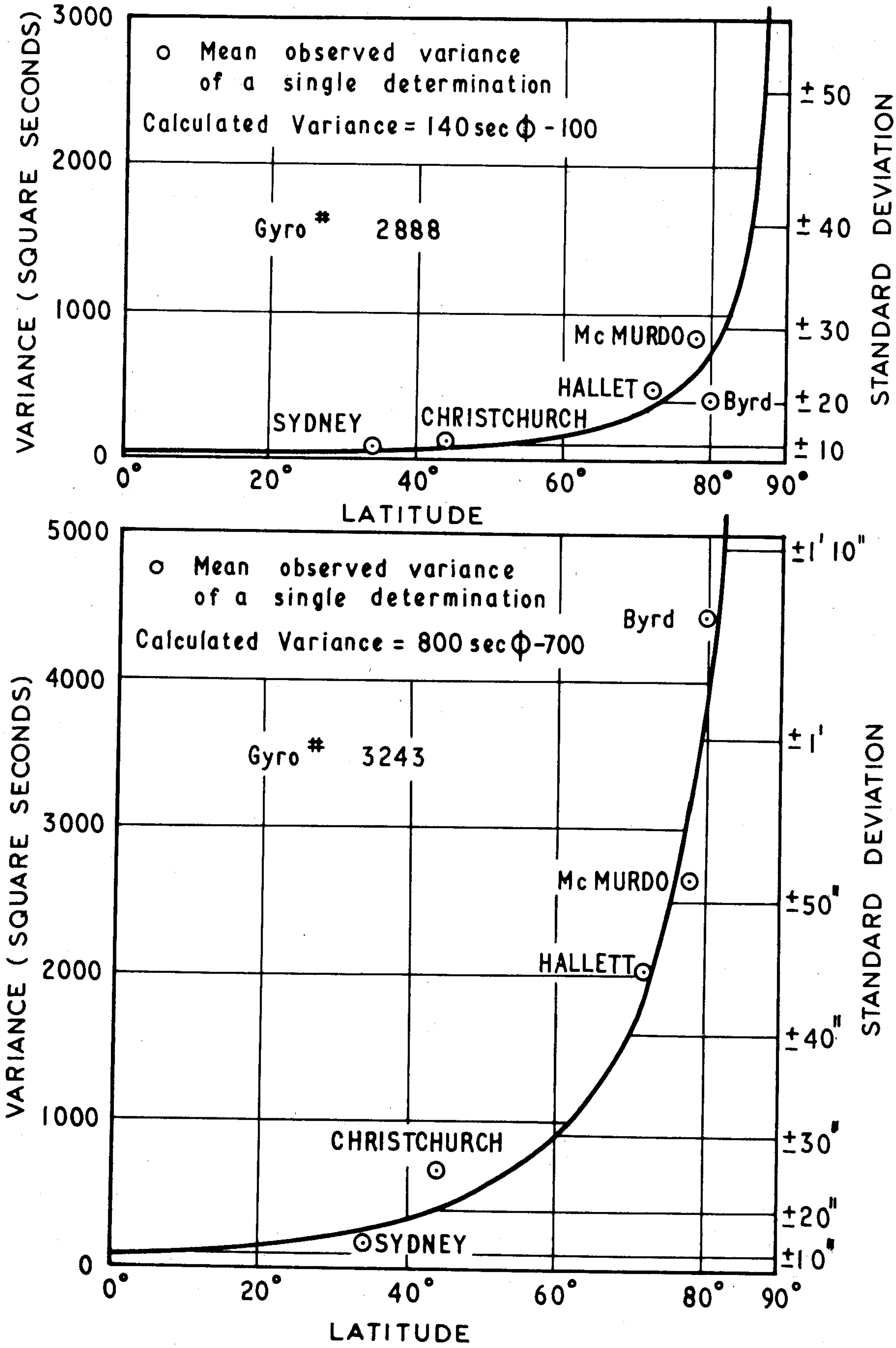


FIG. 7.5: OBSERVED VARIANCES

the observed variance of the turning point and transit methods has been used for each point in the curve fitting. The two fitted relationships, viz.

$$Var._{2888} = 140 \sec. \phi - 100$$

$$Var._{3243} = 800 \sec. \phi - 700$$

show that there is a remarkable difference in performance between the two instruments, e.g. at the equator the standard deviation of a single determination for gyro Number 2888 is ± 7" and for Number 3243 ± 10" but at latitude 75^{o} the standard deviations are ± 21" and ± 49" respectively.

The external accuracy of the results must be judged after the gyro constant, E, has been applied to each gyro-azimuth observation. It has been noted before that E changes with the station position. The cause of this change may be attributed to an unknown couple acting about the vertical axis of the suspended gyroscope. A couple of this nature would be caused by either or both of the following:-

(a) A twisting moment of the tape caused by a non-central rest position of the tape as indicated on the auxiliary scale.

(b) A mass imbalance imparting a horizontal couple at right angles to the gyro spin axis.

The variation with latitude of the effect of both of these couples on the direction of gyro indicated North will be identical. There is no direct means of measuring the mass imbalance but the tape zero position can be estimated by taking readings on the auxiliary scale.

The results of observations for tape zero position are shown and discussed in detail in Chapter 6. In brief the tape zero position for gyro

Number 3243 was approximately + 0.4 divisions and for Number 2888 + 0.2 divisions as deduced from observations made at all stations. A deviation from zero of this magnitude would not account for the observed changes in E, in fact the tape zero position would have to be of the opposite sign and of greater magnitude to account for this change. Because there is no direct means of differentiating exactly between the causes of the effect it will be assumed that the change in E is caused entirely and in the first instance by a constant non-central rest position of the tape.

In order to understand the reasons for the change in E at each station the angular relationships between the principal axes of the gyroscope and theodolite are shown in the plan in Diagrams A and B in Figure 7.6 on Page 227. Referring to Diagram A, it will be assumed that the suspension tape lies in a torque free position and that there is an angular difference $\Delta\ \alpha$ between the gyro optical axis, defined by the projected gyro-mark on the auxiliary scale, and the zero of the auxiliary scale (or vee slot). If we are dealing with an ideal system, $\Delta\ \alpha$ can be determined by observing (with the gyro in a non-spinning state) the extreme positions of oscillation of the gyro-mark on the auxiliary scale and the mean of these positions would give $\Delta\ \alpha$. This determination can be made with the instrument lying in an arbitary orientation. Note that there is a constant angular relationship β, between the gyro optical and spin axes, γ between the gyro spin axis and the upper and lower tape axes, and E_o between the gyro spin axis and the telescope axis (ignoring collimation in azimuth). These mutual relationships are shown in diagram A when the spin axis is aligned North-South, i.e. when the gyro is in the spinning state. Thus we

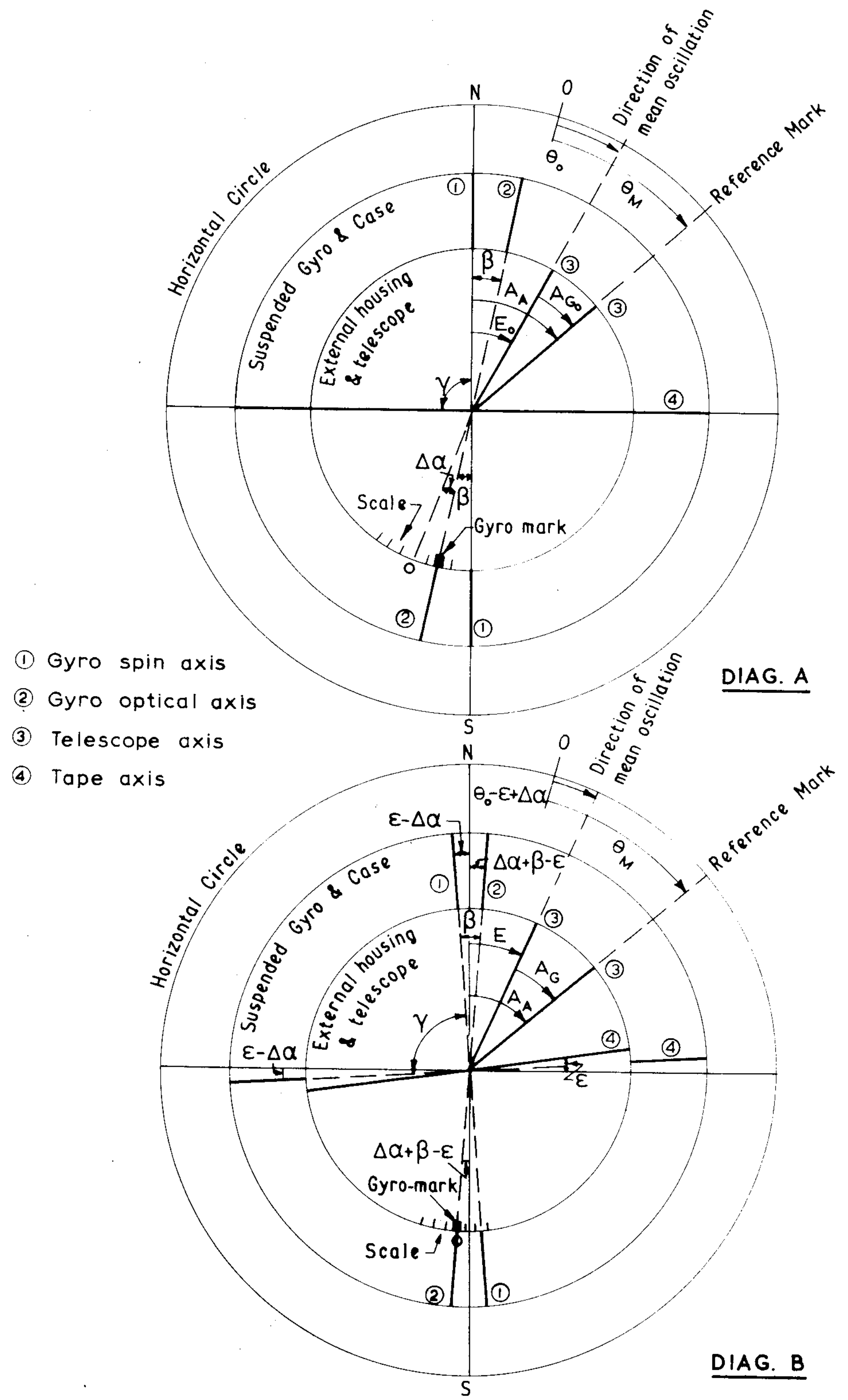


FIG. 7.6: PRINCIPAL AXES OF THE GYROSCOPE

see that the azimuth of the Reference Mark is given by

$$A_A = A_{G_o} + E_o$$

and therefore

$$A_A = \theta_M - \theta_o + E_o$$

where θ_M and θ_o are readings of the horizontal circle.

Referring to Diagram B. If the equilibrium position of the spinning gyro is found when the gyro-mark is centred in the vee slot then the external housing must be rotated by an angle ε, but the gyro spin axis will be rotated by an angle $\varepsilon - \Delta\alpha$. It has been shown in Appendix IV that

$$\varepsilon - \Delta\alpha = \frac{m \mu \Delta \alpha}{J \omega \cos \phi} = K \Delta \alpha$$

bur from Diagram B we see that

$$A_A = A_G + E$$

$$A_A = A_G + E_0 - \varepsilon + \Delta \alpha$$

and $$A_A = \theta_M - (\theta_o - \varepsilon + \Delta \alpha) + E_o - \varepsilon + \Delta \alpha$$

Combining this equation and the previous equation for $\varepsilon - \Delta \alpha$ we obtain

$$A_A = \theta_M - (\theta_o - \varepsilon + \Delta \alpha) + E_o - K \Delta \alpha$$

where θ_M and $(\theta_o - \varepsilon + \Delta \alpha)$ are readings of the horizontal circle. If we adopt the sign convention used on the auxiliary scale of the GAK, i.e. scale readings to the left positive and on the right negative then

$$A_A = \theta_M - (\theta_o - \varepsilon + \Delta \alpha) + E_o + K \Delta \alpha$$

Thus from the aforegoing theory we will define the gyro constant E as

$$E = E_o + K \Delta \alpha$$

where E_o is the angle between the direction corresponding to the mean of the oscillations of the spinning gyro and the direction of North on the condition that the tape lies in a torque free position in the direction of the mean of the oscillations.

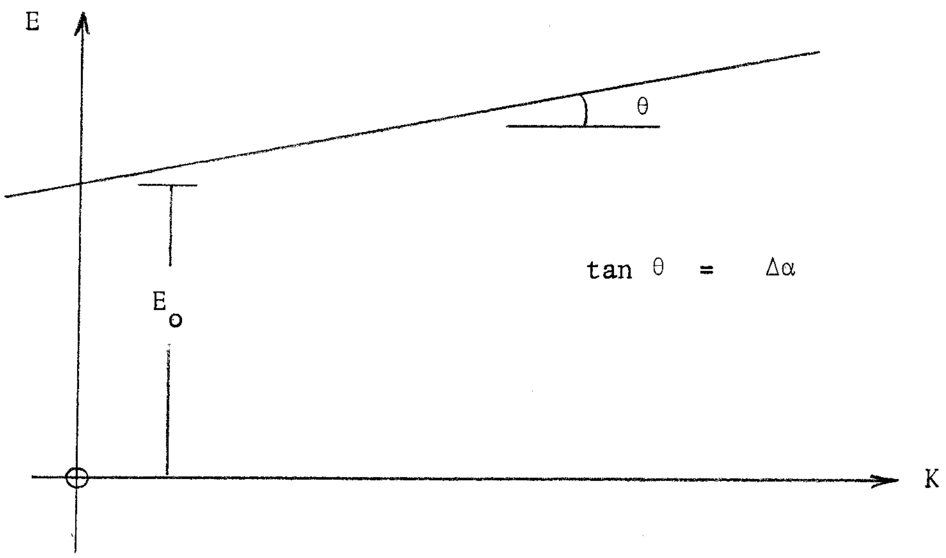
To determine E_o and $\Delta\alpha$, the unknown parameters from the observations, we can write the previous equation in the form of a correction equation as follows

$$E_i + v_i = E_o + K_i \Delta\alpha$$

where E_i has been measured and has an associated correction v_i, and K_i is a calculated quantity considered to be error free. Thus we assume that the n pairs of quantities E_i and K_i satisfy the equation

$$E_i = E_o + K_i \Delta\alpha$$

The problem is represented graphically in the accompanying figure.



If an observation has a variance σ_i^2 then the weight of the preceding correction equation will be $^1/\sigma_i^2$ and thus the reduced parametric equations will be given by

$$\frac{E_o}{\sigma_i} + \frac{K_i \ \Delta\alpha}{\sigma_i} - \frac{E_i}{\sigma_i} = 0$$

the normal equations by

$$\left[\frac{1}{\sigma_i^2}\right] E_o + \left[\frac{K_i}{\sigma_i^2}\right] \Delta\alpha - \left[\frac{E_i}{\sigma_i^2}\right] = 0$$

$$\left[\frac{K_i^2}{\sigma_i^2}\right] \Delta\alpha - \left[\frac{E_i K_i}{\sigma_i^2}\right] = 0$$

and the least squares estimates of E_o and $\Delta\alpha$ by

$$E_o = \frac{\left[\frac{K_i}{\sigma_i^2}\right]\left[\frac{E_i K_i}{\sigma_i^2}\right] - \left[\frac{E_i}{\sigma_i^2}\right]\left[\frac{K_i}{\sigma_i^2}\right]}{\left[\frac{K_i}{\sigma_i^2}\right]^2 - \left[\frac{1}{\sigma_i^2}\right]\left[\frac{K_i^2}{\sigma_i^2}\right]}$$

and $$\Delta\alpha = \frac{\left[\frac{K_i}{\sigma_i^2}\right]\left[\frac{E_i}{\sigma_i^2}\right] - \left[\frac{1}{\sigma_i^2}\right]\left[\frac{E_i K_i}{\sigma_i^2}\right]}{\left[\frac{K_i}{\sigma_i^2}\right] - \left[\frac{1}{\sigma_i^2}\right]\left[\frac{K_i^2}{\sigma_i^2}\right]}$$

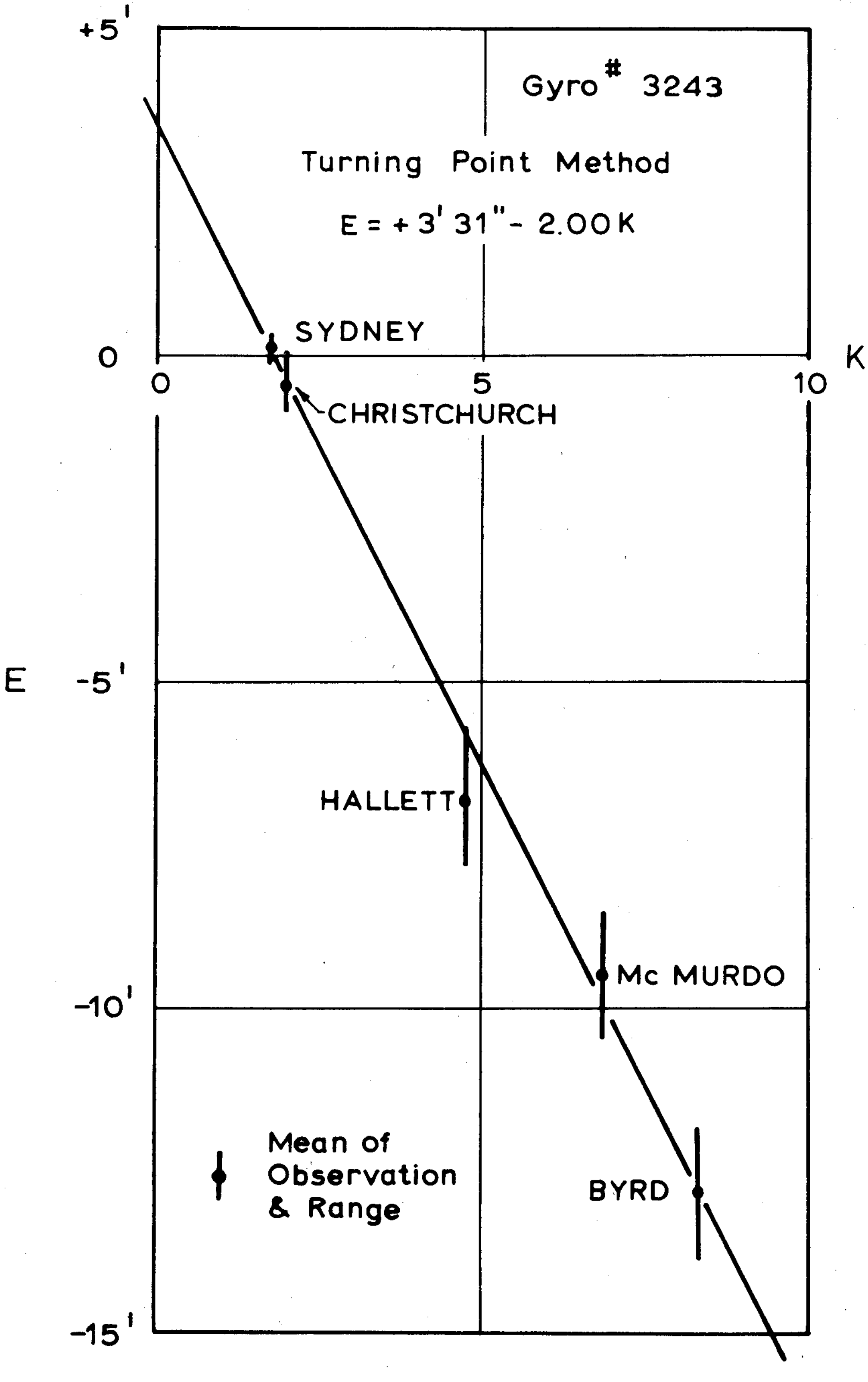
Note: A general treatment of this problem has been given by Tienstra (1956).

Using the values of the variance obtained from the previous curve fittings and the observed values of E, solutions for E_o and $\Delta\alpha$ were found for each method and instrument using a computer programme designed for regression analysis by Mr. J.C. Trinder of the University of New South Wales. Part of this general computer programme uses identical expressions for the solution of E_o and $\Delta\ \alpha$ as given before. The results of these calculations are given in Tables 7.4 to 7.9 on Pages 217 to 222 and are also shown in graphical form in Figures 7.7 to 7.12 on pages 232 to 237 , and support the previous assumption that the cause of the change in E was probably due mainly to a mass imbalance.

For gyro-attachment Number 3243 the line fit departs from the mean observed position by up to nearly one minute of arc and thus the final corrections, V, given by

$$V \quad = \quad \text{Astro Azimuth - Observed Gyro-azimuth} \quad - \quad \text{Calculated E}$$

are larger than the corrections, v. Variances based on the corrections V have not been calculated because the causes of the augmentation of v are not known. For gyro-attachment Number 2888 the line fit is excellent for all stations except Byrd. The departure from linearity is not greater than 2" if the observations at Byrd are not considered and therefore the internal and external accuracies are the same for the four lower latitude stations for this instrument. The cause of the large final corrections, amounting to about 5' at Byrd station is not known but may be due to the uncertainty in the value of the astronomical azimuth. At all stations, with the exception of Byrd, astronomical observations for latitude and longitude were made for the reduction of the astronomical azimuth observations. It was fortunate that the weather and radio reception were clear enough for astronomical observations in the limited time that was available at each station outside Australia. Without this astronomical information the results would have been of limited value.



+5'
Gyro # 3243
Turning Point Method
E = + 3' 31" - 2.00 K
SYDNEY
0
K
0
5
10
CHRISTCHURCH
E
-5'
HALLETT
Mc MURDO
-10'
Mean of
Observation
& Range
BYRD
-15'

FIG. 7.7: GYRO E

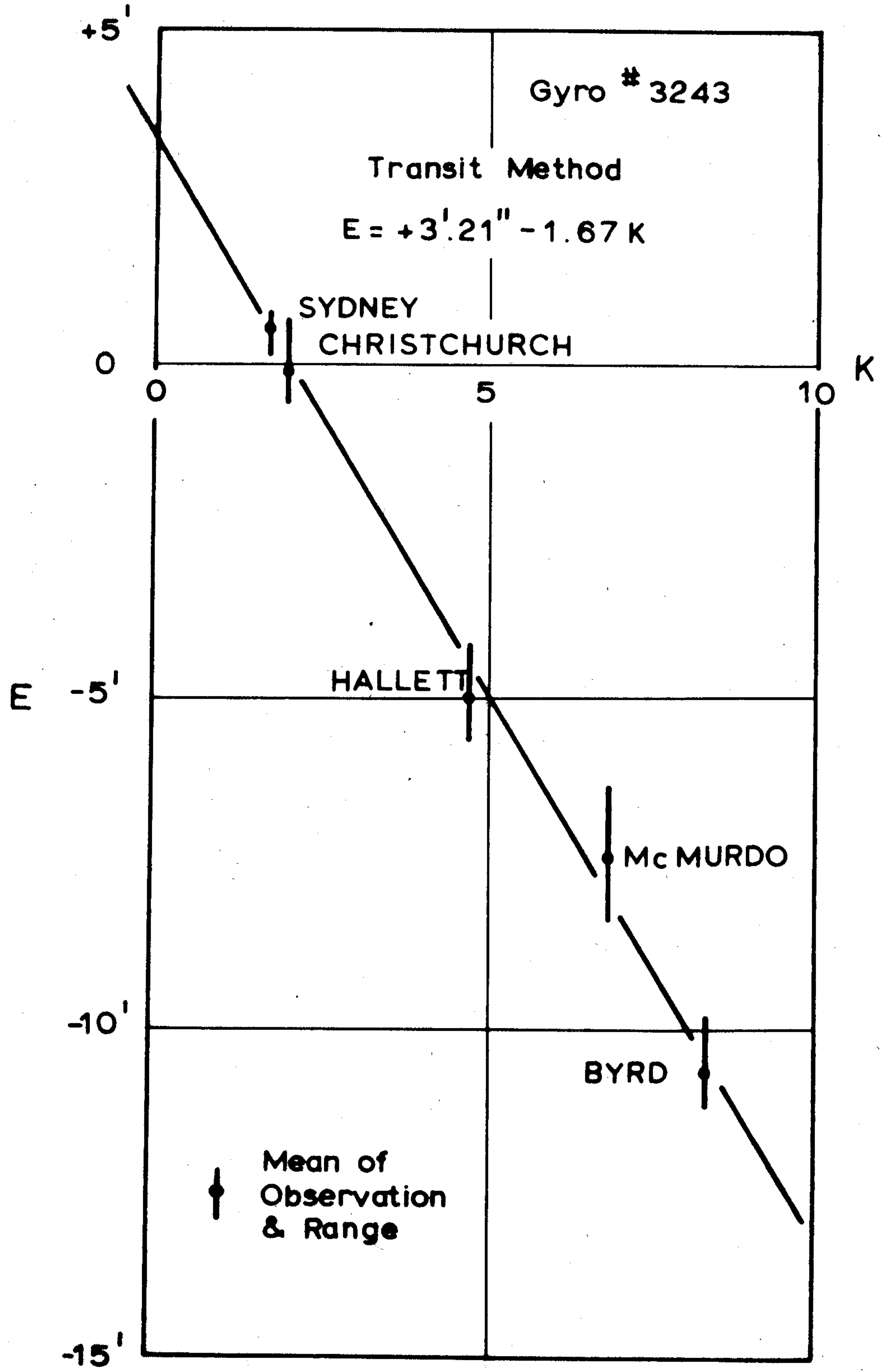

+5'
Gyro # 3243
Transit Method
E = +3'.21" - 1.67 K
SYDNEY
CHRISTCHURCH
0
K
0
5
10
E
-5'
HALLETT
Mc MURDO
-10'
BYRD
Mean of
Observation
& Range
-15'

FIG. 7. 8 : GYRO E

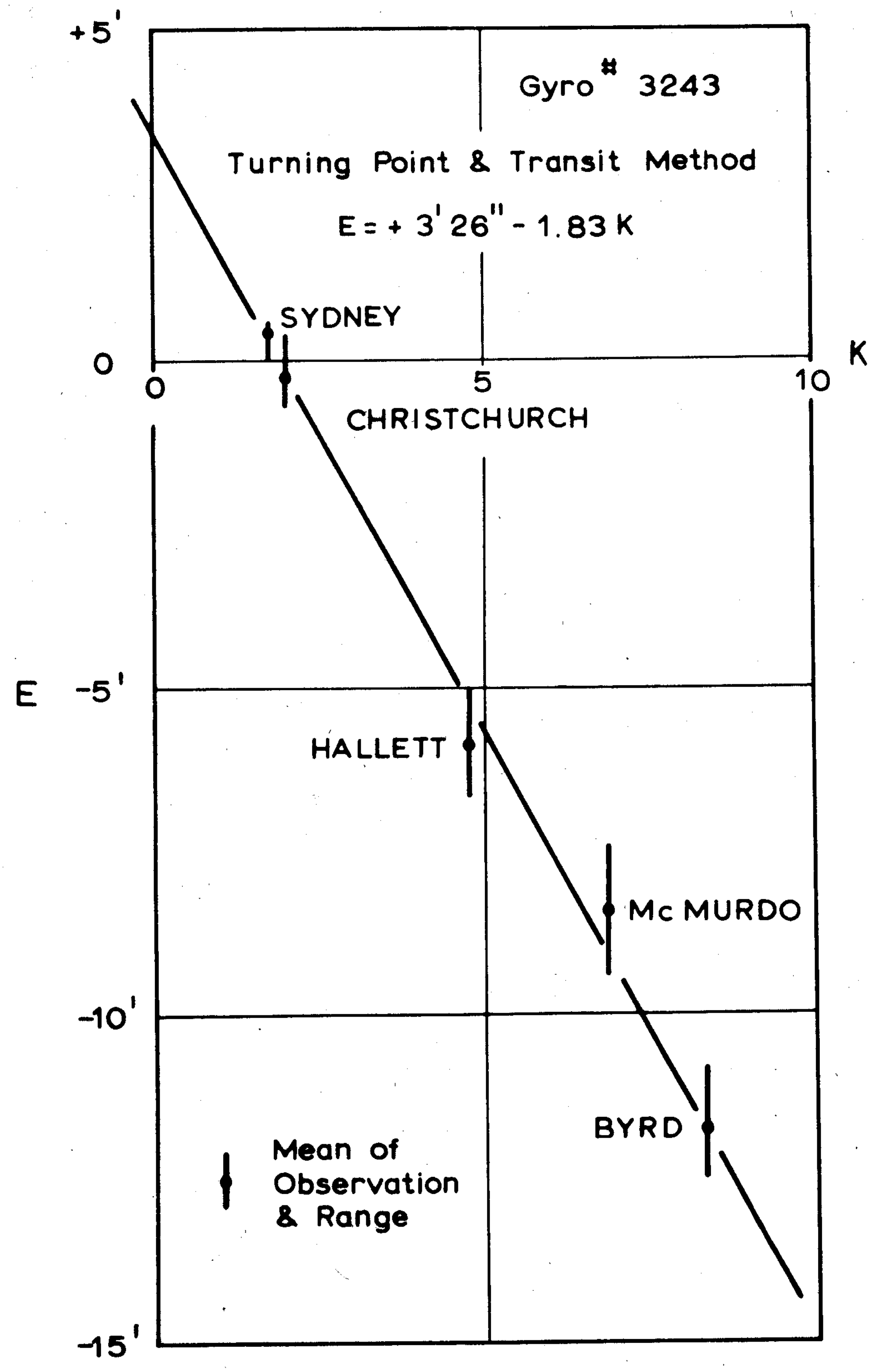

+5'
Gyro # 3243
Turning Point & Transit Method
E = + 3' 26" - 1.83 K
SYDNEY
0
0
5
10
K
CHRISTCHURCH
E
-5'
HALLETT
Mc MURDO
-10'
BYRD
Mean of Observation & Range
-15'

FIG. 7.9 : GYRO E

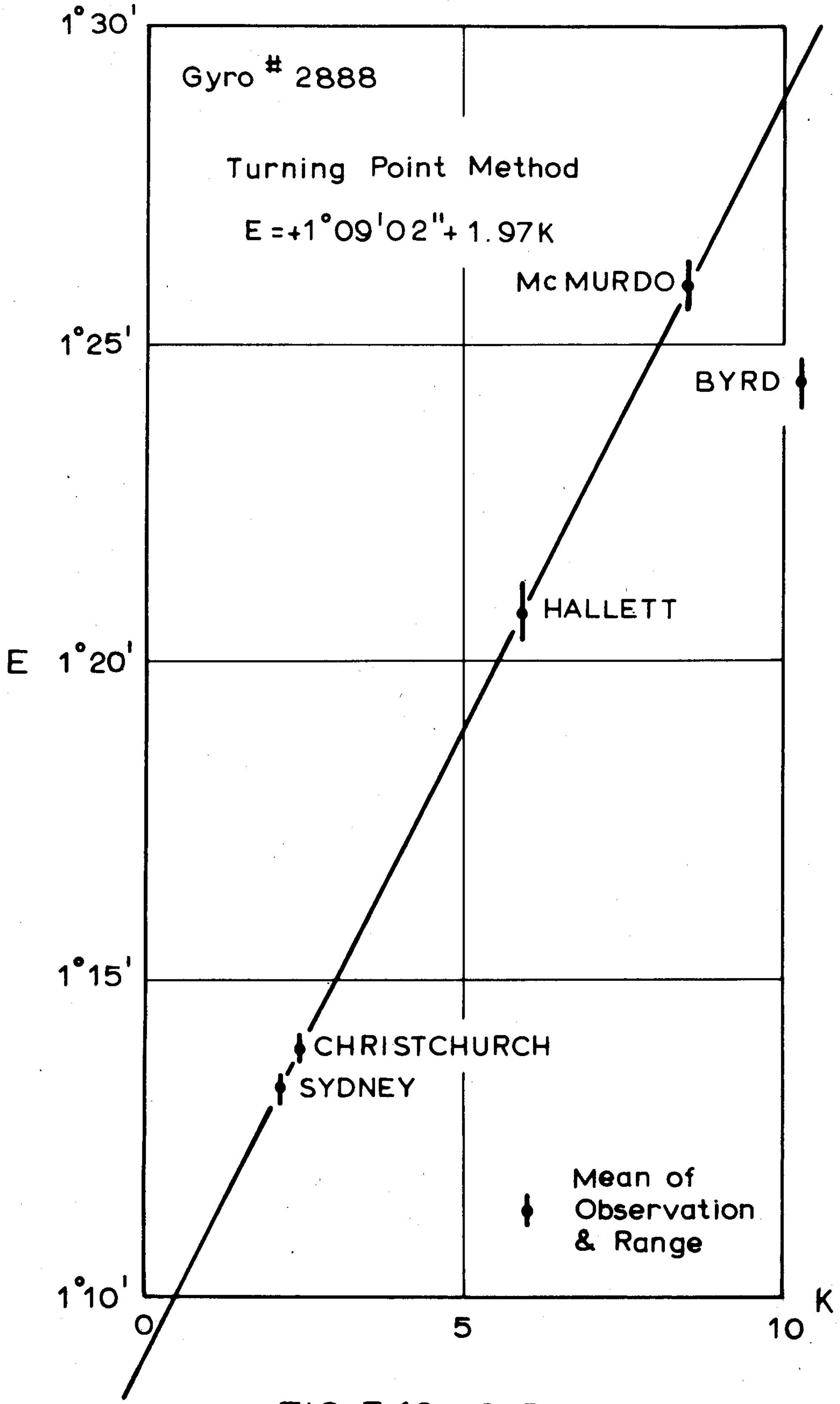


FIG. 7.10: GYRO E

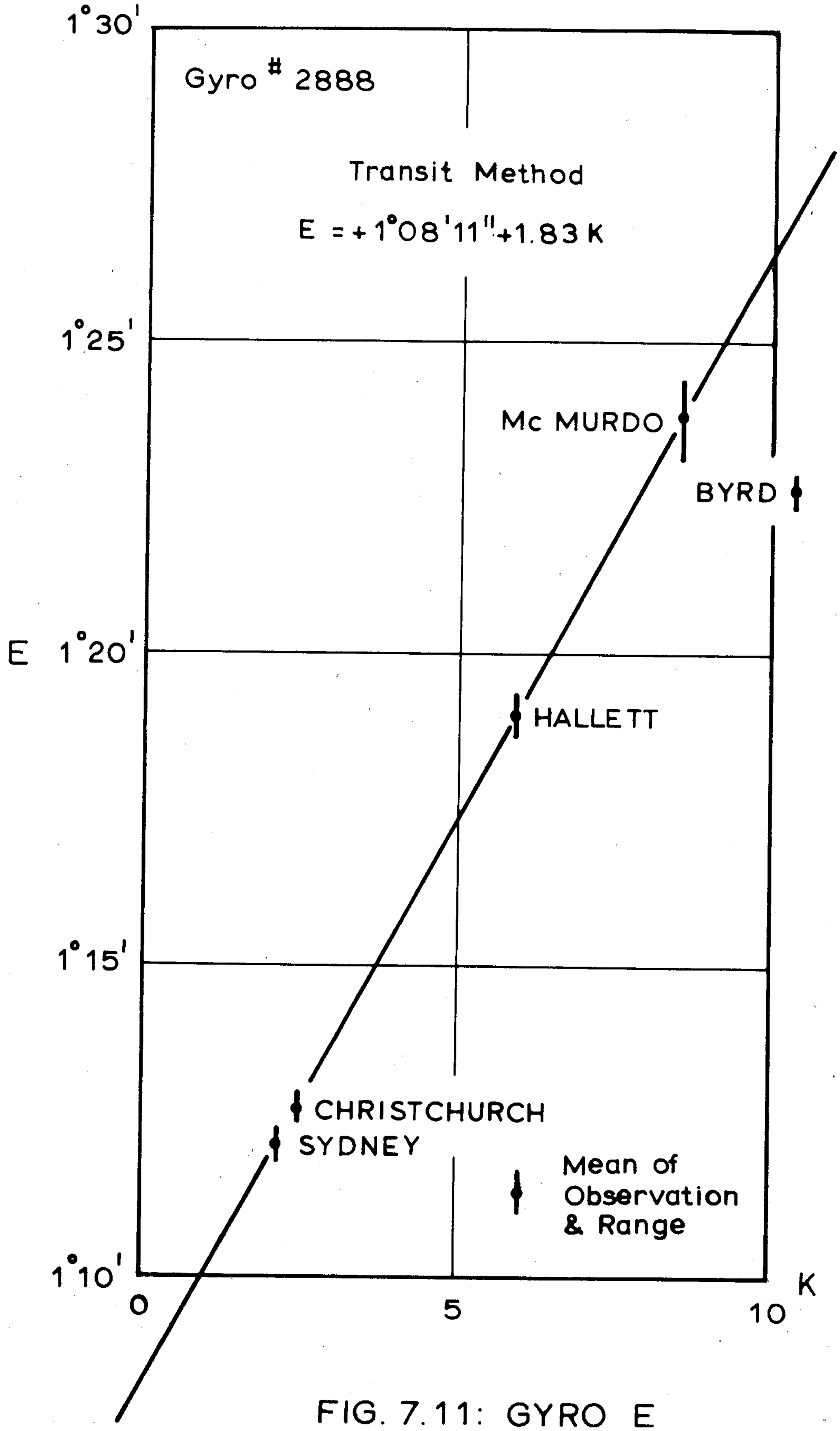


FIG. 7.11: GYRO E

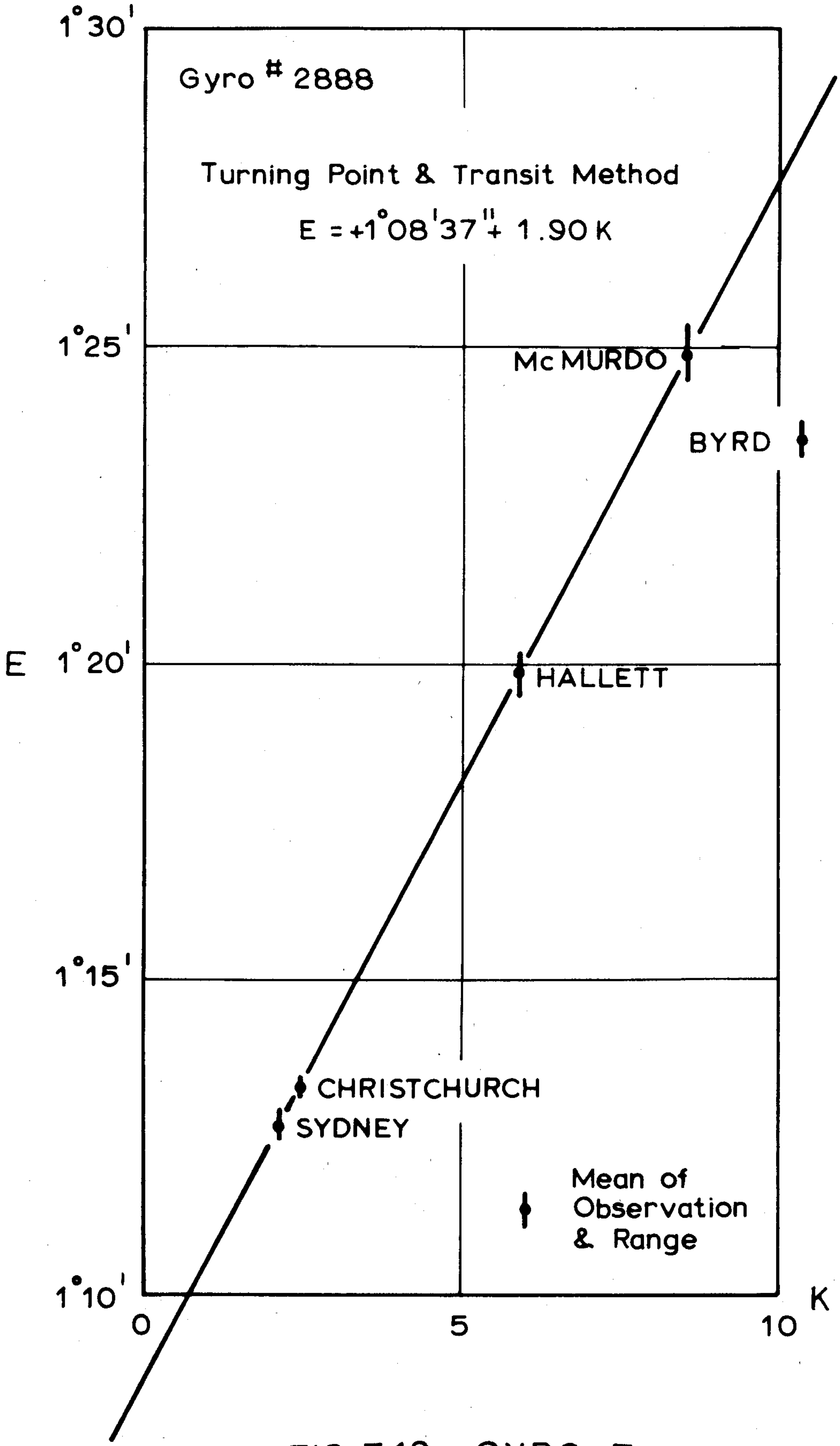


FIG. 7.12: GYRO E

In order to guard against mistakes, redundant observations were always included in the astronomical programmes. At Byrd station only sun observations for azimuth were made. The values of latitude and longitude which were used for the reductions were derived from eccentric measurements made to a station of the U.S. Geological Survey. The co-ordinates for this station, which were supplied for these azimuth calculations were derived from astronomical observations made in 1967-68 austral season, but our recorded description for this station is slightly different from that recorded by the above authority and thus the results from Byrd station must be considered doubtful. If the observed astronomical azimuth at Byrd station is in error by 5' then the results of observations made with gyro Number 3243 would be adversely affected by this amount. It should be noted that the astronomical azimuths used for the derivation of E are not entirely error free. At the low latitude stations, azimuth was derived from observations made to a close circum-polar star and are considered to have an estimated standard deviation of ± 2". For the high latitude stations the azimuth results were based on sun and daylight star observations which gave an estimated standard deviation of about ± 5"-10".

It has been assumed in the previous analysis that the tape zero position was constant for the whole period of observations. However, there is considerable uncertainty in determining this tape zero position from auxiliary scale readings of the non-spinning gyro which is mainly due to residual magnetism in the gyro motor. Because of this uncertainty the stability of the tape zero position was estimated from the difference between the mean gyro-azimuths made at Sydney and Christchurch on the outward and return stages at the beginning and end of the journey. These differences will also contain any variation in the constant E_o. The results of these differences are given below:-

Difference. Outward minus return.

Station	Gyro			
	No. 3243		No. 2888	
	Turning Point	Transit	Turning Point	Transit
Sydney	+ 20"	+ 21"	- 12"	- 7"
Christchurch	+ 30	+ 35	- 17	+ 4

Considering the long distances travelled, with frequent loading and unloading on to aircraft, trucks, sledges etc., these differences are small and indicate that both instruments were remarkably stable. No attempt has been made to apportion these differences amongst the other observations.

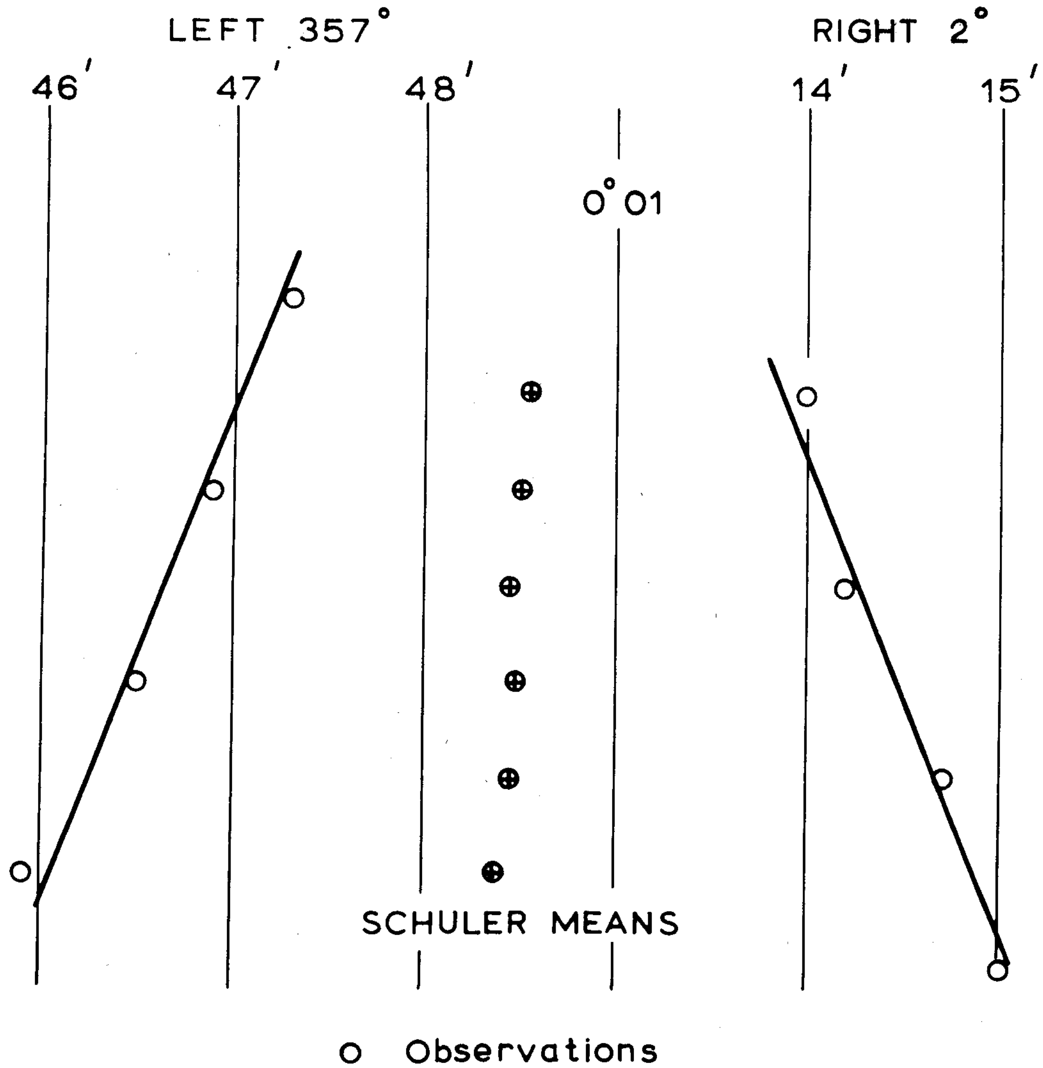
The stability of the tape zero position and the value of E_o are of particular importance if the instrument is to be used for absolute azimuth determinations. Bewanger (1964) reports a change in E of between 2.5' to 3' in a three month period and Peters (1969) also reports a change of about 1' over a month; changes of this magnitude must limit the utility of the instrument.

If E is to remain constant for the GAK gyro-theodolite, the mounting bridge and associated centering pins on the theodolite must always maintain the same position with respect to the line of sight of the telescope. The vee shaped grooves at the ends of the three supporting columns on the gyro-attachment which fit over the centering pins on the mounting bridge must also maintain their position. The construction of this forced centering arrangement is positive and robust but no provision is made for internal checking. For the M.O.M. GiC series, E can be checked by observing an index mark on the gyro-attachment directly by means of a roof prism in front of the telescope objective.

7.7 CONCLUSION.

The two gyro-theodolites which were used for these experiments were not specially selected by this author or by the maker. They were not available for testing and calibration until about two weeks before the departure for New Zealand; gyro Number 3243 was purchased from stock and gyro Number 2888 was borrowed from the University of New South Wales, School of Mining Engineering. It is to the credit of the manufacturer that the only fault which developed during the 98 gyro-azimuth determinations was the failure of a 12 volt light bulb which was replaced in a matter of seconds. A feature of gyro Number 2888 which gave cause for concern during the initial calibration observations in Sydney was the increase in amplitude of the oscillation during the turning point method of observation, which will be seen from the oscillation graph of a typical observation in Figure 7.13 on Page 241. The accompanying solution gives a negative damping factor of 12" per half period for an amplitude of 2° 13'. Negative damping was also present in the observations at all other stations. However, apart from this unusual feature, both gyros functioned quite normally and no great difficulties were encountered in low temperature operations.

The tests which were performed can only be considered as a pilot series. Because of quality variation, it is impossible to deduce the performance characteristics of an instrument which is manufactured by a serial production process by testing one or two instruments. It is only after testing a representative sample group of the product that a reliable estimate of the



Gyro:# 2888

Date: 24th Dec.1967

Line: 707 to B10

FIG. 7.13: OSCILLATION GRAPH

Gyro: No. 2888 Date 24 Dec. 1967

Line: 707 to Bio.

Observations (1)	$\sum_1$	$\sum_2$	$\pm(2y_o-\Sigma_1)$	(2)	$\pm\{(2)-(1)\}$	v	v
357°47'18"				y_o = 27.7"	2°13'09.7"	-2.7"	
	1'18"		-22.6"				
2 14 00		2'12"		y_o-a = 39.8	2 13 20.2		-7.8"
	54		- 1.4				
357 46 54		2 00		y_o+2a= 3.5	2 13 09.5	-2.9	
	1 06		-10.6				
2 14 12		1 48		y_o-3a= 1'04.0	2 13 08.0		+4.4
	42		-13.4				
357 46 30		1 54		y_o+4a= -20.7	2 13 09.3	-3.1	
	1 12		-16.6				
2 14 42		1 48		y_o-5a= 1 28.2	2 13 13.8		-1.4
	36		-19.4				
357 45 54		1 30		y_o+6a= -44.9	2 13 21.1	+8.7	
	54		+ 1.4				
2 15 00				y_o-7a= 1 52.4	2 13 07.6		+4.8

Mean = B = 2 13 12.4 Σ=0 Σ=0

$$y_o = \tfrac{1}{4}\{1'30'' + \frac{7(42+0)-(30+18)+4(18+24)''}{20}\}$$

y_o = 27.7" $2y_o$ = 55.4"

$$a = -\frac{1}{84}\{7(22.6-1.4)+12(1.4+19.4)+15(10.6+16.6)+16x13.4\}''$$

a = -12.1"

performance of the instrument can be established. Nevertheless, the results of these experiments indicate that it would be well worthwhile examining more instruments of the type which was tested and also those of the other manufacturers. Although both of the instruments which were used gave results which are within the manufacturer's specifications up to medium latitudes, it would seem that for successful high latitude operation, instruments should be selected which perform well in low latitudes. With current improvements in gyro technology it is conceivable that the gyro-theodolite could supplant azimuth determinations by astronomical means in the polar regions in the not too distant future.

8 ASTRONOMICAL AZIMUTH

8.1 GENERAL CONSIDERATIONS

In order to check the stability of the gyro-theodolite constant E, it is necessary to make observations periodically on a fixed terrestrial line, which will be referred to as the calibration line. If the instrument is to be used in one locality and our only concern is with the relative transfer of azimuth, then all that is required is that the terminal points of the calibration line be well defined and stable and either that the line be of sufficient length to obviate any need for refined centering of the instrument and target, or if the line is short some means must be provided for the constrained centering of the instrument and target. If gyro-theodolite observations are to be made in several localities then calibration lines should be provided in each locality. An extra check on the stability of the constant E is available for each locality if the azimuth of the calibration line is known. This azimuth data is readily available when geodetic control has been established, especially when control stations are closely spaced. Most of the work which has been done with the gyro-theodolites of this University has been in areas where geodetic control is sparse.

The azimuths of calibration lines can be conveniently determined by astronomical means, the precision of which should be compatible with the type of gyro-theodolite used and the class of work for which azimuths are required. The methods of observation for astronomical azimuth which are to be considered have been chosen for use with a single second glass arc theodolite; a type of instrument which is in general use. The notation used in the following sections will be that proposed by Freislich (1953) and Lee (1953) and given in detail by Bennett and Freislich (1968).

8.2 LOW LATITUDES

According to Roelofs (1950), in low latitudes ($|\phi|<15^o$) it is not possible to observe Polaris (m = 2.1) and σ Octantis (m = 5.5) because of the lack of transparency in the lowest layers of the atmosphere and so an alternative method of azimuth determination is needed. Roelofs has shown that the estimated variance of the mean azimuth derived from four observations on each star of a pair is given by

$$m^2_{\bar{A}} = \frac{1}{8} F\, m^2_T + \frac{1}{8} (m^2_1 + m^2_2) \qquad *$$

where
$$F = \cos^2\phi(\tan\phi - \cos A_1 \tan h_1)^2 + \nu(2 \tan^2\phi + \tan^2 h_1 - 2 \tan\phi \cos A_1 \tan h_1)$$

and
$$\nu = \frac{m^2_1 + m^2_3}{m^2_T}$$

Furthermore Roelofs has shown that F is a minimum under the conditions,

$$A_1 = 0^o \qquad (1)$$

$$A_1 = 180^o \qquad (2)$$

$$\cos A_1 = (1+\nu\sec^2\phi)\tan\phi \cot h_1 \qquad (3)$$

and concludes that in low latitudes a star should be observed near the South (North if $\phi<0$) at an altitude of $h_2 = 15^o$, and a star near the North (South if $\phi<0$) at an altitude h_1 complying with

$$\tan h_1 = 2 \tan|\phi| + \tan 15^o$$

where $F = \nu \tan^2\phi + (\nu+\cos^2\phi)(\tan|\phi| + \tan 15^o)^2$

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* The symbols other than those specified by Bennett and Freislich (1968) are those by Roelofs in Chapters 5, 7, 8 and 10 of "Astronomy Applied to Land Surveying".

and $h_2 = 15^o$ is chosen as a minimum because of the poor atmospheric transparency, as stated before.

Roelofs dismisses an investigation of condition (3) on the grounds that observations made under this condition would be made out of the meridian which would have a disadvantage in the selection of stars (for condition (2) the stars are near meridian transit and selecting a star programme is a relatively simple matter). If the latitude and longitude were known and only azimuth needed to be obtained then the previous approach is justified, on the other hand we are often faced with the problem of determining latitude and longitude, the values of which are needed solely for the azimuth reduction. If this is the case then the latitude and longitude could be conveniently obtained from near meridian observations as with the azimuth determination i.e. circum-meridian altitudes for latitude and meridian transits for longitude. The latitude observations would present no problem but the longitude observations are best made at high altitude - a circumstance which is not convenient with a small theodolite because for accurate transverse levelling, a striding level is required and an "elbow" eyepiece (which reduces magnification) is needed to observe above about 50^o altitude. As an alternative if we consider extra-meridian observations for the longitude determination then it will be worthwhile examining condition (3) which can be satisfied in equatorial latitudes by observing stars near elongation, the locus of which is near the prime vertical.

If we adopt this latter approach then it will be seen that azimuth and longitude could be derived simultaneously from the same stars and thus only one predicted programme would be required. A compromise will be necessary because not all the conditions for minimising systematic and random errors can be met. Consider the observations to be made on a pair of stars near elongation. From the estimated variance of longitude position given by Roelofs as

$$M^2_{\bar{\lambda}\cos\phi} = \frac{1}{2n} m^2_T \cos^2\phi + \frac{\operatorname{cosec}^2 A}{2n} (m^2_4 + m^2_5) + (q^2_m + q'^2_m) \cos^2\phi \, m^2_{\Delta T_o}$$

it will be seen that only the errors of reading the altitude m_4, and vertical pointing m_5, are affected if the stars are not observed in the prime vertical. Note however, that an altitude of about 30^o or greater should be used in order to avoid anomalous refraction effects. The effect of a systematic error in latitude is given by

$$d\lambda = -\frac{\sec\phi}{2} (\cot A_1 + \cot A_2) \, d\phi$$

which will be minimised if $A_1 + A_2 \simeq 360^o$.

The estimated variance of azimuth will be given by

$$m^2_{\bar{A}} = \frac{1}{8} (m^2_1 + m^2_3) \tan^2 h + \frac{1}{8} (m^2_1 + m^2_2)$$

if $h_1 = h_2 = h$. It will be seen that by raising the altitude from an optimum of 15^o to 30^o that $m^2_{\bar{A}}$ is increased. The effect of a systematic error in latitude is given by

$$dA = \frac{\tan h}{2} (\sin A_1 + \sin A_2) \, d\phi$$

which will be minimised if $A_1 + A_2 = 360^{\circ}$, and for a systematic error in clock correction or longitude

$$dA = \frac{\cos\phi}{2}\{2\tan\phi - \tan h(\cos A_1 + \cos A_2)\}\, dt$$

which will be minimised at elongation.

If the pair of stars is chosen such that they are observed in positions symmetrical to the meridian then all of the previous systematic errors will be minimised.

Finally, the effect of this compromise solution on the precision of the longitude and azimuth determination must be examined. For the longitude determination, if we adopt Roelofs' estimate for a pair of stars observed in the prime vertical in latitude 15° with $M_T = 1''$, $n = 8$, $m_4 = 1.8''$, $N = 1$, $m_5 = 2.5''$, $q_m = q'_m = \frac{1}{2}$, $m_{\Delta T_o} = 0.45''$ then $M_{\bar{\lambda}\cos\phi} = 0.9''$. If a pair of stars is observed at elongation in the same latitude at 30° altitude then we find that $M_{\bar{\lambda}\cos\phi} = 1.0''$. For the azimuth the estimated variance for a pair of stars is given by Roelofs as $m_{\bar{A}} = 1.3''$ for $h = 15^{\circ}$ but for $h = 30^{\circ}$, $m_{\bar{A}} = 1.4''$. Thus the net effect of making a compromise observation on the precision of determination of longitude and azimuth in latitudes below 15° is negligible but still retains the advantage of minimising the effects of systematic errors. An additional practical advantage is that critical pointings in azimuth can be made to stars in the vicinity of elongation because the motion of the stars is sensibly vertical.

8.3 MEDIUM LATITUDES

In medium latitudes ($50^{o}>|\phi|>15^{o}$) it has been standard practice for many years to derive astronomical azimuth from observations to close circum-polar stars. For stations in the northern hemisphere α Ursae Minoris (Polaris) m = 2.1 with a polar distance of less than one degree is a convenient body for the purpose. In the southern hemisphere there is a dearth of bright stars in the vicinity of the South Pole but there is a star similarly situated as Polaris namely σ Octantis but which is considerably dimmer i.e. m = 5.5. Roelofs (1950) states that this star can only be picked up with a theodolite of type B (Wild T3 or equivalent) provided it is sufficiently high above the horizon. This previous statement is misleading as numerous surveyors can testify in such countries as Australia, New Zealand and South Africa where σ Octantis is regularly and successfully observed with a theodolite of type A (wild T2 or equivalent), in addition another star, B Octantis (m = 6.5) which is situated about 20' away from σ Octantis, is often observed to as a check but with this star, observing conditions must be clear.

A detailed digression concerning the advantages and an analysis of the random and systematic errors associated with observations to close-circum-polar stars is superfluous as this aspect is well treated in standard texts on field astronomy. However, in brief the two main advantages of observing to these stars are that (1) because of the small polar distance of these stars they appear to be virtually motionless and thus are ideal objects on which to observe for azimuth, and (2) systematic errors in latitude and longitude have little effect on the computed azimuth e.g. in latitude 40^{o} a change in longitude (or clock correction) of 1^{s} leads to a maximum azimuth change of 0.35" and a change of latitude of 10" gives rise to a maximum change of 0.1". Errors of these magnitudes are therefore of little consequence.

In the northern hemisphere, Polaris is a bright star and usually presents no difficulty in its identification. However, in the southern hemisphere we are not so fortunate in that the closest observable circum-polar stars are dim and therefore it is necessary to make instrumental settings to find the stars and then check their identity. Tables of computed altitude and azimuth of Polaris are readily available in most astronomical ephemerides but not for southern circum-polars. An investigation of ways and means of quickly solving for these quantities was made and a small planisphere (see sample in back cover) was constructed. The solution is performed entirely by graphical means and does not require auxiliary tables or a knowledge of sidereal time and its relationship with mean time. The planisphere is similar to a Polaris planisphere described previously by Bennett (1966).

Basically there are two problems to solve in finding the azimuth and altitude of a star. At first the hour angle is found from the standard time of observation, the date, the observer's longitude and the right ascension. Secondly the astronomical triangle is solved for the unknown elements of altitude and azimuth. A graphical solution of the second phase has been described before by Bennett (1963) and is also the basis of a finding diagram for daylight observations on Polaris prepared by Moppett and Blackie (1963).

For the first phase we need to rotate the finding diagram by approximately one degree per day to allow for the changing phase relationship of sidereal and mean solar time and also an index must be provided to allow for the change due to the longitude difference between the observer and the time zone meridian.

The azimuth and altitude obtained from the planisphere will have small errors due to the following assumptions made in the construction:-

$$A = 180^\circ + p \sin t \sec \phi$$

instead of $\sin A = - \cos \delta \sin t \sec h$

and $h = |\phi| + p \cos t$

instead of $\sin h = \sin \phi \sin \delta + \cos \phi \cos \delta \cos t$

where p is the South polar distance, considered positive. For latitudes up to 60° these errors are not serious and for a detailed analysis see reference Bennett (1966).

The planisphere has been constructed for the year 1970 but in a few years time the right ascension and declination will have changed, thus introducing further small errors. It is possible to construct a cursor to take into account these latter changes or alternatively to provide a table of corrections. However, to keep the solution as simple as possible it is proposed that the cursor be replaced every few years. If the planisphere is used solely for finding σ and B Octantis, then all of the errors which have been cited will be of little significance. A useful check on the identity of σ and B may be made by using the planisphere as follows. After the settings have been made, hold the planisphere with the Δh axis upside down, and then the relative position of the two stars in the field of view of an inverting telescope is shown by the relative position of the points of zero latitude on the cursor.

A simple approximate formula for calculating the azimuth of a southern circum-polar star is

$$A = 180^{\circ} + p \sin t \sec h$$

instead of $$\sin A = - \cos \delta \sin t \sec h$$

This formula is surprisingly accurate, see Bennett (1963). The value of the altitude used in the formula need not be known with high accuracy and can usually be obtained at the time of observation. If an altitude is not observed, then its value can be obtained from the planisphere. The error in azimuth resulting from an error in the altitude is given by:-

$$dA = \tan A \tan \phi \, dh$$

e.g. In latitude S40° the maximum azimuth of σ Octantis is 181° 10' and for dh = 1', dA = 1.0".

The planisphere has been used with a high degree of success with all astronomical observations made for establishing azimuth calibration lines for the gyro-theodolite. The planisphere has also proved popular with other surveying agencies and over one hundred of these are now in use in Australia and New Zealand.

Observations in medium latitudes to σ and B Octantis have been made for the purpose of providing check azimuths and calibration lines at the following stations:-

(1) University of N.S.W., Sydney, N.S.W. ($\phi = S34^{o}$)

(2) Kingsford Smith Airport,* Sydney, N.S.W., ($\phi = S34^{o}$)

(3) Broken Hill, N.S.W., ($\phi = S32^{o}$)

  (a) The New Broken Hill Consolidated Mine

  (b) The Broken Hill South Mine

  (c) The Broken Hill North Mine

(4) Aberfoyle Mine, Rossarden, Tasmania ($\phi = S42^{o}$)

(5) University of Canterbury, Christchurch, New Zealand ($\phi = S43^{o}$)

* At this station a calibration line was established for the checking of aircraft gyros in the avionics complex of QANTAS Airways.

A conservative estimate of the accuracy of azimuth determined from σ and B Octantis was ± 2". Errors of this magnitude are of little concern with the GAK 1/T16 gyro-theodolite.

8.4 HIGH LATITUDES.

Besides the problems of designing programmes for determining position and azimuth in high latitudes, astronomical observations in Antarctica during the summer months are further complicated by the condition of perpetual daylight. Therefore the only celestial bodies which are available for observation are those which are bright, namely the sun, moon, planets and the bright stars.

The moon is not an ideal object upon which to observe because -

(1) The right ascension and declination change rapidly and therefore time must be known precisely for the interpolation of these coordinates, notwithstanding the fact that time may not be an important consideration in the observation.

(2) The disc exhibits phase which may restrict observations at times near first and last quarter.

(3) The declination is confined to a range of about $\pm 30^{\circ}$ and it is only when the declination is southerly that the body is in a convenient position for observation in high South latitudes.

Thus the prediction, observation and reduction are complicated.

Observations to the planets are also complicated but to a lesser degree than the moon; declinations are confined to a small band along the ecliptic. The inferior planets are often too near the sun to be observed which leaves only three of the superior planets, namely Mars, Jupiter and Saturn available for observation but it should be noted that these planets have maximum brightness near opposition, which at the time of the austral summer,places them at maximum northern declination and thus they may be below the horizon or appear at low altitudes.

8.4.1 VISIBILITY OF DAYLIGHT STARS.

Gregerson (1963) refers to the work of Tousey and Koomen (1953) on the question of visibility of stars in the daylight sky and quotes a rule as follows:-

"Magnitude 1 or better can be seen throughout the day if they are not closer to the sun than 30° to 40° of arc. Stars with magnitude 1 to 2 require a distance of 60° to 70°. This distance decreases as the sun approaches the horizon. As it drops to about 10° in altitude the stars between magnitudes 2 and 3 also become visible."

The author also stated that in the arctic region, during the period of perpetual sunshine he found that these figures applied fairly well. This last statement can only be taken as a guide because Knoll, Tousey and Hulburt (1946), Hecht (1947) and Tousey and Hulburt (1947, 1948) have shown that the question of visibility is complex and is related to the following parameters:-

Telescope

(1) Magnification (ratio of entrance and exit pupils)

(2) Transmission

(3) Field of view

Star Illumination

(1) Visual magnitude

(2) Zenith distance

(3) Colour

Sky background illumination

(1) Distance from the sun

(2) Altitude of the sun

(3) Atmospheric pressure

(4) Polarisation of sunlight

(5) Reflection from the surrounding terrestrial surface.

Tousey and Hulburt (1948) support their theory with the results of observations made to artificial stars in the laboratory and to real stars in the daylight sky. Charts are also provided which enable the user to estimate the visual thresholds of stars in various parts of a clear sky in terms of the main parameters. The charts are based on data obtained by Blackwell at the Tiffany Foundation and by Tousey and Hulburt at the National Research Laboratory. There is no question that this data is not reliable as it has been obtained from many thousands of observations by nineteen observers at Tiffany and by five observers at N.R.L., but it must be emphasised that this data has been compiled from observations to artificial stars and is therefore authoritative only under the circumstances for which it is valid, i.e. laboratory conditions. Tousey and Hulburt quote the results of *one set* of observations made to *six stars* at Washington, D.C. with three telescopes of differing magnifications on *one day* only. The brightest star was Sirius m = -1.6 and the dimmest Spica m = +1.2

Further remarks by Tousey and Hulburt are of importance for surveyors undertaking these observations, viz.

(1) It is necessary to have the reticule focussed for infinity to aid the eye in maintaining accommodation for infinity.

(2) Near threshold it is necessary to look almost exactly at the star to see it, which means an accurate pointing of the telescope.

(3) When there are ice and snow-covered surfaces around the observer the sky brightness is greater and the polarisation less than when there are dark areas around him, such as forests or the sea.

Most surveyors who have made observations on daylight stars are well aware of the first two remarks but an additional precaution is worth noting and that is that the telescope must also be carefully focussed for infinity otherwise the blurred star image may not be sighted.

The work of Cox (1950) in New Zealand has been directed towards the determination of azimuth from daylight stars. The author has described quick prediction procedures consisting of graphs and tables for use in the area of the Canterbury Plains. The stars chosen for the observations were all brighter than magnitude 2 with the exception of α Pavonis ($m = +2.1$) which was selected so that at least one pair of stars, consisting of one star East and one star West of the meridian, was always available for observation. Cox reports that α Pavonis was easily seen on several occasions. It should be noted that under the circumstances chosen, i.e. near elongation, these stars will seldom be close to the sun and therefore in general will conform with the optimum requirements of visibility proposed by Tousey and Hulburt. It is of interest to note that an example of an observation to α Pavonis quoted by Cox was made when the visual threshold for the position of the star was $m = +2.4$ according to the charts of Tousey and Hulburt. On the basis of this experience it would seem that the work of Tousey and Hulburt may be of practical significance.

A magnitude limit of $m > +1.5$ was chosen in the light of Gregerson's personal experience, this author's personal experience, some observations made by a surveying student in 1967 and the work of Cox.

There are 20 stars which are brighter than magnitude + 1.5, distributed equally between the northern and southern declinations. If we consider observations are to be limited to altitudes above 20° at stations in latitude 70° or higher, then there are only 10 of these bright stars available for observation in each polar region. At the time of our observations only one planet (Jupiter m = -1.7)) complied with the above restrictions and thus the choice of celestial bodies was restricted to the sun, 1 planet and 9 stars (one star, Antares, was within a few degrees of the sun).

8.4.2 OBSERVING METHODS.

It is obvious from the aforegoing limitations imposed on the number of stars available for observation that without undertaking a very extensive astronomical programme, which would require a considerable time spent in prediction to minimise the effects of random and systematic errors, that only an azimuth of limited precision can be obtained. Astronomical observations were of secondary consideration in this project and the time available for observation restricted.

Three methods of observation may be used -

(a) timed altitudes

(b) timed azimuths, and

(c) altitude azimuths.

With method (a) the optimum circumstances are the meridian for the determination of latitude and the prime vertical for the determination of longitude. Under the conditions imposed, the solution of the unknowns can best be made simultaneously by the method of position lines and it should be noted that if possible the observations should be made at similar altitudes preferably above 20^{o} in order to reduce systematic and anomalous refraction effects at low altitudes. The stars should be well distributed in azimuth to give good intersections for the position lines. The precision of the solution for latitude does not deteriorate, in fact, it improves, as the latitude gets higher,but for longitude the time rate of change of altitude decreases in inverse proportion to the cosine of the latitude and thus the precision of the longitude solution will worsen as will be seen from the following differential relationships:-

Latitude $\frac{d\phi}{dh} = \frac{1}{\cos A}$, $\frac{d\phi}{dt} = - \cos \phi \tan A$

Longitude $\frac{dt}{dh} = \frac{1}{\cos \phi \sin A}$

It is then possible to use the values of latitude and longitude obtained by method (a) to solve for azimuth from the observations made by method (b). The effect of a systematic error in latitude is not serious if the stars are well distributed in azimuth as will be seen from

$$\frac{dA}{d\phi} = \tan h \sin A$$

but the effect of a systematic error in longitude can only be eliminated if $\cos \omega$ and/or $\cos \delta$ is zero or minimised if $\cos \omega$ can be made to differ in sign as will be seen from

$$\frac{dA}{dt} = \frac{\cos \omega \cos \delta}{\cos h}$$

All of these circumstances require the altitude to be equal to or greater than the latitude - a circumstance which has been stated before to be not ideal for small instruments. At low to medium altitudes there is roughly a one to one correspondence between the errors in the longitude (and time) and the azimuth, because the diurnal path of the stars are approximately almucantars.

From method (b) alone we may deduce all three unknown elements and for this White (1966) has given a general analytical solution and also a semi-graphical solution. The optimum observational circumstances for this method can be deduced from the general equation and definitions given in Appendix VI which is due to White. However it is not a simple matter to formulate a set of rules for choosing stars to give an optimum solution of the unknowns because the accuracy of the solution depends not only upon the position of the stars but the instrumentation that is employed. We can simplify the problem considerably by finding the conditions under which each unknown can be solved independently and neglect the weighting. The conditions for such an optimum solution are summarised in the following table:-

Unknown	Optimum Position
Latitude	Elongation, preferably away from the pole.
Longitude	Meridian, preferably near the zenith.
Azimuth	Pole.

It will be noticed immediately that all of these positions are at altitudes greater than, or equal to the latitude and therefore the observations suffer from the disadvantages previously referred to. Even though it may not be possible to make observations in the optimum positions, the solutions of the unknowns, although inferior, may have an acceptable level of precision.

These latter remarks also apply to method (c), altitude azimuth observations, for which we can write an equation similar to the previous equation for method (b), viz.:

$$v = -\Delta_o - \frac{1}{\cos\phi \tan t'} d\phi - \frac{1}{\cos h' \tan \omega'} dh + (A' - \psi - 0_o)$$

where A' is calculated from ϕ', δ, and h'.

The approximate optimum observational circumstances for this method can be deduced as follows:-

Unknown	Optimum Position
Latitude	Elongation, preferably near the zenith .
Azimuth	Elongation, preferably near the Pole .

Although from this method only two of the unknowns may be obtained. Again it is possible to use the latitude and longitude obtained by method (a) to solve for azimuth from the observations made by method (c). Once again the effect of a systematic error in latitude is not serious if the stars are well distributed in azimuth as will be seen from

$$\frac{dA}{d\phi} = -\frac{1}{\cos\phi \tan t} \quad ,$$

noting that in high latitudes t behaves in a similar way to A with the sign reversed.

In the light of this analysis it would appear that determining latitude and longitude from (a) would be preferred and then to calculate azimuth from (b) or (c). The question is now to choose between (b) or (c). An examination of the differential relationships

for (a) $\frac{dA}{dt} = - \frac{\cos \omega \sin A}{\sin t}$

and for (c) $\frac{dA}{dh} = - \frac{\cos \omega}{\sin t \cos \phi}$

shows that $\frac{dA}{dt}$ will always be smaller than $\frac{dA}{dh}$ for a given set of circumstances and therefore (b) is preferred. Also for the observer, timed azimuths are easier to observe than simultaneous altitude and azimuth pointings.

8.4.3. PREDICTION AND CALCULATIONS OF ALTITUDE AND AZIMUTH

The problem is to find those bright stars which are available for observation between altitudes 20^{o} and 50^{o} and to calculate (and check) the altitude and azimuth for a specified standard time for each star to an accuracy of 0.1^{o}. It will also be necessary to calculate the azimuth of the sun at a specified instant before the commencement of the star observations so that the horizontal circle can be oriented for finding the stars.

For the selection of stars the "Rude Star Finder and Identifier" (No. 2102-D, U.S. Naval Hydrographic Office) was found to be very convenient. It consists of a sturdy plastic circular base plate, approximately 8" diameter on which the 57 numbered stars listed in the Air and Nautical

Almanacs are plotted, one side for the northern hemisphere and the other side for the southern hemisphere. Other celestial bodies such as the sun, moon and planets may be plotted on the base as well. There are in addition 9 altitude - azimuth templates for each 10^{o} of latitude. The main advantages of this device are that the relative positions of the celestial bodies are seen at a glance and the sequence of observations to the stars may be selected to obtain the best possible altitudes and azimuths. Star charts are also available for this purpose. Biddle (1958) describes the use and construction of the Star Atlas for Land Surveyors which consists of eight charts on the stereographic projection for each of the latitudes 50^{o}N, 10^{o}N and 30^{o}S. Besides the limited latitudes available, star positions are only given for local sidereal time at 3 hour intervals and azimuth can only be measured from the centres of the charts, elsewhere the azimuths are only approximate and according to de Graaff-Hunter (1947) are not particularly suited to the formulation of a star programme. White (1969) has designed star charts on the mercator projection from which the altitudes and azimuths of the 19 navigation stars can be found but these charts are rather clumsy and fragile. Sight Reduction Tables for Air Navigation Vol. I (Published by H.M.S.O. as A.P. 3270 and by U.S.H.O. as Publication No. 249) contains tables of calculated altitudes and azimuths of 7 stars selected from a list of 34 stars for each degree of latitude and two degrees of local hour angle from latitude S89^{o} to N89^{o}. These tables will enable the user to select a star programme quickly but they do not have the flexibility of the Rude Identifier and are somewhat bulky.

The second phase of the problem, that of calculating quickly the altitude and azimuth of the selected stars is one which has intrigued and occupied the minds of some of the world's leading scientists, such as Cassini, Thomson (Lord Kelvin) etc., over the last two centuries. Bowditch (1962) gives an excellent historical review of the work of these men. Weems (1942) considers that of the 20 different methods which have been developed in various ways, only five are now in common use but most of these five must be rejected for the solution of our problem because they do not meet the required accuracy of 0.1°. The tables of Ageton which are based on the division of the astronomical triangle by a perpendicular from the body to the meridian give the required accuracy but according to Comrie (1938) an indeterminate condition "may also arise with a perfectly good observation at any altitude, especially in high latitudes". On the other hand tables based on a perpendicular from the zenith to the celestial meridian through the body do not suffer from the latter defect, although observations made when the hour angle is near 90° become unsatisfactory as with the other method. The latter tables (Hughes, Dreisonstok, Ogura etc.) are restricted to an assumed latitude and hour angle of an integral degree, which for navigators presents no problem but in their present form are not suitable for the problem which was originally posed. It was decided to adopt Comrie's method of solution mainly because the tables are accurate and considerable thought has been given to their layout. However, to suit the needs of our problem it was necessary to extend Table I so that the assumed latitude would be at

10' intervals and thus the maximum error in latitude would be 5' and on the average only 2½'. This required the calculation of the four quantities

$$K = 90^{\circ} - \arctan(\cot\phi \cos t)$$

$$A = 10^5 \times \log \sec \{\arcsin(\cos\phi \sin t)\}$$

$$D = 10^3 \times \log \operatorname{cosec} \{\arcsin(\cos\phi \sin t)\}$$

$$Z_1 = \operatorname{arccot}(\sin\phi \tan t)$$

for t from 0° to 90° in 1° intervals at each ϕ from 0° to $89^{\circ}50'$ at 10' intervals. This entailed the calculation of nearly 200,000 entries on 540 pages, a seeming extravagance for the limited observations that were to be undertaken, but the compilation of these calculations provides a permanent set of tables for all future needs.

The check calculations were performed by an entirely different technique and the Department of Surveying is fortunate in possessing a Bygrave slide rule manufactured by Dennert and Pape. Slide rules have also been made to the designs of Richer, Poor and Bertin. Bowditch describes the slide rule as follows:-

"A cylindrical slide rule designed by the Englishman, Bygrave, to solve the astronomical triangle by dropping a perpendicular from the celestial body to the celestial meridian. This device consists of three concentric tubes. The inner one has a spiral scale of logarithmic tangents, the middle one a spiral scale of logarithmic cosines, and the outer one a pointer for each scale. The solution is simple and relatively fast but one has to take care if the azimuth is near 90° or the meridian angle(t) or declination is very small. An accuracy of 1' or 2' is generally available."

For the work in the Antarctic it was not necessary to take the whole of this extra material of Table I but only the appropriate sheets for the stations which were to be visited because approximate values of the latitude of these stations were known. It was found in practice that the methods of selection and prediction which were chosen were economical and accurate; a star programme could be prepared and checked in about one hour. The computer programme and one page of these extended tables are reproduced in Appendix V together with a sample prediction for the sun and one star and the final astronomical programme for the station McMurdo.

8.4.4. PRACTICAL RESULTS

The first attempt at a daylight star programme was made at station McMurdo on the 23rd November. Orientation was obtained from the sun which was at 32^{o} altitude but only one star, Canopus was sighted. The following table gives a list of the stars, their visual magnitudes taken from the Star Almanac for Land Surveyors and the threshold magnitudes taken from the charts of Tousey and Hulburt.

Star	Magnitude	
	Visual	Threshold
Achernar	0.6	3.3
Spica	1.2	1.0
Canopus	-0.9	2.9
Fomalhaut	1.3	2.8

The reason for failure to sight three of the stars is not known. The orientation of the theodolite was proved by the sighting of Canopus From the table it will be seen that Spica is 0.2 magnitude below threshold and would be difficult to sight but Achernar and Fomalhaut are well above threshold and should present no difficulty. A further programme, see Appendix VI, was attempted on the 29/30 November with complete success but on this occasion the altitude of the sun was only 10°. The following table shows that for this programme all stars are well above threshold. In the case of Fomalhaut for the first programme this star was 1.5 magnitudes above threshold and was not sighted whilst for the the second programme it was 2.0 magnitudes above threshold and no difficulty was experienced.

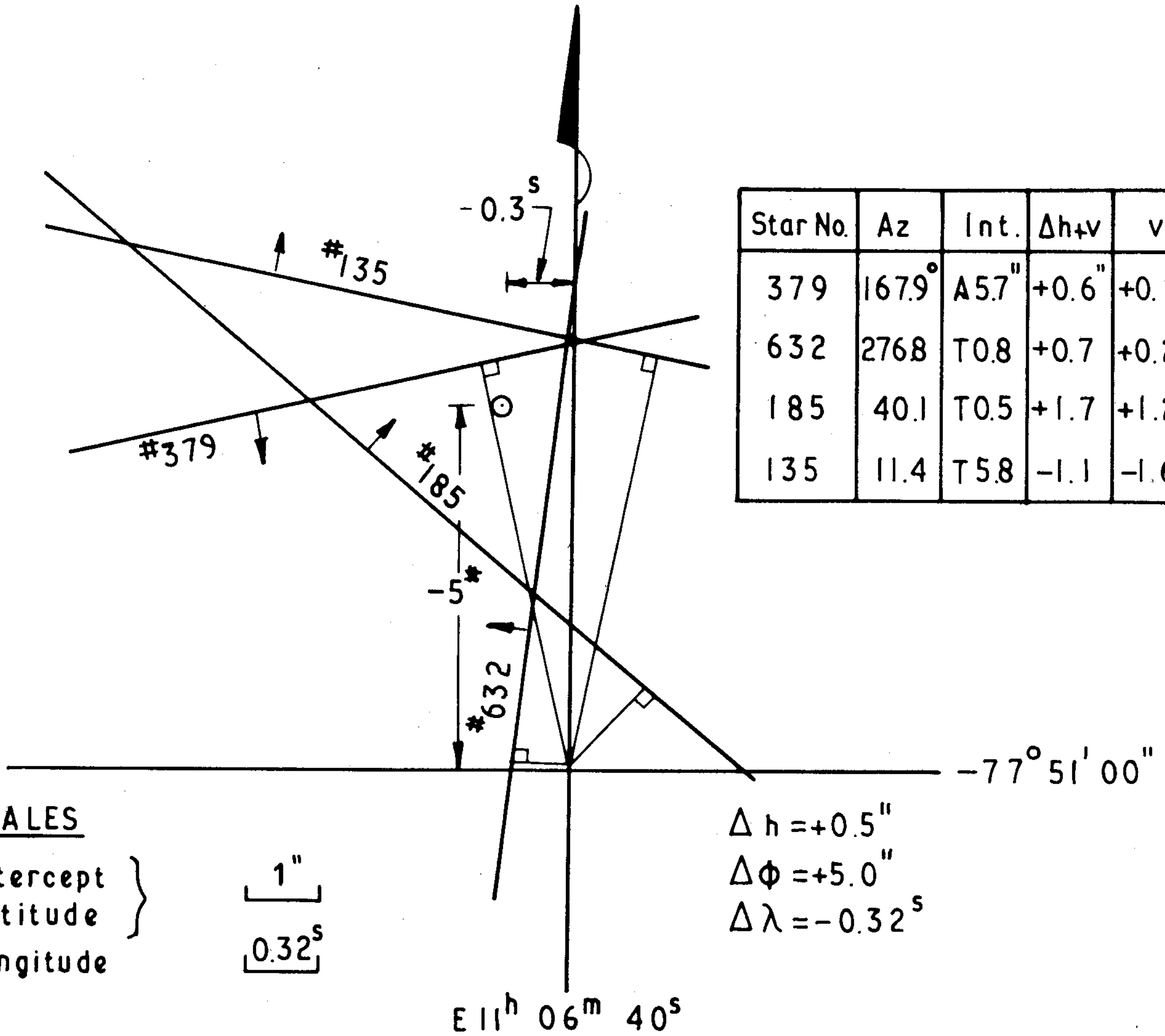
Star	Magnitude	
	Visual	Threshold
Rigil Kentaurus	0.1	1.8
Fomalhaut	1.3	3.3
Sirius	-1.6	1.5
Rigel	0.3	0.8

Another programme at station Hallett was observed without difficulty, but the sun was at a low altitude of 5°. From these results it would appear that some practical investigation into the visibility of daylight stars would be of advantage.

The star observations which were made at station McMurdo will serve to illustrate some of the various points made in the previous study of observing methods. A set of observations for each star was composed of 4 pointings to the R.O. before and after the star observations, which consisted of 6 timed altitude and 6 timed azimuth pointings.

In Figure 8.1 on Page 269 is shown the position line plot for this station derived from the timed altitude observations. It will be seen that the position lines intersect nearly at a point, indicating that the observing errors (v) and any systematic error in the calculated refraction (Δh) were probably small. The relative accuracy of position fixation can be gauged from the scales in latitude and longitude which are 1 : 4.75, showing that the longitude fixation is considerably weaker than that of the latitude. The azimuth was then calculated from the values of latitude and longitude obtained from the position lines and the timed azimuth observations. The mean calculated azimuth from each star is given in the following table:

Star	Azimuth	v	Altitude
Rigil Kentaurus	316° 05' 42"	-14"	49°
Fomalhaut	316 05 21	+ 7	32
Sirius	316 05 22	+ 6	26
Rigel	316 05 28	0	20



Star No.	Az	Int.	Δh+v	v
379	167.9°	A5.7"	+0.6"	+0.1"
632	276.8	T0.8	+0.7	+0.2
185	40.1	T0.5	+1.7	+1.2
135	11.4	T5.8	-1.1	-1.6

Final Position : Station Mc Murdo

ϕ $-77^\circ 50' 55''$ λ E $11^h 06^m 39.7^s$

FIG.8.1: POSITION LINES

The residuals (v) are larger than would be normally expected for similar observations made in low latitudes with a seconds theodolite, which has been attributed to errors in the transverse levelling of the instrument. It was found to be exceedingly difficult to maintain the instrument in a level position (as indicated by the plate bubble) and re-levelling was necessary halfway between each set of observations. The error in azimuth ΔA, caused by a transverse levelling error, e, is given by, $\Delta A = e \tan h$, and it will be noticed that the largest azimuth residuals are for those stars with the highest altitudes. For future work, careful attention should be paid to the shading of the instrument and to providing a firm foundation for the instrument tripod in order to minimise the levelling errors.

In Figure 8.2 on Page 271 is shown the scheme of the semi-graphical adjustment for the solution of latitude, longitude and azimuth according to the technique proposed by White. The adjustment is made in the phases marked I, II and III on the figure.

PHASE I

First of all the differences, designated by $(A'-\psi-0_o)$ are found between the observed azimuths and those calculated using the assumed values of latitude, longitude and azimuth. In this example the assumed values were those derived from the calculations made by methods (a) and (b). From the top (Z) of a vertical line (ZP) of length $\tan \phi$, rays are drawn of length $\tan h'$ for each star according to their calculated azimuths using the line (ZP) as the North-South direction. Lines from

$$x^2 = \frac{1}{\eta^2 + \xi^2}$$

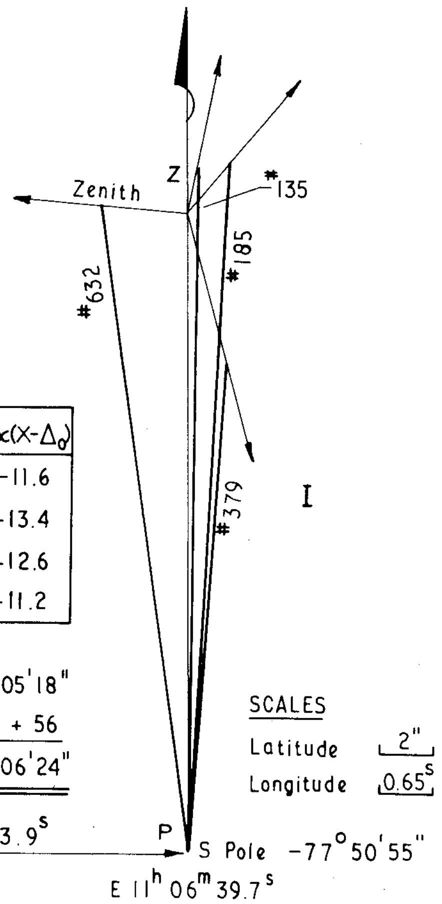
$$X = A' - \psi - O_0$$

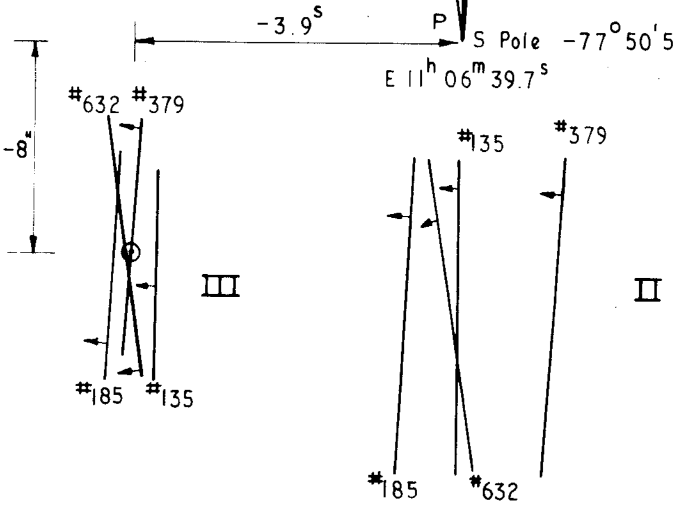
Star No.	x	X	$x.X$	$X-\Delta_0$	$x(X-\Delta_0)$
379	0.283	+15"	+4.2	-41"	-11.6
632	0.210	-8	-1.7	-64	-13.4
185	0.199	-7	-1.4	-63	-12.6
135	0.200	0	0	-56	-11.2

Assumed Azimuth of R.O. 316°05'18"

Δ_0 + 56

Azimuth of R.O 316°06'24"





Final Position Station Mc Murdo

φ -77° 51' 03" λ E 11^h 06^m 35.8^s

FIG. 8.2: ZENIDROMES

(P) to the end of each ray have been called "zenidromes" by White. The length of a zenidrome, $\frac{1}{x}$, is given by

$$\frac{1}{x} = \sqrt{\eta^2 + \xi^2}$$

where $\eta = \tan h' \sin A'$

$\xi = \tan \phi_o - \tan h' \cos A'$

PHASE II

In this phase of the adjustment the effect of the absolute terms is incorporated in the "error figure". Each zenidrome is translated parallel to itself by an amount equal to $x(A'-\psi-0_o)$. The positive direction of this translation (indicated by the small arrow on each zenidrome) is anti-clockwise around the point (P).

PHASE III

For the last phase of the adjustment we need to translate the zenidromes either all in a positive or all in the negative sense by a quantity equal to $\Delta_o x$ (where Δ_o is the unknown orientation correction) until the zenidromes intersect together as closely as possible. This may be accomplished either by trial and error or by drawing "shadow rays" and connecting up corresponding intersections of shadow rays and zenidromes to form a network of lines, the centre of which is the required position. This latter technique is a standard procedure used in the semi-graphic adjustment of minor triangulation which is described in standard survey literature e.g. Close (1925). It will be seen from the example that the final position and orientation is very uncertain because the zenidromes give very flat intersections and all will translate in the same direction for a given value of Δ_o. The situation can be improved slightly by avoiding observations to stars near the meridian but the intersections of the zenidromes for these stars are still flat unless the altitudes are high. Ideally, we need zenidromes which translate in opposite directions which can only be achieved by making observations at altitudes higher than the pole which as has been stated before to be difficult with small instruments. This solution, as White has explained, closely resembles the semi-graphic method of solution of minor triangulation. The poor solution here has a direct analogy with a resection observation made to stations which are grouped in about the same direction from the resection station. An analytical least squares solution is also given in Appendix VI.

The final values of the unknowns are considerably different from those obtained by the combination of methods (a) and (b) which are tabulated as follows: -

Method	Latitude	Longitude	Azimuth
(a) & (b)	S77° 50' 55"	E11^{h}06^{m}39.7^s	316° 05' 28"
(b)	S77 51 03	E11 06 35.8	316 06 24

The plot of the position lines in method (a) show that it is highly improbable that the values of latitude and longitude obtained by method (b) are accurate because to explain discrepancies of the order of 8" in latitude and 3.9^s in longitude we would need to introduce discrepancies of up to 1' in the altitude observations.

From this experience it would appear that the solution of position and azimuth by method (b) should only be attempted if the observations can be made near to the optimum positions given in Section 8.4.2. and then these high altitude observations will reinforce the solution.

8.5 TIME RECORDING

In field astronomy, longitude determinations are particularly prone to the influence of systematic errors in the observations and in the recording apparatus. The effect of a systematic error in the observation to a star can be minimised by either using an impersonal micrometer or by making observations at calibration stations whose longitudes are known from impersonal observations, and then the observed longitudes at the other stations can be corrected by the observed longitude differences at the calibration stations. The use of an impersonal micrometer is usually confined to the larger astronomical instruments such as the Wild T4, Zeiss Theo 003, Kern DKM 3A etc.

Most mechanical chronometers that are used in surveying and allied fields are capable of keeping time to a high precision. The main problems encountered are to compare the chronometer against a standard time signal and then to record precisely the chronometer instant of a random event. The problems are usually overcome either by using a stopwatch synchronised with the audible ticks of the chronometer and signal or, when greater precision is required by using an astronomical detector and amplifier for the time signals and a tape chronograph in conjunction with the seconds contacts on the balance wheel of the chronometer for the recording. In principle both of these methods use a secondary piece of apparatus to further subdivide the time base of one second, provided by the chronometer and signal. The first of these methods may introduce systematic errors of the order of a few tenths of a second. This is overcome in the second method by the use of the tape chronograph, an expensive piece of apparatus requiring a time consuming interpretation of the tape record, although if a printing chronograph is used such as an Omega Time Recorder (OTR 2) then the record can be read directly, but this apparatus is much more expensive than a tape chronograph. A synchronous slave device (termed a "chronostop", see Bennett (1965)) has been used with some success in the past. This instrument is connected to a mechanical chronometer and synchronised electro-mechanically from the circuit breaker on the chronometer balance wheel. The average error in recording the time of random events was 0.005 to 0.011 seconds using this instrument.

In Antarctica and other remote area difficulties and inconvenience have been experienced in the transportation of mechanical chronometers. For the Antarctic observations it was found that the timing observations could be conveniently made without a chronometer by means of a split hand stopwatch and radio only; the stopwatch was started at a noted instant of the time signal and rated at the beginning, during and end of the observations using the secondary hand. The success of the method depends entirely upon the quality of reception of the time signals. However, in recent years with increased transmission power and additional radio stations, astronomical parties seldom experience difficulties with radio reception except in the polar regions. The technique could be improved by the automatic recording of the time signals which would eliminate the systematic error in comparing the watch against the time signal.

In order to record time signals automatically on a stopwatch the stop button has to be depressed by a mechanical plunger or arm which is actuated either by the time signal or by the observer closing a switch. A device called a "timer" has been constructed, see Plates 8.1 and 8.2 on Page 277 which accomplishes this by means of two solenoids which drive a plunger against the secondary stop button. The time signal from a small transistorised short wave radio is detected, amplified (single stage), rectified and the resulting voltage is made to operate the coil of a miniature high speed reed relay which in turn closes the circuit to the solenoids. 1,000 and 1,200 Hertz pot core band pass filters are

PLATE 8.1: TIME RECORDING APPARATUS SHOWING THE TRANSISTOR RADIO AND "TIMER."

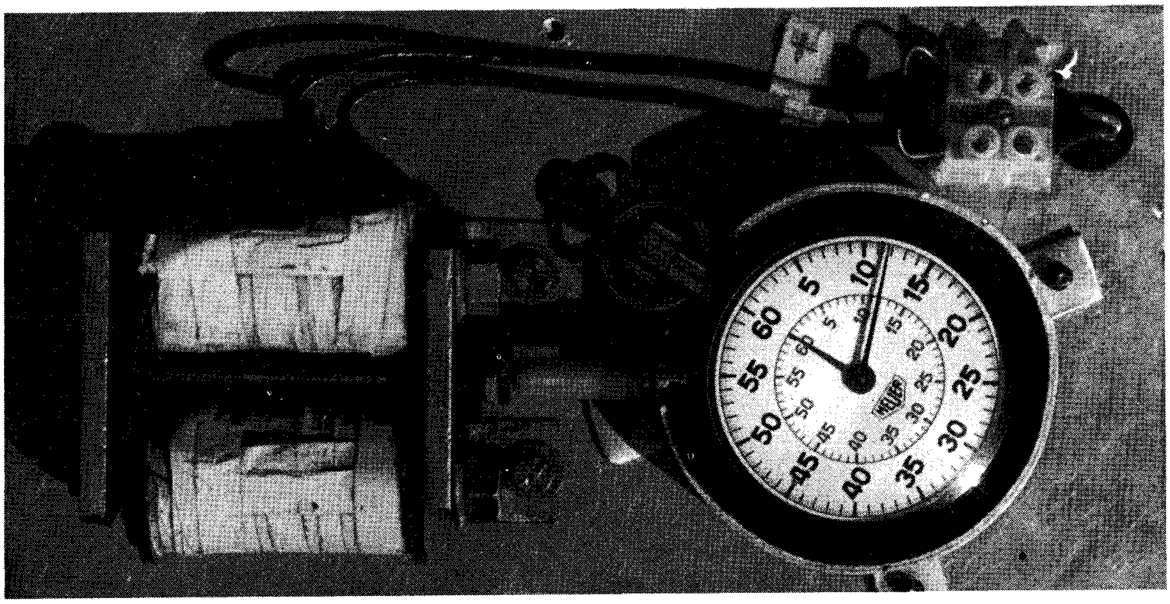


PLATE 8.2: TOP VIEW OF THE "TIMER" SHOWING THE SOLENOIDS WHICH DRIVE THE MECHANICAL PLUNGER AGAINST THE SECONDARY BUTTON OF THE STOP WATCH.

incorporated in the circuit to assist in removing unwanted tones and reducing noise level. These filters have similar characteristics to those described by Bennett (1963b) i.e. narrow bands and low insertion losses. The whole apparatus is quite compact because of the use of transistorised and miniature components with a total weight of less than 5 lbs, including the radio. Power is supplied by a small internal rechargeable battery.

The "timer" has been tested by recording time signals and random observations with the Omega printing chronograph in circuit with the "timer". Four test series were made over periods of between 25 and 45 minutes using time signals from station V.N.G. The Omega chronograph can be read directly to 0.01 sec. and by estimation to 0.001 sec. The record showed that there was no appreciable rate of the chronograph in the test periods and therefore the time instants from the Omega record were taken as absolute for the purpose of comparison. Time signal checks were interspersed with random observations in the test period and any rate fluctuations, which were small, were proportioned out in a linear manner between time checks. The results of one test are shown in detail in Table 8.1 on Page 280 and the results of the four test series are shown in the following table.

Set.	Average d	Estimated Standard Deviation
1	+0.02 sec.	± 0.08 sec.
2	+0.01	± 0.14
3	$+0.00^{5}$	± 0.10
4	+0.03	± 0.13

From the results of these test series it is evident that the timing of an observation is not burdened with a systematic error in the rating of the watch against the time signal, because the average of the differences d, is minute. Thus these differences may be taken as residual errors, distributed about zero mean, and a standard deviation can be estimated directly from these values of d. The average estimated standard deviation of about ± 0.1 sec. is compatible in size with other random errors of observation made with a small theodolite. Therefore if this apparatus is used then the only significant source of systematic error in timing that remains will be the observer's error in estimating the time of observation to the star. The apparatus is considered to be a considerable improvement over the "chronostop" because of the elimination of the chronometer.

TABLE 8.1 COMPARISON TEST BETWEEN OMEGA OTR AND "TIMER"

Omega	Timer	Difference d (0.01 sec.)	Omega	Timer	Difference d (0.01 sec.)
2^m 01.51^s	2^m 01.6^s	- 9	14^m 30.39^s	14^m 30.4^s	- 1
24.29	24.3	- 1	43.36	43.3	+ 6
34.25	34.2	+ 5	58.38	58.5	-12
47.09	47.0	+ 9	18 02.42	18 02.5	- 8
55.95	56.0	- 5	11.15	11.1	+ 5
3 04.50	3 04.5	0	23.42	23.4	+ 2
6 17.94	6 18.0	- 6	38.19	38.2	- 1
28.48	28.4	+ 8	55.34	55.2	+14
40.66	40.6	+ 6	22 09.41	22 09.4	+ 1
48.48	48.5	- 2	19.63	19.7	- 7
57.34	57.4	- 6	29.35	29.3	+ 5
7 07.60	7 07.7	-10	38.30	38.2	+10
10 19.69	10 19.8	-11	43.68	43.6	+ 8
26.34	26.2	+14	51.74	51.8	- 6
34.15	34.0	+15	26 03.71	26 03.7	+ 1
41.93	41.8	+13	12.04	12.0	+ 4
48.90	48.9	0	22.53	22.5	+ 3
55.83	55.9	- 7	33.60	33.5	+10
14 04.03	14 04.0	+ 3	41.54	41.4	+14
11.67	11.7	- 3	26 52.17	26 52.2	- 3
14 20.16	14 20.0	+16			

Average $d = \frac{\Sigma d}{n} = \frac{+0.74}{41} = +0.02$ sec.

Estimated standard deviation of a single observation.

$$= \sqrt{\frac{\Sigma dd}{n-1}}$$

$$= \sqrt{\frac{0.2525}{40}}$$

$$= \pm\ 0.08 \text{ sec.}$$

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APPENDIX I

GYRO TABLE OF K

GYRO THEODOLITE CONSTANT K

A = AMPLITUDE N = SCALE DIVISION C = INSTRUMENT CONSTANT = 0.05000

$K = C\sqrt{A^2 - N^2}$ MEAN = (2*SUM(K)-K(0))/11

AMPLITUDE	SCALE DIVISION 0	1	2	3	4	5	MEAN
10.0	0.500	0.498	0.490	0.477	0.459	0.434	0.474
10.1	0.505	0.503	0.495	0.483	0.464	0.439	0.480
10.2	0.510	0.508	0.501	0.488	0.470	0.445	0.485
10.3	0.515	0.513	0.506	0.493	0.475	0.451	0.490
10.4	0.520	0.518	0.511	0.498	0.480	0.456	0.495
10.5	0.525	0.523	0.516	0.504	0.486	0.462	0.501
10.6	0.530	0.528	0.521	0.509	0.491	0.468	0.506
10.7	0.535	0.533	0.526	0.514	0.497	0.473	0.511
10.8	0.540	0.538	0.531	0.519	0.502	0.479	0.516
10.9	0.545	0.543	0.536	0.524	0.507	0.485	0.522
11.0	0.550	0.548	0.541	0.530	0.513	0.490	0.527
11.1	0.555	0.553	0.546	0.535	0.518	0.496	0.532
11.2	0.560	0.558	0.551	0.540	0.524	0.502	0.537
11.3	0.565	0.563	0.557	0.545	0.529	0.507	0.543
11.4	0.570	0.568	0.562	0.550	0.534	0.513	0.548
11.5	0.575	0.573	0.567	0.556	0.540	0.518	0.553
11.6	0.580	0.578	0.572	0.561	0.545	0.524	0.558
11.7	0.585	0.583	0.577	0.566	0.550	0.529	0.563
11.8	0.590	0.588	0.582	0.571	0.556	0.535	0.569
11.9	0.595	0.593	0.587	0.576	0.561	0.540	0.574
12.0	0.600	0.598	0.592	0.581	0.566	0.546	0.579
12.1	0.605	0.603	0.597	0.587	0.571	0.551	0.584
12.2	0.610	0.608	0.602	0.592	0.577	0.557	0.589
12.3	0.615	0.613	0.607	0.597	0.582	0.562	0.595
12.4	0.620	0.618	0.612	0.602	0.587	0.568	0.600
12.5	0.625	0.623	0.617	0.607	0.593	0.573	0.605

GYRO THEODOLITE CONSTANT K

A = AMPLITUDE   N = SCALE DIVISION   C = INSTRUMENT CONSTANT = 0.05000

$K = C\_/\sqrt{A^2 - N^2}$ MEAN = (2*SUM(K)-K(0))/11

AMPLITUDE	SCALE DIVISION 0	1	2	3	4	5	MEAN
12.5	0.625	0.623	0.617	0.607	0.593	0.573	0.605
12.6	0.630	0.629	0.623	0.612	0.598	0.579	0.610
12.7	0.635	0.634	0.628	0.618	0.603	0.584	0.615
12.8	0.640	0.639	0.633	0.623	0.608	0.590	0.620
12.9	0.645	0.644	0.638	0.628	0.614	0.595	0.626
13.0	0.650	0.649	0.643	0.633	0.619	0.600	0.631
13.1	0.655	0.654	0.648	0.638	0.624	0.606	0.636
13.2	0.660	0.659	0.653	0.643	0.629	0.611	0.641
13.3	0.665	0.664	0.658	0.648	0.635	0.617	0.646
13.4	0.670	0.669	0.663	0.653	0.640	0.622	0.651
13.5	0.675	0.674	0.668	0.659	0.645	0.627	0.657
13.6	0.680	0.679	0.673	0.664	0.650	0.633	0.662
13.7	0.685	0.684	0.678	0.669	0.656	0.638	0.667
13.8	0.690	0.689	0.683	0.674	0.661	0.644	0.672
13.9	0.695	0.694	0.688	0.679	0.666	0.649	0.677
14.0	0.700	0.699	0.693	0.684	0.671	0.654	0.682
14.1	0.705	0.704	0.698	0.689	0.677	0.660	0.687
14.2	0.710	0.709	0.703	0.694	0.682	0.665	0.692
14.3	0.715	0.714	0.708	0.700	0.687	0.670	0.698
14.4	0.720	0.719	0.714	0.705	0.692	0.676	0.703
14.5	0.725	0.724	0.719	0.710	0.697	0.681	0.708
14.6	0.730	0.729	0.724	0.715	0.703	0.686	0.713
14.7	0.735	0.734	0.729	0.720	0.708	0.692	0.718
14.8	0.740	0.739	0.734	0.725	0.713	0.697	0.723
14.9	0.745	0.744	0.739	0.730	0.718	0.702	0.728
15.0	0.750	0.749	0.744	0.735	0.723	0.708	0.733

APPENDIX II

COMPUTER PROGRAMME AND SAMPLE OUTPUT FOR THE ANALYSIS OF THE TURNING POINT METHOD

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FORTRAN PROGRAMME "GYROSIS" -- SVY153
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UNIVERSITY OF NEW SOUTH WALES
DEPARTMENT OF SURVEYING

"GYROSIS" -- ADJUSTMENT AND ANALYSIS OF GYRO THEODOLITE READINGS,
OBSERVED BY THE TURNING POINT METHOD.

PROGRAMMED USING FORTRAN IV, G-LEVEL -- COMPATIBLE WITH WATFOR
COMPILER -- FOR UNSW IBM 360/50.
PREPARED BY F. G. ANDERSON, MARCH 1969.

GYROSIS ACCEPTS INPUT OF EIGHT (MAXIMUM) SUCCESSIVE GYRO THEODOLITE
READINGS IN DEGREES, MINUTES AND SECONDS.  IF LESS THAN EIGHT READINGS
ARE INPUT, ZERO VALUES ARE ADOPTED FOR THE REMAINDER AND PROCESSED.
THE FIRST OBSERVATION OF THE SET MAY BE EITHER LEFT OR RIGHT OF
ASSUMED NORTH:  A RIGHT OBSERVATION BEING DEFINED AS ONE IN WHICH THE
NUMBER OF DEGREES IS LESS THAN 180.

OUTPUT INCLUDES:
1.   OBSERVATIONS
2.   SCHULER MEANS AND THEIR MEAN AND VARIANCES
3.   ADJUSTED OBSERVATIONS AND THEIR VARIANCE:  AND THE STANDARD
     DEVIATION OF A SINGLE OBSERVATION
4.   THE WEIGHTED SCHULER MEAN (THETA) AND ITS STANDARD DEVIATION
5.   THE MIDDLE AMPLITUDE (B) AND ITS STANDARD DEVIATION
6.   DAMPING FACTORS (ALPHA AND A) AND THEIR STANDARD DEVIATION

THIS ANALYSIS IS PERFORMED SEPERATELY FOR EVERY POSSIBLE SET OF
SUCCESSIVE OBSERVATIONS IN GROUPS OF 3,4,5,6,7, AND ALL 8.

THE PROGRAMME RETURNS TO THE BEGINNING, SO THAT ANY NUMBER OF
DATA SETS MAY BE PROCESSED IN ONE JOB
--------------------------------------------------------------------
```

```
C
C        DATA CARDS:
C
C        CARD ONE
C        --------
C                  GYRO SERIAL NO.        COLS  1 -  4          A4
C                                                                1X
C                  DATE:     DAY          COLS  6 -  7           I2
C                          MONTH          COLS  8 -  9           I2
C                           YEAR          COLS 10 - 11           I2
C                                                                1X
C                  JOB IDENTIFICATION     COLS 13 - 24          3A4
C                  OBSERVATIONS   (DEG    COLS 25 - 27           I3
C                                 (MIN    COLS 28 - 29           I2
C                                 (SEC    COLS 30 - 31           I2
C                     -- REPEATED   IN    COLS 32 - 38          (I3,2I2)
C                                         COLS 39 - 45               "
C                                         COLS 46 - 52               "
C                                         COLS 53 - 59               "
C                                         COLS 60 - 66               "
C                                         COLS 67 - 73               "
C                                         COLS 74 - 80               "
C
C        ------------------------------------------------------------
C
C
```

```
      IMPLICIT REAL*8 (A-G,O-Z), REAL*4 (H)
      DIMENSION Y(8),SM(6),SSUM(6),YSUM(6),YOD(6),YEV(6),B(6),TH(5),
     1 SAV(5),TWOB(5),ADJ(5,8),SUM(5),ADJVEC(40),IDAVEC(40),MAVEC(40),
     2 SAVEC(40)
      COMMON SA(5,8),VOB(5,8),STH(5),SB(6),ALFA(6),A(6),SIG(5),SIGTH(5),
     1 SIGB(5),SIGAL(5),SIGA(5),SS(6),VS(5,6),SSAV(5),HDESC(3),HGYRO,
     2 IDEG(8),MIN(8),ISEC(8),IDA(5,8),MA(5,8),IDTH(5),MTH(5),IDB(6),
     3 MB(6),IDS(6),MS(6),IDAV(5),MAV(5),LEFT,N,NSET,IDAY,MONTH,IYR,NTWO
      COMMON/FORMAT/ HFMTA(19),HFMTB(15),HFMTC(14),HFMTD(8),HTABX(2),
     2 HTABY(2),HFMTE(9),HZERO,HONE,HPLUS,HBLNK,LSW(2),NP,NR
C<<<   READ ONE LOT OF DATA                                        >>>
      NDATA=1
  509 READ(NR,101,END=500,ERR=999) HGYRO,IDAY,MONTH,IYR,HDESC,(IDEG(I),
     1 MIN(I),ISEC(I),I=1,8)
  101 FORMAT (A4,1X,3I2,1X,3A4,8(I3,2I2))
      LEFT=1
      N=6
      NSET=3
      IF(IDEG(1).LT.180) LEFT=2
      DO 401 I=1,8
      Y(I)=ISEC(I)+60*MIN(I)+3600*IDEG(I)
  401 CONTINUE
      DO 402 I=LEFT,8,2
      Y(I)=Y(I)-1296000.0
  402 CONTINUE
C<<<   PROCESS GROUP OF THREE                                      >>>
      DO 403 I=1,6
      SM(I)=(Y(I)+2.0*Y(I+1)+Y(I+2))*0.25
      SSUM(I)=SM(I)
      YSUM(I)=Y(I)+Y(I+1)
      YOD(I)=Y(I)+Y(I+2)
      YEV(I)=Y(I+1)
      B(I)=(YOD(I)-2.0*YEV(I))*0.25
      A(I)=(Y(I)-Y(I+2))*0.5
      ALFA(I)=A(I)/B(I)
  403 CONTINUE
      CALL RECONV(SM,6,6,0,IDS,MS,SS)
      CALL RECONV(B,6,6,1,IDB,MB,SB)
      CALL SCRIBE
```

```
C<<<   PROCESS GROUPS FOUR TO EIGHT                                    >>>
      DO 404 NSET=4,8
      N=9-NSET
      LINK=NSET-3
      GO TO (501,502,503,504,505),LINK
  501 DO 405 I=1,N
      TH(I)=(SM(I)+SM(I+1))*0.5
  405 CONTINUE
      GO TO 506
  502 DO 406 I=1,N
      TH(I)=(2.0*(SM(I)+SM(I+2))-SM(I+1))/3.0
      A(I)=(2.0*(Y(I)-Y(I+4))+Y(I+3)-Y(I+1))*0.1
  406 CONTINUE
      GO TO 506
  503 DO 407 I=1,N
      TH(I)=(5.0*(SM(I)+SM(I+3))+SM(I+1)+SM(I+2))/12.0
  407 CONTINUE
      GO TO 506
  504 DO 408 I=1,N
      TH(I)=(3.0*(SM(I)+SM(I+4))-2.0*(SM(I+1)+SM(I+3))+4.0*SM(I+2))/6.0
      A(I)=(3.0*(Y(I)-Y(I+6))+2.0*(Y(I+5)-Y(I+1))+Y(I+2)-Y(I+4))/28.0
  408 CONTINUE
      GO TO 506
  505 DO 409 I=1,N
      TH(I)=(7.0*(SM(I)+SM(I+5))+4.0*(SM(I+2)+SM(I+3))-SM(I+1)-SM(I+4))*
     1 0.05
  409 CONTINUE
C<<<   CALCULATION BASED ON GENERAL FORMULAE                           >>>
  506 NTWO=NSET-2
      SET=NSET
      CONST=(SET+1.0)*0.5
      KSW=MOD(NSET,2)
      SQN=NSET**2
      SQNF=(SQN-4.0)*NSET
      SQND=SQN-1
      DO 410 I=1,N
      SSUM(I)=SSUM(I)+SM(I+LINK)
      SAV(I)=SSUM(I)/NTWO
      DO 411 J=1,NTWO
      VS(I,J)=SAV(I)-SM(I+J-1)
  411 CONTINUE
      K=I+NSET-1
      IF (KSW.EQ.1) GO TO 507
```

```
C<<<   PROCESS EVEN NUMBERED SETS                                            >>>
       CB=1.0/SET
       CA=12.0/SQNF
       CTH=SQNO/SQNF
       YEV(I)=YEV(I)+Y(K)
       B(I)=(YOD(I)-YEV(I))/SET
       TWOB(I)=2.0*B(I)
       YSUM(I)=YSUM(I)+Y(K-1)+Y(K)
       A(I)=2.0*(YSUM(I)/SET-TH(I))
       DO 412 J=1,NSET,2
       ADJ(I,J)=TH(I)+B(I)+A(I)*(CONST-J)
       ADJ(I,NSET+1-J)=ADJ(I,J)-TWOB(I)
  412  CONTINUE
       GO TO 508
C<<<   PROCESS ODD NUMBERED SETS                                             >>>
  507  CB=NSET/SQNO
       CA=12.0/(NSET*SQNO)
       CTH=CB
       C=NSET/2
       DIV=C*SET+C
       YOD(I)=YOD(I)+Y(K)
       B(I)=(C*YOD(I)-(C+1.0)*YEV(I))/DIV
       DO 413 J=1,NSET,2
       ADJ(I,J)=TH(I)+B(I)+A(I)*(CONST-J)
       ADJ(I,NSET+2-J)=TH(I)-B(I)+A(I)*(CONST+1.0-J)
  413  CONTINUE
C<<<   CONTINUE CALCULATION BASED ON COMMON FORMULAE                         >>>
  508  SUM(I)=0.0
       ALFA(I)=A(I)/B(I)
       DO 414 J=1,NSET
       VOB(I,J)=ADJ(I,J)-Y(I+J-1)
       SUM(I)=SUM(I)+VOB(I,J)**2
  414  CONTINUE
       SIGMA=DSQRT(SUM(I)/LINK)
       SIG(I)=SIGMA
       SIGB(I)=DSQRT(CB)*SIGMA
       SIGA(I)=DSQRT(CA)*SIGMA
       SIGTH(I)=DSQRT(CTH)*SIGMA
       SIGAL(I)=SIGA(I)/B(I)
       IF (SIGAL(I).LT.0.0) SIGAL(I)=-SIGAL(I)
  410  CONTINUE
       CALL RECONV(SAV,5,N,0,IDAV,MAV,SSAV)
       CALL RECONV(TH,5,N,0,IDTH,MTH,STH)
       CALL RECONV(B,6,N,1,IDB,MB,SB)
```

```
C<<<   TRANSFER ADJUSTED VALUES TO VECTOR FOR RECONVERSION                    >>>
      DO 415 I=1,N
      DO 415 J=1,NSET
      NORD=I+(J-1)*N
      ADJVEC(NORD)=ADJ(I,J)
  415 CONTINUE
      CALL RECONV(ADJVEC,40,N*NSET,0,IDAVEC,MAVEC,SAVEC)
C<<<   TRANSFER ADJUSTED VALUES TO MATRIX FOR PRINT OUT                       >>>
      DO 416 I=1,N
      DO 416 J=1,NSET
      NORD=I+(J-1)*N
      IDA(I,J)=IDAVEC(NORD)
      MA(I,J)=MAVEC(NORD)
      SA(I,J)=SAVEC(NORD)
  416 CONTINUE
      CALL SCRIBE
  404 CONTINUE
      NDATA=NDATA+1
      GO TO 509
C<<<   TERMINATION MESSAGES                                                   >>>
  999 WRITE(NP,301) NDATA
  301 FORMAT ('1  A DATA TRANSFER ERROR CONDITION HAS OCCURRED -- DATA S
     1ET NO.',I2,'DELETED'//' ',132('X'))
      GO TO 509
  500 WRITE (NP,302)
  302 FORMAT ('1  ALL DATA PROCESSED -- NORMAL TERMINATION'//' ',132('|'
     1))
  510 STOP
      END

      SUBROUTINE RECONV (DATA,NDIM,LOOP,KSW,ID,MN,SC)
C<<<   SUBROUTINE RECONV FOR THE RECONVERSION OF SECONDS OF ARC TO            >>>
C<<<   DEGREES, MINUTES AND SECONDS.   NEGATIVE ANGLES ARE INCREASED BY       >>>
C<<<   360 DEGREES IF KSW=1.                                                  >>>
      IMPLICIT REAL*8 (A-G,O-Z), REAL*4 (H)
      DIMENSION DATA(NDIM),ID(NDIM),MN(NDIM),SC(NDIM)
      DO 431 I=1,LOOP
      X=DATA(I)
      IF (KSW.EQ.0.AND.X.LT.0.0) X=X+1296000.0
      ID(I)=X/3600.0
      IF (X.LT.0.0) X=-X
      SC(I)=DMOD(X,6.0D1)
      IF (SC(I).GE.59.995) SC(I)=SC(I)-0.005
      D=ID(I)
      IF (D.LT.0.0) D=-D
      MN(I)=X/60.0-D*60.0
  431 CONTINUE
      RETURN
      END
```

```
      SUBROUTINE SCRIBE
C<<<   SUBROUTINE SCRIBE PRINTS ALL HEADINGS AND OUTPUT DATA.  VARIABLE      >>>
C<<<   FORMATS ARE STORED IN LABELLED COMMON BLOCK "FORMAT", AND CHANGED     >>>
C<<<   DURING EXECUTION FOR APPROPRIATE LAYOUT OF EITHER LEFT OR RIGHT       >>>
C<<<   READINGS.                                                              >>
      IMPLICIT REAL*8 (A-G,O-Z), REAL*4 (H)
      COMMON SA(5,8),VOB(5,8),STH(5),SB(6),ALFA(6),A(6),SIG(5),SIGTH(5),
     1 SIGB(5),SIGAL(5),SIGA(5),SS(6),VS(5,6),SSAV(5),HDESC(3),HGYRO,
     2 IDEG(8),MIN(8),ISEC(8),IDA(5,8),MA(5,8),TDTH(5),MTH(5),IDB(6),
     3 MB(6),IDS(6),MS(6),IDAV(5),MAV(5),LEFT,N,NSET,IDAY,MONTH,IYR,NTWO
      COMMON/FORMAT/ HFMTA(19),HFMTB(15),HFMTC(14),HFMTD(8),HTABX(2),
     2 HTABY(2),HFMTE(9),HZERO,HONE,HPLUS,HBLNK,LSW(2),NP,NR
      EQUIVALENCE (HTAX,HFMTA(3)),(HTAY,HFMTA(5)),(HTBX,HFMTB(3)),(HTBY,
     2 HFMTB(10)),(HTCX,HFMTC(3)),(HTDX,HFMTD(3)),(HCCB,HFMTB(1)),
     3 (HSCALE,HFMTE(6))
C
      WRITE (NP,351) HGYRO,HDESC,IDAY,MONTH,IYR,NSET
  351 FORMAT ('1',T41,'G Y R O   T H E O D O L I T E   A D J U S T M E N
     1 T'//' ',132('-')/' GYRO: GAK1 - ',A4,15X,'JOB IDENTIFICATION: ',
     2 3A4,15X,'DATE: ',2(I2,'/'),I2,15X,'GROUP OF',I2,' OBSERVATIONS'/
     3 ' ',132('-'))
      WRITE (NP,352)
  352 FORMAT (//T8,'OBSERVATIONS',T49,'ADJUSTED OBSERVATIONS',T82,'SINGL
     1E',T106,'MIDDLE       DAMPING'/T7,'LEFT       RIGHT     SCHULER MEAN
     2   V',9X,'LEFT         RIGHT',9X,'V       OBSN        THETA        AMPLIT
     3UDE       ALPHA        A')
      WRITE (NP,353)
  353 FORMAT (4X,18('-'),2X,12('-'),'  ------    ',24('-'),2(2X,6('-')),
     1 2X,12('-'),2X,11('-'),2X,9('-'),'  ------')
      WRITE (NP,354)
  354 FORMAT ('+',3X,18('_'),2X,12('_'),'  ______    ',24('_'),2(2X,6('_'
     1)),2X,12('_'),2X,11('_'),2X,9('_'),'  ______'/)
      LCC=LEFT
      IF (NSET.EQ.3) GO TO 551
```

```
C<<<    PRINT OUT OF SETS FOUR TO EIGHT                                        >>>
        DO 451 I=1,N
        LOCA=LOC
        HTAX=HTABX(LOC)
        HTAY=HTABY(LOC)
        HCOR=HPLUS
        HSCALE=HZERO
        IF (SIGAL(I).GE.1.1D-2) HSCALE=HONE
        WRITE (NP,HFMTA) I,IDEG(I),MIN(I),ISEC(I),IDA(I,1),MA(I,1),SA(I,1)
       1,VCR(I,1),IDTH(I),MTH(I),STH(I),IDB(I),MB(I),SB(I),ALFA(I),N(I)
        WRITE (NP,HFMTC) SIG(I),SIGTH(I),SIGB(I),SIGAL(I),SIGA(I)
        DO 452 J=1,NTWO
        K=I+J
        KK=K-1
        LOC=LSW(LOC)
        L=J+1
        HTBX=HTABX(LOC)
        HTBY=HTABY(LOC)
        IF (J.EQ.2) HCOR=HBLANK
        WRITE (NP,HFMTB) K,IDEG(K),MIN(K),ISEC(K),IDS(KK),MS(KK),SS(KK),
       1VS(I,J),IDA(I,L),MA(I,L),SA(I,L),VCR(I,L)
  452   CONTINUE
        LOC=LSW(LOC)
        HTAX=HTABX(LOC)
        HTAY=HTABY(LOC)
        K=K+1
        WRITE (NP,HFMTA) K,IDEG(K),MIN(K),ISEC(K),IDA(I,NSET),MA(I,NSET),
       1 SA(I,NSET),VCR(I,NSET)
        WRITE (NP,356) IDAV(I),MAV(I),SSAV(I)
  356   FORMAT (/T19,'MEAN: ',2I4,F6.2)
        WRITE (NP,353)
        WRITE (NP,357)
  357   FORMAT (1H )
        LOC=LSW(LOCA)
  451   CONTINUE
        RETURN
```

```
C<<<    PRINT OUT OF SET OF THREE                                              >>>
  551 DO 453 I=1,6
      HTCX=HTABX(LCC)
      WRITE (NP,HFMTC) I,IDEG(I),MIN(I),ISEC(I),IDS(I),MS(I),SS(I),IDB(I
     1),MB(I),SB(I),ALFA(I),A(I)
      K=I+1
      LOC=LSW(LCC)
      HTCX=HTABX(LOC)
      WRITE (NP,HFMTC) K,IDEG(K),MIN(K),ISEC(K),IDS(I),MS(I),SS(I)
      LOC=LSW(LOC)
      HTCX=HTABX(LOC)
      K=K+1
      WRITE (NP,HFMTC) K,IDEG(K),MIN(K),ISEC(K)
      WRITE (NP,353)
      LCC=LSW(LCC)
  453 CONTINUE
      RETURN
      END
```

```
      BLOCK DATA
      IMPLICIT REAL*8 (A-H,O-Z), REAL*4 (N)
      COMMON/FORMAT/ HFMTA(19),HFMTB(15),HFMTC(14),HFMTD(8),HTABX(2),
     2 HTABY(2),HFMTE(9),HZERO,HONE,HPLUS,HBLNK,LSW(2),NP,NR
C<<<  INITIALIZE STORED FORMATS                                      >>>
      DATA HFMTA/'(1H ','[illegible]','I1,','T15,','[illegible]','T4[illegible]','2I3,','[illegible]','T7',
     1'4,F6','[illegible]','2','(2I3','F6.','[illegible]','2,1X','[illegible]','1','[illegible]','2X',
     2'[illegible]','6.2)'/
      DATA HFMTB/'(1H+','[illegible]','I1,','T14,','2I3,','T25,','2I[illegible]','[illegible]','2X,',
     3'F6.2','[illegible]','16','[illegible]','2I3','F6.','2,','T7','[illegible]')'/
      DATA HFMTC/'(1H ','[illegible]','I1,','T05,','2I3,','T[illegible]','2(2I','[illegible]','2,1',
     1'X),1','PD1','[illegible]','3.2','X,','F','F6.2',')'/
      DATA HFMTD/'(1H ','[illegible]','I1,','T14,','2I3,','T25,','2I3,','F6.','[illegible]')'
     1/
      DATA HFMTE/'(T32','[illegible]','F6.','2,1X','2F1','3.2','PD1','1.3',
     1'OPF8','.2)'/
      DATA HTABX/'T15,','T14,'/, HTABY/'T49','T4 '/
      DATA HZERO,HONE,HPLUS,HBLNK/'OPD1','1PD1','(1H+','(1H '/
      DATA LSW/2,1/
C<<<  SET LOGICAL UNIT NUMBERS FOR IBM 360/50, UNIV. OF N.S.W.      >>>
      DATA NP,NR/3,1/
      END
```

TYPICAL DATA CARD

```
COLUMN    1         2         3         4         5         6         7         8
12345678901234567890123456789012345678901234567890123456789012345678901234567890

2871 123567 UNSW 7[illegible] BI[illegible]  21[illegible]3575106   214[illegible]75148   2141[illegible]575254   21[illegible]6357[illegible]12
```

G Y R O T H E O D O L I T E A D J U S T M E N T

GYRO: GAK1 - 2871   JOB IDENT: UNSW 707 B10   DATE: 2/ 5/67   GROUP OF 3 OBS.

	Observations Left	Observations Right	Schuler Mean	V	Adjusted Observations Left	Adjusted Observations Right	V
1		2 15 6					
2	357 51 6		0 2 58.50				
3		2 14 36					
2	357 51 6						
3		2 14 36	0 3 1.50				
4	357 51 48						
3		2 14 36					
4	357 51 48		0 3 7.50				
5		2 14 18					
4	357 51 48						
5		2 14 18	0 3 19.50				
6	357 52 54						
5		2 14 18					
6	357 52 54		0 3 33.00				
7		2 14 6					
6	357 52 54						
7		2 14 6	0 3 34.50				
8	357 53 12						

G Y R O   T H E O D O L I T E   A N A L Y S I S

GYRO: GAK1 - 2871   JOB IDENT: UNSW 7[illegible]7 BI[illegible]   DATE: 2/ 5/67   GROUP OF 3 OBS.

	SINGLE OBSN	THETA	MIDDLE AMPLITUDE	DAMPING ALPHA	A
VALUE		0 2 58.50	2 11 52.50	1.896D-03	15.00
STANDARD DEVIATION					
VALUE		0 3 1.50	-2 11 34.50	2.660D-03	-21.00
STANDARD DEVIATION					
VALUE		0 3 7.50	2 11 19.50	1.142D-03	9.00
STANDARD DEVIATION					
VALUE		0 3 19.50	-2 10 58.50	4.199D-03	-33.00
STANDARD DEVIATION					
VALUE		0 3 33.00	2 10 39.00	7.654D-04	6.00
STANDARD DEVIATION					
VALUE		0 3 34.50	-2 10 31.50	1.149D-03	-9.00
STANDARD DEVIATION					

G Y R O   T H E O D O L I T E   A D J U S T M E N T

GYRO: GAK1 - 2871 JOB IDENT: UNSW 707 B10 DATE: 2/ 5/67 GROUP OF 4 OBS.

	OBSERVATIONS LEFT	OBSERVATIONS RIGHT	SCHULER MEAN	V	ADJUSTED OBSERVATIONS LEFT	ADJUSTED OBSERVATIONS RIGHT	V
1		2 15 6				2 15 9.00	3.00
2	357 51 6		0 2 58.50	1.50	357 51 9.00		3.00
3		2 14 36	0 3 1.50	-1.50		2 14 33.00	-3.00
4	357 51 48				357 51 45.00		-3.00
		MEAN:	0 3 0.0				
2	357 51 6				357 51 12.00		6.00
3		2 14 36	0 3 1.50	3.00		2 14 42.00	6.00
4	357 51 48		0 3 7.50	-3.00	357 51 42.00		-6.00
5		2 14 18				2 14 12.00	-6.00
		MEAN:	0 3 4.50				
3		2 14 36				2 14 48.00	12.00
4	357 51 48		0 3 7.50	6.00	357 52 0.00		12.00
5		2 14 18	0 3 19.50	-6.00		2 14 6.00	-12.00
6	357 52 54				357 52 42.00		-12.00
		MEAN:	0 3 13.50				
4	357 51 48				357 52 1.50		13.50
5		2 14 18	0 3 19.50	6.75		2 14 31.50	13.50
6	357 52 54		0 3 33.00	-6.75	357 52 40.50		-13.50
7		2 14 6				2 13 52.50	-13.50
		MEAN:	0 3 26.25				
5		2 14 18				2 14 19.50	1.50
6	357 52 54		0 3 33.00	0.75	357 52 55.50		1.50
7		2 14 6	0 3 34.50	-0.75		2 14 4.50	-1.50
8	357 53 12				357 53 10.50		-1.50
		MEAN:	0 3 33.75				

GYRO THEODOLITE ANALYSIS

GYRO: GAK1 - 2871 JOB IDENT: UNSW 707 810 DATE: 2/ 5/67 GROUP OF 4 OBS.

	SINGLE OBSN	THETA	MIDDLE AMPLITUDE	DAMPING ALPHA	A
VALUE		0 3 0.00	2 11 42.00	2.278D-03	18.00
STANDARD DEVIATION	6.00	3.35	3.00	1.38[illegible]D-03	3.00
VALUE		0 3 4.50	-2 11 30.00	1.9[illegible]1D-03	-15.00
STANDARD DEVIATION	12.00	6.71	6.00	[illegible].76[illegible]D-03	6.00
VALUE		0 3 13.50	2 11 3.00	2.671D-03	21.00
STANDARD DEVIATION	24.00	13.42	12.00	1.526D-03	12.00
VALUE		0 3 26.25	-2 10 55.50	2.48[illegible]D-03	-19.50
STANDARD DEVIATION	27.00	15.09	13.50	1.719D-03	13.50
VALUE		0 3 33.75	2 10 34.50	9.57[illegible]D-04	7.50
STANDARD DEVIATION	3.00	1.68	1.50	[illegible].191D-03	1.50

G Y R O   T H E O D O L I T E   A D J U S T M E N T

GYRO: GAK1 - 2871   JOB IDENT: UNSW 707 01   DATE: 2/ 5/67   GROUP OF 5 OBS.

	OBSERVATIONS LEFT	OBSERVATIONS RIGHT	SCHULER MEAN	V	ADJUSTED OBSERVATIONS LEFT	ADJUSTED OBSERVATIONS RIGHT	V
1		2 15 6				2 15 7.60	1.60
2	357 51 6		0 2 58.50	4.00	357 51 13.20		7.20
3		2 14 36	0 3 1.50	1.00		2 14 40.00	4.00
4	357 51 48		0 3 7.50	-5.00	357 51 40.80		-7.20
5		2 14 18				2 14 12.40	-5.60
		MEAN:	0 3 2.50				
2	357 51 6				357 51 9.20		3.20
3		2 14 36	0 3 1.50	8.00		2 14 50.40	14.40
4	357 51 48		0 3 7.50	2.00	357 51 56.00		8.00
5		2 14 18	0 3 19.50	-10.00		2 14 3.60	-14.40
6	357 52 54				357 52 42.80		-11.20
		MEAN:	0 3 9.50				
3		2 14 36				2 14 45.20	9.20
4	357 51 48		0 3 7.50	12.50	357 52 8.40		20.40
5		2 14 18	0 3 19.50	0.50		2 14 20.00	2.00
6	357 52 54		0 3 33.00	-13.00	357 52 33.60		-20.40
7		2 14 6				2 13 54.80	-11.20
		MEAN:	0 3 20.00				
4	357 51 48				357 52 2.00		14.00
5		2 14 18	0 3 19.50	9.50		2 14 30.00	12.00
6	357 52 54		0 3 33.00	-4.00	357 52 38.00		-16.00
7		2 14 6	0 3 34.50	-5.50		2 13 54.00	-12.00
8	357 53 12				357 53 14.00		2.00
		MEAN:	0 3 29.00				

GYRO THEODOLITE ANALYSIS

GYRO: OAK1 - 2871 JOB IDENT: UNSW 7[illegible]7 B1[illegible] DATE: 2/ 5/67 GROUP OF 5 OBS.

	SINGLE OBSN	THETA	MIDDLE AMPLITUDE	DAMPING ALPHA	A
VALUE		0 3 3.5[illegible]	2 11 36.50	1.745D-03	13.8[illegible]
STANDARD DEVIATION	8.76	4.0[illegible]	4.[illegible]	[illegible].351D-03	2.77
VALUE		0 3 11.50	-2 11 15.50	2.971D-03	-23.4[illegible]
STANDARD DEVIATION	17.53	8.[illegible]	8.[illegible]	[illegible].7[illegible]4D-03	5.54
VALUE		0 3 2[illegible].5[illegible]	2 1 59.50	1.6[illegible]3D-03	12.6[illegible]
STANDARD DEVIATION	22.87	1[illegible].44	1[illegible].44	[illegible].92[illegible]D-03	7.23
VALUE		0 3 25.[illegible]	-2 1[illegible] 47.[illegible]	2.294D-[illegible]3	-18.[illegible]
STANDARD DEVIATION	19.29	8.80	8.8[illegible]	[illegible].777D-03	6.1[illegible]

GYRO THEODOLITE ADJUSTMENT

GYRO: GAK1 - 2871 JOB IDENT: UNSW 707 B10 DATE: 2/ 5/67 GROUP OF 6 OBS.

	Observations Left	Observations Right	Schuler Mean	V	Adjusted Observations Left	Adjusted Observations Right	V
1		2 15 6				2 15 19.00	13.00
2	357 51 6		0 2 58.50	8.25	357 51 17.00		11.00
3		2 14 36	0 3 1.50	5.25		2 14 40.00	4.00
4	357 51 48		0 3 7.50	-0.75	357 51 56.00		8.00
5		2 14 18	0 3 19.50	-12.75		2 14 1.00	-17.00
6	357 52 54				357 52 35.00		-19.00
		MEAN:	0 3 6.75				
2	357 51 6				357 51 21.50		15.50
3		2 14 36	0 3 1.50	13.88		2 14 54.50	18.50
4	357 51 48		0 3 7.50	7.88	357 51 56.00		8.00
5		2 14 18	0 3 19.50	-4.13		2 14 20.00	2.00
6	357 52 54		0 3 33.00	-17.63	357 52 30.50		-23.50
7		2 14 6				2 13 45.50	-20.50
		MEAN:	0 3 15.38				
3		2 14 36				2 14 48.50	12.50
4	357 51 48		0 3 7.50	16.13	357 52 9.50		21.50
5		2 14 18	0 3 19.50	4.13		2 14 20.00	2.00
6	357 52 54		0 3 33.00	-9.38	357 52 38.00		-16.50
7		2 14 6	0 3 34.50	-10.88		2 13 51.50	-14.50
8	357 53 12				357 53 6.50		-5.50
		MEAN:	0 3 23.63				

G Y R O   T H E O D O L I T E   A N A L Y S I S

GYRO: GAK1 - 2871   JOB IDENT: UNSW 707 B10   DATE: 2/ 5/67   GROUP OF 6 OBS.

	SINGLE OBSN	THETA	MIDDLE AMPLITUDE	DAMPING ALPHA	A
VALUE		0 3 8.25	2 11 22.00	2.474D-03	19.53
STANDARD DEVIATION	18.44	7.87	7.53	0.585D-03	4.61
VALUE		0 3 16.63	-2 11 12.00	2.191D-03	-17.25
STANDARD DEVIATION	23.26	9.93	9.50	0.739D-03	5.81
VALUE		0 3 21.88	2 10 51.00	1.815D-03	14.25
STANDARD DEVIATION	19.31	8.25	7.88	0.615D-03	4.83

G Y R O   T H E O D O L I T E   A D J U S T M E N T

GYRO: GAKI - 2871   JOB IDENT: UNSW 737 B16   DATE: 2/ 5/67   GROUP OF 7 OBS.

	OBSERVATIONS LEFT	OBSERVATIONS RIGHT	SCHULER MEAN	V	ADJUSTED OBSERVATIONS LEFT	ADJUSTED OBSERVATIONS RIGHT	V
1		2 15 6				2 15 15.86	9.86
2	357 51 6		0 2 58.50	13.50	357 51 26.43		20.43
3		2 14 36	0 3 1.50	10.50		2 14 46.29	10.29
4	357 51 48		0 3 7.50	4.50	357 51 56.00		8.00
5		2 14 18	0 3 19.50	-7.50		2 14 16.71	-1.29
6	357 52 54		0 3 33.00	-21.00	357 52 25.57		-28.43
7		2 14 6				2 13 47.14	-18.86
		MEAN:	0 3 12.00				
2	357 51 6				357 51 21.00		15.00
3		2 14 36	0 3 1.50	17.70		2 14 56.00	20.00
4	357 51 48		0 3 7.50	11.70	357 51 57.00		9.00
5		2 14 18	0 3 19.50	-0.30		2 14 20.00	2.00
6	357 52 54		0 3 33.00	-13.80	357 52 33.00		-21.00
7		2 14 6	0 3 34.50	-15.30		2 13 44.00	-22.00
8	357 53 12				357 53 9.00		-3.00
		MEAN:	0 3 19.20				

G Y R O   T H E O D O L I T E   A N A L Y S I S

GYRO: CAK1 - 2871   JOB IDENT: UNSW 707 B10   DATE: 2/ 5/67   GROUP OF 7 OBS.

	SINGLE OBSN	THETA	MIDDLE AMPLITUDE	DAMPING ALPHA	A
VALUE		0 3 13.75	2 11 17.75	1.877D-03	14.79
STANDARD DEVIATION	21.50	8.21	8.21	0.516D-03	4.06
VALUE		0 3 17.50	-2 11 2.50	2.289D-03	-18.00
STANDARD DEVIATION	20.27	7.74	7.74	0.487D-03	3.83

G Y R O   T H E O D O L I T E   A D J U S T M E N T

GYRO: GAK1 - 2871   JOB IDENT: UNSW 707 B10   DATE: 2/ 5/67   GROUP OF 8 OBS.

	OBSERVATIONS LEFT	OBSERVATIONS RIGHT	SCHULER MEAN	V	ADJUSTED OBSERVATIONS LEFT	ADJUSTED OBSERVATIONS RIGHT	V
1		2 15 6				2 15 19.65	13.65
2	357 51 6		0 2 58.50	17.25	357 51 26.85		20.85
3		2 14 36	0 3 1.50	14.25		2 14 47.55	11.55
4	357 51 48		0 3 7.50	8.25	357 51 58.95		10.95
5		2 14 18	0 3 19.50	-3.75		2 14 15.45	-2.55
6	357 52 54		0 3 33.00	-17.25	357 52 31.05		-22.95
7		2 14 6	0 3 34.50	-18.75		2 13 43.35	-22.65
8	357 53 12				357 53 3.15		-8.85
		MEAN:	0 3 15.75				

G Y R O   T H E O D O L I T E   A N A L Y S I S

GYRO: GAK1 - 2871   JOB IDENT: UNSW 707 B10   DATE: 2/ 5/67   GROUP OF 8 OBS.

	SINGLE OBSN	THETA	MIDDLE AMPLITUDE	DAMPING ALPHA	A
VALUE		0 3 15.23	2 11 8.25	2.040D-03	16.05
STANDARD DEVIATION	19.99	7.24	7.07	0.402D-03	3.16

APPENDIX III

COMPUTER PROGRAMME AND SAMPLE OUTPUT FOR THE LEAST SQUARES ANALYSIS OF THE MODIFIED TRANSIT METHOD

```
/* PL/1 PROGRAMME "GYROTRAN"  (FOR IBM 360/50):    SVY154

   COMPUTES ADJUSTED PARAMETERS OF THE GYRO OSCILLATION EQUATION -- PERIOD
   AND PHASE -- AND THE WIDTH OF THE GYRO MARK, FROM OBSERVATIONS BY A
   MODIFIED TRANSIT METHOD OVER TWO CYCLES.  PARAMETRIC EQUATIONS ARE FORMED
   USING TRANSIT TIME OBSERVATIONS; REDUNDANCIES ARE RESOLVED BY LEAST
   SQUARES SOLUTION.  NORMAL EQUATIONS SOLVED BY INVERSION AND MATRIX
   MULTIPLICATION.  LEAST SQUARES RESIDUALS INCLUDED IN OUTPUT.
              ----------------------------------------------
   PREPARED FOR G. G. BENNETT BY E. G. ANDERSON, DEPT. OF SURVEYING,
   UNIVERSITY OF N.S.W., JUNE 1969.
              ----------------------------------------------
   N = NUMBER OF TRANSITS PER QUARTER PERIOD (EXCLUDING AXIAL TRANSIT)
   A = GRADUATION SPACING (RADIANS)
   IDENT = DATA SET IDENTIFICATION
   SENSE = DIRECTION OF TRAVEL DURING FIRST HALF CYCLE:
           LR = LEFT TO RIGHT,  RL = RIGHT TO LEFT
              ----------------------------------------------
   DATA CARDS:

   CARD TYPE ONE:     "N"        F(2)     COLS  1 - 2,
                      "A"        F(6,2)   COLS  4 - 9,
                      "SENSE"    A(2)     COLS 11 - 12,

   ** WARNING **  THERE MUST BE ONLY ONE BLANK BETWEEN "A" AND "SENSE".

           "IDENT":   GYRO       A(12)    COLS 13 - 24,
                      DATE       A(9)     COLS 25 - 33,
                      PLACE      A(30)    COLS 34 - 63,
                      OBSERVER   A(6)     COLS 64 - 69,
                      RUN UP     A(10)    COLS 70 - 79,

   CARD TYPE TWO:     THEODOLITE    )
                      TIME          )     COLS 13 - 79
                      LINE          )      AS ABOVE
                      RECORDER      )
                      BATTERY       )

   CARD TYPE THREE:   OBSERVATIONS.
                      3N+4 OBSERVATIONS ARE REQUIRED, EACH MUST
                      SPECIFY MINUTES AND SECONDS IN FREE FORMAT.
              ----------------------------------------------  */
```

```
GYROCYC:  PROC OPTIONS (MAIN);
          DCL   (NCOEF(*,*),NCON(*),VAR(*),PROD(*)) CTL DEC FLOAT (16),
                (T,TC,W,WD) DEC FIXED (6,2),
                (TOD,P,TDIF) DEC FIXED (5,1),
                DT DEC FIXED (6,3),
                A DEC FIXED (5,2),
                (FA,FB,ARG,FC,TD,SIGMA) DEC FLOAT (16),
                PI DEC FLOAT (16) INIT (3.141592653589 79),
                CLAS (10) CHAR (12) VAR INIT ('GYRO: ','DATE: ','PLACE: ',
                 'OBSERVER: ','RUN UP: ','THEODOLITE: ','TIME: ','LINE: ',
                 'RECORDER: ','BATTERY: '),
                HEAD (10) CHAR (37) VAR,
                IDENT (10) CHAR (30) VAR,
                SENSE CHAR (2),
                NCOUNT INIT (0),
                RND ENTRY (DEC FLOAT (16),DEC FIXED (2))
                    RETURNS (DEC FLOAT (16)),
                K,N,NF,IMAX,IP,ISIGN,NFIP,I,J;
          ON ENDFILE (SYSIN) GO TO EPILOGUE;
          ON ERROR SNAP SYSTEM;
          ON CONVERSION SNAP
                BEGIN;
                PUT LIST ('MISPUNCHED FIELD:  '||ONSOURCE);
                ONSOURCE='0';
                END;
NEWSET:   GET LIST (N,A);       /*  READ CONTROL DATA  */
          NF=N*2+1;
          IMAX=NF*4;
          WD=1.25;
          GET EDIT (SENSE,IDENT) (A(2),(2)(A(12),A(9),A(30),A(6),A(10),X(13)));
          K=(SENSE='RL')-(SENSE='LR');
          IF K=0
           THEN DO;
                PUT LIST ('DIRECTION OF FIRST PASS INCORRECTLY PUNCHED:  '
                   ||SENSE);
                GET LIST ((DUM DO I=1 TO IMAX*2));
                GO TO NEWSET;
                END;
          PUT PAGE EDIT ('GYRO THEODOLITE SURVEY --- MODIFIED TRANSIT METHOD')
            (X(35),A);
          HEAD=CLAS||IDENT;
          PUT SKIP(2) EDIT (HEAD)(COLUMN(5),A,COLUMN(33),A,COLUMN(52),A,
            COLUMN(93),A,COLUMN(113),A);
```

```
EQUAT:     BEGIN;      /*      FORM PARAMETRIC EQUATIONS      */
           DCL    (COEF(IMAX,4),CONS(IMAX),RESID(IMAX),SQRES(IMAX)) DEC FLOAT(16),
                  TI(IMAX) DEC FIXED (5,1),
                  1 TIME (IMAX),
                    2 MIN DEC FIXED (2),  2 SEC DEC FIXED (3,1);
           GET LIST (TIME);         /*  READ OBSERVATIONS  */
           TI=MIN*60.0+SEC;
           P=TI(N*5+3)-TI(N+1);
           TOD=TI(N+2);
           PUT SKIP(2) EDIT ('N=',N,' A=',A,'SENSE= ',SENSE,'P''=',P,'T''(0)=',
             TOD,'W''=',WD) (X(22),A,F(2),X(6),A,F(6,2),X(6),(2)A,X(12),(2)(A,
             F(7,2),X(6)),A,F(5,2));
           PUT EDIT ((132)'-','OBSERVATIONS','PARAMETRIC  EQUATIONS','RESIDUALS',
             'MIN  SEC','@P','@T','@T(0)','@W','C',(132)'-') (SKIP,A,SKIP,X(3),A,
             X(40),A,X(43),A,SKIP,X(5),A,X(16),A,X(17),A,X(16),A,X(13),A,X(15),
             A,SKIP,A);
           FA=PI*2.0/P;
           FB=FA*A;
COEFF:     DO IP=1 TO 4;
           IF (IP=1)|(IP=3)
            THEN ISIGN=-1;
            ELSE ISIGN=1;
           NFIP=NF*IP;
           DO I=NFIP-NF+1 TO NFIP;
           TDIF=TI(I)-TOD;
           ARG=FA*TDIF;
           FC=FB*COS(ARG);
           COEF(I,1)=FC*TDIF/P;
           COEF(I,2)=K*FB;
           COEF(I,3)=FC;
           COEF(I,4)=ISIGN;
           CONS(I)=A*SIN(ARG)-ISIGN*(NFIP-N-I+WD);
           END COEFF;
```

```
CALL PARNORM(NCOEF,NCON,COEF,CONS);
CALL MINVD(NCOEF);
CALL VMULD(VAR,NCOEF,NCON);
DELT=RND(-4*VAR(2),3);
DT=RND(VAR(2),3);
T=RND(P+VAR(1),2);
TO=RND(TCD+VAR(3),2);
W=RND(WD+VAR(4),2);
CALL VMULD(PROD,COEF,VAR);
RESID=(PROD-CONS)/COEF(*,3);
SQRES=RESID*RESID;
SIGMA=SQRT(SUM(SQRES)/(IMAX-4));
VAR=ROUND(VAR,16);
PUT EDIT ((I,TIME(I),(COEF(I,J) DO J=1 TO 4),CONS(I),RESID(I)
   DO I=1 TO IMAX)) (SKIP,F(2),F(5),F(6,1),F(21,6),(2)F(19,6),F(14,1),
   F(20,6),F(22,6));
PUT EDIT ((132)'-','SOLUTION OF NORMALS:',(VAR(I) DO I=1 TO 4),
    'SIGMA=',SIGMA) (SKIP,A,SKIP,A,F(14,6),(2)F(19,6),F(14,6),X(27),A,
    F(9,6));
PUT SKIP(2) EDIT ('ADJUSTED PARAMETERS:','P=',T,' S.','@T=',DT,
    ' S.','T(0)=',TO,' S.','W=',W,'DEL T=',DELT,(132)'-',(132)' ')
    (A,X(2),A,F(7,2),A,X(7),A,F(6,3),A,X(5),A,F(6,2),A,X(7),A,F(5,2),
    X(7),A,F(6,3),SKIP,(2)(SKIP(0),A));
FREE NCOEF,NCON,VAR,PROD;
END EQUAT;
NCOUNT=NCOUNT+1;
GO TO NEWSET;
```

```
PARNORM:   PROC (A,B,C,D);        /*  FORM NORMAL EQUATIONS*/
           DCL    (A(*,*),B(*)) CTL DEC FLOAT (16),
                  (C(*,*),D(*)) DEC FLOAT (16),
                  M,N,L,J,K;
           M=DIM(C,1);        N=DIM(C,2);         L=DIM(D,1);
           IF M¬=L
            THEN DO;
                  PUT LIST ('MATRIX AND VECTOR INCOMPATIBLE',M,N,L);
                  PUT SKIP(2) LIST (C);
                  PUT SKIP(2) LIST (D);
                  GO TO ENDPAR;
                  END;
           ALLOCATE A(N,N),B(N);
MULT:      DO J=1 TO N;
           B(J)=SUM(C(*,J)*D(*));
           DO K=J TO N;
           IF K=J
            THEN A(J,K)=SUM(C(*,J)*C(*,K));
            ELSE A(K,J),A(J,K)=SUM(C(*,J)*C(*,K));
           END MULT;
ENDPAR:    IF N>M
            THEN PUT SKIP(2) LIST ('MORE VARIABLES THAN EQUATIONS',M,N);
           END PARNORM;


MINVD:     PROC (A);      /*  INVERT NORMAL MATRIX  */
           DCL    (A(*,*),D) DEC FLOAT (16),
                  N,NN,I,K;
           N=DIM(A,1);        NN=DIM(A,2);
           IF N¬=NN
            THEN PUT SKIP(2) LIST ('MATRIX NOT SQUARE',N,NN,A);
           ON ZERODIVIDE
                  BEGIN;
                  PUT LIST ('ZERO ON DIAGONAL') SKIP;
                  GO TO EMINV;
                  END;
PIVOT:     DO K=1 TO N;
           D=A(K,K);
           A(K,K)=1;
           A(K,*)=A(K,*)/D;
           DO I=1 TO K-1,K+1 TO N;
           D=A(I,K);
           A(I,K)=0;
           A(I,*)=A(I,*)-D*A(K,*);
           END PIVOT;
EMINV:     END MINVD;
```

```
VMULD:     PROC (A,B,C);        /*  MATRIX MULTIPLICATION  */
           DCL    A(*) CTL DEC FLOAT (16),
                  (B(*,*),C(*)) DEC FLOAT (16),
                  N,M,MM,I;
           N=DIM(B,1);          M=DIM(B,2);          MM=DIM(C,1);
           IF MM¬=M
            THEN DO;
                  PUT LIST ('MATRIX AND VECTOR INCOMPATIBLE',N,M,MM);
                  PUT SKIP(2) LIST (B);
                  PUT SKIP(2) LIST (C);
                  GO TO ENMUL;
                  END;
           ALLOCATE A(N);
           DO I=1 TO N;
           A(I)=SUM(B(I,*)*C(*));
           END;
ENMUL:     END VMULD;



RND:       PROC (AGT,NDEC) DEC FLOAT (16);
           DCL    AGT DEC FLOAT (16),
                  NDEC DEC FIXED (2);
           RETURN (SIGN(AGT)*(ABS(AGT)+0.5/10**NDEC));
           END RND;

EPILOGUE:  PUT PAGE EDIT (NCOUNT,' DATA SET(S) PROCESSED') (F(10),A);
           END GYROCYC;
```

TYPICAL DATA CARDS, (TWO SETS)

```
COLUMN   1         2         3         4         5         6         7         8
12345678901234567890123456789012345678901234567890123456789012345678901234567890

 5  12.10 LRCAK1/3243   19/7/69  ROOM707, CIVIL ENG.        (1) CCR   88 SECS
           T16/112615  10.20 AM CE 707 TO BIOLOGICAL SCIENCES CCR   EXT.
00 00.0 00 06.2 00 11.6 00 17.1 00 22.1 00 27.1 00 32.4 00 37.3 00 42.5 00 47.8
00 53.0 03 08.7 03 15.0 03 20.4 03 25.9 03 31.2 03 36.3 03 41.3 03 46.6 03 51.8
03 57.1 04 02.2 06 29.9 06 34.4 06 40.0 06 45.0 06 50.4 06 55.4 07 00.6 07 05.9
07 11.3 07 16.5 07 21.5 09 37.0 09 43.0 09 48.6 09 54.2 09 59.5 10 04.7 10 10.0
10 15.0 10 20.0 10 25.4 10 30.3
 4  12.10 LRGAK1/3243   19/7/69  ROOM707, CIVIL ENG.       (1A) CCR   88 SECS
           T16/112615  10.20 AM CE 707 TO BIOLOGICAL SCIENCES CCR   EXT.
00 11.6 00 17.1 00 22.1 00 27.1 00 32.4 00 37.3 00 42.5 00 47.8 00 53.0 03 20.4
03 25.9 03 31.2 03 36.3 03 41.3 03 46.6 03 51.8 03 57.1 04 02.2 06 40.0 06 45.0
06 50.4 06 55.4 07 00.6 07 05.9 07 11.3 07 16.5 07 21.5 09 48.6 09 54.2 09 59.5
10 04.7 10 10.0 10 15.0 10 20.0 10 25.4 10 30.3
```

```
                 GYRO THEODOLITE SURVEY --- MODIFIED TRANSIT METHOD

GYRO:        GAK1/3243                     THEODOLITE:  T16/112615
DATE:        19/7/69                       TIME:        10.20 AM
PLACE:       ROOM 707, CIVIL ENG.   (1)    LINE:        CE 707 TO BIOLOGICAL SCIENCES
OBSERVER:    GOB                           RECORDER:    GOB
RUN UP:      88 SECS                       BATTERY:     EXT.

 N = 5     A = 12.10     SENSE = LR     P' = 388.30     T'(0) = 32.40     W' = 1.25
--------------------------------------------------------------------------------------

                         ADJUSTED PARAMETERS
 P = 388.37 S.    dT = -2.492 S.    T(0) = 31.55 S.     W = 1.34    DEL T = 9.973
======================================================================================
```

OBSERVATIONS	MIN	SEC	PARAMETRIC EQUATIONS ∂P	∂T	∂T(0)	∂W	CONST	RESIDUALS
1	0	0.0	-0.014143	-0.195793	0.169496	-1.0	0.192936	0.372012
2	0	6.2	-0.012041	-0.195793	0.178460	-1.0	0.272595	-0.134375
3	0	11.6	-0.009900	-0.195793	0.184808	-1.0	0.253953	-0.057749
4	0	17.1	-0.007480	-0.195793	0.189924	-1.0	0.284870	-0.240673
5	0	22.1	-0.005122	-0.195793	0.193080	-1.0	0.242652	-0.031464
6	0	27.1	-0.002663	-0.195793	0.195074	-1.0	0.213567	0.110115
7	0	32.4	0.000000	-0.195793	0.195793	-1.0	0.250000	-0.078594
8	0	37.3	0.002463	-0.195793	0.195178	-1.0	0.208382	0.137889
9	0	42.5	0.005025	-0.195793	0.193184	-1.0	0.218721	0.095419
10	0	47.8	0.007525	-0.195793	0.189746	-1.0	0.234108	0.032292
11	0	53.0	0.009815	-0.195793	0.185016	-1.0	0.209064	0.190972
12	3	8.7	-0.064486	-0.195793	-0.160205	1.0	0.706082	-0.007080
13	3	15.0	-0.071534	-0.195793	-0.170828	1.0	0.662424	-0.312248
14	3	20.4	-0.077240	-0.195793	-0.178525	1.0	0.718570	-0.018761
15	3	25.9	-0.082646	-0.195793	-0.184964	1.0	0.718314	-0.047107
16	3	31.2	-0.087390	-0.195793	-0.189785	1.0	0.724620	-0.032583
17	3	36.3	-0.091456	-0.195793	-0.193106	1.0	0.747693	0.074247
18	3	41.3	-0.094906	-0.195793	-0.195087	1.0	0.776679	0.214620
19	3	46.6	-0.097922	-0.195793	-0.195793	1.0	0.740210	0.025537
20	3	51.8	-0.100181	-0.195793	-0.195087	1.0	0.723321	-0.057114
21	3	57.1	-0.101716	-0.195793	-0.192947	1.0	0.694399	-0.197715
22	4	2.2	-0.102414	-0.195793	-0.189549	1.0	0.718480	-0.058765
23	6	28.9	0.156481	-0.195793	0.170439	-1.0	0.294918	-0.167430
24	6	34.4	0.166251	-0.195793	0.178329	-1.0	0.254666	0.031758
25	6	40.0	0.175055	-0.195793	0.184912	-1.0	0.272439	-0.092566
26	6	45.0	0.181847	-0.195793	0.189509	-1.0	0.209004	0.226192
27	6	50.4	0.137959	-0.195793	0.193080	-1.0	0.242652	0.034122
28	6	55.4	0.192411	-0.195793	0.195074	-1.0	0.213567	0.175702
29	7	0.6	0.195743	-0.195793	0.195793	-1.0	0.230421	0.086977
30	7	5.9	0.197713	-0.195793	0.195101	-1.0	0.266924	-0.096142
31	7	11.3	0.198187	-0.195793	0.192920	-1.0	0.315248	-0.337966
32	7	16.5	0.197137	-0.195793	0.189429	-1.0	0.309943	-0.300923
33	7	21.5	0.194707	-0.195793	0.184808	-1.0	0.246047	0.057647
34	9	37.0	-0.224691	-0.195793	-0.160205	1.0	0.706082	0.058507
35	9	43.0	-0.241568	-0.195793	-0.170361	1.0	0.713602	0.055092
36	9	48.6	-0.255532	-0.195793	-0.178395	1.0	0.736416	0.147430
37	9	54.2	-0.267610	-0.195793	-0.184964	1.0	0.718314	0.018480
38	9	59.5	-0.277175	-0.195793	-0.189785	1.0	0.724620	0.032999
39	10	4.7	-0.284689	-0.195793	-0.193159	1.0	0.728380	0.039623
40	10	10.0	-0.290348	-0.195793	-0.195191	1.0	0.698623	-0.120167
41	10	15.0	-0.293765	-0.195793	-0.195793	1.0	0.720631	-0.008857
42	10	20.0	-0.295259	-0.195793	-0.195114	1.0	0.742831	0.108345
43	10	25.4	-0.294663	-0.195793	-0.192947	1.0	0.694399	-0.132128
44	10	30.3	-0.292108	-0.195793	-0.189707	1.0	0.756406	0.206078
SOLUTION			0.065587	-2.493049	-0.848240	0.087431	SIGMA=	0.155317

```
GYRO THEODOLITE SURVEY --- MODIFIED TRANSIT METHOD

GYRO:       GAK1/3243                     THEODOLITE:  T16/112618
DATE:       19/7/69                       TIME:        1 .2  AM
PLACE:      ROOM 707, CIVIL ENG.  (1)     LINE:        CE 707 TO BIOLOGICAL SCIENCES
OBSERVER:   CGB                           RECORDER:    CGB
RUN UP:     88 SECS                       BATTERY:     EXT.

 N = 4    A = 12.10    SENSE = LR    P' = 388.30    T'(0) = 32.40    W' = 1.25
-------------------------------------------------------------------------------
                           ADJUSTED PARAMETERS
P = 388.38 S.    aT = -2.505 S.    T(0) = 32.88 S.    W =  .6     DEL T = 1 . 2?
===============================================================================
```

OBSERVATIONS	MIN	SEC	PARAMETRIC EQUATIONS ∂P	∂T	∂T(0)	∂W	CONST	RESIDUALS
1	0	11.6	-0.011860	-0.195844	0.179143	-1.0	0.360712	-0.051817
2	0	17.1	-0.009651	-0.195844	0.185469	-1.0	0.364060	-0.216673
3	0	22.1	-0.007437	-0.195844	0.189947	-1.0	0.303113	0.007221
4	0	27.1	-0.005076	-0.195844	0.193181	-1.0	0.261456	0.150939
5	0	32.4	-0.002464	-0.195844	0.195228	-1.0	0.291371	-0.047848
6	0	37.3	0.000000	-0.195844	0.195844	-1.0	0.250000	0.151877
7	0	42.5	0.002614	-0.195844	0.195151	-1.0	0.267186	0.082400
8	0	47.8	0.005221	-0.195844	0.193022	-1.0	0.296475	-0.017354
9	0	53.0	0.007666	-0.195844	0.189555	-1.0	0.291763	0.090267
10	3	20.4	-0.072141	-0.195844	-0.171705	1.0	0.569606	-0.073823
11	3	25.9	-0.077915	-0.195844	-0.179399	1.0	0.603434	-0.065886
12	3	31.2	-0.083084	-0.195844	-0.185469	1.0	0.625940	-0.028151
13	3	36.3	-0.087620	-0.195844	-0.190024	1.0	0.677888	0.091603
14	3	41.3	-0.091589	-0.195844	-0.193233	1.0	0.719224	0.234251
15	3	46.6	-0.095212	-0.195844	-0.195253	1.0	0.689105	0.035128
16	3	51.8	-0.098122	-0.195844	-0.195840	1.0	0.671663	-0.064657
17	3	57.1	-0.100369	-0.195844	-0.195011	1.0	0.635274	-0.230735
18	4	2.2	-0.101795	-0.195844	-0.192859	1.0	0.645643	-0.128962
19	6	40.0	0.167614	-0.195844	0.179399	-1.0	0.396566	-0.081958
20	6	45.0	0.175384	-0.195844	0.185162	-1.0	0.308465	0.266287
21	6	50.4	0.182632	-0.195844	0.190024	-1.0	0.322112	0.081346
22	6	55.4	0.188205	-0.195844	0.193233	-1.0	0.280776	0.225624
23	7	0.6	0.192764	-0.195844	0.195228	-1.0	0.291371	0.128006
24	7	5.9	0.196041	-0.195844	0.195840	-1.0	0.328337	-0.071998
25	7	11.3	0.197895	-0.195844	0.194981	-1.0	0.384226	-0.337839
26	7	16.5	0.198210	-0.195844	0.192748	-1.0	0.392918	-0.335377
27	7	21.5	0.197116	-0.195844	0.189313	-1.0	0.348593	-0.028200
28	9	48.6	-0.243845	-0.195844	-0.171705	1.0	0.569606	0.102030
29	9	54.2	-0.257542	-0.195844	-0.179526	1.0	0.585488	0.007026
30	9	59.5	-0.268748	-0.195844	-0.185571	1.0	0.617388	0.045616
31	10	4.7	-0.277965	-0.195844	-0.190176	1.0	0.639868	0.064078
32	10	10.0	-0.285442	-0.195844	-0.193484	1.0	0.622544	-0.095288
33	10	15.0	-0.290638	-0.195844	-0.195301	1.0	0.650049	0.010020
34	10	20.0	-0.293961	-0.195844	-0.195840	1.0	0.671663	0.111197
35	10	25.4	-0.295385	-0.195844	-0.194981	1.0	0.615774	-0.154225
36	10	30.3	-0.294689	-0.195844	-0.192914	1.0	0.664931	0.145639
SOLUTION			0.175853	-2.505423	-4.417319	-0.654177	SIGMA =	0.150985

APPENDIX IV

EQUATIONS OF MOTION FOR THE TRANSIT METHOD

Lauf (1963)* has given a detailed derivation of the equations of motion of a non-spinning gyroscope and for the spinning gyroscope as applied to the turning point method. We may extend Lauf's work to find the equation of motion as applied to the transit method as follows:-

Considering the formation of equation (58)* of Lauf's derivation, the vector of the centre of gravity of the gyroscope with respect to the new axes X''' , Y''' , Z''' is given by

$$V''' = \begin{bmatrix} 0 \\ 0 \\ a \end{bmatrix}$$

To obtain the vector V with respect to the original axes, X,Y, Z we need to transform V''' according to

$$V = R^{-1} V'''$$

where

$$R = R(\xi).\ R(\eta).\ R(\theta)$$

$$R(\xi) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C_3 & S_3 \\ 0 & -S_3 & C_3 \end{bmatrix}$$

$$R(\eta) = \begin{bmatrix} C_2 & 0 & -S_2 \\ 0 & 1 & 0 \\ S_2 & 0 & C_2 \end{bmatrix}$$

---

* Lauf's notation will be used in this appendix with additional symbols defined in the text.

$$R(\theta) = \begin{bmatrix} C_1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and $S_1 = \sin\theta$ $S_2 = \sin\eta$ $S_3 = \sin\xi$

$C_1 = \cos\theta$ $C_2 = \cos\eta$ $C_3 = \cos\xi$

θ, η, and ξ being the Euler rotation angles.

If M = mg, where m is the mass of the gyroscope and g the acceleration due to gravity, the moments about the original axes will be given by

$$M_m = \begin{bmatrix} 0 & -M & V \\ M & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

and therefore the moments about the new axes will be

$$M_m''' = R \begin{vmatrix} 0 & -M & 0 \\ M & 0 & 0 \\ 0 & 0 & 0 \end{vmatrix} R^{-1} V'''$$

Multiplying out we get

$$R = \begin{bmatrix} C_1C_2 & S_1C_2 & -S_2 \\ (C_1S_2S_3 - S_1C_3) & (S_1S_2S_3 + C_1C_3) & C_2S_3 \\ (C_1S_2C_3 + S_1S_3) & (S_1S_2C_3 - C_1S_3 & C_2C_3 \end{bmatrix}$$

It will be seen that

$$R(\theta)^{-1} = R(\theta)^T$$

$$R(\eta)^{-1} = R(\eta)^T$$

$$R(\xi)^{-1} = R(\xi)^T$$

and therefore

$$\begin{aligned} R^{-1} &= \{R(\xi).\ \ R(\eta).\ \ R(\theta)\}^{-1} \\ &= R(\theta)^{-1}.R(\eta)^{-1}.R(\xi)^{-1} \\ &= R(\theta)^{T}.\ R(\eta)^{T}.\ R(\xi)^{T} \\ &= R^{T} \end{aligned}$$

For a detailed exposition of these matrix operations see Thompson (1969).

Multiplying out we get

$$M_m''' = Ma \begin{bmatrix} \{C_1C_2(S_1S_2C_3-C_1S_3) - S_1C_2(C_1S_2C_3+S_1S_3)\} \\ \{(C_1S_2S_3-S_1C_3)(S_1S_2C_3-C_1S_3) - (S_1S_2S_3+C_1C_3)(C_1S_2C_3+S_1S_3)\} \\ \{(C_1S_2C_3+S_1S_3)(S_1S_2C_3-C_1S_3) - (S_1S_2C_3-C_1S_3)(C_1S_2C_3+S_1S_3)\} \end{bmatrix}$$

Because θ, η and ξ are small, we may neglect the products S_iS_j and therefore

$$M_m''' = Ma \begin{bmatrix} -C_1{}^2C_2\ S_3 \\ -C_1{}^2C_3^2\ S_2 \\ 0 \end{bmatrix}$$

and if we put $C_1 = C_2 = C_3 = 1$, then

$$M_m''' = Ma \begin{bmatrix} -\sin\xi \\ -\sin\eta \\ 0 \end{bmatrix} \tag{1}$$

The twisting moment or couple produced by tape torsion will act about the original Z axis. Denoting this moment by $-\mu\theta$ (negative, because it is a restoring couple), where μ is a constant, and transforming to the new axes X''', Y''', Z''', we get

$$M_t''' = R \begin{bmatrix} 0 \\ 0 \\ -\mu\theta \end{bmatrix} = -\mu\theta \begin{bmatrix} -S_2 \\ C_2 S_3 \\ C_2 C_3 \end{bmatrix}$$

and again replacing cosines by unity and neglecting the products θS_2 and θS_3 we get for this couple

$$M_t''' = \begin{bmatrix} 0 \\ 0 \\ -\mu\theta \end{bmatrix} \qquad (2)$$

The motion of the gyroscope will be affected by air friction about all three axes but because the final angular velocities about X‴ and Y‴ are minute we may neglect them and consider only the effect of air friction about the Z‴ axis which may be approximated by $-p \frac{d\theta}{dt}$, where p is the coefficient of friction. Thus the final couple acting on the gyroscope will be obtained by adding equations (1) and (2) and the air friction term giving:-

$$\vec{M} = (-M \sin\xi)\,\vec{i} + (-M \sin\eta)\,\vec{j} + \left(-p \frac{d\theta}{dt} - \mu\theta\right)\vec{k}$$

Following the trend of Lauf's original derivation we obtain the following new simplified differential equations for the motion of the gyroscope

$$I_x \frac{d^2\xi}{dt^2} + M\xi = 0 \qquad (3a)$$

$$I_y \frac{d^2\eta}{dt^2} + J\omega \sin\phi + J \frac{d\theta}{dt} + M\eta = 0 \qquad (3b)$$

$$I_z \frac{d^2\theta}{dt^2} + (J\omega \cos\phi + \mu)\theta - J \frac{d\eta}{dt} + p \frac{d\theta}{dt} = 0 \qquad (3c)$$

Differentiating (3b) with respect to time and neglecting the third derivative of η because it is small in comparison with its first derivative we get

$$J \frac{d^2\theta}{dt^2} + M \frac{d\eta}{dt} = 0 \tag{4}$$

and substituting (4) in (3c) we obtain

$$(I_z + \frac{J^2}{M}) \frac{d^2\theta}{dt^2} + p \frac{d\theta}{dt} + (J\omega\cos\phi + \mu)\theta = 0 \tag{5}$$

The solution of this linear differential equation of the second order gives for the oscillation curve

$$\Theta = Be^{\frac{-pt}{2(I_z+\frac{J^2}{M})}} \cos \left(\frac{\sqrt{4(I_z+\frac{J^2}{M})(J\omega\cos\phi+\mu)-p^2}}{2(I_z+\frac{J^2}{M})} t + \gamma \right) \tag{6}$$

and for the period of oscillation

$$T_D = \frac{4\Pi(I_z+\frac{J^2}{M})}{\sqrt{4(I_z+\frac{J^2}{M})(J\omega\cos\phi+\mu)-p^2}} \tag{7}$$

Furthermore I_z is very small in comparison with $\frac{J^2}{M}$ for the gyroscopes used in gyro-theodolites and as the angular velocity is only of the order of 0.00014 rad. per sec. (for the GAK 1 with an amplitude of 3^o and a period of 6 minutes) the effect of air friction can be neglected to a high degree of approximation. Therefore equation (7) may be written as

$$T_D = 2\pi \sqrt{\frac{J^2}{M(J\omega\cos\phi+\mu)}} \tag{8}$$

and modifying Lauf's equation (67) in a similar way as before, we have for the period of the turning point method

$$T_U = 2\Pi \sqrt{\frac{J^2}{MJ\omega\cos\phi}} \qquad (9)$$

and if T_{U_o} is the period for $\phi = 0$, then

$$T_U = \frac{T_{U_o}}{\sqrt{\cos\phi}} \qquad (10)$$

Combining equations (8) and (9) gives

$$\frac{T_U^2}{T_D^{\,2}} - 1 = \frac{\mu}{J\omega\cos\phi} = \psi \qquad (11)$$

where ψ is the ratio of the tape torque and earth torque. This ratio if expressed as a percentage is called the useful torque ratio by Strasser and Schwendener (1964)

Combining equations (8), (9) and (10) also gives

$$T_D = \frac{T_{U_o}}{\sqrt{\cos\phi + \frac{\mu}{J\omega}}} \qquad (12)$$

Transit method of Schwendener

If the zero of the auto-collimator scale is off the meridian by ΔN and the centre of oscillation is in a position $\Delta N - \alpha$, then considering equation (5) in the rest position, i.e. when $\frac{d^2\theta}{dt^2} = \frac{d\theta}{dt} = 0$, we get that

$$J\omega \cos \phi \, (\Delta N - \alpha) = \mu\alpha$$

and therefore

$$\Delta N = \alpha \left(1 + \frac{\mu}{J\omega\cos\phi}\right)$$

Combining this equation with equation (11) we get

$$\Delta N = \alpha \frac{T_U^2}{T_D^2}$$

$$\text{where } \alpha = \frac{ma\Pi\Delta t}{2T_D}$$

Δt, is the time difference between the excursions of the gyro-mark to the left and right of the zero of the auto-collimator scale.

a, is the amplitude of oscillation as determined from readings made on the auto-collimator scale, in scale divisions.

m, is the arc value of a division of the auto-collimator scale.

Therefore $$\Delta N = \frac{ma\Pi T_U^2 \Delta t}{2T_D^3}$$

and $$\Delta N = ca\,\Delta t$$

where $$c = \frac{m\Pi T_U^2}{T_D^3}$$

The influence of a constant error in the tape zero position.

(a) Turning point method.

If the tape zero position has a constant error $\Delta\alpha$ then the spin axis of the gyro will come to rest at an angle $\varepsilon-\Delta\alpha$ from the meridian, where ε is the rotation angle of the upper tape axis. For this method equation (65) of Lauf's derivation will need to be modified as follows:-

$$(I_z + \frac{J^2}{M}) \frac{d^2\theta}{dt^2} + p \frac{d\theta}{dt} + (J\omega\cos\phi)\ \theta - \mu\ \Delta\alpha = 0$$

Considering this equation in the rest position

i.e. when $\frac{d^2\theta}{dt^2} = \frac{d\theta}{dt} = 0$, we get that

$$J\omega\cos\phi(\varepsilon-\Delta\alpha)-\mu\Delta\alpha = 0$$

$$\varepsilon-\Delta\alpha = \frac{\mu\Delta\alpha}{J\omega\cos\phi}$$

and therefore the required correction is

$$\varepsilon-\Delta\alpha = \frac{\mu\Delta\alpha}{J\omega\cos\phi}$$, where $\Delta\alpha$ is in scale divisions.

(b) Transit method.

If the tape zero position has a constant error $\Delta\alpha$ then equation (5) becomes

$$J\omega\cos\phi\ (\Delta N - \alpha) = \mu(\alpha + \Delta\alpha)$$

$$\therefore \qquad \Delta N = \alpha\left(1 + \frac{\mu}{J\omega\cos\phi}\right) + \frac{\mu\Delta\alpha}{J\omega\cos\phi}$$

and

$$\Delta N = c\ a\ \Delta t + \frac{m\ \mu\Delta\alpha}{J\omega\cos\phi}$$

Thus the required correction is the same as in the turning point method.

APPENDIX V

COMPUTER PROGRAMME AND SAMPLE OUTPUT FOR EXTENDED NAVIGATION TABLES

```
C        PROGRAMME "NAVTAB" -- SVY167
C
C        PRINTS AN EXTENSION OF TABLE I OF HUGHES NAVIGATION TABLES
C        FOR EVERY TEN MINUTES OF LATITUDE FROM 0 DEG. TO 89 DEG. 50 MIN.
C        PROGRAMMED IN FORTRAN IV, G-LEVEL -- COMPATIBLE WITH WATFOR
C        COMPILER -- FOR UNSW IBM 360/50
C        PREPARED BY E. G. ANDERSON, NOVEMBER, 1969
C
C        ------------------------------------------------------------------
C
      IMPLICIT REAL*8 (A-H,O-Z)
      DIMENSION LAT(2,90), ID(91,90), IA(91,90), KD(91,90), JH(91),
     1  KH(91), AKM(91,90), Z(91,90)
      DATA NRCH/4501/, NLAT/90/
      DATA RMIN/0.290888208665722D-3/, INF/'INF'/
      RDEG=60.0*RMIN
      IA(91,1)=100000000
      DO 401 K=1,NRCH,900
      DO 402 L=1,NLAT
      LATM=10*L+K-11
      PHI=LATM*RMIN
      LAT(1,L)=LATM/60
      LAT(2,L)=LATM-LAT(1,L)*60
      ID(1,L)=INF
      Z(91,L)=0.0
      DO 403 IH=1,91
      H=(IH-1)*RDEG
      FINC=DARSIN(DCOS(PHI)*DSIN(H))
      IF(IH.EQ.1) GO TO 501
      ID(IH,L)=DLOG10(1.0/DSIN(FINC))*1.0D3+0.5
  501 IF (IH.EQ.91.AND.LATM.EQ.0) GO TO 502
      IA(IH,L)=DLOG10(1.0/DCOS(FINC))*1.0D5+0.5
  502 IF (LATM.EQ.0) GO TO 503
      AK=90.0-DATAN(DCOTAN(PHI)*DCOS(H))/RDEG
      KD(IH,L)=AK
      AKM(IH,L)=(AK-KD(IH,L))*60.0
      IF (AKM(IH,L).LT.59.95) GO TO 504
      KD(IH,L)=KD(IH,L)+1
      AKM(IH,L)=0.0
  504 IF (IH.EQ.91) GO TO 403
      Z(IH,L)=90.0-DATAN(DSIN(PHI)*DTAN(H))/RDEG
      GO TO 403
```

```
503 KD(IH,I)=0
    AKM(IH,I)=0.0
    Z(IH,I)=90.0
    JH(IH)=IH-1
    KH(IH)=181-IH
403 CONTINUE
402 CONTINUE
    DO 404 I=1,NLAT
    WRITE (3,301) LAT(1,I),LAT(2,I)
301 FORMAT ('1',15X,104('-'),5X,'LATITUDE'/15X,2('|',4X,'H',7X,'K',9X,
   1'A',9X,'D',7X,'Z',11X,'|'),4X,2I4/15X,2('|',51('-'),'|'))
    WRITE (3,302) (JH(J),KD(J,I),AKM(J,I),IA(J,I),ID(J,I),Z(J,I),KH(J)
   1,JH(J+45),KD(J+45,I),AKM(J+45,I),IA(J+45,I),ID(J+45,I),Z(J+45,I),
   2 KH(J+45), J=1,46)
302 FORMAT (15X,'|',I5,I6,F5.1,I9,5X,A3,F8.1,I7,3X,'||',I5,I6,F5.1,2I9
   A,F7.1,I7,3X,'|',4(/15X,2('|',I5,I6,F5.1,2I9,F7.1,I7,3X,'|'))/
   B(15X,2('|',51X,'|')),5(/15X,'|',I5,I6,F5.1,2I9,F7.1,I7,3X,'||',I5,
   CI6,F5.1,3X,I6,3X,I6,F7.1,I7,3X,'|')))
    WRITE (3,303) LAT(1,I),LAT(2,I)
303 FORMAT ('+',14X,2('|',51('-'),'|'),4X,'LATITUDE'/15X,2('|',10X,
   A'180-K',7X,'A',8X,'D',6X,'-Z',6X,'H',4X,'|'),4X,2I4)
404 CONTINUE
401 CONTINUE
    STOP
    END
```

H	K		A	D	Z	
0	77	50.0	0	INF	90.0	180
1	77	50.1	0	2434	89.0	179
2	77	50.4	1	2133	88.0	178
3	77	51.0	3	1957	87.1	177
4	77	51.7	5	1833	86.1	176
5	77	52.7	7	1736	85.1	175
6	77	53.9	11	1657	84.1	174
7	77	55.3	14	1590	83.2	173
8	77	56.9	19	1533	82.2	172
9	77	58.7	24	1482	81.2	171
10	78	0.8	29	1437	80.2	170
11	78	3.0	35	1396	79.2	169
12	78	5.5	42	1358	78.3	168
13	78	8.2	49	1324	77.3	167
14	78	11.1	57	1293	76.3	166
15	78	14.2	65	1263	75.3	165
16	78	17.5	73	1236	74.3	164
17	78	21.0	83	1210	73.4	163
18	78	24.7	92	1186	72.4	162
19	78	28.7	102	1164	71.4	161
20	78	32.8	113	1142	70.4	160
21	78	37.2	124	1122	69.4	159
22	78	41.7	136	1103	68.4	158
23	78	46.5	148	1084	67.5	157
24	78	51.5	160	1067	66.5	156
25	78	56.6	173	1050	65.5	155
26	79	2.0	186	1034	64.5	154
27	79	7.6	200	1019	63.5	153
28	79	13.3	214	1005	62.5	152
29	79	19.3	228	991	61.5	151
30	79	25.4	242	977	60.6	150
31	79	31.8	257	964	59.6	149
32	79	38.3	273	952	58.6	148
33	79	45.0	288	940	57.6	147
34	79	52.0	304	929	56.6	146
35	79	59.1	320	918	55.6	145
36	80	6.3	336	907	54.6	144
37	80	13.8	352	897	53.6	143
38	80	21.5	369	887	52.6	142
39	80	29.3	385	877	51.6	141
40	80	37.3	402	868	50.6	140
41	80	45.5	419	859	49.6	139
42	80	53.8	436	851	48.6	138
43	81	2.4	453	842	47.6	137
44	81	11.1	470	834	46.7	136
45	81	19.9	488	827	45.7	135
	180-K		A	D	-Z	H

H	K		A	D	Z	
45	81	19.9	488	827	45.7	135
46	81	28.9	505	819	44.7	134
47	81	38.1	522	812	43.6	133
48	81	47.5	539	805	42.6	132
49	81	57.0	556	798	41.6	131
50	82	6.6	574	792	40.6	130
51	82	16.4	590	786	39.6	129
52	82	26.3	607	780	38.6	128
53	82	36.4	624	774	37.6	127
54	82	46.7	641	768	36.6	126
55	82	57.0	657	763	35.6	125
56	83	7.5	673	758	34.6	124
57	83	18.2	689	753	33.6	123
58	83	28.9	705	748	32.6	122
59	83	39.8	720	743	31.6	121
60	83	50.8	736	739	30.6	120
61	84	2.0	751	734	29.6	119
62	84	13.2	765	730	28.5	118
63	84	24.6	780	726	27.5	117
64	84	36.1	793	723	26.5	116
65	84	47.6	807	719	25.5	115
66	84	59.3	820	715	24.5	114
67	85	11.1	833	712	23.5	113
68	85	23.0	845	709	22.5	112
69	85	34.9	857	706	21.4	111
70	85	47.0	869	703	20.4	110
71	85	59.1	880	701	19.4	109
72	86	11.3	890	698	18.4	108
73	86	23.6	900	696	17.4	107
74	86	35.9	910	693	16.3	106
75	86	48.4	919	691	15.3	105
76	87	0.9	928	689	14.3	104
77	87	13.4	936	687	13.3	103
78	87	26.0	943	686	12.3	102
79	87	38.7	950	684	11.2	101
80	87	51.4	956	683	10.2	100
81	88	4.1	962	682	9.2	99
82	88	16.9	967	680	8.2	98
83	88	29.7	972	679	7.2	97
84	88	42.5	976	679	6.1	96
85	88	55.4	979	678	5.1	95
86	89	8.3	982	677	4.1	94
87	89	21.2	984	677	3.1	93
88	89	34.1	985	676	2.0	92
89	89	47.1	986	676	1.0	91
90	90	0.0	987	676	0.0	90
	180-K		A	D	-Z	H

APPENDIX VI

EXAMPLE OF PREDICTION FOR SUN AND STAR OBSERVATIONS AND PREDICTED PROGRAMME. ZENIDROMIC SOLUTION, STATION McMURDO

PREDICTION

Station: McMurdo | Latitude: S 77° 51′

Date: 29/30 Nov. 1968 | Longitude: E 166° 40′
E 11h 06m 40s

SUN		STAR	
Standard Time	23h 45m	Standard Time	
Zone	12	Zone	
UT	11 45 (29/11)	UT	
E	12 11 40	R	
dE	-5	dR	
Sum	23 56 35	GST	
λ + E - W	+11 06 40	λ + E - W	
HA Time	11 03 15	LST	
HA Arc E/W	W 166°	- RA	
Corrected Standard Time	23h 45m 45s	HA Time	
		HA Arc E/W	

φ ± −77° 50′

δ ± −21 32

I K(sign as φ) −101 49 → II → A 57 → III → D 1293 → Z_1 −76.3°

K ~ δ 80 17 → B 772 69 → E 767 → III → Z_2 (sign as |K| − |δ|) 89.5

C ← A+B 77326 → D+E 2060 →

Altitude 9 42 ← II

$Z_1 + Z_2$ ~~N~~ S 13.2 ~~E~~ W

Azimuth 193° 12′

PREDICTION

Station: *Mc Murdo* — Latitude: *S 77°51′*

Date: *29/30 Nov. 1968* — Longitude: *E 166°40′*
E 11ʰ06ᵐ40ˢ

SUN		*Rigil K.* STAR			
Standard Time		Standard Time	*0ʰ*	*00ᵐ*	*00ˢ*
Zone		Zone	*12*		
UT		UT	*12*		*(29/11)*
E		R	*4*	*33*	*44*
dE		dR			*00*
Sum		GST	*16*	*33*	*44*
λ + E - W		λ + E - W	*+11*	*06*	*40*
HA Time		LST	*3*	*40*	*24*
HA Arc E/W		- RA	*14*	*37*	*26*
		HA Time	*10*	*57*	*02*
		HA Arc E/W	*E 165°*		
		Corrected Standard Time	*23ʰ*	*57ᵐ*	*00ˢ*

φ ±	*−77°50′*
δ ±	*−60 42*
I K(sign as φ)	*−101 46*
K ~ δ	*41 04*
Altitude	*48 50*

II → A *65*; B *12266*; C ← A+B *12331* II

III → D *1263*; E *9940*; D+E *11203* III → Z_2 (sign as \|K\|-\|δ\|) *86.4*

Z_1 *−75.3°*

Z_1+Z_2 ~~N~~ S *11.1* E ~~W~~

Azimuth *168°54′*

PREDICTED PROGRAMME

Station: McMurdo   Date: 29/30 Nov. 1968

Object	Mag.	Standard Time	Hor. Circle	Vert. Circle (h)
Sun		23^h 45^h 45^s	193^o 12'	9^o 42'
Rigil K.	0.1	23 57 00	168 54	48 50
Fomalhaut	1.3	0 15 31	277 45	32 12
Sirius	-1.6	0 31 19	41 06	26 04
Rigel	0.3	0 44 33	12 36	20 07

ZENIDROMIC SOLUTION : STATION McMURDO

Numerical Adjustment

$$v = -\Delta_o + \eta\, d\phi + \xi\Delta\lambda + (A' - \psi - 0_o)$$

Assumed Latitude $\phi' = -77^{o}50'55''$

Assumed Longitude $\lambda' = E11^{h}06^{m}39.7^{s}$

Assumed Azimuth of R.O. $= 316^{o}05'28''$

Latitude $\phi = \phi' + d\phi$

Longitude $\lambda = \lambda' + d\lambda$

$= \lambda' + \Delta\lambda \sec\phi$

Δ_o Unknown correction to be added to assumed azimuth of R.O. .

ψ Horizontal circle reading on star .

0_o Assumed azimuth of R.O. minus horizontal circle reading on R.O. .

$\eta = \sin A' \tan h'$, $\xi = \tan\phi' - \cos A' \tan h'$.

A' azimuth of star calculated from ϕ', δ, and t' based on λ'.

Star No.	A'	sin A'	cos A'	h'	tan h'	h	ξ
379	166°52'34"	+0.227	-0.974	48°57'	1.148	+0.261	-3.526
632	275 45 45	-0.995	+0.104	31 52	0.622	-0.619	-4.709
185	39 48 23	+0.640	+0.768	26 16	0.494	+0.316	-5.023
135	11 04 21	+0.192	+0.981	20 13	0.368	+0.071	-5.005
Star No.	**ψ**	**0_o**	**$A'-\psi-0_o$**	**$-\Delta_o$**	**$\eta d\phi$**	**$\xi\Delta\lambda$**	**v**
379	166 38 51	+13'28"	+15"	-56.1"	-2.1"	+43.4"	+0.2"
632	275 32 25	+13 28	-8	-56.1	+4.9	+57.9	-1.3
185	39 35 28	+13 28	-7	-56.1	-2.5	+61.8	-3.8
135	10 50 53	+13 28	0	-56.1	-0.6	+61.6	+4.9

Parametric Equations

$-\Delta_o$	$\Delta\phi$	$\Delta\lambda$	Absolute	= 0
1	+0.261	-3.526	+15	
1	-0.619	-4.709	- 8	
1	+0.316	-5.023	- 7	
1	+0.071	-5.005	0	

Normal Equations

$-\Delta_o$	$\Delta\phi$	$\Delta\lambda$	Absolute	= 0
4	+0.029	-18.263	0	
	+0.556	+ 0.052	+ 6.655	
		+84.888	+19.943	

Solution of Normal Equations

$\Delta_o = +56.1''$ $\qquad d\phi = -7.9''$ $\qquad \Delta\lambda = -12.3''$

$$d\lambda = \Delta\lambda \sec \phi = -58.4'' = -3.9^s$$

Final Position

$\phi = \phi' + d\phi = -77^o\ 51'\ 03''$ Latitude

$\lambda = \lambda' + d\lambda = E11^h\ 06^m\ 35.8$ Longitude

Azimuth of R.O.

Assumed Azimuth	316° 05' 28"
Δ_o	+ 56
Azimuth	316 06 24

GYRO-THEODOLITE OBSERVATION AT HALLETT STATION,

ANTARCTICA.

BIOGRAPHICAL NOTES

G. G. BENNETT at present holds the appointment of Senior Lecturer in the Department of Surveying, University of New South Wales, to which he was appointed in 1962. He received a First Class Honours degree in Surveying from the University of Melbourne in 1954. After graduation, Mr. Bennett worked for the Snowy Mountains Hydro-Electric Authority from 1954 to 1959 where he specialised in Geodetic Astronomy. He joined the University of New South Wales in 1959 as a lecturer and completed his Master of Surveying degree at the University of Melbourne in 1962. In 1965 he spent a year as research officer with the Geodetic Surveys Section of the Department of Mines and Technical Surveys, Canada.

Mr. Bennett has published papers on both Geodetic Astronomy and the adjustment of control networks. His current research interests include in addition to the above topics, gyro-theodolites and their applications, as well as all aspects of error theory.

DEPARTMENT OF SURVEYING - UNIVERSITY OF NEW SOUTH WALES

Kensington. N.S.W. 2033.

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* Out of print