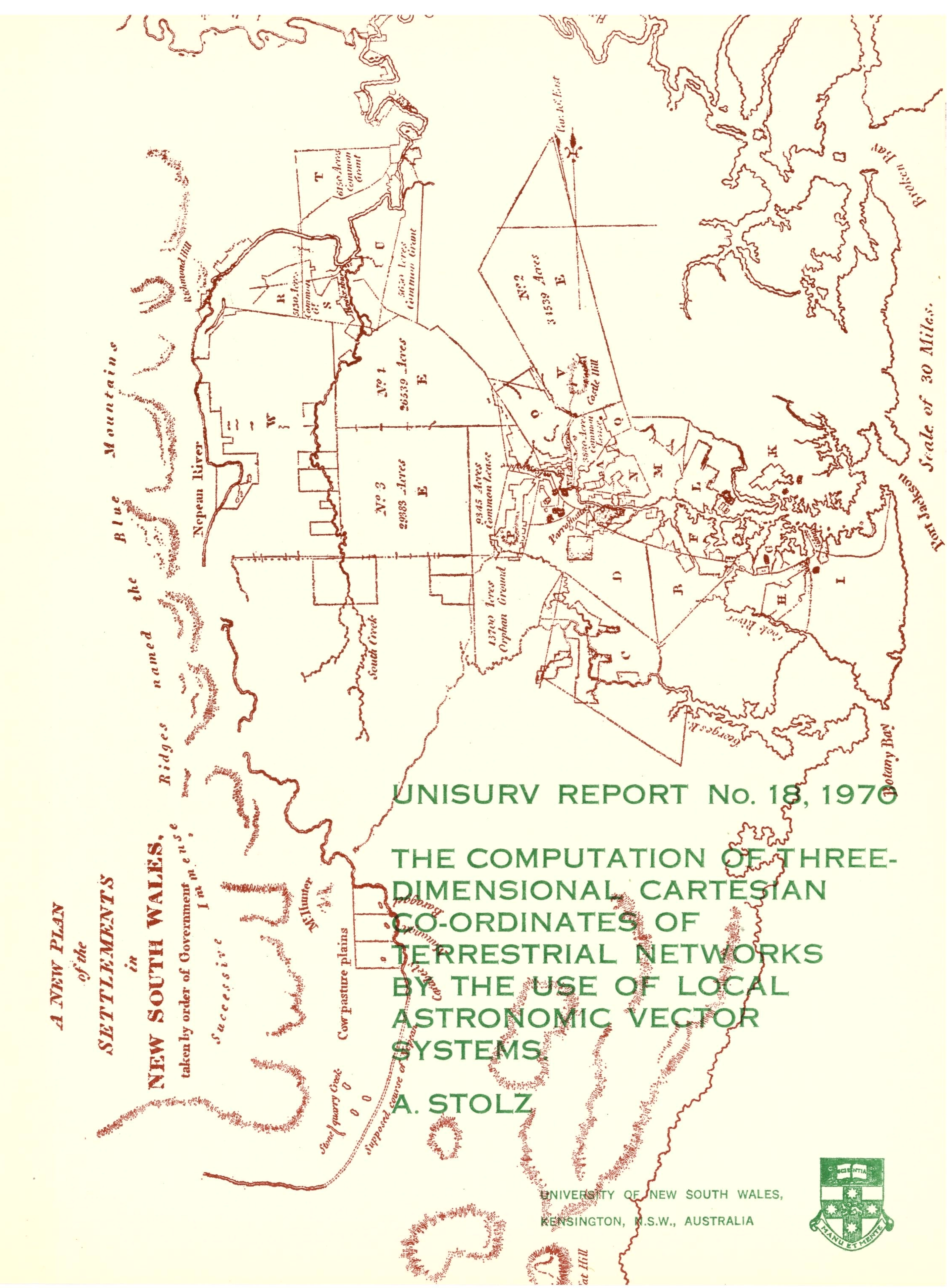

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THE COMPUTATION OF THREE-DIMENSIONAL CARTESIAN CO-ORDINATES OF TERRESTRIAL NETWORKS BY THE USE OF LOCAL ASTRONOMIC VECTOR SYSTEMS.

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ABSTRACT.

The theory of computation of three-dimensional cartesian coordinates of terrestrial networks using local astronomic vector systems, is given. All the usual linear and angular measurements are included in the method as well as astronomic and levelling data, and expressions are derived for the complete definition of the solution. The method presented is not hypothesis free,as the assumption is made that the physical space in which the observations are made may be described as linear Euclidean or flat. It is shown that this is a justifiable assumption provided that all measurements connecting adjacent local vector systems are discrete and do not have to be integrated. Finally a feasibility study is made in the light of data commonly available for modern geodetic networks.

1. INTRODUCTION.

The physical space in which geodetic observations are made may be described in a mathematical sense, as three-dimensional Riemannian with slightly irregular coordinate surfaces. Therefore, in order to achieve a hypothesis free solution to geodetic problems, the computations should also be carried out within this space. The theory of such a method was first proposed by *Marussi (1949)* and later developed by *Hotine (1957, 1959, 1969)* who showed that this type of solution cannot be put to practice, as the higher derivatives of the parameters ϕ, ω and W, respectively the astronomic latitude and longitude, and the geopotential, with respect to the element of arc joining adjacent points, cannot be determined from the measured quantities. For this reason modern research into three-dimensional geodesy has proceeded along theoretical and not practical lines and attempts have been made to fit a mathematical model to the complex physical nature of the level surfaces *(Hotine, 1969)*.

By excluding the third coordinate surface, classical geodesy has adopted the approach of projecting the three-dimensional Riemannian space onto a regular mathematical figure - more precisely an ellipsoid of revolution. Two coordinates of each station are computed on this ellipsoid or on a plane conformal transformation of it. The third geodetic coordinate of height is usually provided by spirit-levelling. Unfortunately, whereas transformed geodetic coordinates refer to an ellipsoid of revolution, the datum of height measurement is an equipotential surface of the earth's gravitational field. Thus within this method the reference surfaces for height and position are not identical, and even though computations are carried out in two dimensions, the method suffers from the additional disadvantage that the computational formulae become more complicated.

An alternative method *(Bruns, 1878)* has been to approximate the three-dimensional Riemannian Space to linear Euclidean or flat space, a method which has the advantage over the classical procedure that it is a truer representation of physical reality.

This paper is concerned with the theory of computation of unique coordinates of terrestrial networks within three-dimensional flat space, the distinction between the theory presented and that treated by other authors *(Bruns, 1878)*, *(Hotine, 1959)*, *(Mather, 1969)*, *(Molodenskii et al, 1962)*, *(Wolf, 1963)* being, that all preliminary computations are made in the natural reference frame of the observations. The natural or astronomic reference frame, which comprises the directions of the astronomic parallel, meridian and zenith, is defined under the assumption that the equipotential surface containing the observing station has no discontinuities at or in the immediate vicinity of the spatial point, implying that a tangent plane to the equipotential surface exists at the point being considered. Within this system, the customary linear and angular measurements completely define the relative positions of points. Moreover all observations are made at discrete points and all but the measured distances are subject to the earth's gravitational field at these points, thus assuming that the earth's gravitational field remains constant over the period of observations, all measurements are affected equally at a point and the earth's gravitational field does not affect the relative positions of points as defined. Orientation of the triads* is achieved by astronomic observations or if these are not available, by the use of gravimetric methods. Naturally, the method does not give any information regarding the centering or absolute position of the system as the potential is no longer a parameter of the solution.

The method is marked by simplicity of computation and, providing care is exercised in defining the direction parameters of the astronomic zenith, the measured quantities may be directly related to perpendicular components in the local astronomic triad which may then be rotated to provide coordinate increments between adjacent point.

It is hoped that the application of the method to an extensive network will provide better results than the conventional procedures. This is the subject of an investigation in which work is continuing.

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* The word *triad* is used throughout this paper to describe a group of three closely related quantities, that is the unit vectors defining the directions of the astronomic parallel, meridian and zenith.

2. NOTATION.

(i) Indices.

The index notation of tensor calculus offers many advantages and is used extensively throughout this paper. Lower case Roman letters i, j, k .... etc. will be used for indices, both as subscripts (a_i) and superscripts (a^i). Thus the three quantities a^1, a^2, and a^3 will be denoted by the symbol a^i. Symbolically this may be written -

$$a^i = (a^1, a^2, a^3)$$

and it must be remembered that a in this relation is a set of three quantities. No operation such as multiplication or division is implied. For i and j ranging from 1 to 3 independently, the symbols a^i, a_{ij} and a^{ij} represent nine quantities.

According to the summation convention, a summation is implied when an index is repeated. For example -

$$b_i c^i_j = b_1 c^1_j + b_2 c^2_j + b_3 c^3_j \qquad (j = 1, 2, 3)$$

A repeated index is called a dummy index, because the value of the term does not depend on the symbol used. An index which is not repeated is called a free index; when a suffix occurs unrepeated in a term, it will be understood that it takes the values from the set (1, 2, 3 ...n) where n is a specified integer called the range. Evaluating $a_i = b_i$ for the range (1, 2, 3) gives

$$a_1 = b_1 \qquad a_2 = b_2 \qquad a_3 = b_3$$

The Kronecker Deltas δ_{ij}, δ^{ij} are symbols carrying two indices and are defined such that $\delta^i_j = 1$ when $i = j$ and $\delta^i_j = 0$ when $i \neq j$. The term δ^j_i becomes $\delta^{(i)}_{(i)}$ when $j = i$ and it is clear that even though a repeated index occurs, the summation convention does not apply.

All derived relationships refer to cartesian systems in which there is no distinction between the contravariant and covariant components of vectors. For this reason and in order to avoid confusion in the notation, the scalar product when used is written in vector form.

A more concise form of notation is employed when deriving expressions that involve the parameters defining the relative positions of points. In such instances the indices a and b are used. These are not summed.

For quantities pertaining to the origin of survey, the suffix o is employed. It is understood that this is not a free index. The following systems of symbols and numbering is adopted throughout this paper.

(ii) Symbols.

A_i - Cartesian components of a unit spatial vector.

B_i - Components of a unit spatial vector in the astronomic triad I_{pi}.

e - Eccentricity of meridian ellipse.

H_p - Orthometric elevation.

I_{pi} - Unit axis vectors of local astronomic reference frame.

I_i - Unit axis vectors of the geodetic reference frame U_j.

J - Unit spatial vector.

N_p - Radius of curvature of meridian ellipse.

U_i - Geodetic cartesian reference frame.

U_{pi} - Geodetic cartesian co-ordinates.

V - Accidental (normally distributed) error.

W_p - Equipotential surface.

X_i - Geocentric cartesian reference frame.

X_{pi} - Geocentric cartesian co-ordinates

α_{ab} - Azimuth

β_{ab} - Zenith angle.

ϕ_p - Astronomic latitude.

ω_p - Astronomic longitude.

ΔU_i^{ab} - Change in U_{pi}.

ΔX_{oi} - Geocentric co-ordinate of survey origin.

(iii) Ranges.

i,j - (1, 2, 3)

p - (1, 2, 3 ......n)

3. SPATIAL CO-ORDINATE SYSTEMS.

Three distinct reference systems are used for the analysis of observational data within the contents of this paper. They are briefly:-

1) A cartesian reference frame integrating all the spatial points under consideration.

2) A celestial system for astronomic work.

3) A local triad in which all the geodetic measurements are made.

These reference systems have to be carefully defined and their precise relationships established so that the spatial concepts involved may be fully exploited.

Theoretically, the cartesian reference system is defined by the mean terrestrial pole of a certain date and the mean meridian of Greenwich, and the celestial system by the mean equator and ecliptic of a certain date. In practice, however, both these systems are defined by a set of co-ordinates given to physical points, as the ideal cartesian reference frame is replaced by a geodetic system that is parallel to the ideal frame and the celestial system by one that is defined with reference to the mean places and proper motions given to stars in a certain catalogue. The two systems can be related when the position of the instantaneous pole, the sidereal time at Greenwich as determined by observation, and the precessional and nutational constants are known.

In addition a system is needed to which the geodetic observations can be referred. For this purpose cartesian components within a curvilinear reference frame will be used in preference to any other system as such a system has several advantages in analytical manipulation and can easily be transformed to any other by matrix multiplication.

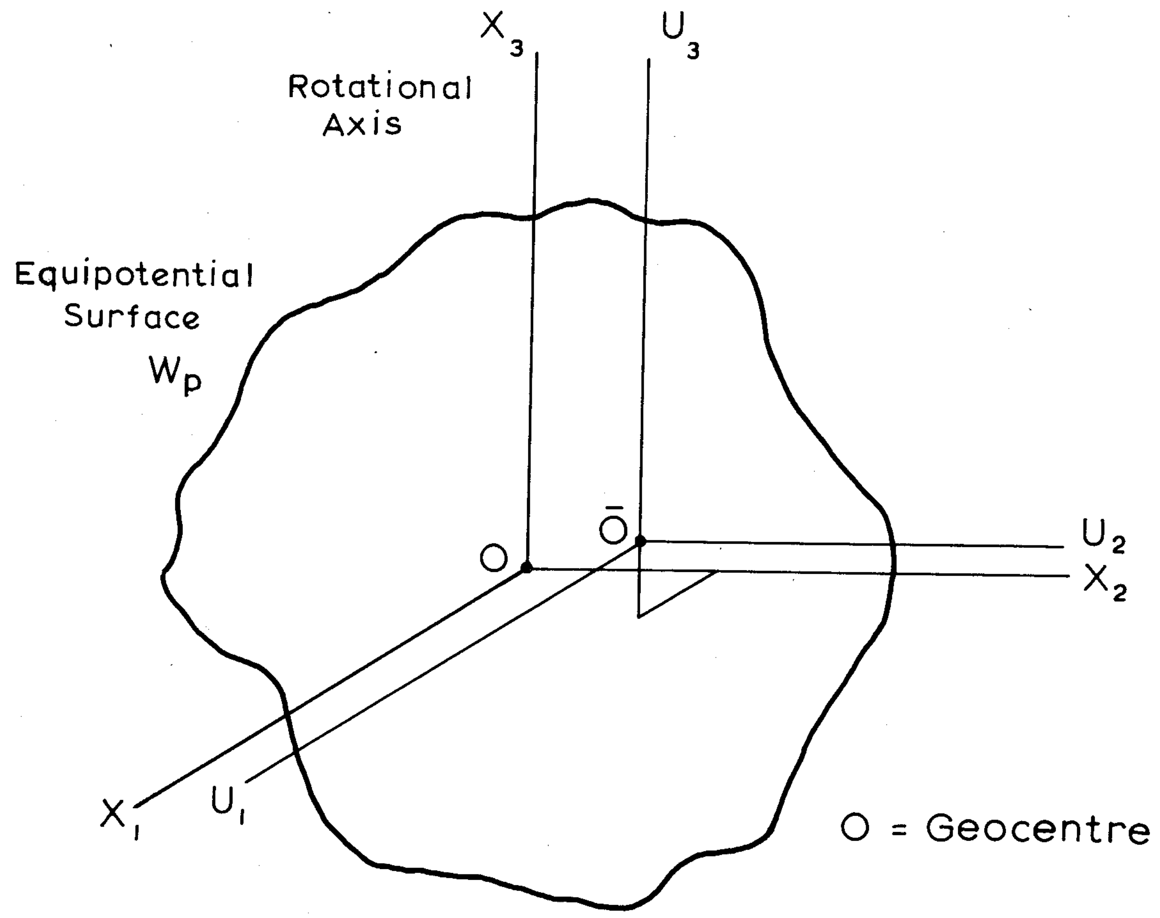

Fig. 1: IDEAL AND GEODETIC CARTESIAN REFERENCE FRAMES

(i) The Cartesian Reference Frame.

The ideal cartesian reference frame for all spatial points is an earth centred equatorial right-handed system X_i defined as follows:-

'The origin is at the earth's centre of gravity and the system is oriented so that the X_3 *- axis is directed towards the mean north pole as defined by the International Polar Motion Service. The* X_1X_3 *- plane is parallel to the mean meridian of Greenwich as defined by the Bureau Internationale de l'Heure, but any other standard meridian is acceptable.'*

This terrestrial co-ordinate system is fixed with respect to the earth's surface and the co-ordinates of any point do not change providing there is no crustal movement. Unfortunately, the centre of gravity of the earth is not known and for this reason, the ideal system is replaced in practice by a rectangular geodetic system U_i in which the axes are respectively parallel to those of the ideal system. Co-ordinates within this system thus remain relative to the origin of the geodetic system until the position of the geocentre with respect to a surface point is known. A translation will then produce geocentric cartesian co-ordinates.

(ii) The Celestial System.

It is necessary to define a celestial system as within it the astronomic observations of latitude, longitude and azimuth are made. The precise aspects of this reference frame, with particular emphasis on permitting a transformation between the ideal cartesian and celestial systems have been investigated by *Veis (1963)*. In order to achieve a one to one correspondence between the two systems the following aspects must be observed:-

1) The star co-ordinates must be referred to a mean system that is independent of time, quite apart from the fact that the co-ordinates given should refer to the same epoch. A time independent system results when the star positions have been corrected for precession, nutation, parallax, aberration and their own movement.

2) Observation dates of position determinations must be freed from the known irregular variations in the speed of rotation of the earth and must be referred to a relatively uniform time system such as UT2.

3) The measured values of latitude and longitude must be transformed from the instantaneous pole at the moment of observation to the mean pole of a certain epoch, as given by the International Polar Motion Service.

(iii) <u>The Local System.</u>

In order to define this system, the assumption is made that the equipotential surface containing a particular spatial point has no discontinuities at and in the vicinity of that point, which implies that a tangent plane to the equipotential surface exists there. An orthogonal vector triad I_{pi}, right-handed in ascending order of i, is then defined at any spatial point p as follows:-

I_{p1} is the direction of the astronomic parallel and lies in the plane tangent to the equipotential surface containing the spatial point.

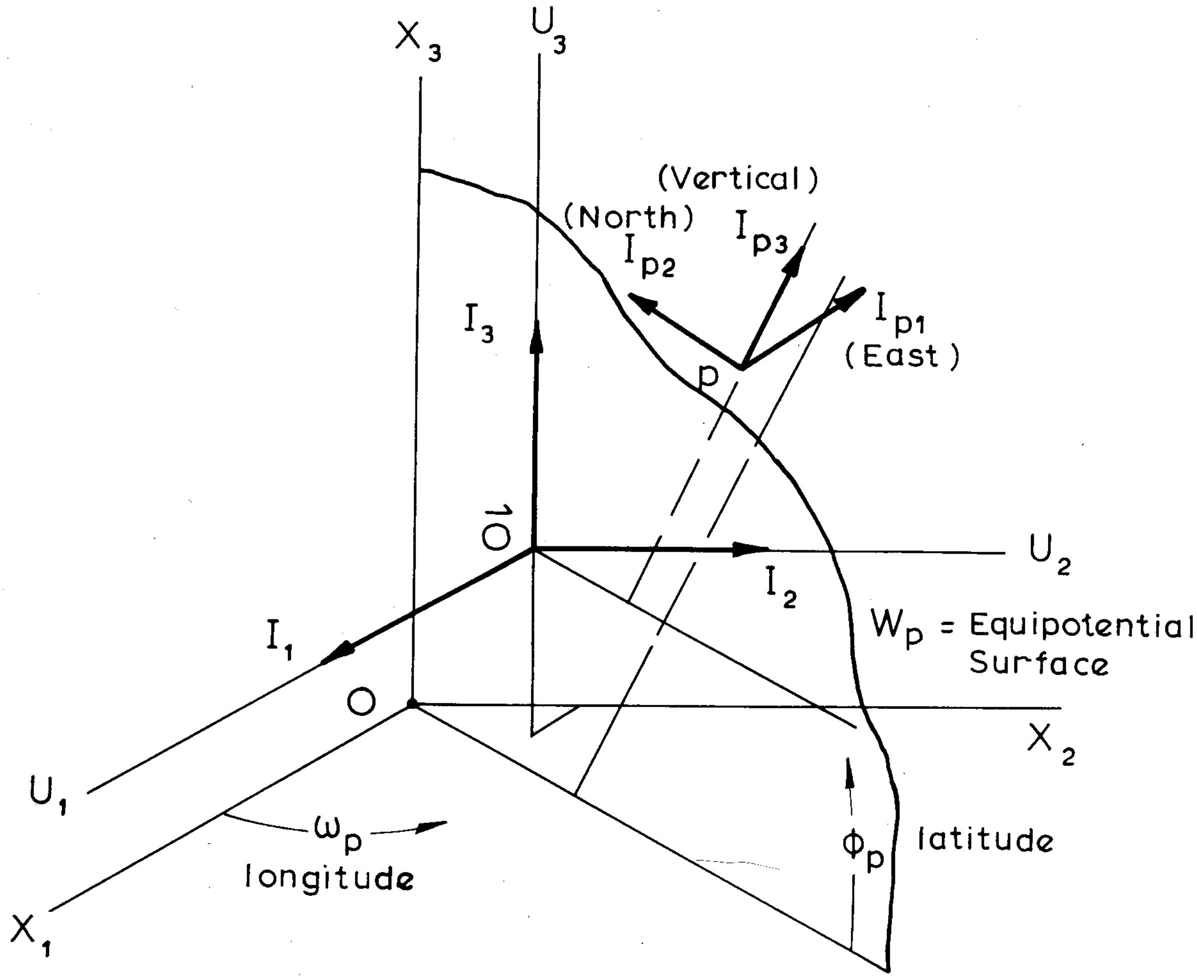

Fig. 2: GEODETIC CARTESIAN AND ASTRONOMIC VECTOR SYSTEMS

I_{p2} is the direction of the astronomic meridian and also lies in the tangent plane.

I_{p3} is the direction of the astronomic zenith and is normal to the tangent plane.

All geodetic measurements are made within this local system.

4. TRANSFORMATIONS.

It remains to establish the precise relationships between the three reference systems previously described as well as those between adjacent local systems. This is best achieved by the use of rotational and translational matrices. Before any transformations are possible, however, it is necessary to define:-

1) A vector triple I_i.
2) The astronomic latitude (ϕ_p) and longitude (ω_p) of a spatial point p using the vectors I_i and I_p.

The vector triple I_i (Fig. 2) is defined so that the I_i are unit vectors in the directions of the corresponding U_i axes.

Astronomic latitude is taken as the angle between the vector I_{p3} and the plane I_1, I_2, positive north. The longitude is the angle between two planes both containing the mean axis of rotation one of which is parallel to I_{p3}, that is the vertical at the observation point whereas the other is parallel to the vertical at some defined point such as the site of the Greenwich transit telescope, or more precisely, as defined by *Bomford (1962, 86-7)* positive east.

These definitions accord with international practice and with astronomic convention for right ascension and local time (but not hour angle which is reckoned positive west). Longitude is thus made a positive rotation in the mathematical sense about the northward rotational axis of the earth.

From the previous definitions, it becomes clear that in the general case, the meridian will fail to pass through the ground point (Fig. 2). However, this is of no consequence as in the method proposed here astronomic measurements of latitude and longitude are used only to define the direction of the vertical, just as in the classical method astronomic measurements are used to orient the reference ellipsoid - and not to define position. The relative positions of points are then given by the other linear and angular geodetic measurements. For the local or astronomic triad this means, providing that an astronomic azimuth has been observed, that the directions of its unit vectors are uniquely defined by the astronomically observed values.

5. TRANSFORMATION RELATIONSHIPS.

Since the vector triads I_{pi} of the local astronomic systems are position dependent but are fixed with respect to the cartesian vectors I_j of the geodetic equatorial system U_i if the spatial point p under consideration is considered as the origin, it is possible by the use of direction cosines to find the transformation formulae from the local system to the geodetic reference frame. The celestial system is included within this transformation as the direction of the local vertical, as given by astronomic observations of latitude and longitude, is defined by it.

Thus -

$$I_{pi} = a_{ij} I_j \qquad \text{...........(1)}$$

where a_{ij} is the matrix of direction cosines which is given by the scalar products I_i, I_{pi} i.e.

$$a_{ij} = \begin{pmatrix} -\sin \omega_a & \cos \omega_a & o \\ -\sin \phi_a \cos \omega_a & -\sin \phi_a \sin \omega_a & \cos \phi_a \\ \cos \phi_a \cos \omega_a & \cos \phi_a \sin \omega_a & \sin \phi_a \end{pmatrix} \qquad \text{..........(2)}$$

and as the matrix a_{ij} is orthogonal the following relationships exist

$a_{ij} \{a_{ij}\}^T = \delta^i_j$ ($\{a_{ij}\}^T$ = Transposed, δ^i_j = Unit Matrix)

and since $\{a_{ij}\} \{a_{ij}\}^{-1} = \delta^i_j$ then $\{a_{ij}\}^{-1} = \{a_{ij}\}^T$

By inversion it is found that

$$I_j = \{a_{ij}\}^{-1} I_{pi} = \{a_{ij}\}^T I_{pi}$$

$$= \begin{vmatrix} -\sin \omega_a & -\sin \phi_a \cos \omega_a & \cos \phi_a \cos \omega_a \\ \cos \omega_a & -\sin \omega_a \sin \phi_a & \cos \omega_a \sin \phi_a \\ 0 & \cos \phi_a & \sin \phi_a \end{vmatrix} I_{pi} \quad \text{....(3)}$$

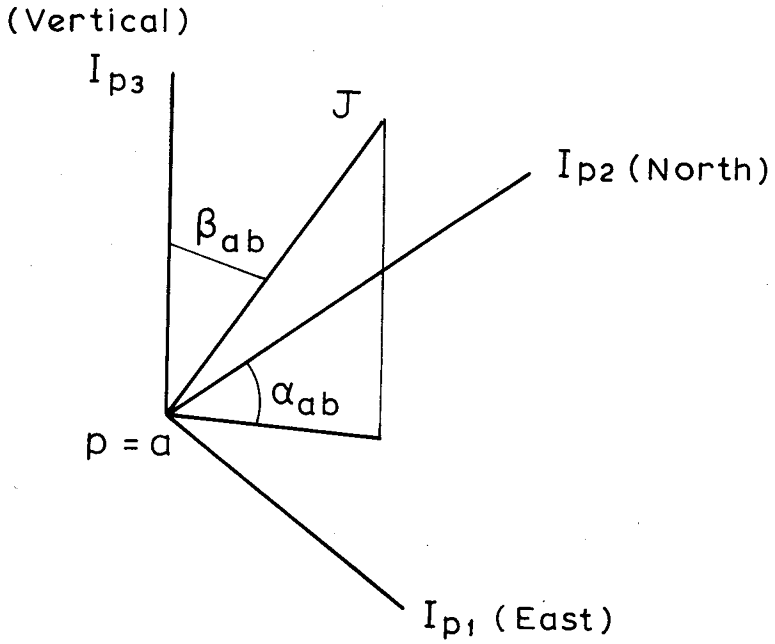
(Vertical)
I_{p3}
J
I_{p2} (North)
β_{ab}
α_{ab}
p = a
I_{p1} (East)

Fig. 3

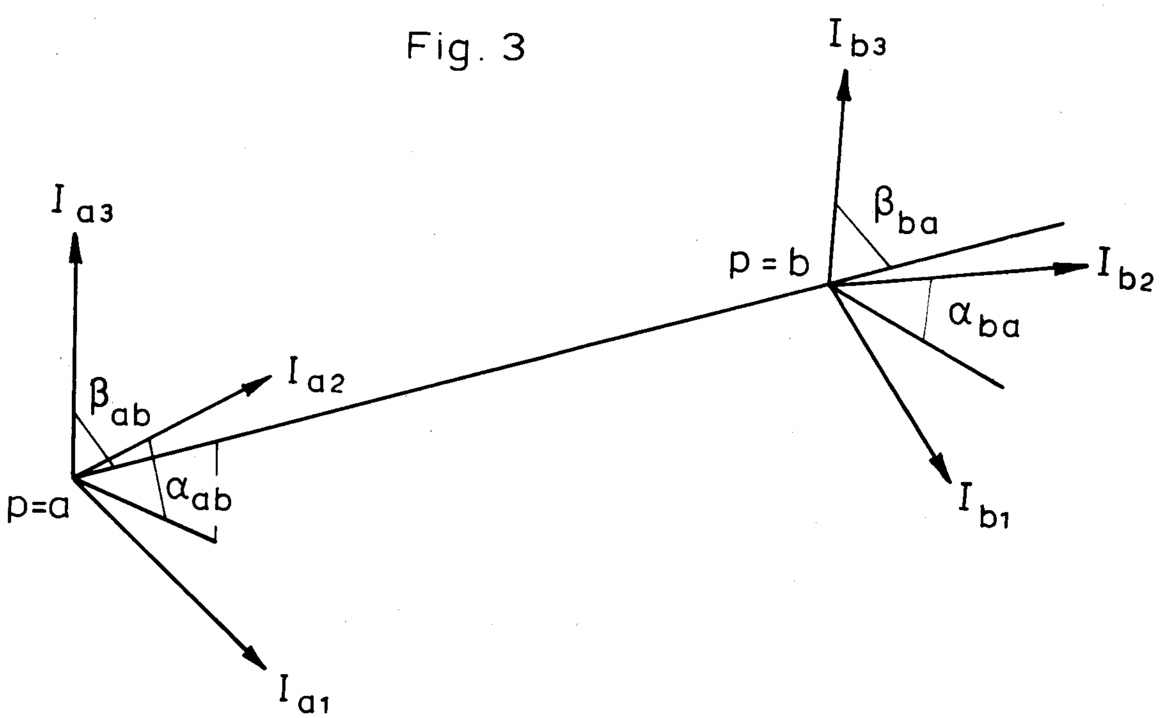
I_{b3}
I_{a3}
β_{ba}
p = b
I_{b2}
α_{ba}
I_{a2}
β_{ab}
p=a
α_{ab}
I_{b1}
I_{a1}

Fig. 4

If I_{bi} denote the unit vectors of the local astronomic system at an adjacent point b with astronomic coordinates ϕ_b, ω_b then the previous relationships will be equally true for the new quantities. However, the vectors I_j remain unchanged for both points, since their directions are fixed with respect to the rotating earth, i.e. they are independent of ϕ_p and ω_p.

Thus

$$I_{bj} = \bar{a}_{ij} \; I_i = \bar{a}_{ij} \; \{a_{ij}\}^T \; I_{ai} = b_{ij} \; I_{ai} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(4)$$

where the elements of the matrix b_{ij} are as given by *(Hotine, 1959, 8)*.

$$\begin{aligned}
b_{11} &= \cos(\omega_b - \omega_a) \\
b_{12} &= \sin\phi_a \sin(\omega_b - \omega_a) \\
b_{13} &= \cos\phi_a \sin(\omega_b - \omega_a) \\
b_{21} &= \sin\phi_b \sin(\omega_b - \omega_a) \\
b_{22} &= \cos\phi_b \cos\phi_a + \sin\phi_b \sin\phi_a \cos(\omega_b - \omega_a) \\
b_{23} &= \sin\phi_b \sin\phi_a - \sin\phi_b \cos\phi_a \cos(\omega_b - \omega_a) \\
b_{31} &= \cos\phi_b \sin(\omega_b - \omega_a) \\
b_{32} &= \cos\phi_a \sin\phi_b - \sin\phi_a \cos\phi_b \cos(\omega_b - \omega_a) \\
b_{33} &= \sin\phi_a \sin\phi_b + \cos\phi_a \cos\phi_b \cos(\omega_b - \omega_a)
\end{aligned}$$

They represent the transformation formulae between adjacent astronomic vector systems.

Previously the quantities I_i were defined as unit axis vectors of the geodetic reference frame U_i. In a similar manner it is possible to consider the elements I_{pi} as unit axis vectors of a local cartesian system. If J is any unit vector fixed in space and A_i, B_i are its respective components in the U_i and I_{pi} systems, then the following transformation formulae exist between the component groups

$$B_i = \{a_{ij}\}^T A_j \qquad \text{..................(5)}$$

and conversely, by inversion

$$A_i = a_{ij} B_j \qquad \text{..................(6)}$$

If the unit vector J is now considered in the local coordinate system I_{pi} (Fig. 3) with azimuth α_{ab} and zenith distance β_{ab}, then it can be visualized that

$$B_i = \begin{Bmatrix} \sin\alpha_{ab} \sin\beta_{ab} \\ \cos\alpha_{ab} \sin\beta_{ab} \\ \cos\beta_{ab} \end{Bmatrix}$$

$$= \{a_{ij}\}^T A_j \qquad \text{..................(7)}$$

and by inversion and multiplication of matrices

$$A_j = \begin{Bmatrix} -\sin\omega_a \sin\alpha_{ab} \sin\beta_{ab} - \sin\phi_a \cos\omega_a \cos\alpha_{ab} \sin\beta_{ab} + \cos\phi_a \cos\omega_a \cos\beta_{ab} \\ \cos\omega_a \sin\alpha_{ab} \sin\beta_{ab} - \sin\phi_a \sin\omega_a \cos\alpha_{ab} \sin\beta_{ab} + \cos\phi_a \sin\omega_a \cos\beta_{ab} \\ \cos\phi_a \cos\alpha_{ab} \sin\beta_{ab} + \sin\phi_a \cos\beta_{ab} \end{Bmatrix} \ldots (8)$$

The equation sets (7) and (8) are not independent, since from two equations in each set the third follows, i.e.

$$A_i A_i = B_i B_i = 1 \qquad \text{..................(9)}$$

Alternatively if the same vector J were to originate at an adjacent point b (ω_b, ϕ_b) and has there an azimuth α_{ba} and zenith distance β_{ba} then similar formulae will exist between these quantities. However, the components A_j will be the same in the relationships established for the new quantities as they refer to the same cartesian system. Thus for b

$$\begin{Bmatrix} \sin\alpha_{ba}\sin\beta_{ba} \\ \cos\alpha_{ba}\sin\beta_{ba} \\ \cos\beta_{ba} \end{Bmatrix} = \{\bar{a}_{ij}\}^T A_j \qquad \text{.................(10)}$$

and by substitution of Equation (8) into Equation (10) it is found, after some manipulation *(Hotine, 1959, 9)* that

$$\begin{Bmatrix} \sin\alpha_{ba}\sin\beta_{ba} \\ \\ \cos\alpha_{ba}\sin\beta_{ba} \\ \\ \cos\beta_{ba} \end{Bmatrix} \begin{Bmatrix} \sin\alpha_{ab}\sin\beta_{ab}\cos\Delta\omega_{ba} \begin{matrix} +\cos\alpha_{ab}\sin\beta_{ab}\sin\phi_a\sin\Delta\omega_{ba} \\ -\cos\beta_{ab}\cos\phi_a\sin\Delta\omega_{ba} \end{matrix} \\ \\ -\sin\alpha_{ab}\sin\beta_{ab}\sin\phi_b\sin\Delta\omega_{ba} \begin{matrix} +\cos\alpha_{ab}\sin\beta_{ab}(\cos\phi_a\cos\phi_b+\sin\phi_a\sin\phi_b\cos\Delta\omega_{ba}) \\ +\cos\beta_{ab}(\sin\phi_a\cos\phi_b-\cos\phi_a\sin\phi_b\cos\Delta\omega_{ba}) \end{matrix} \\ \\ \sin\alpha_{ab}\sin\beta_{ab}\cos\phi_b\sin\Delta\omega_{ba} \begin{matrix} +\cos\alpha_{ab}\sin\beta_{ab}(\cos\phi_a\sin\phi_b-\sin\phi_a\cos\phi_b\cos\Delta\omega_{ba}) \\ +\cos\beta_{ab}(\sin\phi_a\sin\phi_b+\cos\phi_a\cos\phi_b\cos\Delta\omega_{ba}) \end{matrix} \end{Bmatrix}$$

$$\text{.................(11)}$$

where $\Delta\omega_{ba} = \omega_b - \omega_a$

These three equations are again not independent, as from a pair the third follows. They give the transformation formulae for azimuth and zenith distance of any space vector from one local system into an adjacent one. Equation (11) gives the azimuth and zenith distance of the vector J at the point b as functions of the observed azimuth and zenith distance at the point a. This is made possible by moving the vector J parallel to itself into the b system. It must be remembered, however, that the quantities α_{ba} and β_{ba} so obtained refer to the same sense of the I_{bi} vectors as do the quantities α_{ab} and β_{ab} (Fig. 4). Thus, to obtain back directions β_{ba} must be subtracted from 180° and 180° should be added to α_{ba}.

The difference in orientation between adjacent astronomic systems will usually be small and if

$$\begin{pmatrix} \phi_b \\ \omega_b \\ \alpha_{ba} \\ \beta_{ba} \end{pmatrix} = \begin{pmatrix} \phi_a + \delta\phi \\ \omega_a + \delta\omega \\ \alpha_{ab} + \delta\alpha \\ \beta_{ab} + \delta\beta \end{pmatrix} \qquad \text{.................(12)}$$

then to the first order Equations (11) reduce to

$$\delta\alpha = \sin\phi_a \delta\omega + \cot\beta_{ab}(\sin\alpha_{ab}\delta\phi - \cos\alpha_{ab}\cos\phi_a\delta\omega) \qquad \text{.................(13)}$$

$$\delta\beta = -\cos\phi_a \sin\alpha_{ab}\delta\omega - \cos\alpha_{ab}\delta\phi \qquad \text{.................(14)}$$

If in addition $\beta_{ab} \simeq 90^{\circ}$, a legitimate assumption for terrestrial geodetic measurements, then Equation (13) becomes

$$\delta\alpha = \sin\phi_a \delta\omega \qquad \text{.................(15)}$$

This is the so-called Laplace azimuth equation which is used in the classical method to orient the spheroid.

The relationships previously derived are the very fundamentals of the spatial geodetic method proposed in this paper.

6. ORIGIN CONDITIONS.

As with all geodetic datums, the coordinates of the origin must be defined. A convenient value to adopt would be, as is customary in the classical procedure, based on astronomic observations for latitude and longitude, and the spirit-levelled height. A set of rectangular cartesian coordinates would then be given by the well-known transformation formulae

$$U_{oi} = \begin{pmatrix} \overline{N_o + H_o} \cos\phi_o \cos\omega_o \\ \overline{N_o + H_o} \cos\phi_o \sin\omega_o \\ \overline{N_o(1-e^2) + H_o} \sin\phi_o \end{pmatrix} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

If this approach is adopted, the coordinates U_{oi} would depend upon an ellipsoid of revolution that has its axis of revolution parallel to the earth's mean rotational axis, its zero meridian plane parallel to that of Greenwich and its centre displaced from the earth's centre of mass by amounts ΔX_{oi}. Moreover, the chosen ellipsoid would be tangential to the geoid at a point H_o units vertically below the origin. The coordinates of the origin with respect to the geocentre would then become

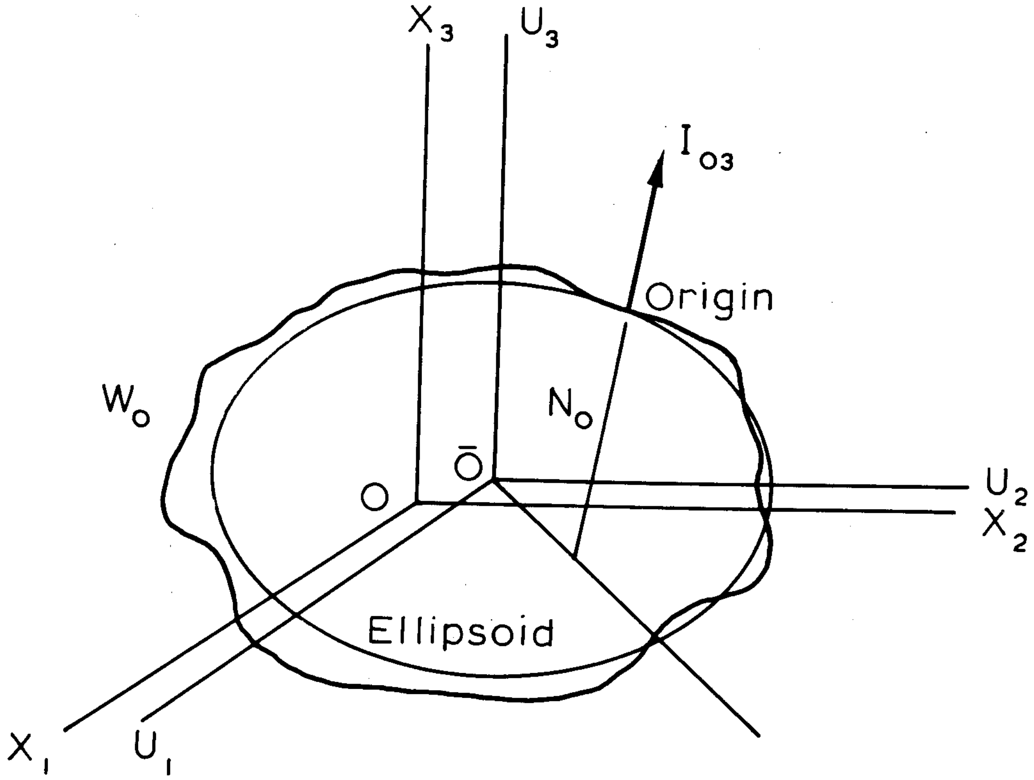

Fig. 5

$$X_{oi} = U_{oi} + \Delta X_{oi} \quad \text{................(17)}$$

The ellipsoid may also be used to give a visual representation of the orientation of the scheme of survey, and is convenient to use for this purpose because astronomic determinations of latitude and longitude are essentially a set of curvilinear coordinates. Thus, for this purpose an ellipsoid of revolution tangential, at the origin of survey to the equipotential surface containing the origin of survey, is introduced (Fig. 5).

The rectangular cartesian coordinates of the origin of survey relative to the geocentre then become

$$X_{oi} = \Delta X_{oi} + N_o \begin{pmatrix} \cos\phi_o \cos\omega_o \\ \cos\phi_o \sin\omega_o \\ (1-e^2) \sin\phi_o \end{pmatrix} \quad \text{................(18)}$$

The previous considerations only concern the positioning of the survey origin with respect to the geocentre. It remains to orient the scheme of the survey with respect to the cartesian system. This may be achieved in two ways by the manipulation of the astronomic triad at the origin of survey.

In general a coordinate system is said to be properly oriented in space when any one vector is uniquely defined within it and no rotation can be effected about this vector. Thus with respect to the cartesian reference frame and the local astronomic triad at the initial point of a survey, two aspects must be considered:-

1) The direction of a vector in the astronomic triad must be fixed with respect to the cartesian system.

2) No rotation must be possible about the vector chosen in 1).

These aspects and their implication will be considered in sequence for the two separate methods that are available.

The natural vector to hold fixed in direction, with respect to the cartesian system at the origin of survey would be the local vertical as its direction cosines are defined by astronomic latitude and longitude observations. Finally, the system would become completely rigid, when the direction of another vector within the local system is held fixed. The most convenient method by which to achieve this, is to observe the astronomic azimuth of a line emanating from the origin - a procedure which is customarily adopted.

The principles of an alternative method of orienting the local astronomic system have already been treated *(Hotine, 1959, 10-12)*, *(Näbauer, 1965, 24-29)*, *(Heibronner, 1968, 46-48)* and will therefore be only considered briefly.

The parameters (α = azimuth and β = refraction free zenith distance) of the unit spatial vector J, can be used to define the local coordinate system to a rotation about the vector J. This degree of freedom may be eliminated by the fixation of a second spatial direction $\bar{J}$ emanating from the same point. The method is best explained using geometric concepts.

Thus when the astronomic triad is rotated about the vector J (parameters α, β) the local vertical I_{p3} will generate a circular cone with apex at the origin. To spatially fix I_{p3} Hotine *(1959)* and Näbauer *(1965)* allow this cone to be intersected or touched by another cone (parameter $\bar{\beta}$). The same effect is achieved when instead of a second cone, a plane (parameter $\bar{\alpha}$) that contains the apex of the previous cone and the vector I_{p3} is chosen. To avoid ambiguity Heilbronner *(1968)* points out that the difference between the azimuth α and $\bar{\alpha}$ should not be near or equal to 90°.

Thus in this latter method, it is possible to fix the unit vectors of the spatial triad I_{pi} by means of the parameters $\alpha, \beta, \bar{\beta}$ or $\alpha, \beta, \bar{\alpha}$ of two independent directions in that triad.

As both the former methods lead to the same result, field conditions determine the method that is preferable. The refraction free zenith distance is not generally known and even if the spirit-levelled height between the origin and an adjacent point had been determined, additional information such as astronomic position observations or gravity data, is required to calculate it. Overall therefore the initially described method is preferable.

7. ERRORS IN ASTRONOMIC DATA AND THEIR EFFECT UPON ORIENTATION.

The rectangular cartesian coordinates X_{oi} of the origin, relative to the geocentre were defined in Equations (17) and (18). It is legitimate to assume, that the astronomic determinations, even though of the highest precision, will contain errors.

Thus

$$\begin{bmatrix} \phi_o \\ \\ \omega_o \end{bmatrix} = \begin{bmatrix} \bar{\phi}_o + \Delta\phi_o \\ \\ \bar{\omega}_o + \Delta\omega_0 \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (19)$$

and

$$X_{oi} = \Delta X_{oi} + N_o \begin{bmatrix} \cos\ (\bar{\phi}_o + \Delta\phi_o)\ \cos\ (\bar{\omega}_o + \Delta\omega_o) \\ \cos\ (\bar{\phi}_o + \Delta\phi_o)\ \sin\ (\bar{\omega}_o + \Delta\omega_o) \\ (1 - e^2)\ \sin\ (\bar{\phi}_o + \Delta\phi_o) \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (20)$$

Providing $\Delta\phi_o$ and $\Delta\omega_o$ are small so that the terms containing the products of these elements may be neglected, then to the first order

$$X_{oi} = \Delta X_{oi} + N_o \begin{bmatrix} \cos\bar{\phi}_o \cos\bar{\omega}_o \\ \cos\bar{\phi}_o \sin\bar{\omega}_o \\ (1-e^2) \sin\bar{\phi}_o \end{bmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (21)$$

$$+ N_o \begin{bmatrix} -\sin\bar{\phi}_o \cos\bar{\omega}_o & -\sin\bar{\omega}_o \cos\bar{\phi}_o \\ -\sin\bar{\phi}_o \sin\bar{\omega}_o & \cos\bar{\omega}_o \cos\bar{\phi}_o \\ (1-e^2) \cos\bar{\phi}_o & 0 \end{bmatrix} \begin{bmatrix} \Delta\phi_o \\ \Delta\omega_o \end{bmatrix}$$

Comparing Equations (20) and (21) it can be seen that the geodetic system U_i has its origin translated by

$$\Delta U_{oi} = \begin{bmatrix} -\sin\bar{\phi}_o \cos\bar{\omega}_o & -\cos\bar{\phi}_o \sin\bar{\omega}_o \\ -\sin\bar{\phi}_o \sin\bar{\omega}_o & \cos\bar{\phi}_o \cos\bar{\omega}_o \\ (1-e^2) \cos\bar{\phi}_o & 0 \end{bmatrix} \begin{bmatrix} N_o\ \Delta\phi_o \\ N_o\ \Delta\omega_o \end{bmatrix} \quad \ldots\ldots\ldots (22)$$

from the centre of mass. The axes of the geodetic system and the ideal reference frame will therefore remain parallel.

The orientation of the local triad is, as was previously mentioned, partly achieved by astronomic azimuth observations. Unfortunately such observations are often strongly influenced by systematic observational errors. The effect of introducing an erroneous azimuth at the datum point will be to rotate the local triad, and thus the introduced ellipsoid, about the local vertical at that point. This means that the geodetic system U_i is rotated so that its axes are no longer parallel to those of the ideal system.

Thus, if an approximate azimuth $\bar{\alpha}$ is introduced at the datum point so that

$$\alpha = \bar{\alpha} + \Delta\alpha \qquad \ldots\ldots\ldots\ldots\ldots\ldots(23)$$

then the geocentric coordinates of the origin of survey are given by

$$X_{oi} = U_{oi} + \Delta X_{oi} + \Delta U_{oi} + R_o \qquad \ldots\ldots\ldots\ldots\ldots\ldots(24)$$

where R_o is the rotational matrix brought about by the introduction of an erroneous azimuth at the datum point. This rotational matrix is given by

$$R_o = \begin{pmatrix} 1 & \Delta\alpha \sin\bar{\phi}_o & -\Delta\alpha\cos\bar{\phi}_o \sin\bar{\omega}_o \\ -\Delta\alpha\sin\bar{\phi}_o & 1 & \Delta\alpha\cos\bar{\phi}_o\cos\bar{\omega}_o \\ \Delta\alpha\cos\bar{\phi}_o\sin\bar{\omega}_o - & \cos\bar{\phi}_o\cos\bar{\omega}_o & 1 \end{pmatrix} \qquad (25)$$

providing $\Delta\alpha$ is small so that one may write

$$\begin{pmatrix} \cos\Delta\alpha \\ \sin\Delta\alpha \end{pmatrix} = \begin{pmatrix} 1 \\ \Delta\alpha \end{pmatrix},$$

The previous analysis demonstrates that, even with observations of the highest precision, one can only hope to achieve parallelism between the axes of the ideal and geodetic reference frames. Centering the geodetic frame presents another problem, and is, as has been shown, a function of the knowledge of the geocentre and errors in astronomic position determinations. However, this aspect is of no consequence as the present interest lies in unique cartesian coordinates relative to the origin of survey, although it must be remembered that a new rotational matrix R_p is introduced for every station where another azimuth is observed, the overall effect of which would be to produce a series of non-parallel reference frames.

8. SPIRIT LEVELLING.

In order to incorporate the results of spirit-levelling into a three-dimensional cartesian reference frame *(Stolz & Gilliland, 1969)*, it is necessary to define the vector C_p as the vector joining the origin $\bar{0}$ of the geodetic system and a surface point p (Fig. 6).

In Fig. 7, G is the physical surface of the earth and p = a and p = b are arbitary points on G. C_a is the vector previously defined and bC = - dh is the infinitesimal levelling increment between a and b and is measured normal to the equipotential surface W_a = const. through a. Denoting the projection of ab = s onto the surface C_a = const. by ds and the infinitesimal increase bD in the vector C_a as $-dC_a$ gives to the first order that Cb = bE, or

$$- |dC_a| + dh = \delta\ ds \qquad \qquad ....(26)$$

where δ is the component of the angle between the vectors C_a and $-I_{a3}$ as measured in the azimuth α_{ab} of the vertical plane containing a and b.

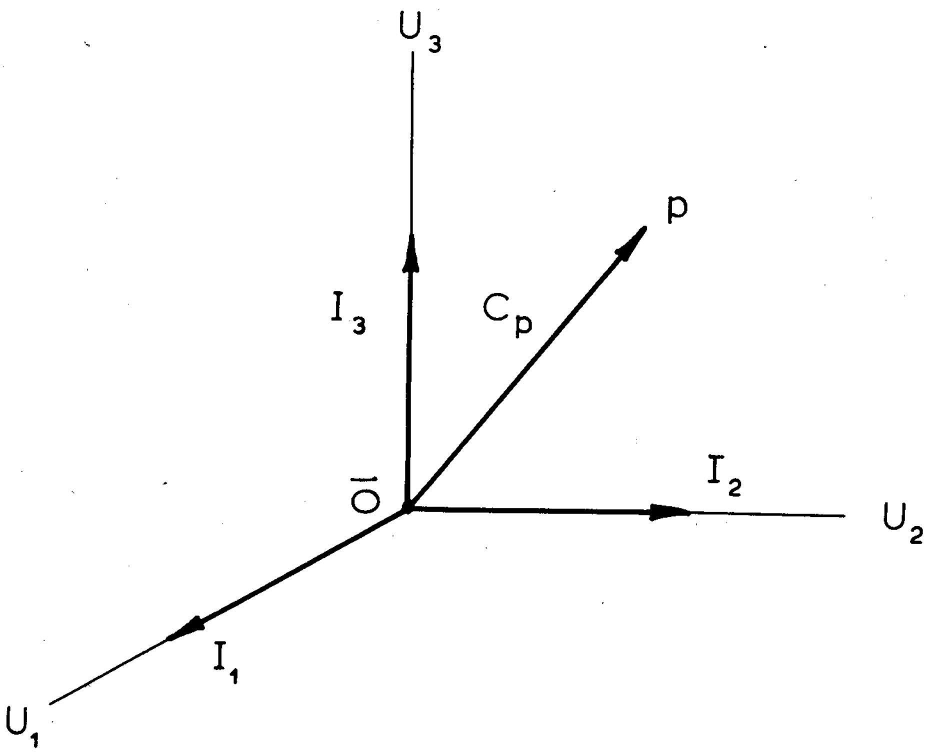
U3
p
I3
Cp
O
I2
U2
I1
U1

Fig. 6

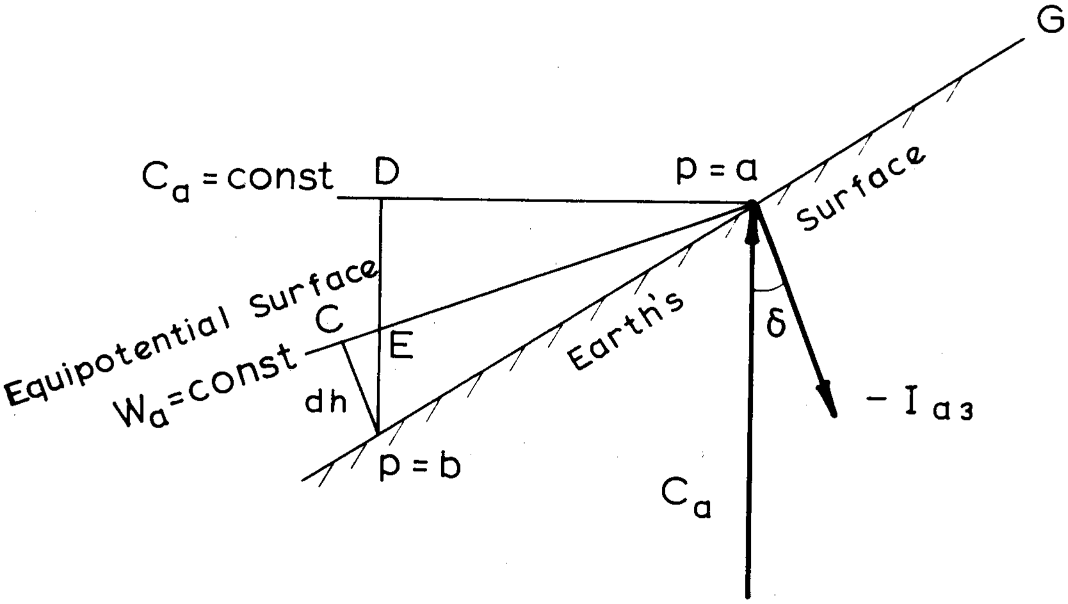
G
Ca = const
D
p = a
Surface
Equipotential Surface
C
E
Earth's
δ
Wa=const
dh
- I a3
p = b
Ca

Fig. 7

Thus

$$|dC_a| = dh - (\xi_a^1 \cos\alpha_{ab} + \eta_a^1 \sin \alpha_{ab})ds \qquad \ldots\ldots\ldots\ldots\ldots\ldots(27)$$

where ξ_a^1 and η_a^1 are respectively the components of the angle δ in the plane of the meridian and of the prime vertical.

Equation (27) establishes the connection between an infinitesimal levelling increment and the increment in the vector C_a. The difference in magnitude of the vectors C_a and C_b at a and b respectively, which are a finite distance apart is expressed by the equation

$$|C_b| - |C_a| = \lambda = \int_a^b dh - \int_a^b (\xi^1 \cos \alpha_{ab} + \eta^1 \sin \alpha_{ab})ds \ldots\ldots(28)$$

and since the integrals depend on the path of levelling, these must be calculated along the same path.

To determine ξ^1 and η^1 a knowledge is required of the direction of the vertical and the direction of the vector C_p at each point. The direction of the vertical may be established by astronomic latitude and longitude observations or may be interpolated by the use of gravity data, and the relative direction of the vector C_p may be obtained from Equation (8).

The problem of unifying the results of spirit - levelling and all other geometric geodetic measurements thus reduces to the determination of the magnitude and direction of the vector C_b. This vector is completely defined by the equation

$$C_b = (U_{ai} + s A_i) I_i \qquad \ldots\ldots\ldots\ldots\ldots\ldots(29)$$

where the A_i are computed from Equation (8) and the results of spirit-levelling have not been included.

All the observations defining the A_i are made at discrete points and all but the measured distances are subject to the earth's gravitational field at that point. However assuming that the earth's gravitational field remains constant over the period of observation, all observations are equally affected at a point. For this reason Equation (29) completely defines the vector C_b.

Considering the results of spirit-levelling another vector C_b may be obtained using the direction established by Equation (29) and the magnitude λ calculated in Equation (28) or

$$\overline{C}_b = (|C_a| + \lambda)\ L \qquad \ldots\ldots\ldots\ldots\ldots\ldots (30)$$

where L is the unit vector in the direction of C_b and is defined by

$$L = \frac{C_b}{|C_b|} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (31)$$

The position vector $\overline{C}_b$ will give rise to another set of coordinate differences $s\,\overline{A}_i$ where in general

$$\overline{A}_i \neq A_i \qquad \ldots\ldots\ldots\ldots\ldots\ldots (32)$$

and since the direction of the two position vectors $\overline{C}_b$ and C_b are identical, this additional set of coordinate differences $s\,\overline{A}_i$ gives rise to a refraction-free zenith distance β_{ab} between connected points. This new zenith distance is defined by

$$\cos \beta_{ab} = I_{a3} \cdot \overline{J} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (33)$$

where $\overline{J}$ is the unit vector in the direction of ab as obtainable from

$$\overline{J} = \frac{C_a - \overline{C}_b}{|C_a - \overline{C}_b|} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (34)$$

Equation (33) incorporates spirit-levelling into a three-dimensional cartesian reference frame.

The relationship between this computed zenith distance and the observed zenith distance remains somewhat obscure and the question arises whether it is possible to do without observed zenith distances when all or some points of a spatial geodetic network have been connected by spirit-levelling. Previously it was stated that Equation (29) completely defined the vector C_b when the results of spirit-levelling were not considered and that an additional refraction free zenith distance β_{ab} could only be calculated using the direction as established by this equation and the magnitude $|C_a| + \lambda$ as given by spirit-levelling. This gives the impression that the two are algebraically related. It can be shown however, that in order to determine the A_i it is not a prerequisite to know the observed zenith distance when the points a and b have been connected by spirit-levelling and that therefore the two quantities are distinct.

From Fig. 8 it is seen that the point b is uniquely defined by the intersection of the spatial segments ab, $\overline{0b}$ with the vertical plane containing the line segment ab (parameter α). Two sets of solutions A_i are obtained from the three conditions that are to be satisfied, i.e.

$$A_i \; A_i - 1 = 0 \qquad \text{................(35)}$$

$$\overline{A}_i \; \overline{A}_i - L_i \; L_i = 0 \qquad \text{................(36)}$$

$$\text{Tan } \alpha_{ab} = \frac{A_2 \cos \omega_a - A_1 \sin \omega_a}{A_3 \cos \omega_a - A_1 \sin \phi_a \cos \omega_a - A_2 \sin \phi_a \sin \omega_a} \qquad \text{..........(37)}$$

where the L_i are the cartesian components of the unit vector L that was defined in Equation (31).

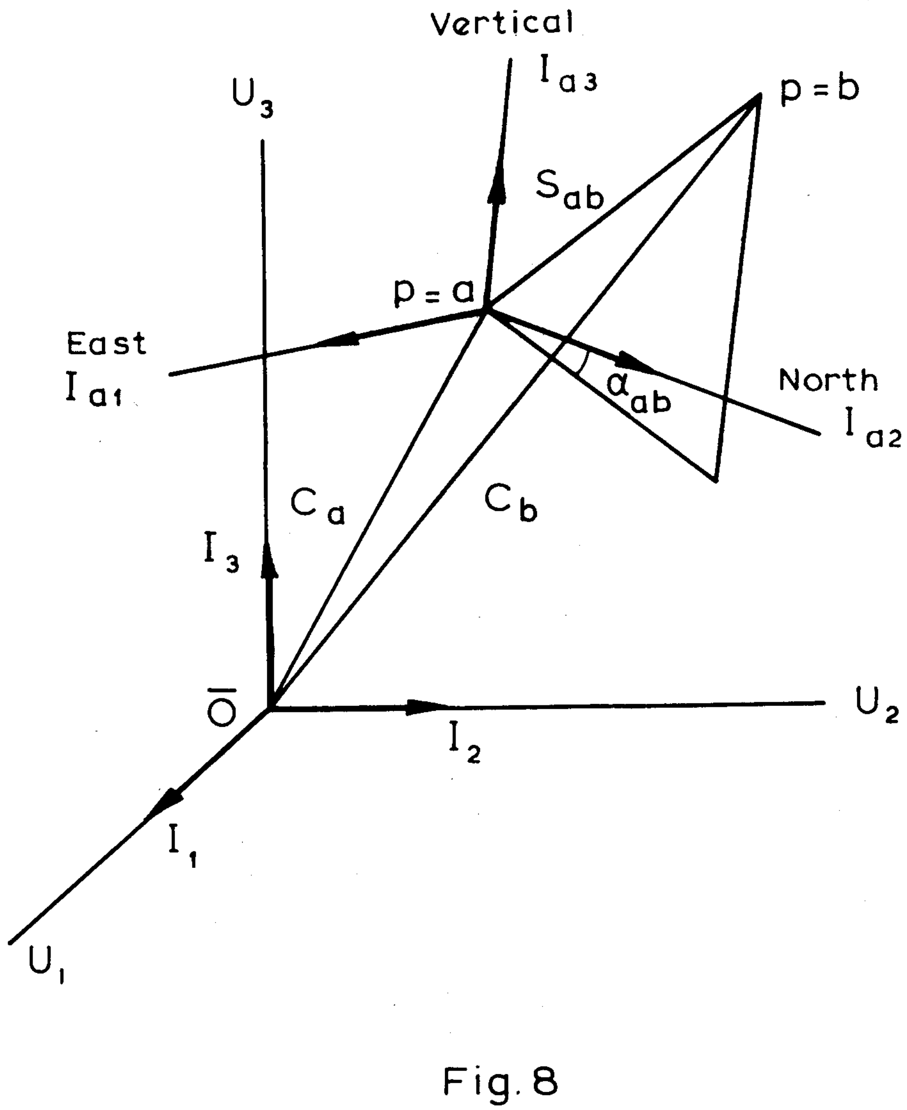
Vertical
I_{a3}
U_3
$p = b$
S_{ab}
$p = a$
East
I_{a1}
α_{ab}
North
I_{a2}
C_a
C_b
I_3
$\bar{O}$
U_2
I_2
I_1
U_1

Fig. 8

The required solution may then be extracted, as the direction of the line segment ab is known, and it can be seen that the measured and calculated zenith distances are independent.

9. DIFFERENTIAL DISPLACEMENTS.

It is now proposed to find the changes in length azimuth and zenith angle resulting from a change in the end coordinates $d\,U_{pi}$ of a line segment with unit vector J.

Multiplying both sides of Equation (5) by s and remembering that

$$s\ A_i = \Delta U_i^{ab} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(38)$$

where the $\Delta\,U_i^{ab}$ represent the changes in cartesian coordinates between adjacent surface points ab, then

$$\begin{Bmatrix} \sin\alpha_{ab}\ \sin\beta_{ab} \\ \cos\alpha_{ab}\ \sin\beta_{ab} \\ \cos\beta_{ab} \end{Bmatrix} = \{a_{ij}\}^T\ \Delta U_i^{ab} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(39)$$

These relationships can be used to calculate the measured values s_{ab}, β_{ab} and α_{ab}, i.e.

$$s_{ab} = \{\Delta\,U_i^{ab}\ \Delta\,U_i^{ab}\}^{\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(40)$$

$$\mathrm{Tan}\ \alpha_{ab} = \frac{B_1}{B_2} = \frac{\Delta U_2^{ab}\cos\omega_a - \Delta U_1^{ab}\sin\omega_a}{\Delta U_3^{ab}\cos\omega_a - \Delta U_1^{ab}\sin\phi_a\cos\omega_a - \Delta U_2^{ab}\sin\phi_a\sin\omega_a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(41)$$

$$\cos \beta_{ab} = \frac{B_3}{s_{ab}} = \frac{\Delta U_1^{ab} \cos \phi_a \cos \omega_a + \Delta U_2^{ab} \cos \phi_a \sin \omega_a + \Delta U_3^{ab} \sin \phi_a}{s_{ab}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(42)$$

Upon differentiating Equations (40), (41) and (42) one obtains

$$ds_{ab} = \frac{1}{s_{ab}} \Delta U_i^{ab} (dU_{ai} - dU_{bi}) \quad \ldots\ldots\ldots\ldots\ldots\ldots(43)$$

$$\begin{aligned} d\beta_{ab} = {} & \frac{\Delta U_1^{ab} \cos \beta_{ab} - s_{ab} \cos \phi_a \sin \omega_a}{s_{ab}^2 \sin \beta_{ab}} (dU_{a1} - dU_{b1}) + \\ & \frac{\Delta U_2^{ab} \cos \beta_{ab} - s_{ab} \cos \phi_a \sin \omega_a}{s_{ab}^2 \sin \beta_{ab}} (dU_{a2} - dU_{b2}) + \\ & \frac{\Delta U_3^{ab} \cos \beta_{ab} - s_{ab} \sin \phi_a}{s_{ab}^2 \sin \beta_{ab}} (dU_{a3} - dU_{b3}) + \\ & \cos \alpha_{ab} \, d\phi_a - \cos \phi_a \sin \alpha_{ab} \, d\omega_a \end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots(44)$$

$$\begin{aligned} d\alpha_{ab} = {} & \frac{\sin \phi_a \cos \omega_a \sin \alpha_{ab} - \sin \omega_a \cos \alpha_{ab}}{s_{ab} \sin \beta_{ab}} (dU_{a1} - dU_{b1}) + \\ & \frac{\sin \phi_a \sin \omega_a \sin \alpha_{ab} + \cos \omega_a \cos \alpha_{ab}}{s_{ab} \sin \beta_{ab}} (dU_{a2} - dU_{b2}) + \\ & \frac{\cos \phi_a \sin \alpha_{ab}}{s_{ab} \sin \beta_{ab}} (dU_{a3} - dU_{b3}) + \\ & \cos \beta_{ab} \sin \alpha_{ab} \, d\phi_a + (\sin \phi_a - \cos \alpha_{ab} \cos \phi_a \cos \beta_{ab}) \, d\omega_a \end{aligned} \quad \ldots\ldots\ldots\ldots(45)$$

and these represent the expressions sought.

10. NETWORK ADJUSTMENT.

The formulae derived in the previous sections are those that are required for the computation of provisional coordinates from observed quantities which are all subject to observational error. The observed values will be used in this section to set up observation equations from which corrections to the provisional coordinates can be deduced in order to obtain estimates of the most probable values of these quantities.

In modern geodetic work, it is customary to measure the following:-

1) horizontal directions
2) spatial distances
3) vertical angles
4) astronomic azimuths, latitudes and longitudes
5) spirit-levelled differences in elevation
6) gravity data , and

in practise, the number of observations made always exceeds the minimum required for the unique determination of the unknown quantities. These redundant observations serve to guard against blunders and to obtain a statistically more precise estimate of the unknowns.

One way to deal with the problem of adjustment is to form condition equations, one for each redundant observation, with the corrections to the observed quantities as unknowns. Assuming a normal distribution of errors, the most probable set of corrections is given, according to the principles of least squares, by that set which minimizes the sum of the weighted squares of the residuals, while satisfying the condition equations. A different approach is to assume initial values for all the unknown quantities and to express the effect of small changes in the observed quantities upon these values. The most probable corrections to the initial estimate of the unknowns are then those which make the sum of the weighted squares of the changes in the observed quantities a minimum. As opposed to the adjustment by "conditions" this method represents an adjustment by "parameters or differential displacements."

On account of its suitability for programmed computation, only the method of differential displacements is considered throughout this paper. Moreover, the least squares procedures and error analysis are simpler and quicker.

The procedure is commonly commenced by assuming approximations for the required unknowns. If the displacements resulting from the adjustment are too large, then the corrected values of the unknowns may be used as second approximations and the process reapplied.

The main disadvantage of the method is the inability of the procedure to detect mistakes, either in the data or the results, at the completion of one cycle of the adjustment. The observational mistakes can only be found if the adjustment is preceded by the setting out of some of the circuit closures and testing whether the observed quantities satisfy them within reasonable limits. Similar tests must be applied after adjustment, to verify the correctness of the result.

When the unknowns are the geographical or cartesian coordinates, the differential displacement method is known by the name of "variation of coordinates."

(i) Variation of Coordinates.

The method of variation of coordinates is readily adaptable to the simultaneous adjustment of a spatial network on a digital computer. The required provisional coordinates may either be obtained by a "forward computation" or from the geographical coordinates of a prior adjustment and may then be used in conjunction with the iterative procedure previously mentioned.

The unknowns of the observation equations are dU_{pi}, the corrections to the provisional cartesian coordinates U_{pi} of the network stations and $d\phi_p$, $d\omega_p$ the corrections to astronomic latitude and longitude respectively. It is necessary to introduce ϕ_p and ω_p as unknowns because all observations with the exception of the measured distances are subject to the direction of gravity.

As is customary, the approximations -

$$\begin{pmatrix} U_{pi} \\ \phi_p \\ \omega_p \\ o_p \\ k_p \end{pmatrix} = \begin{pmatrix} \overline{U}_{pi} + dU_{pi} \\ \overline{\phi}_p + d\phi_p \\ \overline{\omega}_p + d\omega_p \\ \overline{o}_p + do_p \\ \overline{k}_p + dk_p \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots (46)$$

are introduced, where do_p is the unknown optimum correction to the round of theodolite directions at station p and dk_p is the correction to be applied to the assumed value of the coefficient of refraction.

Denoting the unknowns dU_{pi}, $d\phi_p$ and $d\omega_p$ of the observation equations by the vector dY_{pk}, the changes $d\alpha_{ab}$ in azimuth, $d\beta_{ab}$ in zenith angle and ds_{ab} in spatial distance - resulting from displacements dU_{ai}, dU_{bi} in the provisional coordinates of the two end stations and rotations $d\omega_a$, $d\omega_b$, $d\phi_a$, $d\phi_b$ about the vectors I_{a3}, I_{b3} - are given by equations of the form

$$d\alpha_{ab} = a_k\{dY_{ak} - dY_{bk}\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(47)$$

$$d\beta_{ab} = b_k\{dY_{ak} - dY_{bk}\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(48)$$

$$ds_{ab} = c_k\{dY_{ak} - dY_{bk}\} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(49)$$

The interim result will be the matrix of coefficients A and the vector of right-hand sides b of the observation equations

$$AY = b \qquad \ldots\ldots\ldots\ldots\ldots\ldots(50)$$

The equations that must be satisfied by the unknowns, subject to V^TWV being a minimum, may be of the following types:-

1) Horizontal directions = α_{ab}

$$V_{\alpha_{ab}} = -do_a + a_k\{dY_{ak} - dY_{bk}\} + \overline{\overline{\alpha}_{ab} - \alpha_{ab} + \overline{o}_a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(51)$$

where

$$\tan\overline{\alpha}_{ab} = \frac{\Delta\overline{U}_2^{ab}\cos\overline{\omega}_a - \Delta\overline{U}_1^{ab}\sin\overline{\omega}_a}{\Delta\overline{U}_3^{ab}\cos\overline{\Phi}_a - \Delta\overline{U}_1^{ab}\sin\overline{\Phi}_a\cos\overline{\omega}_a - \Delta\overline{U}_2^{ab}\sin\overline{\Phi}_a\sin\overline{\omega}_a} \qquad \ldots\ldots(52)$$

$$dY_{b4} = d\phi_b = dY_{b5} = d\omega_b = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(52a)$$

and

$$a_1 = (\sin\phi_a \cos\omega_a \sin\alpha_{ab} - \sin\omega_a \cos\alpha_{ab}) / s_{ab} \sin\beta_{ab}$$

$$a_2 = (\sin\phi_a \sin\omega_a \sin\alpha_{ab} + \cos\omega_a \cos\alpha_{ab}) / s_{ab} \sin\beta_{ab}$$

$$a_3 = -\cos\phi_a \sin\alpha_{ab} / s_{ab} \sin\beta_{ab} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (53)$$

$$a_4 = \cot\beta_{ab} \sin\alpha_{ab}$$

$$a_5 = \sin\phi_a - \cos\alpha_{ab} \cos\phi_a \cot\beta_{ab}$$

2) Zenith angles $= \beta_{ab}$

$$V_{\beta_{ab}} = -s_{ab}\, dk_a + b_k\{dY_{ak} - dY_{bk}\} + \overline{\overline{\beta}_{ab} - \beta_{ab} + s\overline{k}_a} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (54)$$

where

$$\cos\overline{\beta}_{ab} = \frac{1}{\overline{s}_{ab}} (\Delta\overline{U}_1^{ab} \cos\overline{\phi}_a \cos\overline{\omega}_a + \Delta\overline{U}_2^{ab} \cos\overline{\phi}_a \sin\overline{\omega}_a + \Delta\overline{U}_3^{ab} \sin\overline{\phi}_a) \qquad \ldots\ldots\ldots\ldots\ldots\ldots (55)$$

$$dY_{b4} = d\phi_b = dY_{b5} = d\omega_b = 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots (55a)$$

and

$$\overline{s}_{ab} = \{\Delta\overline{U}_i^{ab} \; \Delta\overline{U}_i^{ab}\}^{\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (56)$$

At this stage it is irrelevant whether an average dk_p is assumed for a region or whether several partial regions are considered or alternatively whether the methods proposed by Hotine *(1959, 23)* or Hradilek *(1968)* who adopt an unknown dk_p for each observation station are followed, as long as the degree of over determination remains sufficiently large.

$$b_1 = \frac{\Delta\overline{U}_1^{ab} \cos \beta_{ab} - s_{ab} \cos \overline{\phi}_a \cos \overline{\omega}_a}{s_{ab}^2 \sin \beta_{ab}}$$

$$b_2 = \frac{\Delta\overline{U}_2^{ab} \cos \beta_{ab} - s_{ab} \cos \overline{\phi}_a \sin \overline{\omega}_a}{s_{ab}^2 \sin \beta_{ab}}$$

$$b_3 = \frac{\Delta\overline{U}_3^{ab} \cos \beta_{ab} - s_{ab} \sin \overline{\phi}_a}{s_{ab}^2 \sin \beta_{ab}} \qquad \text{.................(57)}$$

$$b_4 = - \cos \alpha_{ab}$$

$$b_5 = - \cos \overline{\phi}_a \sin \alpha_{ab}$$

3) Spatial Distances $= s_{ab}$

$$V_{s_{ab}} = c_k\{dY_{ak} - dY_{bk}\} + \overline{\overline{s}_{ab} - s_{ab}} \qquad \text{.................(58)}$$

where $dY_{a4} = dY_{b4} = 0$ ....(58a)

and $c_i = \dfrac{\Delta\overline{U}_i^{ab}}{s_{ab}}$ ....(59)

4) Astronomic Azimuths $= \alpha^*_{ab}$

$$V_{\alpha^*_{ab}} = a_k\{dY_{ak} - dY_{bk}\} + \overline{\overline{\alpha}_{ab} - \alpha^*_{ab}} \qquad \text{.................(60)}$$

where

$$dY_{b4} = d\phi_b = dY_{b5} = d\omega_b = 0 \qquad \text{.................(60a)}$$

and the a_j were defined in Equation (53)

5) Astronomic Latitudes $= \phi_p$

$$V_{\phi_p} = dY_{p4} + (\overline{\phi}_p - \phi_p) \qquad \text{.................(61)}$$

6) Astronomic Longitudes - ω_p

$$V_{\omega_p} = dY_{p5} + (\bar{\omega}_p - \omega_p) \qquad \text{................(62)}$$

where in all cases the barred quantities pertain to the approximately known values and the asterisk is used to distinquish directions from astronomic azimuths.

(ii) Formation of Normal Equations.

According to the least squares principle, the observation equations

$$AY = b + V \qquad \text{................(63)}$$

with a matrix of weights W, are to be solved so that $V^T WV$ becomes a minimum. Here it should be realized that the matrix W is diagonal if the observations are not correlated. Otherwise and in general

$$W = \sigma^{-1} \qquad \text{................(64)}$$

where σ is the estimated variance covariance matrix of the observed quantities. Substituting B and C for $W^{\frac{1}{2}}A$ and $W^{\frac{1}{2}}b$ respectively, then

$$V^T W V = (BY - C)^T (BY - C) \qquad \text{................(65)}$$

and

$$\frac{\partial (V^T WV)}{\partial Y} = \frac{\partial (Y^T B^T BY - Y^T B^T C - C^T BY + C^T C)}{\partial Y} = 0 \qquad \text{........(66)}$$

The respective differentiation of the quadratic and bilinear forms with respect to the column vector Y is then *(Thompson, 1962)*

$$\frac{\partial (Y^T B^T B Y)}{\partial Y} = 2 B^T B Y \qquad \text{................(67)}$$

and

$$\frac{\partial(Y^T\ B^T\ C)}{\partial Y} = \frac{\partial(C^T\ B\ Y)}{\partial Y} = B^T\ C \qquad \text{................(68)}$$

Hence

$$\frac{\partial(V^T\ W\ V)}{\partial Y} = 2\ B^T\ B\ Y - B^T\ C = 0 \qquad \text{................(69)}$$

and the normal equations are

$$B^T\ B\ Y = B^T\ C \qquad \text{................(70)}$$

or $$A^T\ W\ A\ Y = A^T\ W\ b \qquad \text{................(71)}$$

Adopting a shorter notation

$$N\ Y = d \qquad \text{................(72)}$$

from which the unknowns dU_{pi}, $d\phi_p$, $d\omega_p$, do_p and dk_p may be extracted. An error analysis will then give the precision of the points in space and that of the adjusted observations.

11. CONCLUSION.

The ideal theoretical requirements of the astronomic vector method presented in this paper are:-

1) Astronomic determinations of latitude and longitude of every station as well as the observed azimuth of one line in each local triad.

2) The connection of adjacent station by spirit-levelling.

Astronomic position determinations are necessary to define the direction of the local vertical and an azimuth is needed to orient the astronomic triad - although it must be noted that in order to transfer azimuth between adjacent vector systems, only the astronomic position coordinates of the terminal stations need to be known (see Equation (11)).

The connection of trigonometric stations by spirit-levelling is desirable because for average terrestrial networks, the measured zenith distances are known to be strongly influenced by refraction errors and thus the precision of the vertical angle measurement is not compatible with that of other geodetically determined quantities.

Apart from giving theoretically sound results, compliance with the abovementioned conditions will, in conjunction with other geodetic measurements, ensure a high degree of over determination and thus a good estimate of the precision of the results.

Unfortunately, departures from such an ideal system are the rule rather than the exception and it seems that the proposed method is only a dream that cannot be realized. However, most modern terrestrial networks, of which the Australian Geodetic Datum is an example, comprise a fairly regular pattern of Laplace stations and trigonometric points where both the latitude and longitude have been observed. Levelling connections are also made where practicable and at fairly regular intervals and apart from an abundance of geometric data, a good coverage of gravity data is commonly available.

Thus providing a gravity survey around the station exists, the problem of the deficiency of astronomic position determinations may be overcome by gravimetric interpolations of the deviations of the vertical. A precision of $\pm 0\rlap{.}''5$ is possible by this method and this is approximately the order of accuracy of the astronomic observations.

An alternative approach would be to introduce the geodetically computed values of latitude and longitude of a prior adjustment into the new adjustment as 'quasi-observations' with a systematic error, the magnitude of which would be identical to the deflections of the vertical. These 'quasi-observations' would be of a high precision and theoretically must be introduced with their variance - covariance matrix. Such a method can only be considered if the degree of over determination of the adjustment is not prejudiced.

One possibility of reducing the effect of refraction or measured zenith distances would be to adjust the level network separately and to recompute 'new'zenith distances from the trigonometric heights derived from an additional adjustment in which the points connected by spirit-levelling have been held fixed.

A better method is to compute the zenith distances between points connected by spirit-levelling according to the proposals within this paper and to incorporate this computed zenith distance into the adjustment with its appropriate weight coefficient. This proposal would also include the correction to be applied to the assumed value of the coefficient of refraction as an unknown - a concept that remains somewhat obscure in the previous suggestion.

Although the method presented is theoretically sound, only test networks computed according to the principles within this paper can provide positive conclusions as to its value. It is clear however that, providing the interest lies in unique three-dimensional cartesian coordinates, the geodetic curvilinear reference frame to which observations are customarily projected before transformation, is not a prerequisite to the solution of geodetic problems as these computations can all be performed within the natural reference frame in which the observations are made and it is surprising that such a method has not been previously used.

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