THE THEORY AND GEODETIC USE OF SOME COMMON PROJECTIONS

by

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P R E F A C E

The contents of this monograph have been put together primarily for the use of students following degree courses in Surveying at the University of New South Wales, in the absence of a suitable text.

The writer has based his approach in the latter sections on P.D. Thomas' *Conformal Projections in Geodesy & Cartography* which provides a comprehensive summary of earlier work, while the initial material is considerably influenced by J.E. Jackson's lectures on the subject at the University of Cambridge.

A study of the table of contents will show that the first three sections are developed from the point of view of a spherical reference surface. Emphasis is placed on the classification of projections and related basic principles which enable *any* projection to be developed, *once a handful of common basic rules are postulated.* In addition, the concept of *families* of projections is stressed, as many common projections comprising the conical family, are developed from the same basic set of rules.

The projection of an ellipsoidal surface of reference is dealt with in the last two sections. The development is influenced by the needs of undergraduate students with an emphasis on the *minimum set of concepts* necessary for *complete* formulation. The necessary ellipsoidal geometry is developed in the appendix which can be considered to be self-contained, if preceded by about two hours of lectures in basic differential geometry.

The second edition removes certain anomalies in the development, as presented in the first edition, and provides a consistent development of the chord to arc correction and the line scale factor for the transverse Mercator, correct to order 10^{-9}.

The writer is grateful to Messrs. A.H.W. Kearsley and M.Maughan for bringing inconsistencies and suggestions for improvement to his attention.

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ctd.

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A GUIDE TO NOTATION

1. Commonly used notation

$x_i = x_i(u_1,u_2),\ i=1,3$	$\equiv$	x_1, x_2 and x_3 are functions of the variables u_1 and u_2.
$\left(\frac{\partial x_1}{\partial x_2}\right)_{\sigma=\sigma_o}$	$\equiv$	the value of $\left(\frac{\partial x_1}{\partial x_2}\right)$ at $\sigma = \sigma_o$
σ_0	$\equiv$	the value of $\sigma\big(= \sigma(s)\big)$ at $s = 0$
$x \simeq 10^{-3}$	$\equiv$	order of magnitude of x is 10^{-3}
$x \doteqdot 0.002$	$\equiv$	x is approximately equal to 0.002 but not 0.0020.
$o\{x^{-6}\}$	$\equiv$	the terms neglected have a maximum magnitude of x^{-6}
x^1	$\equiv$	$\frac{dx}{ds}$
$(x^2)^2$	$\equiv$	$\left(\frac{d^2x}{ds^2}\right)^2$
$\overset{(3.44)}{=}$	$\equiv$	equals, on applying equation (3.44)

2.

Equations are derived in sections 4 to 6 for the general case of the line P_1P_2. Quantities capable of evaluation at both ends of the line are assigned the subscript $_{12}$ if evaluated at P_1, and $_{21}$ if at P_2.

3. Commonly used symbols

Symbol		Meaning	See Section
a	=	equatorial radius of reference ellipsoid	
C	=	constant, usually of integration	
dx	=	differential change in x	
E	=	East rectangular Cartesian projection co-ordinate	3.6
e	=	eccentricity of meridian ellipse	
f	=	flattening of meridian ellipse	
h	=	linear separation of chord from arc	4.2
h_s	=	elevation above ellipsoid	
k	=	the scale; point scale factor for conformal projections	
k_c	=	constant of cone	1.4.2
k_ℓ	=	line scale factor (ℓ_p/ℓ)	4.1
k_o	=	central scale factor	3.5(ii)
$k_{\phi(\lambda)}$	=	scale at a point along projected meridian (parallel)	1.3.3

3. Commonly used symbols (ctd)

Symbol		Meaning	See Section
ℓ	=	length of geodesic on reference surface	4.1
ℓ_p	=	length of projected geodesic	4.1
ℓ_c	=	length of chord on projection	4.1
m	=	linearisation factor; i.e., the metric tensor for an orthogonal system of isometric surface co-ordinates	5.3(ii)
n	=	linear displacement along normal	4.4
N	=	North rectangular Cartesian projection co-ordinate	3.6
P	=	the general point	
r	=	radius of projected parallel in conic projections; alternately, distance from equator	3
r	=	ν/ρ	6
R	=	radius of spherical reference surface	
t	=	linear displacement along tangent	4.4
t	=	$\tan\phi$	6
u_2	=	isometric latitude	3.5.3
u_i	=	surface parameters	
x_i	=	rectangular Cartesian co-ordinates	
α	=	azimuth	
β	=	grid bearing of projected geodesic	4.6
γ	=	grid convergence	1.3.2
δ	=	chord to arc correction	4.1
Δx	=	a change in x	
θ	=	co-latitude	2.2
θ	=	grid bearing of chord on projection; called plane bearing	4.4
λ	=	longitude, positive East	
ρ	=	radius of curvature in the prime vertical normal section on ellipsoid	
ν	=	radius of curvature in meridian normal section on ellipsoid	
σ	=	curvature of projected geodesic	4.2
$\sum_{i=1}^{n} x_i$	=	the sum $(x_1 + x_2 + x_3 + \ldots + x_n)$	
ϕ	=	latitude, positive North	
ϕ_f	=	foot point latitude	3.6.2
ψ	=	the angular distance on the reference surface between two points on it	
i	=	$\sqrt{(-1)}$	5 & 6
$M\{x\}$	=	the mean value of x	App

4. Suffixes

			See Section
$_f$	=	quantities evaluated at the foot point latitude ϕ_f	3 & 6
$_o$	=	related to the standard parallel or the centre of the projection, as the case may be	1 - 3
$_o$	=	evaluated at the initial terminal of the general line P_1P_2	3 - 6
$_r$	=	values for the reverse line at a station	4
$_t$	=	co-ordinates referring to the transverse Mercator projection	3 - 4
$_\phi$	=	evaluated along a meridian	1 - 3
$_\lambda$	=	evaluated along a parallel	1 - 3
$_{12}$	=	value at P_1 for the line P_1P_2	4 - 6
*	=	the equivalent quantity on a projection	

5. Terminology

It has been conventional to use the term *oblate spheroid* and/or *spheroid* in Commonwealth countries, including Australia, to describe the solid of revolution obtained by rotating an ellipse about its minor axis. It is also current geodetic practice to call such a figure an *ellipsoid of revolution* and/or *ellipsoid*. The latter practice is followed in the revision of this monograph, the terms being considered to be interchangeable.

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by

R. S. Mather

1. INTRODUCTION

1.1 Co-ordinate systems for the representation of position on the Earth's surface

Two co-ordinate systems are available for the representation of position on the Earth's surface. The first is that afforded by a three dimensional rectangular Cartesian co-ordinate system. An older and more common reference frame is that provided by a closed mathematical surface and its normals. In the case of the Earth, such a surface approximating to *mean sea level* (MSL) deviates from a sphere by one part in 300. The physical sea level datum, called the *geoid*, deviates from the shape of an ellipsoid of revolution (oblate spheroid) by magnitudes of the order of f^2(e.g., Mather 1969, fig 4), where f is the flattening of the reference ellipsoid. Current parameters adopted by the International Association of Geodesy (IAG) are those defining *Reference Ellipsoid 1967* (IAG Resolutions 1967, p.367), where the equatorial radius a and the flattening f are, for most practical purposes, given by

$$a = 6\ 378\ 160\ \text{m} \quad ; \quad f^{-1} = 298.25 \qquad (1.1)$$

Thus the magnitude of the deviations of the geoid from a well fitting reference ellipsoid are of the order af^2($\simeq$ 60 m), while those from a sphere of mean radius R are of order Rf($\simeq$ 18 km).

Position on the Earth's physical surface (point P in figure 1.1) can be represented by the co-ordinates of the point P_o on the reference surface, the surface normal at which passes through P, together with the displacement $PP_o(h_s)$ along this normal.

The co-ordinates of a point P_o on the surface of a solid of revolution are best defined by the following parameters shown in figure 1.1. They are

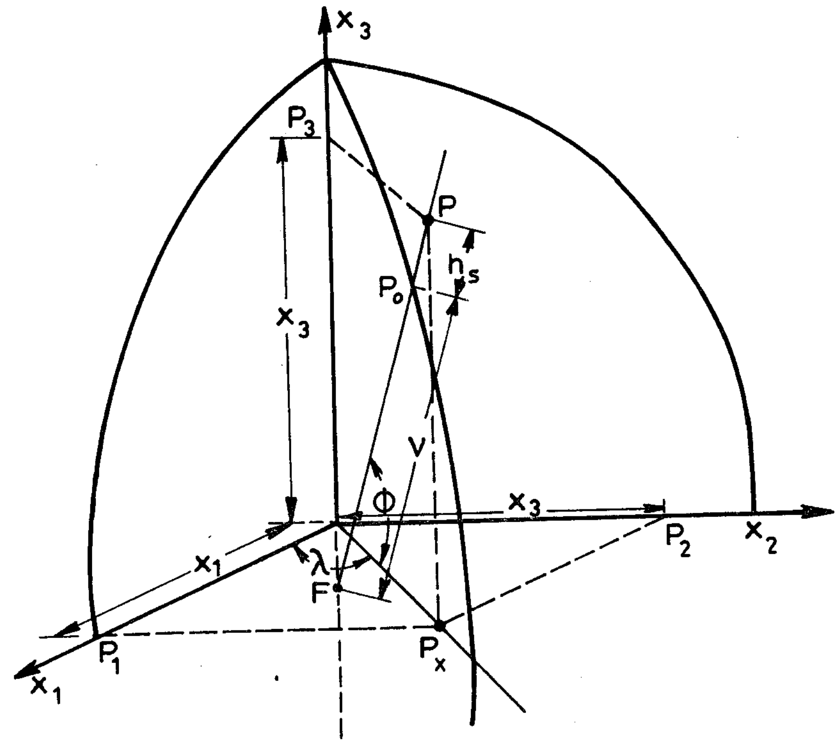


FIG. (1·1)

The Ellipsoid of Revolution

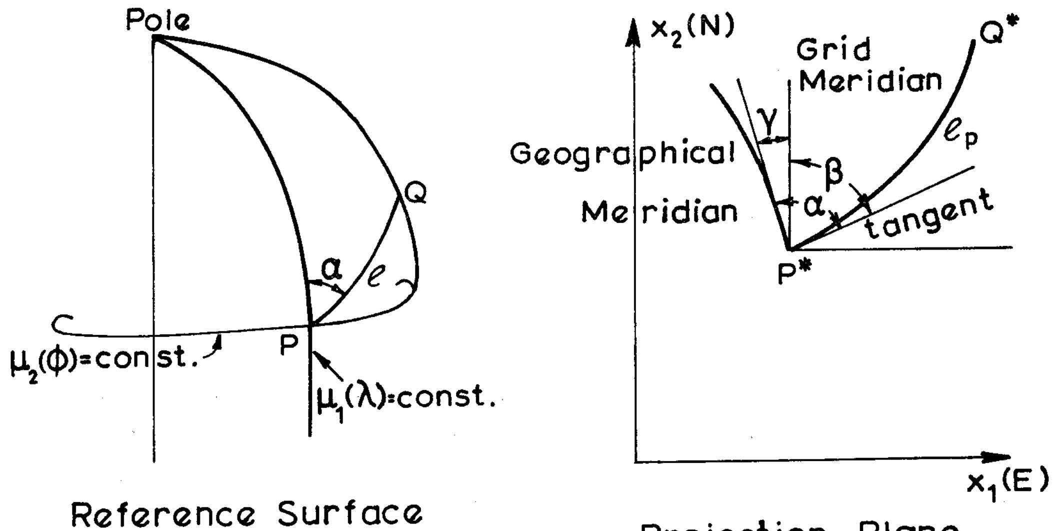


FIG. (1·2)

Relative Displacements on Reference Surface & Projection Plane

(a) the angle λ between the plane containing P_o and the rotation axis and a similar plane through some point of reference P_r;

and (b) the complement of the angle between the rotation axis and the normal to the surface at P_o.

The latter quantity is called the latitude ϕ of the point P_o while λ is termed the longitude. These parameters comprise a set of co-ordinates which, in the case of an ellipsoid, completely define the position of P_o, provided the dimensions (a,f) of the figure are specified.

The locus of points on the surface of an ellipsoid of revolution given by the equation

$$\lambda = \text{constant}$$

is an ellipse, called the meridian ellipse, while that of points satisfying

$$\phi = \text{constant}$$

is a circle of radius $\nu \cos \phi$, called a parallel of latitude; ν being the length P_oF in figure 1.1. The rotation axis Ox_3 is normal to the plane of all parallels of latitude.

Thus the set of curvilinear co-ordinates (ϕ,λ,h_s) completely defines the location of P in Earth space if the parameters defining the ellipsoid are known. Any change in the values a and f will alter the values (ϕ,λ,h_s). However, once the curvilinear co-ordinates have been established for a particular ellipsoid, it can be computed for any other reference system using formulae for the transformation of co-ordinates. The system postulated above is commonly used because the quantities ϕ, λ and h_s can be either measured or defined directly in terms of observed quantities.

The $x_1x_2x_3$ rectangular Cartesian co-ordinate system, where the x_3 axis conventionally coincides with the Earth's axis of rotation, the x_1 and x_2 axes being in the equatorial plane, the former lying in some reference meridian and the latter at right angles to it, is capable of affording a reference frame with practical applications. From a consideration of equation 13 in appendix 1 and figure 1.1, direct resolution gives

$$Ox = (\nu + h_s)\cos \phi,$$

where Ox is the intersection of the equatorial plane with the meridian plane at P. Further resolution gives

$$x_1 = (\nu + h_s)\cos \phi \cos \lambda \qquad (1.2a)$$

$$x_2 = (\nu + h_s)\cos \phi \sin \lambda \qquad (1.2b).$$

If PP_3 is parallel to Ox,

$$x_3 = (\nu(1-e^2) + h_s)\sin\phi \qquad 1.2c$$

The rectangular Cartesian co-ordinates of the point P cannot be observed directly, but can be deduced from either the curvilinear co-ordinate set (ϕ,λ,h_s) or from differences in x_1, x_2 and x_3 computed from field observations, provided those of one of the terminals is known.

It is conventional in classical geodesy, to express final co-ordinates in the (ϕ,λ,h_s) system but the rectangular Cartesian system (x_1,x_2,x_3) is of importance in deriving expressions for the properties of curves on the ellipsoid and their projection onto a plane.

1.2 Map projections

The Earth's topographical features can be most easily comprehended from models. The ideal physical representation is a globe showing tactual relief. Such a model, while affording the *possibility* in theory, of accurate representation without distortion, is not a practical proposition because of its unwieldiness at representations used for normal large scale mapping (e.g., 1 part in 50,000), as well as its limitations with regard to accuracies attainable in mensuration. Consequently, man from the time of Ptolemy, has been concerned with the problem of representing the Earth's physical surface on a plane. The task has been dealt with in two stages using the system of curvilinear co-ordinates defined in section 1.1. The first is the projection of points onto the reference surface along the relevant normal (P to P_o in figure 1.1), the representation of the altimetry (or normal displacements) being treated separately from that of position on the reference surface. The former can be represented in the form of contours on representations of surface co-ordinates which are most conveniently expressed on a plane.

Closed surfaces cannot be represented on a plane without undergoing some distortion. These distortions can, however, be controlled by means of mathematical formulae in order that some property fundamental to the surface representation is retained. The set of mathematical rules which enables the reference surface to be unambiguously represented on a plane is called a *map projection*. An infinite number of map projections are possible in theory. The type of projection adopted in practice will depend on

(a) the shape and extent of the area to be mapped;

and (b) the specific property of the reference surface which is to be retained on the projection.

The position of points transferred on projection from the reference surface to the plane can be uniquely defined by means of a two dimensional Cartesian system. The definition of *both* the projection and the projection co-ordinates of a point on the Earth's surface provide the necessary and sufficient conditions for the unique definition of position on the reference surface.

1.3 Definition of fundamental terms

1.3.1 The mapping equations

The basic relations necessary to completely define the general point P, whose co-ordinates on the reference surface are (u_1,u_2), on the x_1x_2 projection plane, are of the form

$$x_i = x_i(u_1,u_2) \;,\; i=1,2 \qquad 1.3.$$

In common practice, equation 1.3 can be interpreted as

$$x_1 = E \quad ; \quad x_2 = N \qquad 1.4$$

where E is a co-ordinate axis near enough to the local prime vertical and N is the other axis of the rectangular Cartesian set on the projection plane which approximately coincides with the local meridian and is called the *grid meridian*. It is also possible to interpret u_1 and u_2 by the equations

$$u_1 = \lambda \quad ; \quad u_2 = \phi \qquad 1.5,$$

but caution should be exercised in assuming such definitions to be the only ones possible. Equations 1.3, as interpreted in terms of equations 1.4 and 1.5, comprise the final form desirable for plotting individual points on the projection.

A second problem is also of considerable importance in practical geodesy. In this case, positions of points are fixed relative to one another using a system of local polar co-ordinates. The most common system is afforded by the azimuth α and length ℓ of a geodetic line on the reference surface. The required mapping equations in such a case are of the form

$$x_{i2} - x_{i1} = \Delta x_i = \Delta x_i(\alpha,\ell) \qquad 1.6.$$

The *reverse problem* is also of relevance. In this case, it is required to find the geographical co-ordinates or the azimuth and length of a line from the projection co-ordinates, the required relations being of the form

$$u_i = u_i(x_1,x_2), \qquad i=1,2 \qquad 1.7$$

in the first case, and

$$\ell = \ell\left((x_{ij},\ j=1,2),\ i=1,2\right)$$
$$\alpha = \alpha\left((x_{ij},\ j=1,2),\ i=1,2\right) \qquad \ldots\ldots\ldots \quad 1.8.$$

1.3.2 Convergence on a projection (grid convergence)

The geographical meridian on the reference surface can be plotted element by element onto the projection plane using either equation 1.3 or 1.6. The resultant curve will be complex in the general case. The angle between the tangent to the projected meridian and the local grid meridian is called the *grid convergence* γ. A study of figure 1.2 gives

$$\tan\gamma = -\left(\frac{dx_1}{dx_2}\right)_{\lambda=\text{const}} .$$

As λ is a constant along a given meridian,

$$\tan\gamma = -\frac{\left(\frac{\partial x_1}{\partial\lambda}\right)d\lambda + \left(\frac{\partial x_1}{\partial\phi}\right)d\phi}{\left(\frac{\partial x_2}{\partial\lambda}\right)d\lambda + \left(\frac{\partial x_2}{\partial\phi}\right)d\phi} = -\frac{\left(\frac{\partial x_1}{\partial\phi}\right)}{\left(\frac{\partial x_2}{\partial\phi}\right)} \qquad \ldots\ldots\ldots \quad 1.9.$$

Other expressions are also possible for the definition of γ. For example, along the projected meridian,

$$d\lambda = \sum_{i=1}^{2}\left(\frac{\partial\lambda}{\partial x_i}\right)dx_i = 0.$$

Thus

$$\tan\gamma = -\left(\frac{dx_1}{dx_2}\right)_{\lambda=\text{const}} = \frac{\left(\frac{\partial\lambda}{\partial x_2}\right)}{\left(\frac{\partial\lambda}{\partial x_1}\right)} \qquad \ldots\ldots \quad 1.10.$$

Equation 1.10 is of relevance when the expressions available for relating the two sets of co-ordinates are of the form

$$\lambda = \lambda(x_1,x_2) \quad ; \quad \phi = \phi(x_1,x_2) \qquad \ldots\ldots \quad 1.11,$$

which is the reverse case. The convergence is of importance in determining the relation between the grid bearing β of the projected curve and the azimuth α^* on projection. Refer figure 1.2.

$$\alpha^* = \beta + \gamma \qquad \ldots\ldots \quad 1.12.$$

It should be noted that α^* is not necessarily equal to the azimuth α of the line. This would only occur in those projections which are characterized by small areas retaining their shape.

1.3.3 The scale *(Point scale factor in conformal projections)*

The scale is defined as the ratio of an elemental distance on the projection plane $d\ell_p$ to the distance $d\ell$ between corresponding points on the reference surface. Thus

$$k = \frac{d\ell_p}{d\ell} \qquad \ldots\ldots \quad 1.13,$$

and is, in general, given by equations of the form

$$k = k(\phi,\lambda,\alpha) = k(x_1,x_2,\alpha^*) \qquad \ldots\ldots \quad 1.14.$$

In the case of projections where small areas retain their shape, outlines on the projection in the limit are *geometrically* similar to their equivalents on the reference surface, resulting in k being independent of α. For other projections which do not have this property, k can be completely defined by two components k_ϕ in the meridian and k_λ at right angles to it. The changes in x_1 and x_2 as a result of changes $d\phi$ in ϕ and $d\lambda$ in λ can be expressed as

$$dx_i = \left(\frac{\partial x_i}{\partial \phi}\right) d\phi + \left(\frac{\partial x_i}{\partial \lambda}\right) d\lambda \quad , \ i=1,2 \qquad \ldots\ldots \quad 1.15.$$

If the particular curve is the projected meridian, $d\lambda = 0$ and

$$d\ell_p = \left(\sum_{i=1}^{2} (dx_i)^2\right)^{\frac{1}{2}} = \left(\sum_{i=1}^{2} \left(\frac{\partial x_i}{\partial \phi}\right)^2\right)^{\frac{1}{2}} d\phi \ .$$

If the reference surface is an ellipsoid, the equivalent length $d\ell$ in the meridian is $\rho \ d\phi$. Thus

$$k_\phi = \frac{d\ell_p}{d\ell} = \frac{\left(\sum_{i=1}^{2} \left(\frac{\partial x_i}{\partial \phi}\right)^2\right)^{\frac{1}{2}}}{\rho} \qquad \ldots\ldots \quad 1.16.$$

Similar consideration of the projected parallel along which $d\phi = 0$, gives

$$k_\lambda = \frac{d\ell_p}{d\ell} = \frac{\left(\sum_{i=1}^{2} \left(\frac{\partial x_i}{\partial \lambda}\right)^2\right)^{\frac{1}{2}}}{\nu \cos \phi} \qquad \ldots\ldots \quad 1.17.$$

In those projections where small areas retain their shape,

$$k_\phi = k_\lambda = k \qquad \ldots\ldots \quad 1.18,$$

where k is called the *point scale factor*. If small extents are to retain their *area* on projection,

$$k_\phi \cdot k_\lambda = 1 \qquad \ldots\ldots \quad 1.19.$$

1.4 A classification of common projections

1.4.1 Introduction

Projections may be classified in terms of either distinct characteristics used in the mechanical construction of the projection or properties of the reference surface which are considered desirable and retained on the plane representation. It should be borne in mind that each projection is merely an adopted set of rules which can be represented mathematically and, in some cases, by simple geometrical concepts. These rules can then be "bent" to retain certain desirable properties of the reference surface. Unless the projection adopted is totally unsuited for the mapping being undertaken, there will always be at least one point on the projection where

$$k_\phi = k_\lambda = 1.$$

Such a point or locus of points is called the centre or central (standard) curve, as the case may be, of the projection and represents locations where the reference surface is mapped on the plane without distortion of any kind. As the distance of the region being mapped increases from the centre or central curve, distortion of one kind or another occurs and, in the general case,

$$k_\phi \neq k_\lambda \neq 1.$$

If, on the other hand, restrictions are introduced in the rules governing the mapping, in order that certain properties of interest on the reference surface are retained on projection, it is possible to control the resulting distortion to suit the needs of the mapping being undertaken.

The properties of interest are as follows.

1. Restricted regions retain their area on projection.

   The resulting projection is called *equal area* or *authalic*. Such projections are of interest in instances where comparative studies of different areas on the projection are to be made; e.g., demographic investigations.

2. Restricted regions retain their shape on projection.

   The projection obtained in this case is called a *conformal* or *orthomorphic* projection. In such projections, small features like lakes and bays retain their shape but regions distant from the centre may have larger areas than on the reference surface. Conformal projections are therefore of interest in geodesy and surveying as observed directions over geodetic lines, if correctly interpreted, are transferred without significant distortion to the projection plane.

These properties can be superimposed on any set of rules comprising a simple projection. Such projections have been given the title *aphylactic*. The specific characteristics of the common projections are defined with comparative ease for cases where the reference surface is a sphere. This assumption is valid so long as large scale maps are not involved. In addition, various simple stratagems are available for extending spherical concepts to allow for the ellipsoidal nature of the reference surface at larger scales so that accuracy is not lost in the process.

Projections can be classified as follows for a spherical reference surface.

1. *Perspective projections*

   These are purely geometrical projections of a spherical reference surface onto a plane from a centre of projection; e.g., the gnomonic projection.

2. *Quasi perspective projections*

   Such projections do not have a common centre of projection. Instead, some curve on the sphere (e.g., a parallel of latitude) is entirely projected onto an enveloping surface from a centre of projection. This results in each of the family of curves so projected having its own centre of projection. Some references (e.g., Deetz & Adams 1945,p.27) are inclined to treat this class under 1 above but this gives rise to unnecessary ambiguities. The entire family of conical projections, obtained by the use of an enveloping cone as the surface of projection, belong to this class.

   Conical projections require curves orthogonal to the family being projected to remain orthogonal on projection with no change in scale. Such cases are called *equidistant projections*. Either the equal area or conformal property may be introduced by varying the scale along the orthogonal family of curves to satisfy the characteristic property required.

3. *Pseudo projections*

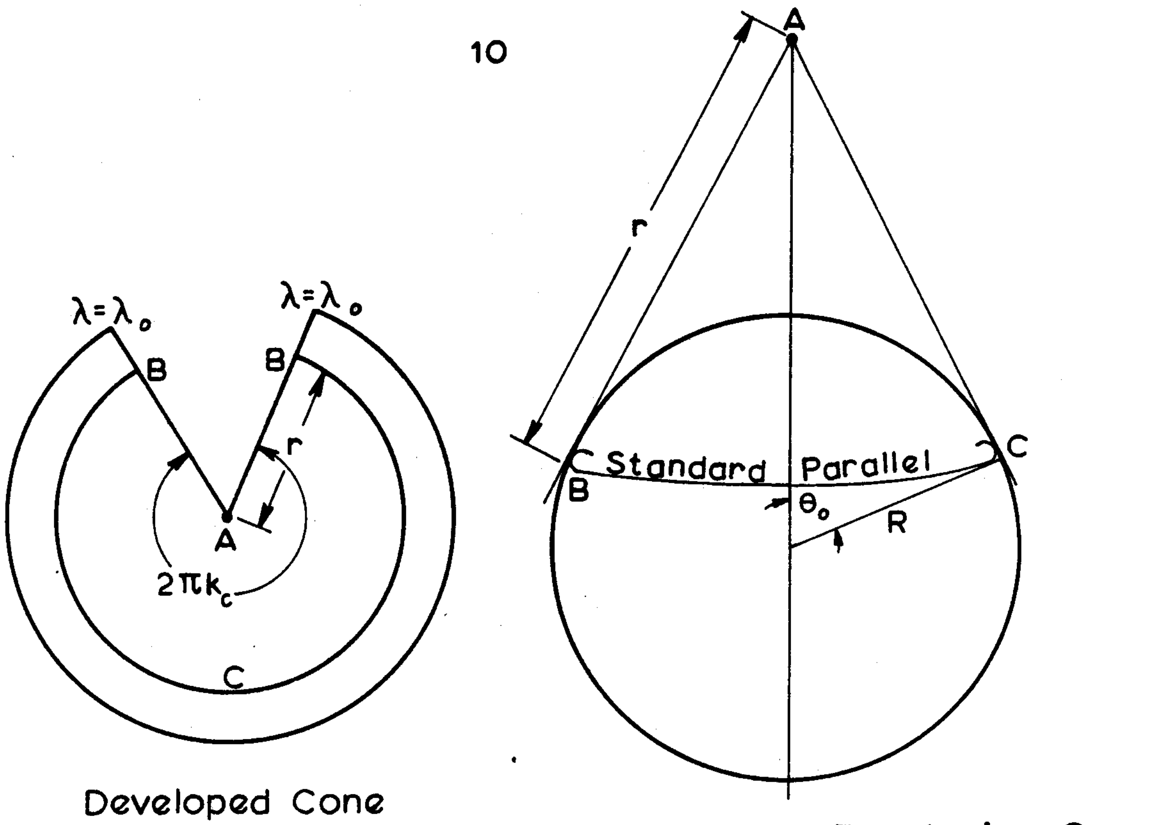
   This class of projection is of the quasi perspective type but without a simple relationship to an enveloping surface. It would, in fact, be simpler to consider both types 2 and 3 as convenient sets of rules for mapping the sphere onto a plane.

4. *Non-perspective projections*

   Projections in this class are used primarily for atlas and world maps and will not be covered in the present development.

1.4.2 The cone as an enveloping surface

A right circular cone is said to envelop the spherical reference surface when it rests on the latter. The cone has the advantage of being a *developable surface* in that it can be cut along a line joining the apex to its circular base (slant height) and its surface area spread

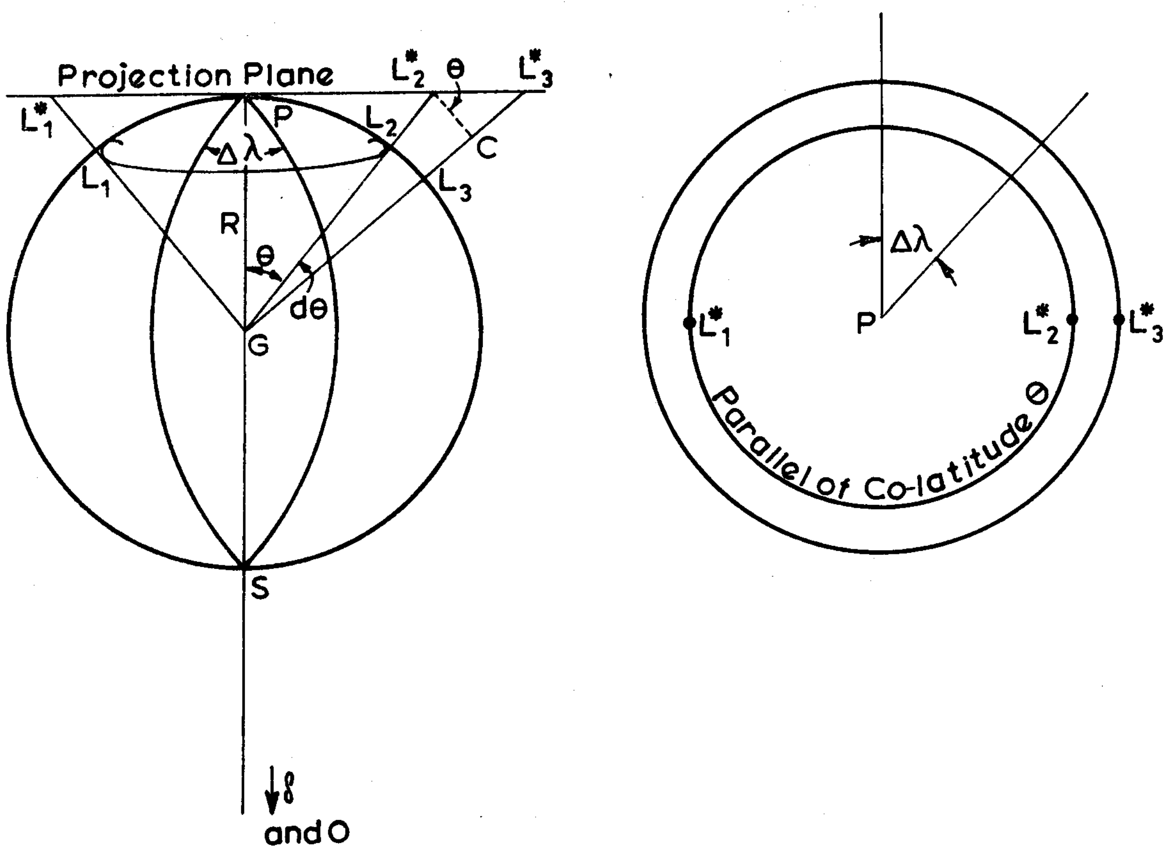


Developed Cone

Reference Surface & Enveloping Cone

FIG (1.3)

The Enveloping Cone as a Projection Plane



Reference Sphere

Projection Plane

FIG. (2·1)

Perspective Projections

out, without distortion, on a plane. Consider the case when the axis of the cone passes through the poles of a spherical reference surface. The sphere and the cone will have a series of common tangent planes along one parallel of co-latitude θ_o. This co-latitude is related to the slant height r of the cone and the radius R of the sphere by the direct trigonometrical relation

$$r = R \tan \theta_o \qquad \ldots\ldots \quad 1.20.$$

The parallel so defined is called the *standard parallel*. The developed cone is a sector of a circle. The sphere is projected geometrically onto the cone from a common perspective centre for *each* parallel as described in section 3.1. In the case of the standard parallel, this centre is coplanar with the curve on the sphere and lies on the rotation axis. Thus the standard parallel of length $2\pi R \sin \theta_o$ projects as a circular arc BCB in figure 1.3, of length $2\pi k_c r$, the meridian AB being represented by two lines, on development of the cone. As these lengths are equal by definition,

$$k_c r = R \sin \theta_o \qquad \ldots\ldots \quad 1.21.$$

The combination of equations 1.20 and 1.21 gives

$$k_c = \cos \theta_o \qquad \ldots\ldots \quad 1.22.$$

k_c is called the *cosntant of the cone*. When $k_c = 0$, the standard parallel coincides with the equator, as $\theta_o = \frac{1}{2}\pi$ and the cone becomes a cylinder, as its apex is at infinity.

When $k_c = 1$, the cone becomes a plane tangential at the pole as $\theta_o = 0$.

Thus the general theory of conical projections covers the following three classes of projections

1. Cylindrical projections when $k_c = 0$; $\theta_o = \frac{1}{2}\pi$;
2. Conic projections when $0 < k_c < 1$; $0 < \theta_o < \frac{1}{2}\pi$;

and 3. Azimuthal projections when $k_c = 1$; $\theta_o = 0$.

This particular classification is of significance as nearly all mapping projections in current use fall into one of the above categories. Further, the basic mathematical concepts for defining all the above projections are identical,stemming from a common general theory for a spherical reference surface.

A classification of some common map projections is given in table 1.1 under the headings

A. Perspective;

B. Quasi perspective

C. Pseudo;

and D. Non-perspective.

Type		Conformal $k_\phi = k_\lambda$	Equal Area $k_\phi . k_\lambda = 1$	Equidistant $k_\phi = 1$	Other	Remarks
A :- PERSPECTIVE PROJECTIONS						
					Gnomonic	{ from centre
		Stereographic				{ from end diameter
					Orthographic	{ from infinity
B :- QUASI PERSPECTIVE PROJECTIONS						
1. Conical projections						
Conic	$0<k_c<1$					
Tangent cone	$0<\theta_o<\frac{1}{2}\pi$	Lambert		Simple Conic		1 std. parallel
Secant cone		Lambert	Albers	de L'Isle		2 std. parallels
Cylindrical	$k_c=0$					
Equatorial tangency	$\theta_o=\frac{1}{2}\pi$	Mercator	Equal area	Plate Carré		
Transverse case		Transverse Mercator		Cassini-Soldner		Meridian tangency
Azimuthal (Zenithal)	$k_c=1$ $\theta_o=0$	Conformal	Equal area	Equidistant		
C :- PSEUDO PROJECTIONS						
			Bonne			
			Sanson-Flamsteed			
			Werner			
				Polyconic		
			Mollweide			
D :- NON PERSPECTIVE PROJECTIONS						
			Hammer-Aitoff			

TABLE 1.1
A classification of some common projections

The present development is confined to perspective and quasi perspective projections only.

2. PERSPECTIVE PROJECTIONS

2.1 Introduction

Attention will be confined to those perspective projections where the point from which the projecting lines are drawn (*the point of projection* or *the perspective centre*) is on the diameter through the point at which the projection plane is tangential to the reference sphere (*the centre of projection*).

The three common cases are

1. The gnomonic projection
   with point of projection at G in figure 2.1;
2. The stereographic projection
   with perspective centre at S;

and 3. The orthographic projection
   with point of projection at infinity (O).

The centre of projection can be at any point on the spherical reference surface. When at the pole, it can be seen that meridian great circles project as straight lines in all three cases. This same characteristic is possessed by all great circles passing through the centre of projection in non-polar cases, no angular distortion occurring. Further, all small circles whose centres lie on the axis of projection which is the line joining the point of projection to the centre of projection, plot as concentric circles.

It follows that the projected meridians and parallels form an orthogonal, though not necessarily conformal system of curves in the polar case. In non-polar cases, a similar system will be formed by azimuthal great circles which plot as straight lines, while curves of equi-angular distance ψ from the centre of projection plot as concentric circles. This is equivalent to the (ϕ,λ) system being replaced by a (ψ,α) system of surface co-ordinates.

2.2 The gnomonic projection

Point of projection	:-	Centre of sphere
Special property	:-	All great circles project as straight lines
Primary use	:-	Navigation
Details	:-	See (Deetz & Adams 1945,p.147 et seq)

2.2.1 The polar case

Centre of projection	:- Pole
Radius of parallel of latitude on projection (figure 2.1)	= R tan θ
Radius of parallel of latitude on sphere	= R sin θ

Thus

$$k_\lambda = \frac{R \tan \theta}{R \sin \theta} = \sec \theta \qquad \ldots\ldots 2.1$$

The meridional element of length $R\,d\theta$ (L_2L_3 in figure 2.1) plots as a straight line $L_2^*L_3^*$ on projection. If L_2^*C is perpendicular to GL_3^*, as $L_2^*C = R \sec\theta\, d\theta$,

$$L_2^*L_3^* = L_2^*C \sec\theta = R \sec^2\theta\, d\theta \;.$$

Thus

$$k_\phi = \frac{R \sec^2\theta\ d\theta}{R\ d\theta} = \sec^2\theta \qquad \ldots\ldots\ 2.2.$$

Notes:-

1. The equator plots at infinity, the scale along meridians increasing from unity at the centre of the projection to 50% greater (i.e., 1.50) at $\theta \doteqdot 35^\circ$. Thus the gnomonic projection is of limited value in measuring azimuths and distances from projection values at any significant distances from the centre of projection.

2. The gnomonic projection is of considerable antiquity, being ascribed to the period of Thales (c. B.C. 550).

3. It has advantages in great circle route navigation.

2.2.2 The non-polar case

Centre of projection :- Any point $P(\phi_o,\lambda_o)$ on the Earth's surface.

The meridians in the polar case are replaced by great circles of constant azimuth α and the parallels by small circles of equi-angular distance ψ from the centre of projection. These circles of equi-angular distance ψ, by analogy with the polar case, plot as concentric circles of radius $R \tan\psi$, with centre at P.

The general point Q on this circle at an azimuth α from P, will project as the point Q* on the projection plane, as shown in figure 2.2. The angle α is unaffected by perspective projection as the azimuthal great circles pass through the centre of projection. If P is also the origin of the rectangular Cartesian system of reference on the projection plane, such that the N axis is coincident with the projected meridian at P and the E axis with the prime vertical, the co-ordinates (ϕ,λ) of Q in terms of the (ψ,α) polar system of co-ordinates are given by

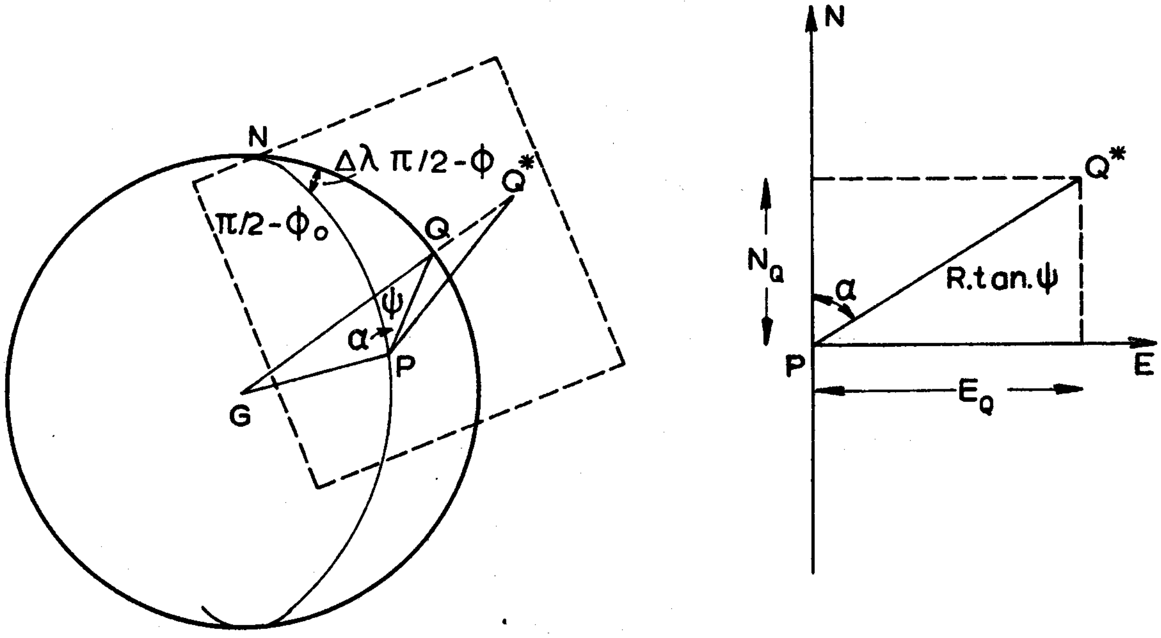
$$\begin{aligned} N_Q &= R \tan\psi \cos\alpha \\ E_Q &= R \tan\psi \sin\alpha \end{aligned} \qquad \ldots\ldots\ 2.3$$

The (ψ,α) co-ordinate system at P can be related to the normal surface co-ordinate system (ϕ,λ) by the solution of spherical triangle NPQ on the reference surface when

$$\psi = \cos^{-1}(\sin\phi_o \sin\phi + \cos\phi_o \cos\phi \cos\Delta\lambda\ ),$$

where

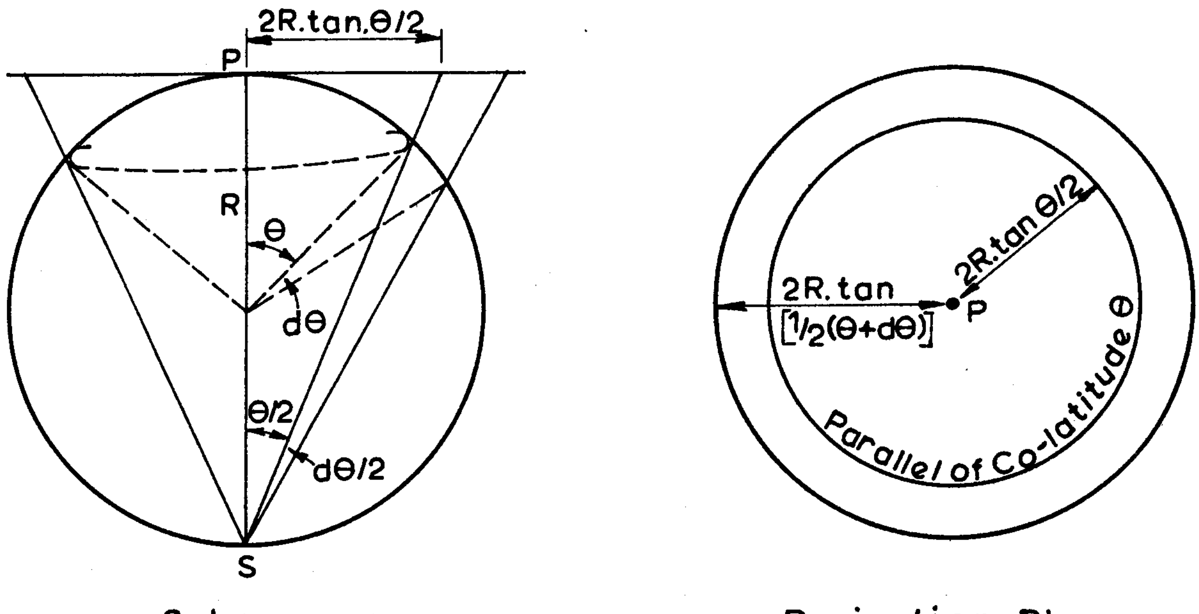
$$\Delta\lambda = \lambda - \lambda_o\ ,$$



Reference Sphere　　　Projection Plane

FIG. (2·2)

Gnomonic Projection — Non-polar case.



Sphere　　　Projection Plane

FIG. (2·3)

The Stereographic Projection

and

$$\alpha = \sin^{-1}(\cos\phi \sin\psi \operatorname{cosec}\Delta\lambda), \quad \text{if} \quad \Delta\lambda \neq 0$$

or

$$\alpha = \cos^{-1}\left(\frac{\sin\phi - \sin\phi_o \cos\psi}{\cos\phi_o \sin\psi}\right), \quad \text{if} \quad \phi_o \neq \tfrac{1}{2}\pi$$

and the case is non-trivial.

Notes:-

1. Formulae for manual computations are given by Deetz & Adams (1945,p.149 et seq.).

2. The gnomonic projection is also classified as an azimuthal projection (e.g.,Robinson 1960,p.69) as lines of constant azimuth from a specified point (the centre) project as straight lines, retaining the angle between them on the projection.

3. The scale at any point on the non-polar case can be obtained by analogy with the polar case as $\sec^2\psi$ along lines of constant azimuth from the centre, and $\sec\psi$ tangential to circles of equal ψ. k_ϕ and k_λ can be obtained from vectorial considerations of these quantities.

4. The meridians plot as straight lines passing through the pole and intersect the equator which, in the non-polar case, is also a straight line on the projection.

5. The projection is symmetrical about the N axis.

2.3 The stereographic projection

Perspective centre	:-	diametrically opposite centre of projection
Centre of projection	:-	pole
Special property	:-	conformality
Principal uses	:-	star charts; mapping polar regions

A study of figure 2.3 shows that the parallel of co-latitude θ plots as a circle of radius $2R\tan\tfrac{1}{2}\theta$. Thus,

$$k_\lambda = \frac{2R\tan\tfrac{1}{2}\theta}{R\sin\theta} = \sec^2\tfrac{1}{2}\theta.$$

The parallel of co-latitude $\theta+d\theta$ plots as a circle of radius $2R\tan\tfrac{1}{2}(\theta+d\theta)$. Therefore

$$k_\phi = \frac{2R\{\tan\tfrac{1}{2}(\theta+d\theta) - \tan\tfrac{1}{2}\theta\}}{R\,d\theta} = \frac{2\{\tan\tfrac{1}{2}\theta + \tfrac{1}{2}d\theta\sec^2\tfrac{1}{2}\theta - \tan\tfrac{1}{2}\theta\}}{d\theta}$$

$$= \sec^2\tfrac{1}{2}\theta$$

As $k_\phi = k_\lambda$, the projection is conformal.

Notes:-

1. The projection is an azimuthal one, all meridians and parallels having geometrical characteristics similar to those exhibited in the gnomonic projection.

2. Non polar cases are possible and, in all instances, circles on the sphere project as circles. This property is unique to the stereographic projection, making it suitable for the preparation of star charts and other representations of spherical position requiring ready interconversion between two different sets of surface co-ordinates (e.g., $\{\phi,\lambda\}$ to $\{\psi,\alpha\}$).

3. The scale exaggeration over a hemisphere is easily verified as being 100% thereby making it an acceptable conformal representation of a hemisphere.

2.4 The orthographic projection

Point of projection	:-	At infinity
Centre of projection	:-	Pole
Special property	:-	Scale exaggeration decreases from unity at the centre
Uses	:-	Mapping hemispheres; lunar charts

All parallels plot true to scale as circles of radius $R \sin \theta$. Thus,

$$k_\lambda = 1 .$$

The scale along the meridians is obtained by considering two parallels of co-latitude θ and $(\theta+d\theta)$ when

$$k_\phi = \frac{R\{\sin(\theta+d\theta) - \sin\theta\}}{R\,d\theta} = \frac{\sin\theta + d\theta\cos\theta - \sin\theta}{d\theta}$$

$$= \cos\theta \leq 1.$$

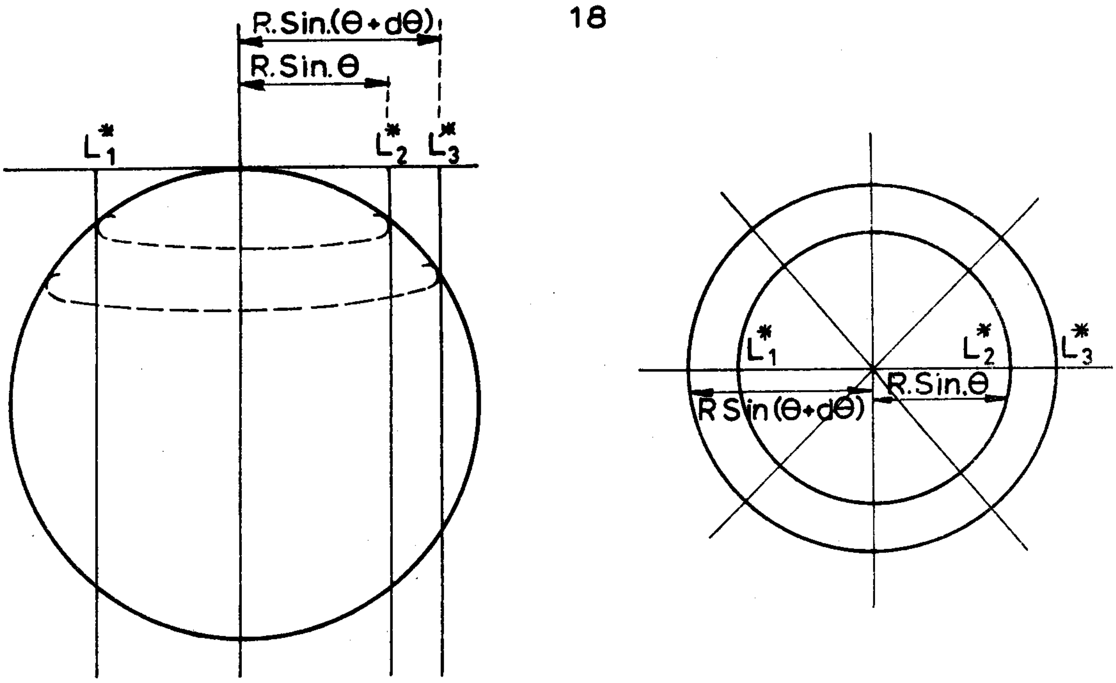
Note that $k_\phi \to 0$ as $\theta \to \frac{1}{2}\pi$. Consequently, regions near the great circle at an angular distance $\frac{1}{2}\pi$ from the centre of projection are badly distorted.

3. CONICAL PROJECTIONS

3.1 Introduction

The cone as an enveloping surface has been dealt with in section 1.4.2. Three major classes of projections are possible in the case where the axis of the right circular cone coincides with the diameter of the spherical reference surface passing through the poles. These are set out below in table 3.1.

The rules for constructing the equidistant case of all conical projections are identical. They are as follows.

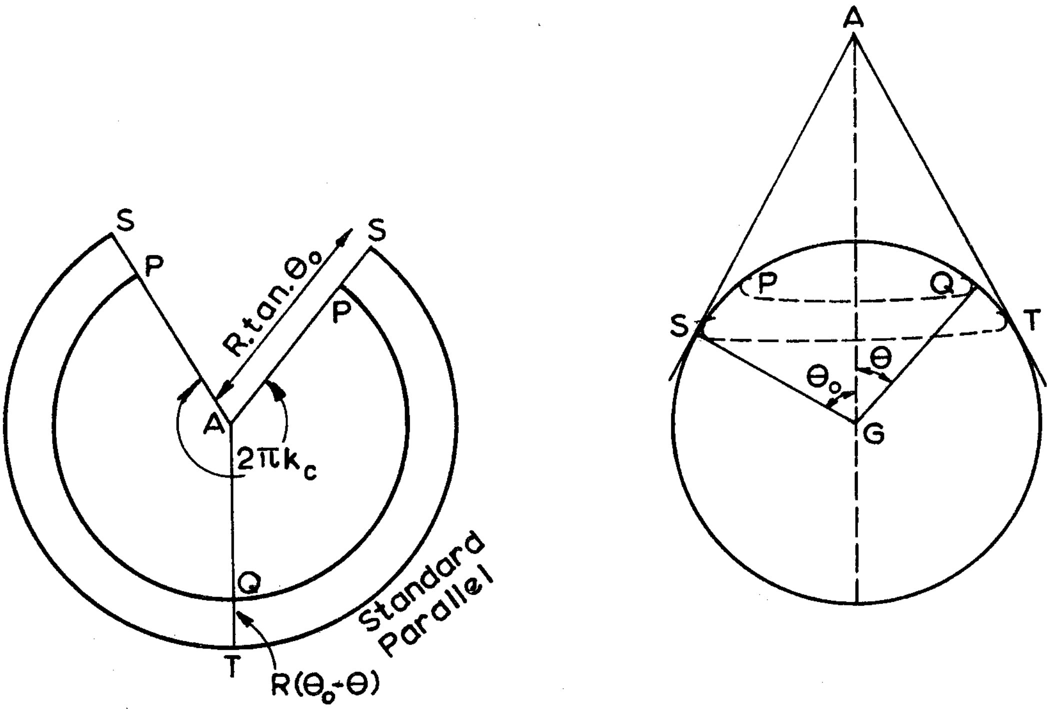


From ∞ Sphere

Projection Plane

FIG. (2·4)

The Orthographic Projection.



Developed Cone

Sphere & Tangent Cone

FIG. (3·1)

Projections on a Tangent Cone — Equidistant Case.

Standard parallel θ_o	k_c	Class of conical projection
$\theta_o = 0$	$k_c = 1$	Azimuthal (Zenithal)
$0 < \theta_o < \frac{1}{2}\pi$	$1 > k_c > 0$	Conic
$\theta_o = \frac{1}{2}\pi$	$k_c = 0$	Cylindrical

TABLE 3.1

A classification of conical projections

1. Plot the standard parallel, whose co-latitude is θ_o, as a circular arc of radius $R \tan \theta_o$ (see equation 1.20), and length $2\pi R \sin \theta_o$.

2. Plot all parallels as concentric circular arcs of radius r, given by

$$r = R\{ \tan \theta_o - (\theta_o - \theta)\} \qquad \ldots\ldots \quad 3.1,$$

and length ℓ, defined by the equation

$$\ell = 2\pi k_c R\{\tan \theta_o - (\theta_o - \theta)\} \qquad \ldots\ldots \quad 3.2,$$

where k_c is given by equation 1.21 as

$$k_c = \cos \theta_o .$$

3. Represent the 2π variation in longitude on the sector of a circle formed by the developed cone, as described in section 1.4.2; the bounding radii of the sector representing the same meridian on the sphere, as shown in figure 3.1.

4. The meridians are straight lines joining the centre of this circle, which is the apex of the cone, to the appropriate point on the standard parallel, divided true to scale.

Step 1 is equivalent to perspectively projecting the standard parallel onto the enveloping cone from the centre of the sphere. Step 2 can be interpreted as perspectively projecting each individual parallel from its own perspective centre C on the common axis, whose distance x from the centre G of the sphere can be obtained by the geometrical consideration of figure 3.2. As triangles P*P*'C and PP'C are similar,

$$\frac{x + R\{\cos \theta_o + (\theta_o - \theta)\sin \theta_o\}}{r \cos \theta_o} = \frac{x + R \cos \theta}{R \sin \theta} .$$

On using the relations set out in the key to figure 3.2,

$$x\{\sin \theta_o - (\theta_o - \theta)\cos \theta_o - \sin \theta\} = R\big(\{\cos \theta_o + (\theta_o - \theta)\sin \theta_o\}\sin \theta - \cos \theta\{\sin \theta_o - (\theta_o - \theta)\cos \theta_o\}\big) .$$

The re-grouping of terms, together with the use of elementary trigonometry gives

KEY TO FIG. (3·2)

$AS = R \tan.\theta_0$

$SP^* = R(\theta_0 - \theta)$

$AP^* = r = R[\tan.\theta_0 - (\theta_0 - \theta)]$

$P^*P^{*1} = r \cos\theta_0$

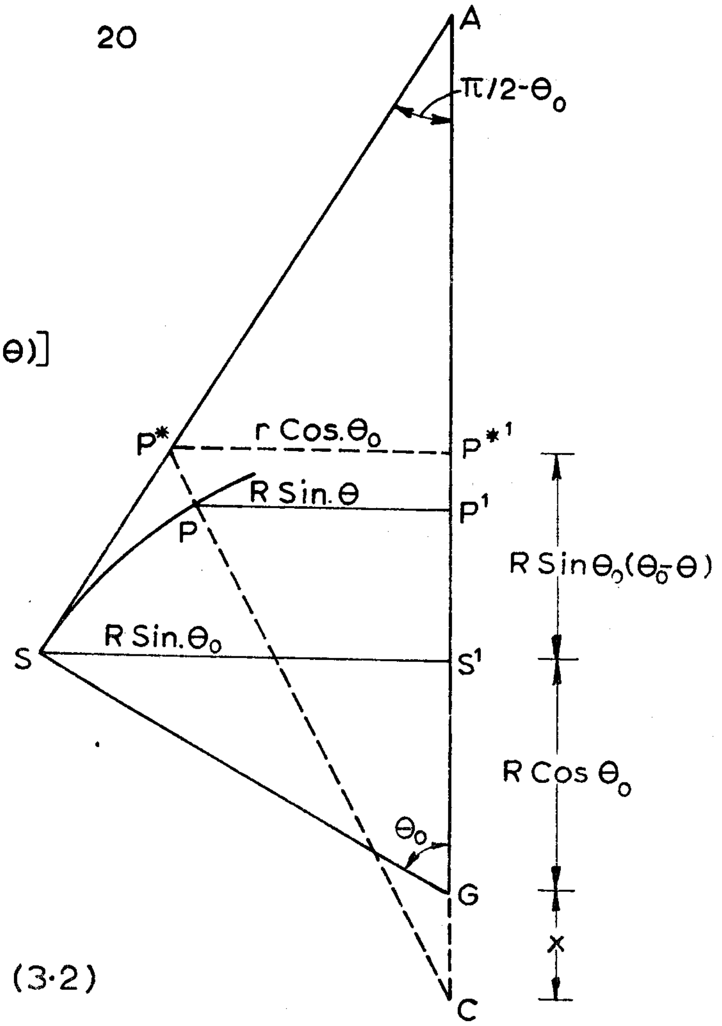


FIG. (3·2)

The Quasi-perspective nature of Conical Projections

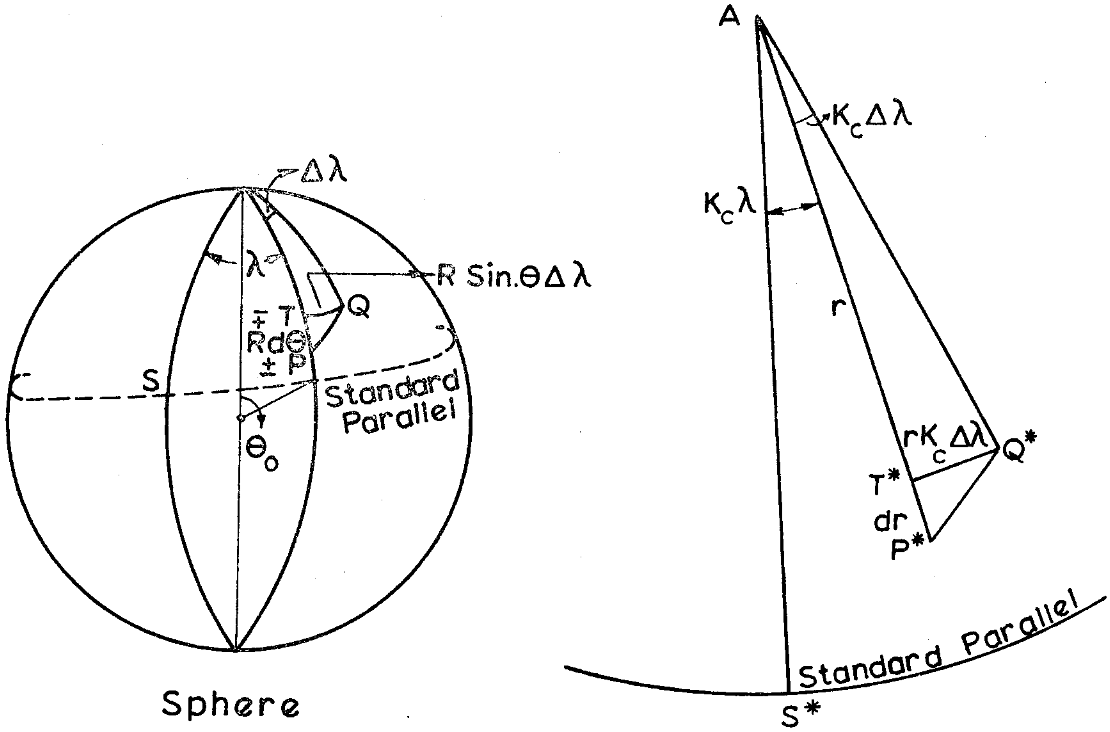


FIG. (3·3)

Conic Projections – Equal Area and Orthomorphic Cases

$$x = \frac{R\left(\cos(\theta_o-\theta)\{\theta_o-\theta\} - \sin(\theta_o-\theta)\right)}{\sin\theta_o - \sin\theta - \{\theta_o-\theta\}\cos\theta_o} \qquad \ldots\ldots \quad 3.3.$$

The properties of orthomorphism and equal area are introduced by varying the distance between the parallels on the projection, i.e., they are no longer equidistant. The rules 1 to 4 are fundamental to all conical projections on a tangent cone.

Summary:-

The mathematical equations governing the equidistant cases of *all* conical projections on a tangent cone are

$$k_c = \cos\theta_o \qquad \ldots\ldots 1.21$$

$$r = R\{\tan\theta_o - (\theta_o-\theta)\} \qquad \ldots\ldots 3.1 \qquad \ldots\ldots \; 3.4;$$

$$\ell = 2\pi k_c r \qquad \ldots\ldots 3.2$$

all parallels being concentric circular arcs, while the meridians are straight lines radiating from the common centre to the truly divided standard parallel.

Equal area and conformal cases are obtained by varying the spacing between the parallels, k_ϕ no longer being unity.

3.2 Conic projections on a tangent cone

$$0 < k_c < 1.$$

Along the standard parallel,

$$r_o k_c = R\sin\theta_o \qquad \ldots\ldots \quad 3.5,$$

where r_o is the radius of the standard parallel on the projection plane.

3.2.1 The equidistant case

This case is described in section 3.1 and illustrated by figure 3.1. Equation 3.4 applies. By definition,

$$k_\phi = 1.$$

From equation 3.2,

$$k_\lambda = \frac{\ell}{2\pi R\sin\theta} = \frac{k_c\{\tan\theta_o - (\theta_o-\theta)\}}{\sin\theta}.$$

The use of equation 1.21 gives

$$k_\lambda = \frac{\sin\theta_o - (\theta_o-\theta)\cos\theta_o}{\sin\theta}.$$

The variations in k_λ can be studied by putting

$$\Delta\theta = \theta_o - \theta$$

and replacing θ by functions of Δθ when the use of the first three terms in the expansion of the resulting denominator by Taylor's theorem gives

$$k_\lambda = \frac{\sin\theta_o - \Delta\theta\cos\theta_o}{\sin(\theta_o - \Delta\theta)} = \left(1 - \Delta\theta\cot\theta_o\right)\left(1 - \Delta\theta\cot\theta_o - \frac{(\Delta\theta)^2}{2}\right)^{-1}.$$

Expansion of the right hand term using the binomial theorem and restricting attention to terms of order $(\Delta\theta)^2$ or greater,

$$k_\lambda = \left(1 - \Delta\theta\cot\theta_o\right)\left(1 + \Delta\theta\cot\theta_o + \tfrac{1}{2}(\Delta\theta)^2 + (\Delta\theta)^2\cot^2\theta_0\right)$$

$$= 1 - \Delta\theta\cot\theta_o + \Delta\theta\cot\theta_o + \tfrac{1}{2}(\Delta\theta)^2 - (\Delta\theta)^2\cot^2\theta_o + (\Delta\theta)^2\cot^2\theta_o$$

$$= 1 + \tfrac{1}{2}(\Delta\theta)^2 + o\{(\Delta\theta)^3\} .$$

The scale at points not on the standard parallel will therefore always be too great, increasing with distance from it.

3.2.2 The equal area case

1. The scale is true, as before, along the standard parallel and hence

$$r_o k_c = R\sin\theta_o \qquad \ldots\ldots \qquad 3.5.$$

2. All parallels are concentric circular arcs.
3. All meridians are straight lines radiating from the centre to the truly divided standard parallel.
4. The spacing between parallels is varied to introduce the equal area property.

Hence,

a. the standard parallel no longer plots as a circular arc of radius $R\tan\theta_o$;

and b. $k_c \neq \cos\theta_o$

The general expression for r is obtained by considering the projection of an elemental triangle on the sphere, as shown in figure 3.3, and consisting of an elemental length $R\,d\theta$ of the meridian and an equivalent displacement $R\sin\theta\,d\lambda$, corresponding to the arc of the great circle joining the adjacent points $P(\theta,\lambda)$ and $Q(\theta+d\theta,\lambda+d\lambda)$.

If the parallel through Q projects as a circular arc of radius r and that through P as one of radius $r+dr$, the projection will be equal area if the two equivalent triangles PQT on the sphere and P*Q*T* on the projection plane, have the same area. It is therefore necessary that the differential equation

$$R^2 \sin\theta \; d\theta \; d\lambda = r k_c d\lambda \; dr \qquad \ldots\ldots \quad 3.6$$

be satisfied. The required expression for r is obtained on integrating equation 3.6, when

$$- R^2 \cos\theta = \tfrac{1}{2} r^2 k_c + C .$$

The constant of integration is assigned by choosing a value for r at a specific co-latitude. In practice, the use of the condition

$$r = 0 \qquad \text{when} \qquad \theta = 0$$

gives simple expressions for r and k_c as

$$C = -R^2 .$$

The earlier equation then becomes

$$2R^2 (1 - \cos\theta) = r^2 k_c$$

Simple trigonometrical manipulation gives

$$r^2 = 4 k_c^{-1} R^2 \sin^2 \tfrac{1}{2}\theta$$

or

$$r = 2 k_c^{-\frac{1}{2}} R \sin \tfrac{1}{2}\theta \qquad \ldots\ldots \quad 3.7.$$

Along the standard parallel, $\theta = \theta_o$, and

$$r_o^2 k_c^2 = R^2 \sin^2\theta_o$$

$$k_c \left(4R^2 \sin^2 \tfrac{1}{2}\theta_o\right) = R^2 \left(4 \sin^2 \tfrac{1}{2}\theta_o \cos^2 \tfrac{1}{2}\theta_o\right)$$

Therefore,

$$k_c = \cos^2 \tfrac{1}{2}\theta_o \qquad \ldots\ldots \quad 3.8.$$

Thus the conic equal area projection with one standard parallel can be defined by the following set of equations.

$$k_c = \cos^2 \tfrac{1}{2}\theta_o$$

$$r = 2R \sec \tfrac{1}{2}\theta_o \sin \tfrac{1}{2}\theta \qquad \ldots\ldots \quad 3.9$$

$$\ell = 2\pi k_c r$$

The scale is given by the expressions

$$k_\phi = \frac{dr}{R\,d\theta} = \frac{2R \sec \tfrac{1}{2}\theta_o \cos \tfrac{1}{2}\theta \; \tfrac{1}{2} d\theta}{R\,d\theta} = \sec \tfrac{1}{2}\theta_o \cos \tfrac{1}{2}\theta \; \ldots\ldots \quad 3.10$$

and

$$k_\lambda = \frac{r\,k_c}{R \sin\theta} = \frac{2R \sec \tfrac{1}{2}\theta_o \sin \tfrac{1}{2}\theta \cos^2 \tfrac{1}{2}\theta_o}{2R \sin \tfrac{1}{2}\theta \cos \tfrac{1}{2}\theta} = \cos \tfrac{1}{2}\theta_o \sec \tfrac{1}{2}\theta \;\ldots\ldots\; 3.11.$$

Note than $k_\phi . k_\lambda = 1$ which confirms that the projection is equal area. Also $k_\phi = k_\lambda = 1$ when θ is put equal to θ_o in equations 3.10 and 3.11. The characteristics of scale on this projection can be directly verified from these same equations and summarised as

$$k_\phi \gtrless 1 \quad \text{when} \quad \theta \lesseqgtr \theta_o \quad \text{while} \quad k_\lambda \gtrless 1 \quad \text{when} \quad \theta \gtrless \theta_o .$$

3.2.3 Conformal conic with one standard parallel

Conditions 3.2.2(1) to 3.2.2(3) hold in this case too. Condition 3.2.2(4) is amended in order that the spacing between the parallels is now varied to introduce the *conformal* property. This condition will be satisfied if the two elemental triangles PQT on the sphere and P*Q*T* on the projection in figure 3.3 are *similar*. The required differential equation in this case is

$$\frac{R\ d\theta}{R\ \sin\theta\, d\lambda} \doteq \frac{dr}{r k_c\, d\lambda} \qquad \ldots\ldots \quad 3.12$$

Integration gives

$$\log r = k_c \int \frac{d\theta}{\sin\theta} = k_c \int \frac{d\theta}{2\sin\tfrac{1}{2}\theta\ \cos\tfrac{1}{2}\theta} = k_c \int \left(\frac{\cos\tfrac{1}{2}\theta\ \tfrac{1}{2}d\theta}{\sin\tfrac{1}{2}\theta} - \frac{(-\sin\tfrac{1}{2}\theta\ \tfrac{1}{2}d\theta)}{\cos\tfrac{1}{2}\theta} \right)$$

Thus

$$\log r = k_c \log\{\tan\tfrac{1}{2}\theta\} + C ,$$

where the constant of integration C is defined by adopting a value for r at a particular co-latitude θ. For the conformal conic with one standard parallel, it is conventional to adopt

$$r_o = R \tan\theta_o$$

as in the case of the conic equidistant projection (e.g., Thomas 1952, p.123; Jordan-Eggert 1962, III sec 75). In this case,

$$k_c = \cos\theta_o$$

and C can be evaluated at $\theta = \theta_o$, when

$$C = \log r_o - k_c \log\{\tan\tfrac{1}{2}\theta_o\}$$

and

$$\log\left(\frac{r}{r_o}\right) = k_c \log\left(\frac{\tan\tfrac{1}{2}\theta}{\tan\tfrac{1}{2}\theta_o}\right),$$

which can be written as

$$r = r_o \left(\frac{\tan\tfrac{1}{2}\theta}{\tan\tfrac{1}{2}\theta_o}\right)^{k_c} = R\tan\theta_o \left(\frac{\tan\tfrac{1}{2}\theta}{\tan\tfrac{1}{2}\theta_o}\right)^{(\cos\theta_o)} \quad \ldots\ldots \quad 3.13.$$

For limited extents of latitude, equation 3.13 can be written as (Jordan-Eggert 1962, III sec 74)

$$dr = r - r_o$$
$$= r_o \cot \theta_o \Delta\theta + \frac{1}{6} r_o \cot \theta_o (\Delta\theta)^3 - \frac{1}{24} r_o \cot^2\theta_o (\Delta\theta)^4$$
$$= R\ \Delta\theta + \frac{1}{6} R(\Delta\theta)^3 - \frac{1}{24} R \cot \theta_o (\Delta\theta)^4 \quad \ldots\ldots 3.14.$$

Equation 3.13 should pose no serious problems for computer use. The scale on this projection is given by the expressions

$$k_\phi = \frac{dr}{R\ d\theta} = \frac{r_o . k_c \left(\tan \tfrac{1}{2}\theta\right)^{(k_c - 1)} \sec^2 \tfrac{1}{2}\theta\ \tfrac{1}{2} d\theta}{\left(\tan \tfrac{1}{2}\theta_o\right)^{k_c} . R\ d\theta}$$

$$= \frac{r\ k_c}{R \sin \theta}$$

$$k_\lambda = \frac{r\ k_c}{R \sin \theta} \overset{(3.15)}{=} k_\phi$$

which verifies the conformal property.

3.2.4 Summary of projections on a tangent cone

Projections on a tangent cone are completely defined by relations of the form

$$k_c = k_c(\theta_o)$$

and $\ldots\ldots$ 3.16..

$$r = r(\theta_o, \theta)$$

Rectangular projection or grid co-ordinates (N,E) are defined by a Cartesian system with its origin at the intersection of the standard parallel and a meridian of longitude λ_o which is central to the region being mapped. The rectangular co-ordinates (N,E) of the general point $P(\phi,\lambda)$ can be defined in the case of all three projections on the tangent cone, by equations of the form given below.

$$N = r_o - r \cos(k_c d\lambda)$$
$$E = r \sin(k_c d\lambda) \quad \ldots\ldots 3.17$$

where

$$d\lambda = \lambda - \lambda_o \quad \ldots\ldots 3.18.$$

The above formulae can be improved in instances where $d\lambda$ is small by replacing the trigonometrical functions in equations 3.17 by the larger terms in the series expansions of sine and cosine which are of the form

$$\cos(k_c d\lambda) = \sum_{i=0}^{n} \frac{(-1)^i (k_c d\lambda)^{2i}}{(2i)!}$$

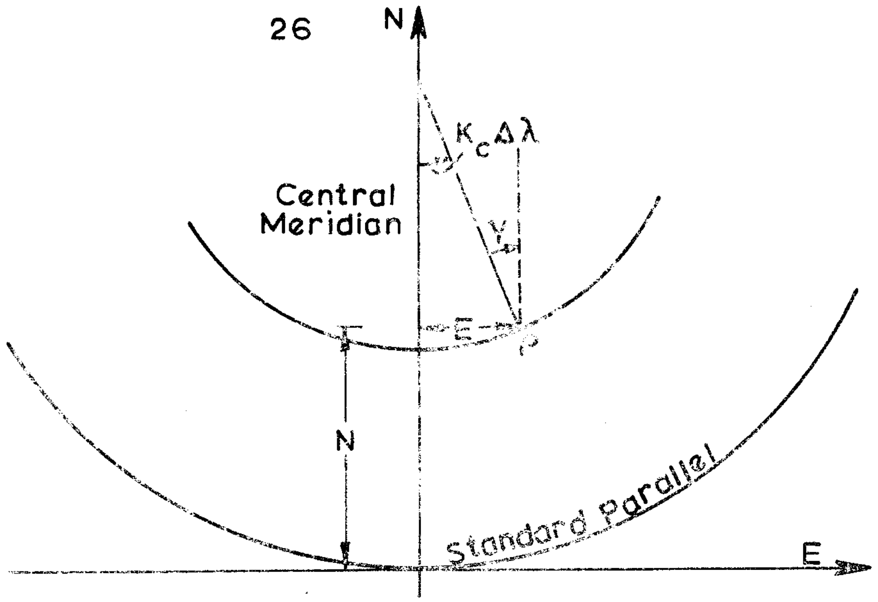


FIG.(3-4)

Rectangular Cartesian Co-ordinates for Conic Projections

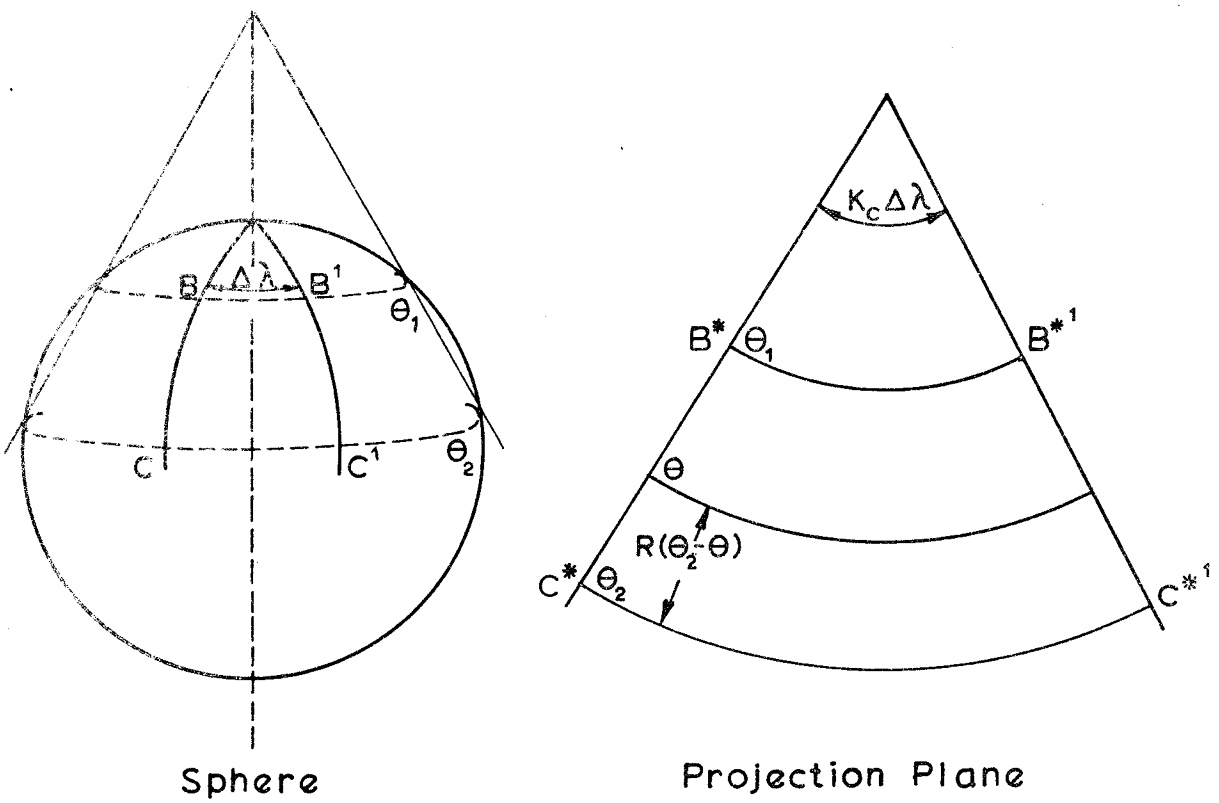


FIG. (3-5)

The Concept of Secant Cone Projections

and

$$\sin(k_c d\lambda) = \sum_{i=0}^{n} \frac{(-1)^i (k_c d\lambda)^{2i+1}}{(2i+1)!}$$

The properties of the projection are summarised in table 3.2.

Projection	$k_c(\theta_o)$	$r(\theta_o,\theta)$	k_λ	k_ϕ
Equidistant	$\cos\theta_o$	$R\{\tan\theta_o-(\theta_o-\theta\}$	$\frac{\sin\theta_o-(\theta_o-\theta)\cos\theta_o}{\sin\theta}$	1
Equal area	$\cos^2\frac{1}{2}\theta_o$	$2R\sec\frac{1}{2}\theta_o\sin\frac{1}{2}\theta$	$\cos\frac{1}{2}\theta_o\sec\frac{1}{2}\theta$	$\sec\frac{1}{2}\theta_o\cos\frac{1}{2}\theta$
Conformal	$\cos\theta_o$	$R\tan\theta_o\left(\frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\theta_o}\right)^{\cos\theta_o}$	$\frac{\sin\theta_o}{\sin\theta}\left(\frac{\tan\frac{1}{2}\theta}{\tan\frac{1}{2}\theta_o}\right)^{k_c}$	as at left

TABLE 3.2

Projections on a tangent cone

The grid convergence γ in all cases is given by

$$\gamma = k_c\, d\lambda \qquad \ldots\ldots \quad 3.17a.$$

3.3 Conic projections on a secant cone

$$0 < k_c < 1.$$

Projections on a tangent cone result in the scale being true along the circle of common tangency but too great at all other points in the equidistant case. It is desirable, when using projections for large scale mapping, to control scale errors in order that the reduction of common field observations will not be affected by their existence. This is achieved in the case of conical projections by introducing the concept of two circles along which the scale is true. In the case of the common conic projections, where the axis of the cone coincides with the polar axis of the reference surface, the two circles become two standard parallels and this class of projection is referred to as the *conic projection with two standard parallels*. This can, in concept, be visualised as a quasi perspective projection on a secant cone, as illustrated in figure 3.5.

Along the standard parallels of co-latitude θ_1 and θ_2 which project as concentric circles of radii r_1 and r_2,

$$k_c r_i = R \sin\theta_i \ , \ i=1,2 \qquad \ldots\ldots \quad 3.19.$$

3.3.1 The equidistant case

Also known as

1. Simple conic projection with two standard parallels;

and 2. the de L'Isle projection.

As the projection is equidistant,

$$k_\phi = 1.$$

From figure 3.5, it can be seen that

$$BC = B^*C^* = R(\theta_2 - \theta_1) = r_2 - r_1 \qquad \ldots\ldots 3.20.$$

The combination of equations 3.19 and 3.20 gives

$$k_c(r_2 - r_1) = R(\sin\theta_2 - \sin\theta_1) = k_c R(\theta_2 - \theta_1).$$

Thus

$$k_c = \frac{\sin\theta_2 - \sin\theta_1}{\theta_2 - \theta_1} \qquad \ldots\ldots 3.21.$$

The specification of θ_1 and θ_2 will therefore define r_1 and r_2 through equations 3.19 and 3.21. The parallel of co-latitude which projects as a concentric circular arc of radius r is defined by the equation

$$r = r_1 - R(\theta_1 - \theta) = \frac{R\sin\theta_1(\theta_2-\theta_1)}{\sin\theta_2 - \sin\theta_1} - R(\theta_1-\theta)$$

$$= R\left(\frac{\sin\theta_1(\theta_2-\theta) - \sin\theta_2(\theta_1-\theta)}{\sin\theta_2 - \sin\theta_1}\right) \qquad \ldots\ldots 3.22.$$

The simple conic projection with two standard parallels is therefore defined completely by equations 3.21, 3.22, 3.2 and 3.17. The scale along the parallels is given by

$$k_\lambda = \frac{r\,k_c}{R\sin\theta} = \frac{\sin\theta_1(\theta_2-\theta) - \sin\theta_2(\theta_1-\theta)}{\sin\theta(\theta_2 - \theta_1)} \qquad \ldots\ldots 3.23.$$

Direct verification of equation 3.23 will show that $k_\lambda < 1$ between the standard parallels and $k_\lambda > 1$ outside this region.

Notes:-

1. The use of the secant cone instead of the tangent cone has the advantage of smaller scale errors over larger extents.

This type of projection is suited for areas which have a considerable variation in longitude but only a limited variation in latitude. It is therefore adequate for the small scale mapping of regions like Australia, Canada, the Soviet Union, the United States, etc. The conic equidistant projection has been used for the production of the 1:2,500,000 map of Australia (1968) with standard

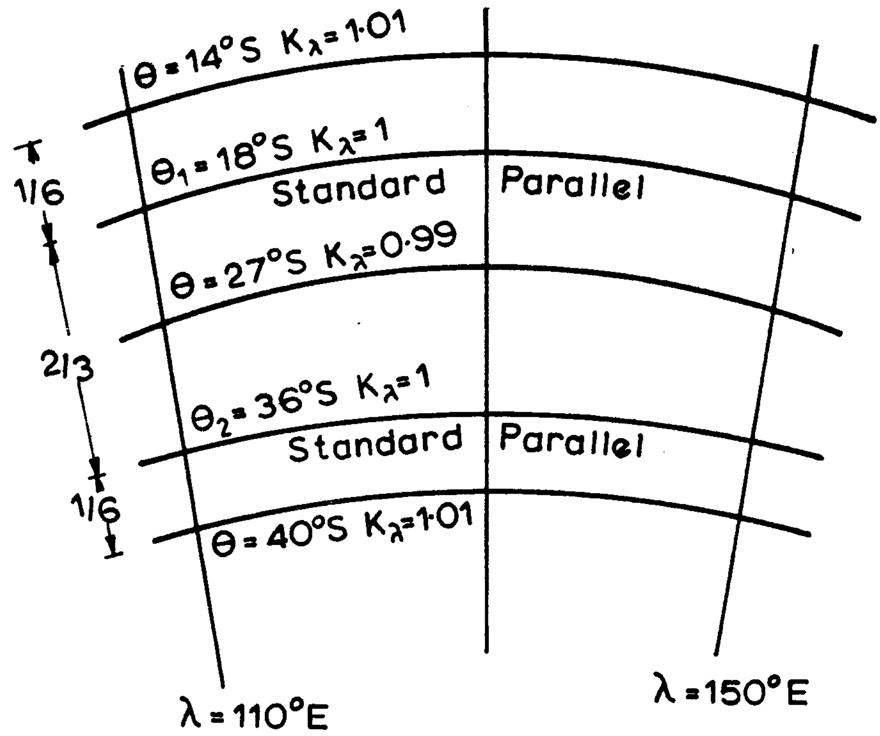


FIG. (3·6)

Spacing of Standard Parallels on an Equidistant Secant Cone Projection for Australia

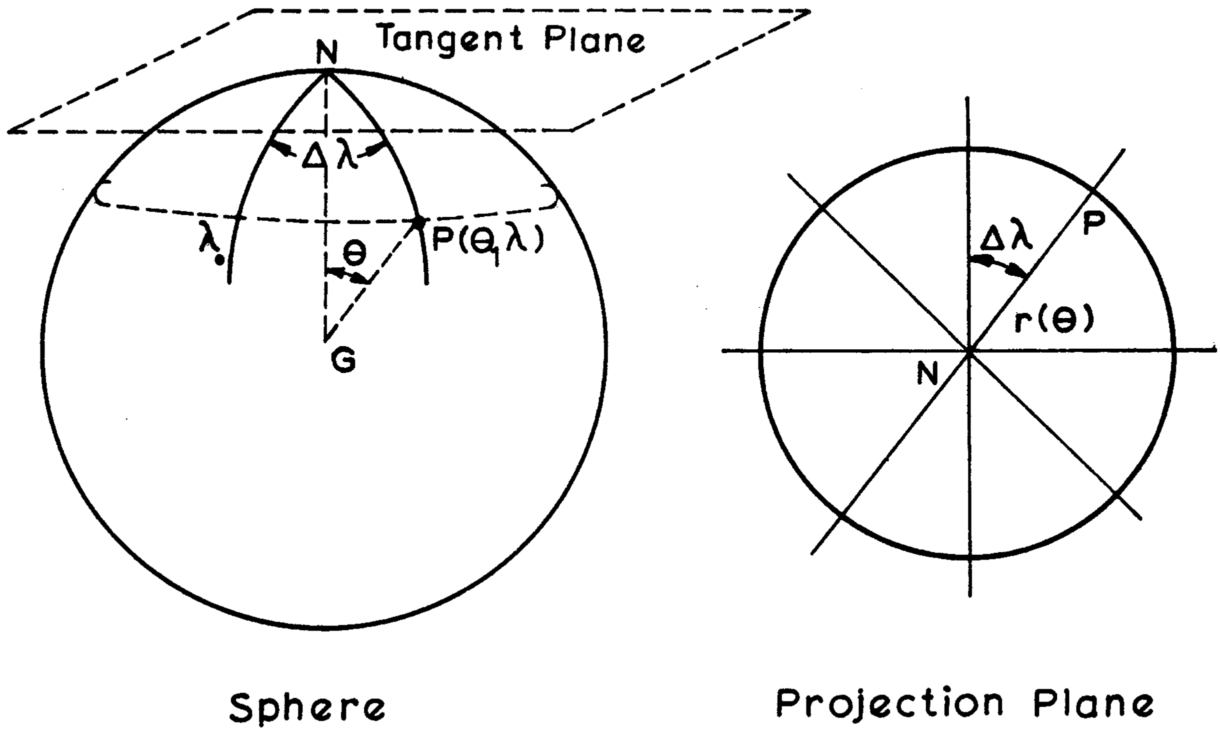


FIG. (3·7)

parallels at 18°S and 36°S. This is in keeping with common practice for areas of this extent where the standard parallels are placed one sixth way in from the north and south limits of the area being mapped. The maximum scale error within the standard parallels is less than 1¼% ($k_\lambda = 0.987$), while the scale errors on the peripheries has approximately the same magnitude ($k_\lambda = 1.01 \doteq 1\%$ at both 14°S and 40°S). Scale errors of 1-1½% can be considered to be acceptable for projections of this type.

3.3.2 The equal area case

Also known as Albers projection

The relation between this projection and the equidistant case is the same as that between the equivalent projections on a tangent cone. The elemental triangles on both the reference sphere and the projection plane are defined as in section 3.2.2. Thus equations 3.6 and 3.19 apply. If r_1 and r_2 are the radii of the concentric circular arcs representing the standard parallels of co-latitude θ_1 and θ_2 on the projection,

$$r_1 k_c = R \sin \theta_1$$
$$r_2 k_c = R \sin \theta_2 \qquad \ldots\ldots 3.19.$$

The integral of equation 3.6 gives

$$k_c r^2 = 2\{C - R^2 \cos \theta\} \qquad \ldots\ldots 3.24,$$

where C is the constant of integration. Squaring and differencing the equations at 3.19 gives

$$k_c\{k_c r_2^2 - k_c r_1^2\} = R^2\{\sin^2\theta_2 - \sin^2\theta_1\}.$$

The use of equation 3.24 on the left hand side gives

$$k_c = \frac{R^2(\sin\theta_2 - \sin\theta_1)(\sin\theta_2 + \sin\theta_1)}{2R^2(\cos\theta_1 - \cos\theta_2)}$$

$$= \frac{\left(2\cos\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_2-\theta_1}{2}\right)\left(2\sin\frac{\theta_1+\theta_2}{2}\cos\frac{\theta_1-\theta_2}{2}\right)}{2\left(2\sin\frac{\theta_1+\theta_2}{2}\sin\frac{\theta_2-\theta_1}{2}\right)}$$

$$= \frac{\cos\theta_1 + \cos\theta_2}{2}.$$

The constant of integration C in equation 3.24 is evaluated by the use of equation 3.19 at either standard parallel when

$$C = \tfrac{1}{2}\,k_c r_1^2 + R^2 \cos\theta_1 = R^2\left(\frac{\sin^2\theta_1}{2k_c} + \cos\theta_1\right)$$

$$\overset{(3.25)}{=} \quad \frac{R^2(1 + \cos\theta_1\cos\theta_2)}{\cos\theta_1 + \cos\theta_2} \qquad \ldots\ldots \quad 3.26.$$

The pole no longer plots as a point. Albers' projection is completely defined by the equations

$$k_c = \tfrac{1}{2}(\cos\theta_1 + \cos\theta_2)$$

$$C = \frac{R^2(1 + \cos\theta_1\cos\theta_2)}{2k_c} \qquad \ldots\ldots \quad 3.27.$$

$$r = \left\{\frac{2}{k_c}(C - R^2\cos\theta)\right\}^{\frac{1}{2}}$$

The scale along the meridian, as seen from figure 3.3, is given by

$$k_\phi = \frac{dr}{R\,d\theta} = \frac{\tfrac{1}{2}\left(\frac{2}{k_c}\right)^{\frac{1}{2}} R^2\sin\theta\,d\theta}{R(C - R^2\cos\theta)^{\frac{1}{2}}d\theta} = \frac{R\sin\theta}{k_c r} = \frac{1}{k_\lambda}\,.$$

Notes:-

1. The variation in scale with co-latitude has the same characteristics as in the equidistant case, the extent of scale error reaching the same magnitudes.

2. For the ellipsoidal case and notes on the construction of a grid, see Deetz & Adams (1945, pp. 96 et seq).

3. The projection which has been used for demographic maps of regions with a considerable east-west extent, but a limited extent in latitude (30 - 35 degrees), is attributed to H.C. Albers (c. 1805).

3.3.3 The conformal case

Commonly titled

the Lambert conformal conic projection with two standard parallels.

As the projection is conformal, the relevant equations are those governing the conformality of elemental triangles (equation 3.12) and the condition for standard parallels (equation 3.19). Thus

$$r_1 k_c = R\sin\theta_1$$

$$r_2 k_c = R\sin\theta_2 \qquad \ldots\ldots \quad 3.19,$$

where r_1 and r_2 are the radii of the concentric circular arcs on the projection plane which represent the standard parallels of co-latitude

θ_1 and θ_2 respectively. The conformal condition, as developed in section 3.2.3, on integration, gives

$$\log r = k_c \log\{\tan \tfrac{1}{2}\theta\} + C' \qquad \ldots\ldots \quad 3.28.$$

The constant k_c of the cone is evaluated by applying equation 3.28 to the two standard parallels and differencing, when

$$\log r_2 - \log r_1 = k_c\left(\log\{\tan \tfrac{1}{2}\theta_2\} - \log\{\tan \tfrac{1}{2}\theta_1\}\right) .$$

On consideration of the equations at 3.19 in logarithmic form, differencing and substitution in the left hand side of the above equation give

$$k_c = \frac{\log\{\sin \theta_2\} - \log\{\sin \theta_1\}}{\log\{\tan \tfrac{1}{2}\theta_2\} - \log\{\tan \tfrac{1}{2}\theta_1\}} \qquad \ldots\ldots \quad 3.29.$$

Equation 3.28 can be expressed as

$$r = C \tan^{k_c} \tfrac{1}{2}\theta .$$

Evaluation on the standard parallels give

$$C = \frac{r_1}{(\tan \tfrac{1}{2}\theta_1)^{k_c}} = \frac{r_2}{(\tan \tfrac{1}{2}\theta_2)^{k_c}}$$

$$= \frac{R \sin \theta_1}{k_c(\tan \tfrac{1}{2}\theta_1)^{k_c}} = \frac{R \sin \theta_2}{k_c(\tan \tfrac{1}{2}\theta_2)^{k_c}} .$$

The second equality can easily be verified as being equation 3.29. Therefore the Lambert conformal conic projection with two standard parallels θ_1 and θ_2 can be completely defined by the equations

$$k_c = \frac{\log\{\sin \theta_2\} - \log\{\sin \theta_1\}}{\log\{\tan \tfrac{1}{2}\theta_2\} - \log\{\tan \tfrac{1}{2}\theta_1\}}$$

$$C = \frac{R \sin \theta_1}{k_c\left(\tan \tfrac{1}{2}\theta_1\right)^{k_c}} = \frac{R \sin \theta_2}{k_c\left(\tan \tfrac{1}{2}\theta_2\right)^{k_c}} \qquad \ldots\ldots \quad 3.30.$$

$$r = C\left(\tan \tfrac{1}{2}\theta\right)^{k_c}$$

$$k_\phi = k_\lambda = \frac{r\,k_c}{R \sin \theta}$$

Notes :-

1. The Lambert conformal conic projection with two standard parallels is used by many countries as the basis for a plane co-ordinate system though, at the present time, there is a tendency to favour the Universal Transverse Mercator (UTM) system for this purpose. It has been used recently in Australia for 1:1,000,000 mapping.

2. The projection is attributed to J.H. Lambert and was first published in 1772.

3. For details regarding the drawing of grids for small scale maps, see (Deetz & Adams 1945,pp.85 et seq).

4. The use of this projection for large scale mapping has resulted in considerable attention being paid to the characteristics of curves on the reference ellipsoid, on transformation to the Lambert projection. This is not covered in the present development as the principles of such representation is covered in full for the more commonly used UTM projection in section 4.

3.3.4 Summary of projections on a secant cone

Projections on a secant cone are completely defined by equations of the form

$$k_c = k_c(\theta_1,\theta_2)$$
$$r = r(\theta,\theta_1,\theta_2) \qquad \ldots\ldots \quad 3.31,$$

where θ_1 and θ_2 are the co-latitudes of the standard parallels which are placed one sixth way in from the north and south extremities of the region to be mapped. The basic characteristics of the three common cases are listed in table 3.3. The scale error for an area the size of Australia which has a range of latitude in excess of 30 degrees, is never greater than 1½%.

The plane projection co-ordinates for the general point $P(\theta,\lambda)$ are obtained from figure 3.4, on using formulae at 3.17, as

$$N = r_o - r\cos(k_c d\lambda)$$

and $\qquad \ldots\ldots \quad 3.32,$

$$E = r\sin(k_c d\lambda)$$

where r_o is the radius of curvature of the parallel of co-latitude θ_o on which lies the origin of co-ordinates (θ_o,λ_o), and

$$d\lambda = \lambda - \lambda_o \qquad \ldots\ldots \quad 3.33.$$

The grid convergence is given as before by

$$\gamma = k_c d\lambda \ .$$

Name	$k_c(\theta_1, \theta_2)$	$C(\theta_1, \theta_2)$
de L'Isle	$\dfrac{\sin\theta_2 - \sin\theta_1}{\theta_2 - \theta_1}$	-
Albers	$\frac{1}{2}(\cos\theta_1 + \cos\theta_2)$	$\dfrac{R^2(1 + \cos\theta_1 \cos\theta_2)}{2k_c}$
Lambert	$\dfrac{\log\{\sin\theta_2\} - \log\{\sin\theta_1\}}{\log\{\tan\frac{1}{2}\theta_2\} - \log\{\tan\frac{1}{2}\theta_1\}}$	$\dfrac{R\sin\theta_1}{k_c(\tan\frac{1}{2}\theta_1)^{k_c}}$

T A B L E 3.3 ctd. below

Name	$r(\theta,\theta_1,\theta_2)$	k_λ	k_ϕ
de L'Isle	$\dfrac{R\{\sin\theta_1(\theta_2-\theta) - \sin\theta_2(\theta_1-\theta)\}}{\sin\theta_2 - \sin\theta_1}$	$\dfrac{\sin\theta_1(\theta_2-\theta) - \sin\theta_2(\theta_1-\theta)}{\sin\theta(\theta_2-\theta_1)}$	1
Albers	$\frac{2}{k_c}(C - R^2\cos\theta)^{\frac{1}{2}}$	$\dfrac{k_c r}{R\sin\theta}$	$\dfrac{1}{k_\lambda}$
Lambert	$C(\tan\frac{1}{2}\theta)^{k_c}$	$\dfrac{k_c r}{R\sin\theta}$	k_λ

T A B L E 3.3 ctd from above

Summary of properties on secant cone projections

3.4 Projections on a tangent plane

Alternative headings

(i) Azimuthal projections

(ii) Zenithal projections

Characteristics

$k_c = 1; \quad \theta_o = 0.$

3.4.1 The equidistant case

Alternative title :- the azimuthal equidistant projection

As $\theta_o = 0$, the standard parallel is a point which, in the case where the axes of the two solids coincide, is the pole.

This can be treated as a special case of the equidistant projection on a tangent cone, developed in section 3.2.1, in the instance when $\theta_o = 0$. The parallels thus project as concentric circles of radius r given by

$$r = R\theta \qquad \ldots\ldots \quad 3.35a.$$

The length ℓ of the parallel on the projection is defined by

$$\ell = 2\pi r = 2\pi R\theta \qquad \ldots\ldots \quad 3.35b.$$

$k_\phi = 1$ by definition.

$$k_\lambda = \frac{\theta}{\sin\theta} = \theta\left(\theta^{-1} + \frac{\theta}{6} + \frac{7\,\theta^3}{360} + o\{\theta^5\}\right) > 1$$

The scale error reaches 1% when $\theta \doteqdot 14^o$.

Notes:-

1. The use of the concept of a central scale factor as developed in section 3.5.1.*2* , can extend the region over which the scale error is less than 1% to $\theta \doteqdot 20^o$.

2. Azimuthal projections are suited to polar regions. Non-polar cases have the characteristic property that great circles of constant azimuth from the centre of projection, plot as straight lines.

3.4.2 The equal area case

Commonly titled the Lambert azimuthal equal area projection.

Refer to section 3.2.2. The equal area property is introduced by varying the spacing between the parallels which still project as concentric circles in view of the fact that θ_o is zero and k_c is unity. The modified forms of the equations at 3.9 become

$$r = 2R \sin \tfrac{1}{2}\theta \qquad \ldots\ldots \quad 3.36$$

and

$$\ell = 2\pi r .$$

$$k_\phi \overset{(3.10)}{=} \cos \tfrac{1}{2}\theta \quad ; \quad k_\lambda \overset{(3.11)}{=} \sec \tfrac{1}{2}\theta .$$

3.4.3 The conformal case

On applying the development in section 3.2.3 to the polar case, the evaluation of the the basic integral when $k_c = 1$ gives

$$k_c = C \tan \tfrac{1}{2}\theta .$$

In theory, it is possible to choose C such that either

a. some parallel of co-latitude θ_c is at a distance $R\theta_c$ from the pole ;

or

b. a parallel central to the region being mapped is true to scale.

In case a,

$$r_c = R\theta_c \quad = C \tan \tfrac{1}{2}\theta_c$$

when ...... 3.37.

$$C = R\theta_c \cot \tfrac{1}{2}\theta_c$$

In case b, if the selected parallel is of co-latitude θ_c,

$$r_c = R \sin \theta_c = C \tan \tfrac{1}{2}\theta_c$$

when ...... 3.38.

$$C = 2R \cos^2 \tfrac{1}{2}\theta_c$$

In both cases it can be seen that

$$C = 2R\left(1 + o\{\theta^2\}\right) ,$$

and when $C = 2R$,

the projection reduces to the stereographic projection, developed in section 2.3.

Notes:-

1. The scale error in the case when $C = 2R$ reaches 1% at $\theta \doteqdot 11\tfrac{1}{2}^{o}$.

Non-polar cases are possible when azimuthal great circles from the centre of projection, plot as straight lines. In addition, points with equi-angular distance ψ from the centre of projection retain this property on projection. As illustrated in figure 2.2, the general point $Q(\phi,\lambda)$ is related to the centre of projection $P(\phi_o,\lambda_o)$ on the projection plane by the equations

$$N = C \tan \tfrac{1}{2}\psi \cos \alpha$$

and 3.39,

$$E = C \tan \tfrac{1}{2}\psi \sin \alpha$$

where ψ and α are given by equations at 2.4 and the N and E axes are taken along the meridian and prime vertical directions at P.

3.5 Projections on a tangent cylinder (Cylindrical projections)

In this case,

$$\theta_o = \frac{\pi}{2} \qquad \text{when} \qquad k_c = 0.$$

The cone becomes a cylinder tangential at the equator. The apex of the cone, which is the centre of the concentric circular arcs representaing the parallels, is at infinity. Consequently, the parallels project as parallel straight lines. The meridians which radiate from infinity to the truly divided standard parallels, which in this case is the equator, will be a family of straight lines orthogonal to the projected parallels.

Cylindrical projections, illustrated in figure 3.8, will therefore exhibit the following characteristics.

(i) All parallels are of the same length $2\pi R$ on projection.

(ii) The projected meridians and parallels form a rectangular grid.

Notes:-

1. The cylinder can, in general, have common tangency with any great circle on the sphere.

The rules of cylindrical projections can therefore be generalised as follows.

1. The great circle of common tangency projects as a straight line.

2. All orthogonal great circles project as straight lines perpendicular to and correctly spaced on the great circle of common tangency.

3. All small circles parallel to the great circle of common tangency, project as parallel straight lines of the same length $2\pi R$.

3.5.1 The equidistant case

Alternate names

1. Square plate projection
2. Square projection

The equator plots as a straight line, true to scale. All parallels of latitude project as parallel straight lines of the same length ℓ given by

$$\ell = 2\pi R$$

The meridians divide the equator truly and the other parallels in correct proportion. As the projection is equidistant, the spacing r of parallels from the equator is given by

$$r = R\phi \quad , \quad \text{where } \phi \text{ is the latitude}$$

and

$$k_\phi = 1.$$

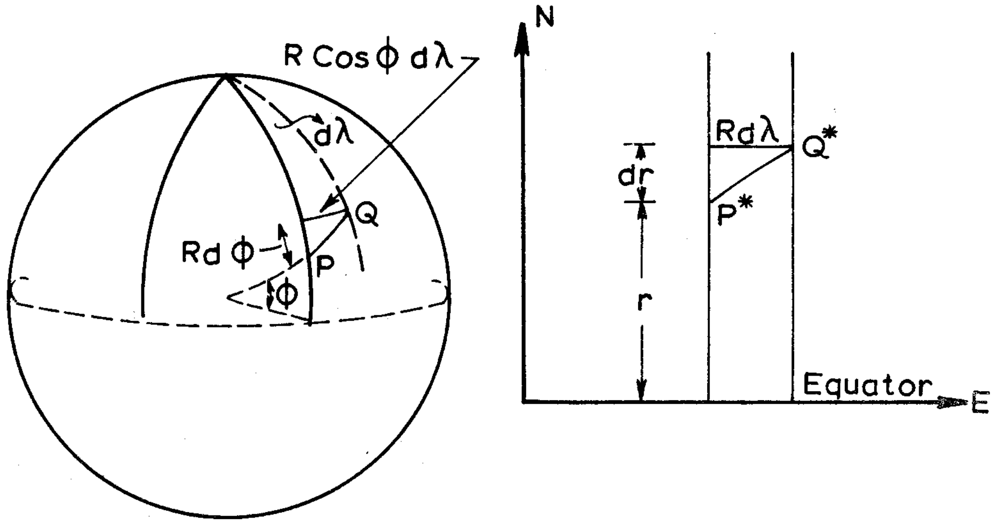
$$k_\lambda = \frac{R}{R\cos\phi} = \sec\phi \qquad \ldots\ldots \; 3.40.$$

Notes:-

1. k_λ increases from 1 to ∞ between $\phi = 0$ and $\phi = \frac{1}{2}\pi$, the scale error reaching 1% at $\phi \doteqdot 8^\circ$.

2. *The central scale factor* k_o

Projections on a tangent cylinder can only map a band of approximately 16° in latitude, symmetrical about the equator, if the resulting scale errors are to be comparable in magnitude with those in secant cone projections covering a thirty degree variation in latitude.



Reference Sphere . Projection Plane

FIG.(3·8)

Projections on a tangent Cylinder

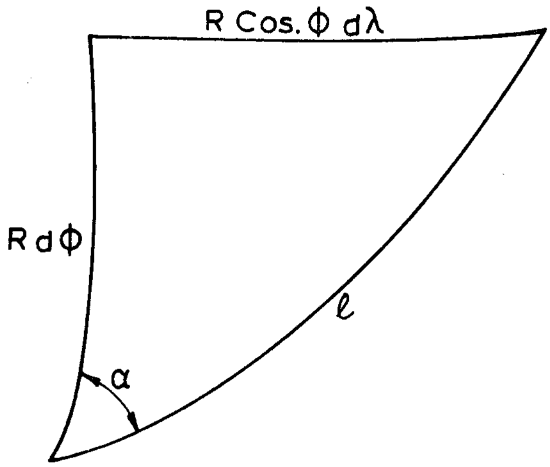


FIG.(3·9)

Elemental Triangle

The same concept can be introduced in the case of cylindrical projections. Let the cylinder cut the reference surface along two parallels $(+\phi_o, -\phi_o)$ symmetrical with respect to the equator. Its radius r_c will then be given by

$$r_c = Rk_o \qquad , \qquad \text{where} \quad k_o \leq 1.$$

Along parallels of common tangency,

$$2\pi R \cos \phi_o = 2\pi R\ k_o.$$

Thus,

$$k_o = \cos \phi_o.$$

If all other characteristics of the projection are retained, the quantities ℓ, r, k_ϕ, k_λ defined above are given by the equations

$$\ell = 2\pi R\ k_o$$

$$r = k_o R\phi.$$

$$k_\phi = \cos \phi_o \qquad ; \qquad k_\lambda = \frac{\cos \phi_o}{\cos \phi}.$$

If $\phi_o = 8^o$; at the equator, $k_\phi = k_\lambda = 0.99$;
at parallel 8^o, $k_\phi = 0.99$, $k_\lambda = 1.00$;
at parallel 12^o, $k_\phi = 0.99$, $k_\lambda = 1.01$.

This technique enables an area of 24^o, symmetrical about the equator, to be projected without scale errors in excess of 1%.

Conceptually, the quantity k_o can be considered to be a constant scale factor, called the *central scale factor*, which is used to scale down all linear quantities on the reference surface prior to projection. This has the effect of replacing scale errors e_m of the form

$$e_{m\ell} \geq e_m \geq 0 \qquad \text{over the latitude range} \qquad \phi_{\ell_1} \geq |\phi| \geq 0$$

by scale errors of the form

$$e_{m\ell} \geq e_m \geq -e_{m\ell} \qquad \text{over the range} \qquad \phi_{\ell_2} \geq |\phi| \geq 0,$$

where

$$\phi_{\ell_2} - \phi_{\ell_1} > 0.$$

The purely positive scale errors in tangent cylinder projections are now replaced by both positive and negative scale errors by the use of a secant cylinder, thereby extending the range of latitude over which the projection has scale errors e_m less than some prescribed limit $e_{m\ell}$.

3. The plane projection co-ordinates (N,E) of the point $P(\phi,\lambda)$ with respect to an origin $P_o(\phi_o,\lambda_o)$ are given by

$$N = R\phi$$

$$E = R\ d\lambda \qquad \qquad ......3.41.$$

$$d\lambda = \lambda - \lambda_o$$

$$\gamma = 0$$

The central scale factor should be applied to these relations when relevant.

4. The square plate projection, like all equatorial cylindrical projections, is used to produce world maps, as the grid is simple to construct. The scale error near the poles is excessive. A hemisphere has an area $\pi^2 R^2$ on this projection, as compared with the true surface area of $2\pi R^2$, the exaggeration being approximately 50%.

The projection can therefore be said to be satisfactory for the mapping of equatorial belts with a limited north-south extent.

3.5.2 The equal area case

The equal area property is introduced by varying the distance on the projection between the family of straight lines orthogonal to the projected meridians, and representing the parallels of latitude. The elemental triangles are as shown in figure 3.8. The equal area condition is obtained from the same considerations used in setting up equation 3.6 when

$$R^2 \cos\phi \, d\phi \, d\lambda = R \, dr \, d\lambda .$$

The imposition of the limiting condition $r = 0$ when $\phi = 0$ after integration, gives

$$r = R \sin\phi .$$

$$k_\phi = \frac{dr}{R \, d\phi} = \cos\phi = \frac{1}{k_\lambda} ,$$

k_λ being the same as in the equidistant case.

Plane projection co-ordinates in this case are given by

$$N = R \sin\phi \quad ; \quad E = R \, d\lambda \quad ; \quad \gamma = 0 \qquad \ldots . \; 3.43.$$

Notes:-

1. As cylindrical projections tend to gross scale exaggerations at high latitudes in the absence of imposed conditions, the introduction of the equal area property results in the considerable distortion of shape at high latitudes (e.g., see Robinson 1960, p.75). In all other respects, observations similar to those described in section 3.5.1 for the square plate projection, are of relevance.

3.5.3 The conformal case

Commonly called the Mercator projection.

Originator :- Gerhard Kramer (c.1569). For a historical summary, see Deetz & Adams (1945,p.103 et seq).

The property of conformality is introduced as in section 3.5.2 by varying the distance between the parallels on the projection using the elemental triangles defined in figure 3.8, on lines similar to those utilised in setting up equation 3.12, when

$$\frac{dr}{R} = \frac{d\phi}{\cos\phi} .$$

$$\frac{d\phi}{\cos\phi} = \frac{d\phi}{\sin(\frac{1}{2}\pi+\phi)} = \frac{\frac{1}{2}d\phi}{\sin(\frac{1}{4}\pi+\frac{1}{2}\phi)\cos(\frac{1}{4}\pi+\frac{1}{2}\phi)} = \frac{\cos(\frac{1}{4}\pi+\frac{1}{2}\phi)\ \frac{1}{2}d\phi}{\sin(\frac{1}{4}\pi+\frac{1}{2}\phi)} - \left(\frac{-\sin(\frac{1}{4}\pi+\frac{1}{2}\phi)\ \frac{1}{2}d\phi}{\cos(\frac{1}{4}\pi+\frac{1}{2}\phi)}\right)$$

Integration gives

$$\int \frac{d\phi}{\cos\phi} = \log\{\tan(\frac{\pi}{4} + \frac{\phi}{2})\} = \frac{r}{R}$$

or

$$r = R \log\{\tan(\frac{\pi}{4} + \frac{\phi}{2})\} \qquad \ldots\ldots \quad 3.44.$$

Also,

$$k_\phi = \frac{dr}{R\ d\phi} = \frac{1}{\cos\phi} = k_\lambda .$$

Notes:-

1. *An isometric system of surface co-ordinates*

An isometric system of surface co-ordinates is afforded by two curvilinear parameters (u_1, u_2) such that the linear displacements $(d\ell_1, d\ell_2)$ equivalent to unit changes in u_1 and u_2 satisfy

$$d\ell_1 = d\ell_2 .$$

Alternatively, if h_1 and h_2 are quantities which satisfy

$$d\ell_i = h_i du_i , \quad i=1,2 ,$$

the parameters (u_1, u_2) form an isometric system if

$$h_1 = h_2 = m ,$$

the elements $d\ell_1$ and $d\ell_2$ being perpendicular to one another. The (u_1, u_2) system is then an orthogonal one. The quantity m is called the linearisation factor for the isometric system of orthogonal surface co-ordinates .

The (ϕ,λ) system of surface co-ordinates on a sphere do not form an isometric system as

$$d\ell_\lambda = R \cos\phi\ d\lambda$$

while

$$d\ell_\phi = R\ d\phi .$$

Thus, if $u_1 = \lambda$ and $u_2 = \phi$,

$h_1 = R \cos\phi$; $h_2 = R \neq h_1$.

If, on the other hand, ϕ was replaced by the parameter u_2, given by

$$du_2 = \frac{d\phi}{\cos\phi} \qquad \ldots\ldots \quad 3.46,$$

the same linear meridian displacement $d\ell_\phi$ $(= d\ell_{u_2})$ can be written as

$$d\ell_\phi = R\ d\phi = R \cos\phi\ \frac{d\phi}{\cos\phi} = R \cos\phi\ du_2 .$$

Thus the (λ, u_2) system of surface co-ordinates is an orthogonal isometric one in that

1. the same increment (e.g., one arcsec) in both λ and u_2 produce a common linear displacement $R \cos\phi\ (1")^{rad}$;

and 2. $h_1 = h_2 = R\cos\phi$.

The quantity u_2, which is similar in nature to the geographical latitude ϕ is called the *isometric latitude*.

As

$$du_2 = \frac{d\phi}{\cos\phi} ,$$

$$u_2 = \int \frac{d\phi}{\cos\phi} \overset{(3.44)}{=} \log\{\tan(\frac{\pi}{4} + \frac{\phi}{2})\} \quad \ldots\ldots \quad 3.47.$$

The plane projection co-ordinates in terms of the isometric orthogonal set of surface co-ordinates are given by

$$N = R u_2 \quad ; \quad E = R\, d\lambda \quad ; \quad \gamma = 0 .$$

2. *Rhumb lines*

Loxodromes are curves on the reference surface which cross successive meridians at the same azimuth. If two adjacent points P and Q on a loxodrome of azimuth α have a difference of latitude $d\phi$ and a difference of longitude $d\lambda$,

$$\tan\alpha = \frac{R\cos\phi\, d\lambda}{R\, d\phi} \quad \ldots\ldots \quad 3.48.$$

The equation of the loxodrome is obtained by the integration of equation 3.48, holding α as a constant, when

$$\lambda_o - \lambda = \tan\alpha \int_0^{\phi} \frac{d\phi}{\cos\phi} ,$$

where λ_o is the meridian at which the loxodrome crosses the equator. The evaluation of this integral on the lines set out in the derivation of equation 3.44 gives

$$\lambda_o - \lambda = \tan\alpha \log\{\tan(\frac{\pi}{4} + \frac{\phi}{2})\} \quad \ldots\ldots \quad 3.49.$$

On the Mercator projection, defined by equation 3.45, the surface variables (ϕ,λ) can be replaced by the projection co-ordinates (N,E) when the above equation becomes

$$E = K_1 N + K_2 ,$$

where K_1 and K_2 are constants. This is the equation of a straight line, as α is a constant for a given loxodrome. The plot of a loxodrome on a Mercator chart, and called a *rhumb line*. is therefore a straight line and has resulted in the considerable use of Mercator charts for navigation. Deetz & Adams (1945,p.110) estimate the usage at around the 90% mark of all nautical charts.

3.6 Transverse cylindrical projections

3.6.1 Introduction

Tangent cones become tangent cylinders in the limiting case when $k_c = 0$. The projections which result when the circle of common tangency is a meridian, are called transverse cylindrical

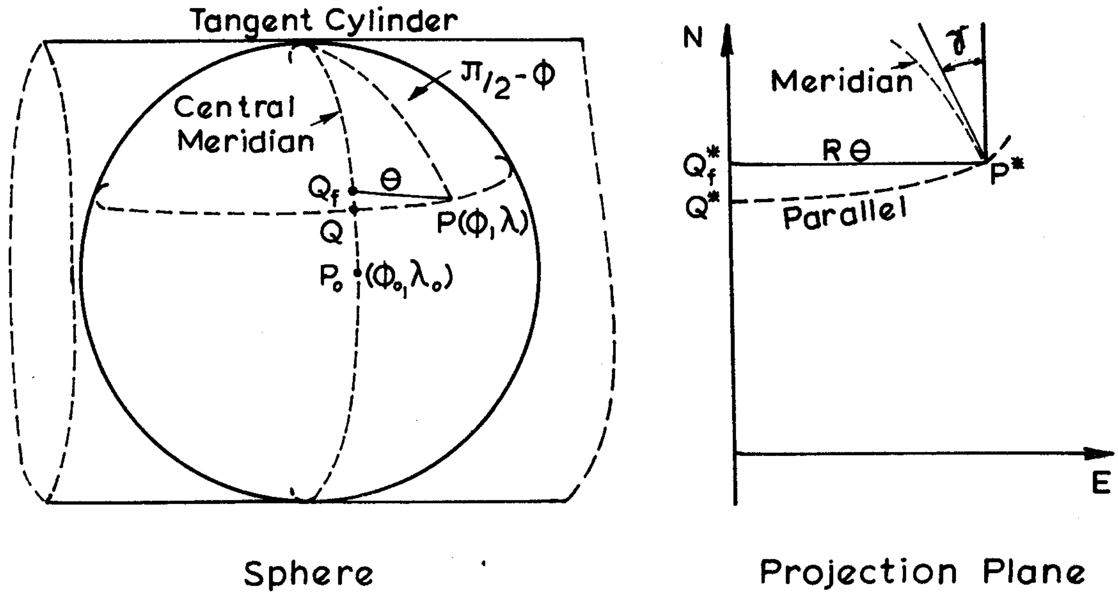


FIG. (3·10)

Transverse Cylindrical Projections

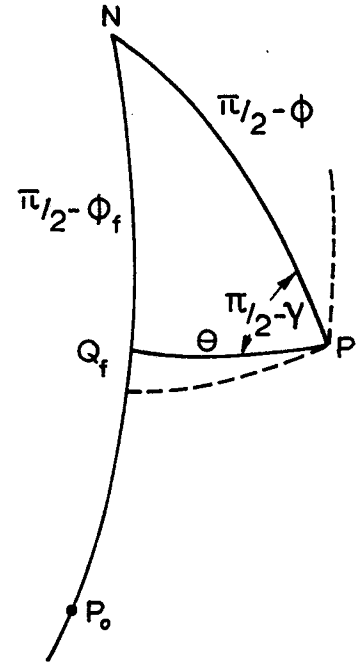


FIG. (3·11)

Transverse Cylindrical Projections – Basic Triangle projections.

Cylindrical projections, as shown in section 3.5, have the property that the great circle of common tangency projects

i. as a straight line ;

and ii. trueto scale.

The family of great circles orthogonal to the circle of common tangency, project as straight lines perpendicular to the latter. In addition, small circles parallel to the great circle of common tangency, plot as parallel straight lines, equal in length to the latter. In the case of transverse cylindrical projections where the circle of common tangency is a meridian central to the region being mapped, and called the *central meridian*, two families of curves forming an orthogonal set on the sphere, project as orthogonal sets of straight lines on the projection plane. The central meridian and all small circles lying in planes parallel to it, comprise the first set. It is conventional for the grid meridian to be chosen parallel to the projected central meridian as it simplifies computations. Hence the family of curves referred to, project as the set of straight lines given by

$$E = \text{constant.}$$

The second set of curves is made up of the family of great circles orthogonal to the central meridian. These curves project as straight lines perpendicular to the projected central meridian, and are given by the equation

$$N = \text{constant.}$$

In the equidistant case, the projected small circles correctly divide the second set of curves described above. Thus PQ_f in figure 3.11 is equal to $P^*Q^*_f$ on the projection in figure 3.10. As such a system affords a set of co-ordinates where the position of a point is fixed by measuring arc lengths along two mutually perpendicular great circles, it is said to provide a system of *rectangular spherical co-ordinates*. The direction of the N axis is commonly called that of *grid north*.

Equal area and conformal properties are introduced by varying the distance between the projected positions of small circles parallel to the central meridian. Only the conformal case will be covered in the present development. This forms the basis of the *transverse Mercator projection* which is the most commonly used projection for large scale mapping in the world today.

3.6.2 The equidistant case

Common names:-

1. Rectangular spherical system of co-ordinates;
2. Cassini-Soldner projection.

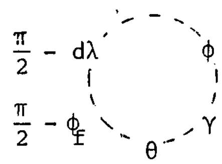
If the origin of co-ordinates $P_o(\phi_o,\lambda_o)$ lies on the central meridian of longitude λ_o, which plots as a straight line true to scale, let the general point be $P(\phi,\lambda)$, as shown in figure 3.11. The projection co-ordinates, as defined in section 3.6.1, are given by

$$N = P_oQ_f \quad ; \quad E = PQ_f \qquad \ldots\ldots \quad 3.40,$$

where Q_f is the point at which the great circle through P and orthogonal to the central meridian, intersects the latter. The latitude ϕ_f of Q_f is called the *foot point latitude*. The significance of this quantity is discussed in section 3.6.3.*4*. As the arc PQ_f is perpendicular to the central meridian and therefore, to any direction parallel to it at P, and if the convergence γ is defined as in section 1.3.2,

$$N\hat{P}Q_f = \frac{\pi}{2} - \gamma \ ; \quad NQ_f = \frac{\pi}{2} - \phi_f \ ; \quad NP = \frac{\pi}{2} - \phi,$$

where N is the pole. Let the arc $PQ_f = \theta$. The application of Napier's rule of circular parts to the spherical triangle NPQ_f, right angled at Q_f,



which states that

$$\sin(\text{any part}) = \Pi\{\tan(\text{adjacent parts})\} = \Pi\{\cos(\text{opposite parts})\} \qquad \ldots\ldots \quad 3.41,$$

provides a solution for the projection co-ordinates of P.

Problem :- Given the geographical co-ordinates of the origin $P_o(\phi_o,\lambda_o)$ and the general point $P(\phi,\lambda)$, determine the projection co-ordinates (N,E) and the convergence γ at P.

$$N = R(\phi_f - \phi_o) = R(\phi - \phi_o) + R(\phi_f - \phi) \qquad \ldots\ldots \quad 3.42,$$

where $(\phi_f - \phi)$ is a small angle if

$$d\lambda = \lambda - \lambda_o \qquad \ldots\ldots \quad 3.43$$

is of limited magnitude. The application of equation 3.41 to the the right angles spherical triangle NPQ gives

$$\sin(\tfrac{1}{2}\pi - d\lambda) = \tan\phi \tan(\tfrac{1}{2}\pi - \phi_f) \qquad \ldots\ldots \quad 3.44$$

or

$$\tan \phi_f = \sec d\lambda \tan \phi$$

and

$$\sin \theta = \cos \phi \sin d\lambda \qquad \ldots\ldots \quad 3.45.$$

As the projection is equidistant, the scale is true along

i. the central meridian whose equation is $E = 0$;

and ii. the family of straight lines N = constant.

A study of the development in section 3.5.1 shows that the scale along the lines

$E = k$, where k is a non-zero constant,

increases with E, analogous to the square plate projection; the increase being a function of the arc length θ ($=PQ_f$).

Consider two adjacent points P_1 and P_2 which are both at angular distance θ away from the central meridian along orthogonal great circle arcs (P_1Q_{f1} and P_2Q_{f2} in figure 3.12).

The distance P_1P_2 on sphere is equal to $R(\phi_{f2} - \phi_{f1})\cos \theta$, where $\phi_{fi}(i=1,2)$ are the foot point latitudes of the points $P_i(i=1,2)$.

The projection distance $P_1^*P_2^*$ is equal to the spherical distance $Q_{f1}Q_{f2} = R(\phi_{f2} - \phi_{f1})$.

Thus the scale error e_m along the arc P_1P_2 is given by

$$e_m = \frac{1}{\cos \theta} - 1 = 1 + \tfrac{1}{2}\theta^2 + \{\theta^4\} - 1$$

$$= \tfrac{1}{2}\theta^2 \simeq \tfrac{1}{2}(d\lambda)^2.$$

It has been recent practice to adopt a value of 1 in 10^3 as the maximum scale error acceptable for day-to-day cadastral work. The introduction of a central scale factor as described in section 3.5.1.2 will reduce this to about half the magnitude (i.e., 5 parts in 10^4) which would enable all field work apart from the establishment of control, to be computed on the projection without the application of corrections for scale in all but large scale work. The adoption of these criteria will restrict the range of $d\lambda$ to satisfy

$$\tfrac{1}{2}(d\lambda)^2 \not> 1\times10^{-3} \quad \text{or} \quad d\lambda \not> 5 \times 10^{-2} (\simeq 3^\circ).$$

It will therefore be assumed in the subsequent development that transverse cylindrical projections are being formulated for cadastral purposes where the scale error, prior to the application of the central scale factor, should not exceed 10^{-3}. This would, in effect, restrict the region being mapped to a zone of 6° width in longitude, symmetrical with respect to a central meridian.

As θ, $d\lambda$ are small angles ($\simeq 5\times10^{-2}$ or less), it is convenient from a computational viewpoint, even in this computerised era, to replace the relevant trigonometrical functions by the larger order terms in the series expansions , as both

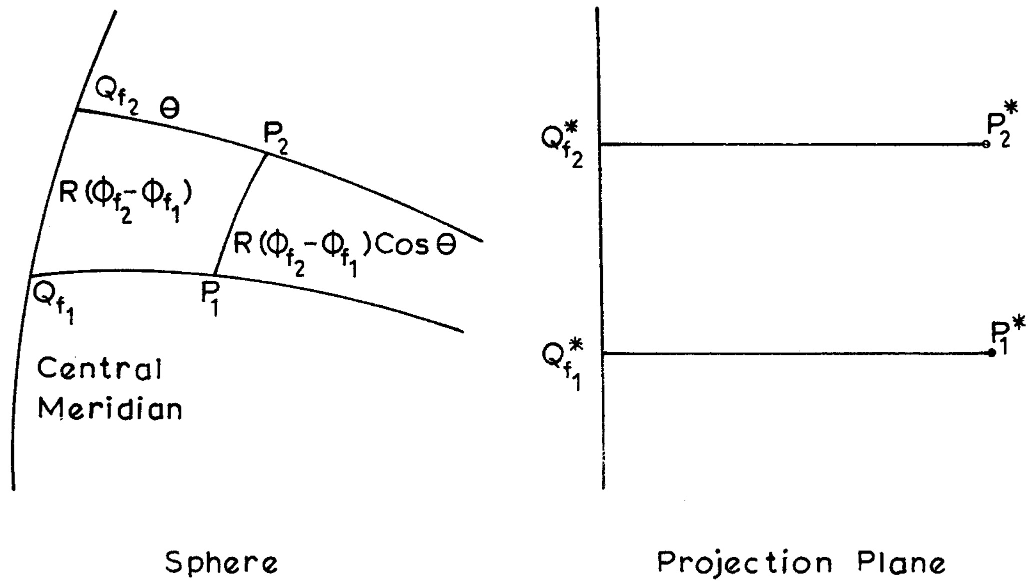


FIG.(3·12)

Basic relations on Cassini Soldner Projection

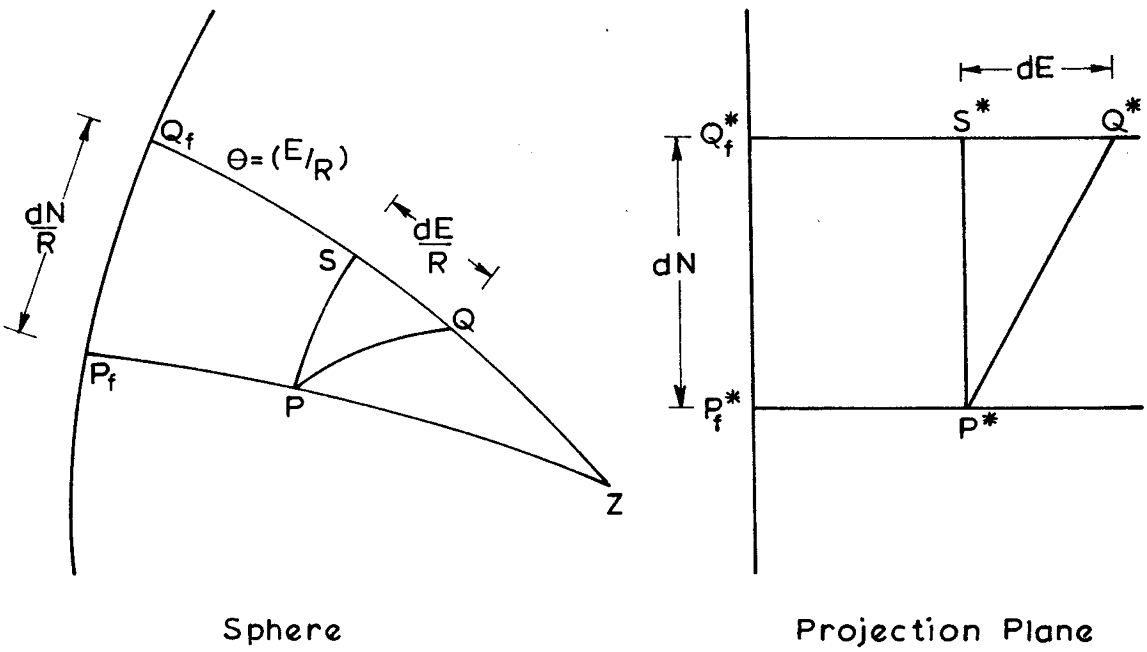


FIG.(3·13)

The transverse Mercator Projection

$R(d\lambda)^6$ and $R\theta^6 \simeq$ 10 cm or less.

The expression for N

Refer to equations 3.42 and 3.44. As $(\phi_f - \phi)$ is a small angle, consider the relation

$$\tan(\phi_f - \phi) = \frac{\tan \phi_f - \tan \phi}{1 + \tan \phi_f \tan \phi}$$

$$\overset{(3.44)}{=} \frac{\tan \phi(\sec d\lambda - 1)}{1 + \tan^2\phi \sec d\lambda} = \frac{\tan \phi(1 - \cos d\lambda)}{\cos d\lambda + \tan^2\phi} .$$

The above relation is simplified by noting that

1. $\tan \phi$ is $(\sin \phi/\cos \phi)$;

2. $1 - \cos d\lambda = 2 \sin^2 \tfrac{1}{2} d\lambda = 2\left(\frac{d\lambda}{2} - \frac{1}{6}\frac{(d\lambda)^3}{8}\right)^2 = \frac{(d\lambda)^2}{2}\left(1 - \frac{1}{12}(d\lambda)^2\right) + o\{\frac{(d\lambda)^4}{24}\}$;

and 3. $\cos d\lambda = 1 - \tfrac{1}{2}(d\lambda)^2 + o\{\frac{(d\lambda)^4}{24}\}$.

Further

$$\tan(\phi_f - \phi) \simeq \tfrac{1}{2}(d\lambda)^2 \simeq 10^{-3}.$$

Thus

$$\tan(\phi_f - \phi) = (\phi_f - \phi) + o\{10^{-9}\} ,$$

where $R \times 10^{-9} <$ 1 cm and can be neglected, and

$$\phi_f - \phi = \tfrac{1}{2} \sin \phi \cos \phi (d\lambda)^2 \left\{1 - \frac{(d\lambda)^2}{12}\right\}\left\{\cos^2\phi\{1 - \tfrac{1}{2}(d\lambda)^2\} + \sin^2\phi\right\}^{-1}$$

$$= \tfrac{1}{2} \sin \phi \cos \phi (d\lambda)^2 \left\{1 - \frac{(d\lambda)^2}{12}\right\}\left\{1 + \frac{(d\lambda)^2}{2}\cos^2\phi\right\}$$

$$= \tfrac{1}{2} \sin \phi \cos \phi (d\lambda)^2 \left\{1 - \frac{(d\lambda)^2}{12}(1 - 6\cos^2\phi)\right\} + o\{\frac{(d\lambda)^6}{10^3}\}.$$

The use of equation 3.42 gives

$$N = R(\phi - \phi_o) + \tfrac{1}{2} R \sin \phi \cos \phi (d\lambda)^2 - \frac{1}{24} R \sin \phi \cos \phi (1 - 6 \cos^2\phi)(d\lambda)^4 + o\{R(d\lambda)^6 \times 10^{-3}\} \quad ...3.46.$$

If $\lambda \ngtr 3^o$; $\phi \ngtr 45^o$, the second and third terms in equation 3.46 will be of order 6×10^3 metres and 10^0 metres respectively at the most.

The expression for E

E is obtained by the use of equations 3.40 and 3.45 as

$$E = R\theta \quad ...... 3.47,$$

where

$$\sin \theta = \cos \phi \sin d\lambda.$$

As θ, $d\lambda$ are small angles, the trigonometrical function sine is unstable and it is convenient to replace it by the larger terms in the series expansion when terms of order $R(d\lambda)^6\times 10^{-3}$ are negligible for most practical purposes. As

$$\sin^{-1}x = x + \frac{1}{2}\frac{x^3}{3} + \frac{1.3}{2.4}\frac{x^5}{5} + o\{x^7\} \qquad \ldots\ldots \quad 3.48$$

and

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} + o\{x^7\} \qquad \ldots\ldots \quad 3.49,$$

$$\theta = \sin^{-1}(\cos\phi \sin d\lambda)$$

$$= \cos\phi \sin d\lambda + \frac{1}{6}\cos^3\phi \sin^3 d\lambda + \frac{3}{40}\cos^5\phi \sin^5 d\lambda.$$

As

$$\sin^3 d\lambda = (d\lambda)^3\left(1 - \frac{(d\lambda)^2}{6}\right)^3 = (d\lambda)^3\left(1 - \tfrac{1}{2}(d\lambda)^2\right) + o\{(d\lambda)^7\}$$

and

$$\sin^5 d\lambda = (d\lambda)^5 + o\{(d\lambda)^7\},$$

$$\theta = \cos\phi\left(d\lambda - \frac{1}{6}(d\lambda)^3 + \frac{1}{120}(d\lambda)^5\right) + \frac{1}{6}\cos^3\phi\,(d\lambda)^3\left(1 - \frac{(d\lambda)^2}{2}\right) +$$

$$\frac{3}{40}\cos^5\phi\,(d\lambda)^5$$

$$= \cos\phi\, d\lambda - \frac{1}{6}\cos\phi(1 - \cos^2\phi)(d\lambda)^3 + \frac{\cos\phi}{120}\left(1 - 10\cos^2\phi + 9\cos^4\phi\right)(d\lambda)^5.$$

As

$$1 - 10\cos^2\phi + 9\cos^4\phi = \sin^2\phi - 9\cos^2\phi(1 - \cos^2\phi)$$

$$= \sin^2\phi\cos^2\phi(\sec^2\phi - 9) = \sin^2\phi\cos^2\phi(\tan^2\phi - 8),$$

the use of equation 3.47 gives

$$E = R\cos\phi\, d\lambda - \frac{1}{6}R\cos\phi\sin^2\phi(d\lambda)^3 + \frac{1}{120}R\cos^3\phi\sin^2\phi(\tan^2\phi - 8)(d\lambda)^5 +$$

$$o\{R(d\lambda)^7\times 10^{-3}\} \ldots \quad 3.50.$$

The maximum order of magnitudes of terms in equation 3.50 for $d\lambda \not> 3^0$; $\phi \not> 45^0$ are 3×10^5, 10^2 and 10^{-1} metres respectively.

The expression for γ.

The application of equation 3.41 in triangle NPQ_f gives

$$\sin\phi = \tan\gamma \cot d\lambda .$$

Thus

$$\gamma = \tan^{-1}(\sin\phi \tan d\lambda).$$

As

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} + o\{x^7\} \qquad \ldots\ldots \quad 3.51$$

and

$$\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + o\{x^7\} \quad \ldots\ldots \quad 3.52,$$

$$\gamma = \sin\phi\left(d\lambda + \frac{(d\lambda)}{3} + \frac{2(d\lambda)}{15}\right) - \frac{1}{3}\sin^3\phi(d\lambda)^3\left(1+(d\lambda)^2\right) + \frac{\sin\phi}{5}(d\lambda)^5$$

$$= \sin\phi\, d\lambda + \frac{1}{3}\sin\phi(1-\sin^2\phi)(d\lambda)^3 + \frac{\sin\phi}{15}\left(2-5\sin^2\phi + 3\sin^4\phi\right)(d\lambda)^5 .$$

As

$$2 - 5\sin^2\phi + 3\sin^4\phi = 2\cos^2\phi - 3\sin^2\phi\cos^2\phi$$

$$= \cos^4\phi\,(2\sec^2\phi - 3\tan^2\phi) = \cos^4\phi(2 - \tan^2\phi),$$

$$\gamma = \sin\phi\, d\lambda + \frac{1}{3}\sin\phi\cos^2\phi(d\lambda)^3 + \frac{\sin\phi}{15}\cos^4\phi\,(2-\tan^2\phi)(d\lambda)^5 + o\{(d\lambda)^7\} \quad \ldots\ldots \quad 3.53.$$

The maximum order of magnitude of terms in equation 3.53 for $d\lambda \not> 3^o$; $\phi \not> 45^o$ are 3×10^{-2}, 10^{-5} and 10^{-7} radians respectively.

3.6.3 The conformal case

Alternative titles:-

i. The transverse Mercator projection;

ii. The Gauss conformal projection.

The ellipsoidal version has been given the titles

i. The Gauss Schreiber projection (no longer used);

and ii. The Gauss Kruger projection.

The title "transverse Mercator" is currently used to describe the ellipsoidal case.

As explained in section 3.6.1, the property of conformality is introduced by varying the distance between the small circles parallel to the circle of common tangency, as in the case of the Mercator projection described in section 3.5.3.

Consider the general elemental triangle (PQS in figure 3.13) on the sphere, where P and S are the same distance from the central meridian, measured along the great circle arcs $ZQSQ_f$ and ZPP_f, orthogonal to the central meridian. The N co-ordinates remain unchanged on introducing the conformal property by varying the spacing between the curves E = constant. Thus

$$P^*_fQ^*_f = P_fQ_f = P^*S^* = dN = N_Q - N_P ,$$

where N has exactly the same meaning as in section 3.6.2. Further,

$$SQ = dE = E_Q - E_P,$$

where E too, has the same significance as in section 3.6.2. The projection is made conformal by expanding the length S*Q* on the projection in relation to the length SQ on the sphere in order that the triangle P*Q*S* on the projection is similar to triangle PQS on the sphere. Differentiating between the east co-ordinate E on the Cassini-Soldner projection, which is the spherical great circle arc length, and that on the new conformal projection (E_t), then $S^*Q^* = dE_t$. The triangles P*Q*S* and PQS are similar if

$$\frac{PS}{P^*S^*} = \frac{SQ}{S^*Q^*} ,$$

or

$$dE_t = \frac{P^*S^*}{PS}\, dE$$

The ratio P*S*/PS is obtained by noting that P*S* and P_fQ_f are of the same length (= dN). By analogy with the pole-equator-parallel system, it can be seen that

$$PZ = \frac{\pi}{2} - \frac{E}{R} \quad ; \qquad P\hat{Z}Q = \frac{dN}{R}$$

and

$$PS = R\left(\frac{dN}{R} \cos\frac{E}{R}\right) + o\{5\text{-}6 \text{ mm}\}$$

or

$$\frac{P^*S^*}{PS} = \sec\frac{E}{R} = 1 + \frac{E^2}{2R^2} + \frac{5}{24}\left(\frac{E}{R}\right)^4 \qquad \ldots\ldots \quad 3.55.$$

Also,

$$\frac{PS}{P^*S^*} = \frac{PS}{dN} = \cos\frac{E}{R} = 1 - \frac{1}{2}\left(\frac{E}{R}\right)^2 + \frac{1}{24}\left(\frac{E}{R}\right)^4.$$

Substitution in equation 3.54 gives

$$dE_t = \left[1 + \frac{1}{2}\left(\frac{E}{R}\right)^2 + \frac{5}{24}\left(\frac{E}{R}\right)^4\right] dE.$$

Integration with respect to E gives

$$E_t = E\left[1 + \frac{1}{6}\left(\frac{E}{R}\right)^2 + \frac{1}{24}\left(\frac{E}{R}\right)^4\right]$$

$$\overset{(3.50)}{=} R\cos\phi\, d\lambda - \frac{1}{6} R\cos\phi \sin^2\phi (d\lambda)^3 + \frac{R\sin^2\phi}{120}\cos^3\phi\,(\tan^2\phi - 8)(d\lambda)^5 + \frac{1}{6} R\cos^3\phi (d\lambda)^3 \left(1 - \tfrac{1}{2}\sin^2\phi (d\lambda)^2\right) + \frac{R}{24}\cos^5\phi (d\lambda)^5.$$

On re-arrangement,

$$E_t = R\cos\phi\, d\lambda + \frac{1}{6} R\cos^3\phi\, \{1-\tan^2\phi\}(d\lambda)^3 + \frac{R\cos^5\phi}{120}\left(\tan^2\phi(\tan^2\phi - 8) - 10\tan^2\phi + 5\right)(d\lambda)^5 .$$

Thus

$$E_t = R\cos\phi\, d\lambda + \frac{R}{6}\cos^3\phi\,(1-\tan^2\phi)(d\lambda)^3 + \frac{R}{120}\cos^5\phi\,(\tan^4\phi - 18\tan^2\phi + 5)(d\lambda)^5 + o\{R(d\lambda)^7 \times 10^{-3}\}. \quad 3.57.$$

Also, $N_t (= N)$ is given by equation 3.46 as

$$N_t = R(\phi - \phi_o) + \frac{R}{2}\sin\phi\cos\phi(d\lambda)^2 - \frac{R}{24}\sin\phi\cos^3\phi(\tan^2\phi - 5)(d\lambda)^4 + o\{R(d\lambda)^6 \times 10^{-3}\} \ldots\ldots \quad 3.46.$$

As $\gamma_t = \gamma$ from the properties of the projection, the former is given by equation 3.53 as

$$\gamma_t = \sin\phi\, d\lambda + \frac{1}{3}\sin\phi\cos^2\phi(d\lambda)^3 + \frac{\sin\phi}{15}\cos^4\phi(2-\tan^2\phi)(d\lambda)^5 + o\{(d\lambda)^7\} \quad \ldots\ldots\ 3.53.$$

The above equations completely define the projection co-ordinates on the transverse Mercator projection and the direction of the grid meridian in relation to the true meridian, provided the *geographical* co-ordinates of the point and the origin are available.

The point scale factor k on this conformal projection is given by

$$k = \frac{dE_t}{dE} \overset{(3.55)}{=} 1 + \frac{1}{2}\left(\frac{E}{R}\right)^2 + o\left\{\left(\frac{E}{R}\right)^4\right\} \quad \ldots\ldots\ 3.58.$$

Notes:-

1. The scale on the Cassini - Soldner projection along the E axis (k_E) and the N axis (k_N) by the relations

$$k_E = 1 \quad ; \quad k_N = \frac{dN}{PS} \overset{(3.55)}{=} 1 + \frac{1}{2}\left(\frac{E}{R}\right)^2 + o\left\{\left(\frac{E}{R}\right)^4\right\}.$$

Thus the transverse Mercator projection is merely the Cassini-Soldner, converted to conformality by increasing the scale along the E axis.

2. The plot of the scale error e_m, given by

$$e_m = k - 1$$

for the conformal case, against E , will be parabolic, as shown in curve i in figure 3.14. e_m will take a maximum value of 0.0011 at E = 300 km (= E_{max}).

The introduction of the concept of a central scale factor k_o, as defined in section 3.5.1.*2*, which reduces the scale factor *uniformly* over the entire projection, can be used to reduce the magnitude of the scale error on the final representation. In this case, the uniform reduction of the scale error by the constant factor

$$\frac{1}{2}\left(\frac{E^2_{max}}{2R^2}\right),$$

where $|E_{max}|$ is the largest value of E as a consequence of the zone width adopted. If

$$k_o - 1 = \frac{1}{2}\left(\frac{E^2_{max}}{2\ R^2}\right) = 0.0005 \quad \ldots\ldots\ 3.59,$$

then

$$k_o = 0.9995$$

and the scale error is reduced as shown in curve ii in figure 3.14. This has the effect of replacing the irrelevant characteristic of having only positive scale errors by that of both positive and negative ones, but with reduced magnitudes. This is equivalent to computing on a sphere of radius k_oR.

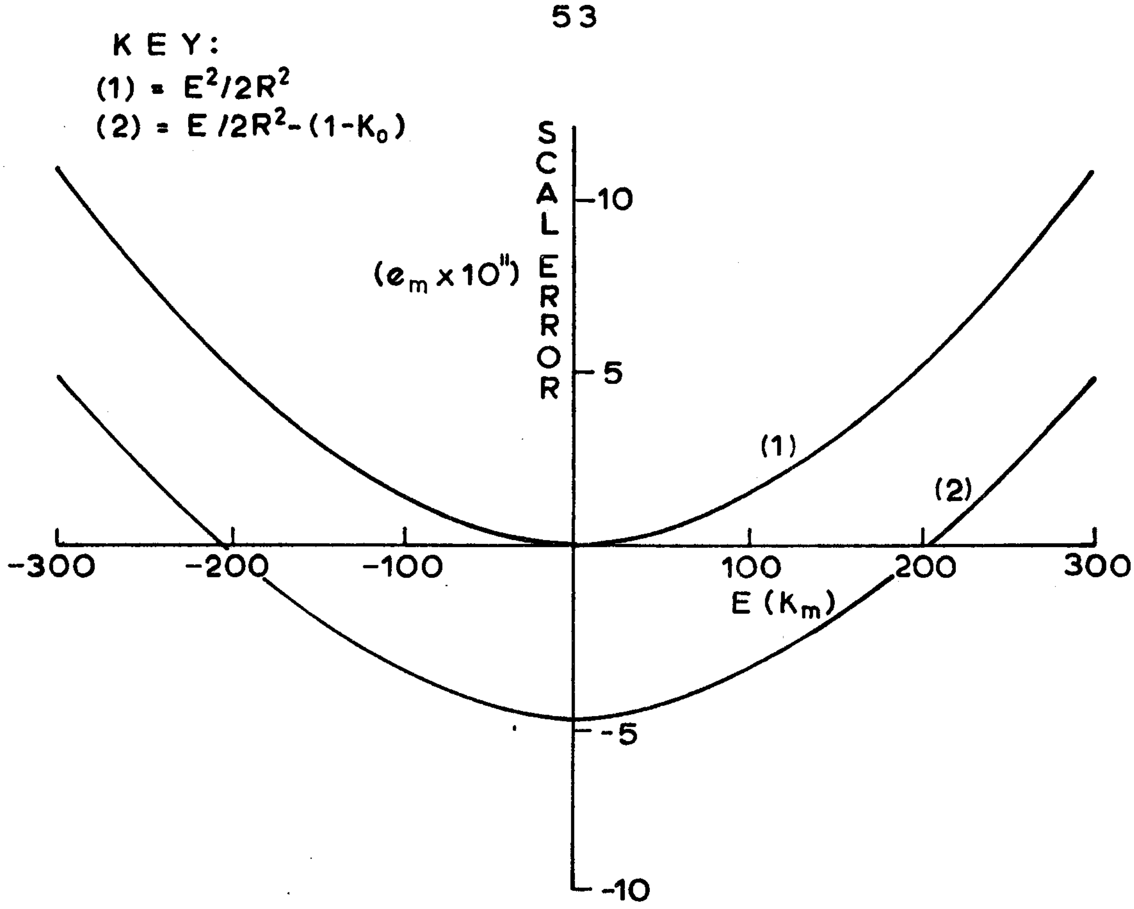


FIG. (3·14)

The accumulation of scale error (e_m) on the transverse Mercator Projection

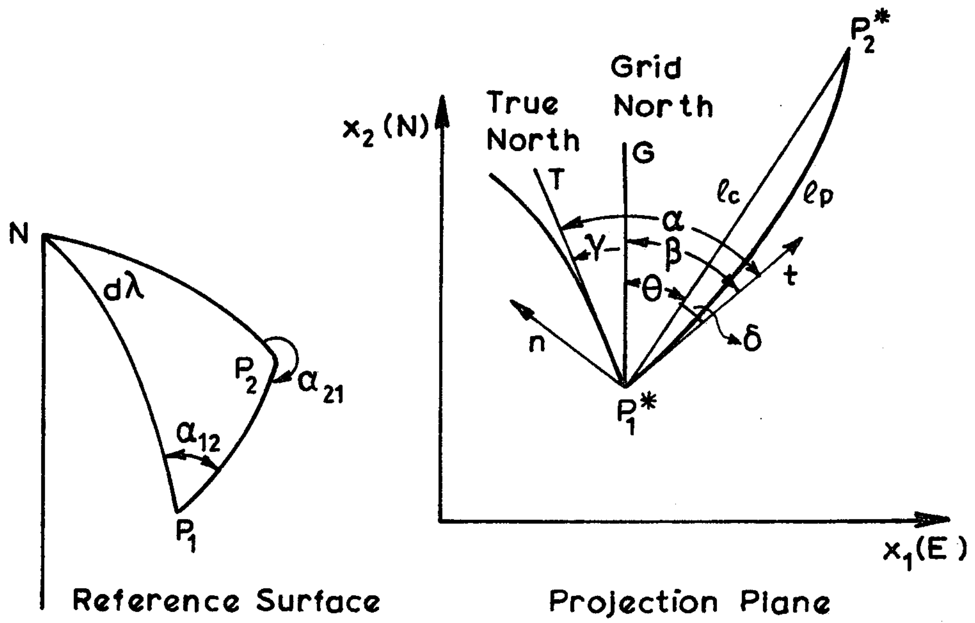


FIG. (4·1)

Relation between Projected curve and the chord

3. *The Universal Transverse Mercator (UTM) system of co-ordinates*

The Universal Transverse Mercator (UTM) system of co-ordinates has been adopted on a global basis as a means of providing a unique conformal representation of the Earth for day to day mapping and cadastral purposes. The entire globe, with the exception of the polar regions beyond latitudes $+84^{\circ}$N and -80°N, is divided into 6° zones, whose central meridian (λ_o°) is related to the *zone number* N by the relation

$$\lambda_o^{\circ} = -177 + (N-1)\times 6 \quad \ldots\ldots \quad 3.60.$$

The UTM zone number of any local zone can be obtained from the following equation as the truncated portion of given by

$$N = \frac{180 + \lambda^{\circ}}{6} + 1 \quad \ldots\ldots \quad 3.61,$$

where λ° is the local longitude in degrees. Thus the Australian continental region is entirely covered by zones 49 to 56. Continuity between zones is maintained by having half degree overlaps between adjacent zones.

The *origin* for each zone is the intersection of the central meridian and the equator.

UTM co-ordinates (N_{ut}, E_{ut}), which are in metric units,

a. are based on a central scale factor of

$$k_o = 0.9996$$

(AMG 1969, section 2.1);

and b. have the dubious characteristic of being referred to a *false origin* such that

$$N_{ut}^{(met)} = N_t^{(met)} + \begin{cases} 0 & \text{if } \phi \geq 0 \\ 10{,}000{,}000 & \text{if } \phi < 0 \end{cases}$$

and $\quad \ldots\ldots \quad 3.62.$

$$E_{ut}^{(met)} = E_t^{(met)} + 500{,}000$$

It should be borne in mind that UTM co-ordinates, corrected for the false origin, are 4 parts in 10^4 smaller than true transverse Mercator co-ordinates. The regeneration of properties characteristic of the projection, should not be attempted till this has been allowed for. This is achieved by using

$k_o R$ for R and $k_o \ell$ for measured distances ℓ

in all calculations both to the reference surface from the projection plane and vice versa.

4. *Convergence from projection co-ordinates*

An expression for the convergence from projection co-ordinates is of considerable importance. Reference to figure 3.10 and the equation for circular parts at 3.41, gives

$$\sin\theta = \tan\gamma \cot\phi_f.$$

$$\gamma = \tan^{-1}(\sin\theta \tan\phi_f)$$

$$\overset{(3.51)}{\doteq} \tan\phi_f \sin\theta - \frac{1}{3}\tan^3\phi_f \sin^3\theta + \frac{1}{5}\tan^5\phi_f \sin^5\theta.$$

From equations 3.47 and 3.56,

$$\theta = \frac{E}{R} \quad ,$$

where E is the rectangular spherical co-ordinate. Then

$$\theta \overset{(3.56)}{=} \frac{E_t}{R}\left[1 + \frac{1}{6}\left(\frac{E}{R}\right)^2 + \frac{1}{24}\left(\frac{E}{R}\right)^4\right]^{-1} = \frac{E_t}{R}\left[1 - \frac{1}{6}\left(\frac{E}{R}\right)^2 - \frac{1}{72}\left(\frac{E}{R}\right)^4\right].$$

To a first approximation,

$$\frac{1}{6}\left(\frac{E}{R}\right)^2 = \frac{1}{6}\frac{E_t^2}{R^2}\left[1 - \frac{1}{3}\left(\frac{E}{R}t\right)^2\right] .$$

Whence

$$\theta = \frac{E}{R}t\left[1 - \frac{1}{6}\left(\frac{E}{R}t\right)^2 + \left(\frac{1}{18} - \frac{1}{72}\right)\left(\frac{E}{R}t\right)^4\right]$$

$$= \frac{E}{R}t\left[1 - \frac{1}{6}\left(\frac{E}{R}t\right)^2 + \frac{1}{24}\left(\frac{E}{R}t\right)^4\right] \quad \ldots\ldots \quad 3.63.$$

Further,

$$\sin\theta = \theta - \frac{\theta^3}{6} + \frac{\theta^5}{120}$$

$$= \frac{E}{R}t - \frac{1}{6}\left(\frac{E}{R}t\right)^3 + \frac{1}{24}\left(\frac{E}{R}t\right)^5 - \frac{1}{6}\left(\frac{E}{R}t\right)^3\left[1 - \frac{1}{2}\left(\frac{E}{R}t\right)^2\right] + \frac{1}{120}\left(\frac{E}{R}t\right)^5$$

$$= \frac{E}{R}t\left[1 - \frac{1}{3}\left(\frac{E}{R}t\right)^2 + \frac{2}{15}\left(\frac{E}{R}t\right)^4\right].$$

Thus

$$\sin^3\theta = \left(\frac{E}{R}t\right)^3\left[1 - \left(\frac{E}{R}t\right)^2\right] \quad ; \quad \sin^5\theta = \left(\frac{E}{R}t\right)^5 .$$

Hence,

$$\gamma = \tan\phi_f\left(\frac{E}{R}t - \frac{1}{3}\left(\frac{E}{R}t\right)^3 + \frac{2}{15}\left(\frac{E}{R}t\right)^5\right) - \frac{1}{3}\tan^3\phi_f\left(\frac{E}{R}t\right)^3\left[1 - \left(\frac{E}{R}t\right)^2\right] + \frac{1}{5}\tan^5\phi_f\left(\frac{E}{R}t\right)^5$$

$$= \frac{E}{R}t\tan\phi_f\left[1 - \frac{1}{3}(1+\tan^2\phi_f)\left(\frac{E}{R}t\right)^2 + \frac{1}{15}(2+5\tan^2\phi_f+3\tan^4\phi_f)\left(\frac{E}{R}t\right)^4\right] + o\left\{\left(\frac{E}{R}t\right)^7\right\} \quad \ldots\ldots \quad 3.64.$$

Equation 3.64 enables the convergence γ to be computed at a point P on the transverse Mercator projection whose co-ordinates (N_t, E_t) are known.

The foot point latitude ϕ_f, which is the latitude of the point on the central meridian which has the same N_t co-ordinate as the point in question, is a known quantity once N_t is specified, as can be seen from figure 3.11, being given by equation 3.42 as

$$\phi_f = \frac{N_t}{R} + \phi_o \quad \ldots\ldots \quad 3.65.$$

4. THE RELATION OF OBSERVATIONS TO THE PROJECTION PLANE

4.1 Fundamental terms

a. *On the reference surface*

Measurements made in the field can be reduced and/or corrected to give the azimuth α_{12} and the length ℓ_{12} of the curve P_1P_2 on the reference surface.

The length ℓ_{12} is unambiguous for a sphere, being the great circle distance. The problem is more complex for the ellipsoid. A unique definition is afforded by the geodesic, described in appendix g, which, incidentally, is the great circle on a sphere. From a practical point of view, it is more meaningful to define ℓ_{12} in the normal section, described in appendix d. The lengths of the geodesic and the normal section can be considered to be equal for all practical purposes over normal geodetic distances which seldom exceed 125 km.

The azimuth α_{12} is the angle between the plane of the meridian at P and that containing the curve P_1P_2, if the latter is planar. In the case of a skew curve like the geodesic on the ellipsoid, the latter plane would be the osculating plane at P_1. The azimuth α_{12} is therefore unique only when the nature of the curve P_1P_2 is defined.

b. *On the projection plane*

Any curve on the reference surface can be plotted, element by element, on the projection plane, giving a *projected curve* with a variable curvature σ. The geodesic on the reference surface does not transform into a geodesic (straight line) on the projection plane due to variations in scale along the line.

Consider the projection of the meridian NP_1^* and the curve P_1P_2, as shown in figure 4.1. Let P_1^*t be the tangent to the projected curve at P_1^* and P_1^*n the associated normal. If P_1^*G is the grid meridian at P_1 and the tangent to the projected meridian is P_1^*T; the angle

$$T\hat{P}_1^*t = \alpha$$

would be equal to α_{12} if the projection were conformal.

The grid convergence γ is given by

$$\gamma = T\hat{P}_1^*G \quad .$$

The grid bearing β of the projected curve is given by

$$\beta = G\hat{P}_1^*t,$$

while the grid bearing θ of the chord, also called the *plane bearing of* P_1P_2, is given by

$$\theta = \widehat{GP_1^*P_2^*}.$$

The chord-to-arc correction δ is defined by the equation

$$\delta = \widehat{P_2^*P_1^*t} .$$

It is related to the plane bearing by the relation

$$\theta = \beta - \delta \qquad \ldots\ldots \quad 4.1.$$

Also,

$$\alpha = \gamma + \beta = \gamma + \theta + \delta \qquad \ldots\ldots \quad 4.2.$$

If the projection is conformal,

$$\alpha = \alpha_{12} = \gamma + \beta = \gamma + \theta + \delta \qquad \ldots\ldots \quad 4.3.$$

The curve P_1P_2 of length ℓ on the reference surface, projects as a curve of length ℓ_p. Let the chord $P_1^*P_2^*$ on the projection be of length ℓ_c. If the curvature of the projected curve is extremely small, i.e.,

$$\ell_p = \ell_c + o\{10^{-8}\},$$

these two lengths are, for all practical purposes, equal. The angular separation δ between the chord and the curve has still to be considered as it still has a significant magnitude in such cases.

The *line scale factor* k_ℓ

$$k_\ell = \frac{\ell_p}{\ell} \qquad \ldots\ldots \quad 4.4.$$

It should be noted that

$$\lim_{\ell \to 0} k_\ell = k$$

where k is the point scale factor in the case of conformal projections. Conversely, for most practical purposes,

$$k_\ell \doteq \frac{1}{\ell}\int_0^\ell k \, d\ell \qquad \ldots\ldots \quad 4.5.$$

Notes:-

1. *Plane rectangular co-ordinates*

A sequence of lengths ℓ_c and plane bearings θ which satisfy

$$\Delta N = \ell_c \cos\theta \quad ; \quad \Delta E = \ell_c \sin\theta \qquad \ldots\ldots \quad 4.6$$

conforms to conventional concepts when computing in plane rectangular co-ordinates. Such a system is afforded by ℓ_c and θ defined in section 4.1. The field observations ℓ and α_{12} can be made compatible with such a system through equations 4.1 to 4.5, if the convergence γ, the chord-to-arc correction δ and the line scale factor k_ℓ are known.

2. Sets of equations which would completely define the conversion of field measurements on the reference surface, to differences of projection co-ordinates Δx_i are as follows.

a. *Using the measured length ℓ and the grid bearing β of the projected curve*

$$\gamma = \gamma(\phi,\lambda)$$

$$\beta_{21} = \beta_{12} + \pi + \Delta\beta(\ell,\beta_{12}) \quad \ldots\ldots \quad 4.7,$$

$$\Delta x_i = \Delta x_i(\ell,\beta_{12}) \quad , \ i=1,2$$

where β_{12} is the grid bearing of the projected curve P_1P_2 and β_{21} is that of P_2P_1. As shown in figure 4.1 and the subsequent development,

$$x_1 = E \quad ; \quad x_2 = N \quad \ldots. \quad 4.8.$$

b. *Using plane bearings and distances*

$$\gamma = \gamma(\phi,\lambda)$$

$$\delta = \delta(x_{11},x_{12},\ell,\beta) \quad \ldots\ldots \quad 4.9a$$

$$k_\ell = k_\ell(x_{11},x_{12},\ell,\beta)$$

$$\theta = \beta - \delta$$

$$\ell_c = \ell + (k_\ell - 1)\ell \quad \ldots\ldots \quad 4.9b$$

$$\Delta x_i = \ell_c \cos\theta_i \quad , \ i=1,2,$$

where

$$\theta_1 = \theta \quad ; \quad \theta_2 = \tfrac{1}{2}\pi - \theta \quad \ldots. \quad 4.10.$$

3. As nearly all projections used for mapping purposes are conformal, it will be assumed in the subsequent development, that the scale at a point is independent of azimuth, as discussed in section 1.3.3 and can therefore be represented by the point scale factor k.

In addition, the projected curve will be assumed to be a geodesic on the reference surface as its characteristics on the latter are clearly defined.

4.2 The curvature of the projected geodesic

Consider the representation of a geodesic of length ℓ on the reference surface, on the projection plane. If $P_i(x_{ji}, j=1,2)$, $i=1,2$ are its terminals on the latter, the projected length being ℓ_p. Let P_1t be the tangent to the projected geodesic at P_1 with a grid bearing β_{12}.

Let the grid bearing of the general element of length ds at the point $P(x_1,x_2)$ on the projected geodesic which is a distance s from P_1, be β. If $Q(x_1+dx_1,x_2+dx_2)$ is the other terminal of the element of length,

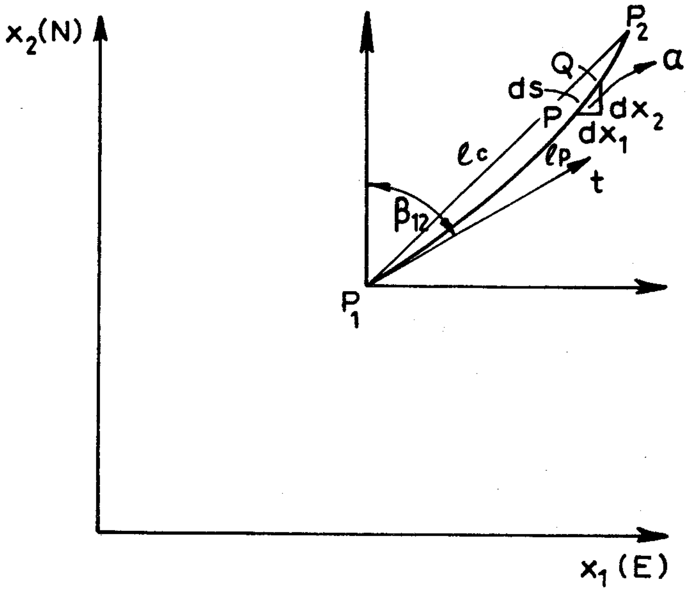


FIG. (4·2)

The Projected Geodesic

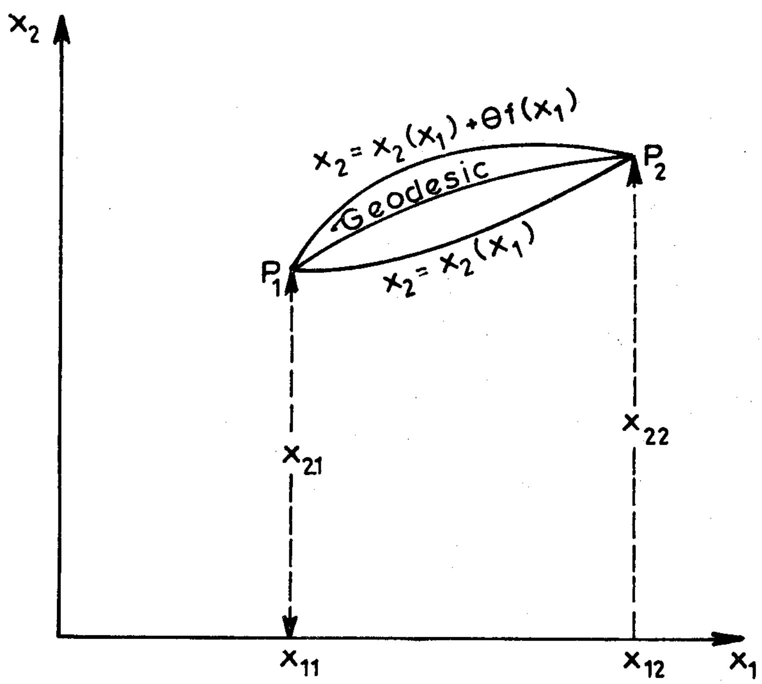


FIG. (4·3)

Relation of Geodesic to other curves

$$\tan \beta = \frac{dx_1}{dx_2} \quad ; \qquad \cos \beta = \frac{dx_2}{ds} \quad ; \qquad \sin \beta = \frac{dx_1}{ds} \quad \ldots(4.11).$$

The curvature σ of the projected geodesic at P is given by

$$\sigma = -\frac{d\beta}{ds} \qquad \ldots\ldots \qquad 4.12.$$

If $d\ell$ is the corresponding element of length on the reference surface, the consideration of equation 1.13 gives

$$d\ell = \frac{1}{k}\,ds = \frac{1}{k}\left((dx_1)^2 + (dx_2)^2\right)^{\frac{1}{2}}$$

$$= \frac{1}{k}\left(1 + \left(\frac{dx_2}{dx_1}\right)^2\right)^{\frac{1}{2}} dx_1.$$

Therefore

$$\ell = \int_{x_{11}}^{x_{12}} \mathrm{l}\; dx_1 \qquad \ldots\ldots \qquad 4.13,$$

where

$$\mathrm{l} = \frac{1}{k}\left(1 + (x_2^1)\right)^{\frac{1}{2}} = \mathrm{l}(x_1, x_2, x_2^1) \ldots\ldots \qquad 4.14,$$

as

$$k = k(x_1, x_2).$$

The quantity x_2^1 in equation 4.14 is defined by the relation

$$x_2^1 = \frac{dx_2}{dx_1}.$$

Equation 4.13 will apply to any curve on the reference surface after projection. The geodesic has the characteristic of being the shortest distance between the terminals on the reference surface. In such a case, the appropriate expression for l must give a minimum value for ℓ. Assuming that l has continuous partial derivatives with respect to x_1, x_2 and x_2^1, l can be interpreted as being a function of the *general element of length along any of the infinite number of projected curves possible between P_1 and P_2.*

This family of curves, as shown in figure 4.3, can be defined by equations of the form

$$x_2 = x_2(x_1).$$

If a curve in the vicinity of the geodesic on the projection plane can be defined by an equation of this form, any adjacent curve can be represented by an equation of the form

$$x_{2o} = x_2 + dx_2$$
$$= x_2(x_1) + \theta\, f(x_1) \qquad \ldots\ldots \qquad 4.15,$$

where θ is a parameter; different values of θ defining the family of curves between P_1 and P_2 in the neighbourhood of the geodesic. ℓ will take a maximum or minimum value when

$$\frac{d\ell}{d\theta} \overset{(4.13)}{=} \frac{d}{d\theta}\left(\int_{x_{11}}^{x_{12}} l(x_1, x_2, x_2^1)\, dx_1\right) = 0 \;\ldots. \qquad 4.16.$$

On changing the variables for the region under investigation using equation 4.15, the above equation becomes

$$\frac{d}{d\theta}\left(\int_{x_{11}}^{x_{12}} l\left[x_1,\; x_2+\theta f(x_1),\; \frac{dx_2}{dx_1} + \theta\frac{df}{dx_1}\right] dx_1\right.$$

$$= \int_{x_{11}}^{x_{12}} \left(\frac{\partial l}{\partial x_2^1}\frac{df}{dx_1} + \frac{\partial l}{\partial x_2} f(x_1)\right) dx_1.$$

The integration of the first term by parts gives

$$\int_{x_{11}}^{x_{12}} \left(\frac{\partial l}{\partial x_2^1}\frac{df}{dx_1} + \frac{\partial l}{\partial x_2} f(x_1)\right) dx_1 = \left[f(x_1)\frac{\partial l}{\partial x_2^1}\right]_{x_{11}}^{x_{12}} +$$

$$\int_{x_{11}}^{x_{12}} f(x_1)\left(\frac{\partial l}{\partial x_2} - \frac{d}{dx_1}\left(\frac{\partial l}{\partial x_2^1}\right)\right) dx_1 = 0.$$

The nature of the function $\partial l/\partial x_2^1$ is such that its value is zero at both P_1 and P_2, all the curves being coincident at these points (e.g., Jeffreys & Jeffreys 1962,p.315). Thus the integral in equation 4.13 is stationary if the equation

$$\frac{\partial l}{\partial x_2} - \frac{d}{dx_1}\left(\frac{\partial l}{\partial x_2^1}\right) = 0. \qquad \ldots\ldots \qquad 4.17$$

is satisfied. This condition must be satisfied by all projected geodesics. Referring back to equations 1.14, 1.15 and 4.14, it can be seen that k is a function of x_1 and x_2. As

$$l(x_1, x_2, x_2^1) = \frac{1}{k(x_1, x_2)}\left(1 + (x_2^1)^2\right)^{\frac{1}{2}},$$

the use of equation 4.11 gives

$$\frac{\partial l}{\partial x_2} = -\frac{1}{k^2}\frac{\partial k}{\partial x_2}\; \text{cosec}\,\beta \qquad \ldots\ldots \qquad 4.18,$$

$$\frac{\partial l}{\partial x_2^1} = \frac{1}{k}\frac{x_2^1}{\left(1 + (x_2^1)^2\right)^{\frac{1}{2}}} = \frac{1}{k}\frac{\cot\beta}{\text{cosec}\,\beta} = \frac{1}{k}\cos\beta$$

and

$$\frac{d}{dx_1}\left(\frac{\partial l}{\partial x_2}\right) = -\frac{1}{k^2}\left(\frac{\partial k}{\partial x_1} + \frac{\partial k}{\partial x_2}\cot\beta\right)\cos\beta - \frac{1}{k}\sin\beta\frac{d\beta}{ds}\operatorname{cosec}\beta \qquad \ldots\ldots \quad 4.19.$$

The substitution of equations 4.18 and 4.19 in equation 4.17, gives

$$\sigma = -\frac{d\beta}{ds} = -\frac{1}{k}\left(\frac{\partial k}{\partial x_2}\operatorname{cosec}\beta(1-\cos^2\beta) - \frac{\partial k}{\partial x_1}\cos\beta\right).$$

Hence the curvature of the projected geodesic can be expressed by the equation

$$\sigma = \frac{1}{k}\left(\frac{\partial k}{\partial x_1}\cos\beta - \frac{\partial k}{\partial x_2}\sin\beta\right) \qquad \ldots\ldots \quad 4.20.$$

Notes:-

1. It can be shown that equation 4.20 expresses the curvature as the derivative of the scale factor in a direction normal to the curve (Thomas 1952,p.78).

2. In the case of the transverse Mercator projection of a sphere, k is given by equation 3.58. 1/k can be obtained from equation 3.55. Thus, as

$$k = 1 + \frac{1}{2}\left(\frac{E}{R}\right)^2 + o\left\{\left(\frac{E}{R}\right)^4\right\},$$

$$\frac{\partial k}{\partial x_1} = \frac{\partial k}{\partial E} = \frac{1}{R}\left(\frac{E}{R} + o\left\{\left(\frac{E}{R}\right)^3\right\}\right) \quad \text{and} \quad \frac{\partial k}{\partial x_2} = \frac{\partial k}{\partial N} = 0.$$

Therefore

$$\sigma = \left(1 - \frac{1}{2}\left(\frac{E}{R}\right)^2 + o\left\{\left(\frac{E}{R}\right)^4\right\}\right)\frac{E}{R^2}\left(1 + o\left\{\left(\frac{E}{R}\right)^2\right\}\right)\cos\beta$$

or

$$\sigma = \frac{E}{R^2}\left(1 + o\left\{\left(\frac{E}{R}\right)^2\right\}\right)\cos\beta \qquad \ldots\ldots \quad 4.21.$$

The evaluation of terms in the case of six degree zone widths (i.e., $E/R \not> 5\times10^{-2}$) indicates that the curvature of the projected geodesic is of order 10^{-5} km^{-1}.

3. The order of magnitude of the difference $(\ell_p - \ell_c)$, as illustrated in figure 4.1, can be obtained from this estimate of σ. If the projected geodesic is assumed to be circular arc of curvature σ, the difference in length between the arc of length ℓ_p and the chord ℓ_c is given by the standard arc-to-chord correction

$$\ell_p - \ell_c = \frac{\ell_p^3}{24}\sigma^2.$$

On allowing a maximum length of 100 km for normal geodetic lines,

$$\frac{\ell_p - \ell_c}{\ell_c} \simeq 4\times10^{-8} \doteqdot 4 \text{ mm} \quad \ldots\ldots \quad 4.22$$

and can be neglected.

4.3 The line scale factor on a conformal projection

Also referred to as the finite distance scale (e.g., Bomford 1962,p.168).

Its difference from unity, on multiplication by the length of the line is commonly titled the (s-S) correction.

While the point scale factor k is independent of azimuth at a point on a conformal projection, the line scale factor k_ℓ, given by the equation 4.4 as

$$k_\ell = \frac{\ell_p}{\ell}$$

is dependent on the location on both terminals of the line being projected. Considering an element of length $d\ell_p$ on the projection, assumed conformal, the point scale factor is given by

$$k = \frac{d\ell_p}{d\ell} ,$$

where $d\ell$ is the element of length on the reference surface which is being projected. As k is position dependent, its magnitude will vary with length s along the line whose length on the projection is ℓ_p. Its value at any point can be expressed by the Maclaurin's series

$$k = k_o + \sum_{i=1}^{n} \frac{\ell^i}{i!} \left(\frac{d^i k}{d\ell^i}\right)_{\ell=0} ,$$

where k_o is the value of k at $\ell = 0$. This is a convenient form in that

$$\ell_p = \int_0^\ell k \, d\ell .$$

This facility is offset by the fact that k is a property of the projection plane and best expressed as derivatives with respect to the length s of the projected geodesic, the increment ds being equivalent to $d\ell_p$. Thus

$$\left(\frac{dk}{d\ell}\right)_{\ell=0} = \left(\frac{dk}{d\ell_p}\right)_{\ell_p=0} \left(\frac{d\ell_p}{d\ell}\right)_{\ell_p=0} \overset{(1.13)}{=} k_o^1 \, k_o$$

and

$$\left(\frac{d^2k}{d\ell^2}\right)_{\ell=0} = \left\{\left(\frac{d^2k}{d\ell_p^2}\right)\left(\frac{d\ell_p}{d\ell}\right)^2 + \left(\frac{dk}{d\ell_p}\right)^2 \frac{d\ell_p}{d\ell}\right\}_{\ell_p=0} = k_o^2 (k_o)^2 + (k_o^1)^2 \, k_o .$$

Further differentiation gives

$$\left(\frac{d^3k}{d\ell^3}\right)_{\ell=0} = \left[\frac{d^3k}{d\ell_p^3} \left(\frac{d\ell}{d\ell}p\right)^3 + 2\frac{d^2k}{d\ell_p^2}\left(\frac{d\ell}{d\ell}p\right)^2 \frac{dk}{d\ell_p} + 2\frac{dk}{d\ell_p}\frac{d^2k}{d\ell_p^2}\left(\frac{d\ell}{d\ell}p\right)^2 + \left(\frac{dk}{d\ell_p}\right)^3 \frac{d\ell}{d\ell}p\right]_{\ell_p=0}$$

$$= k_o^3(k_o)^3 + 4\, k_o^2\, k_o^1 (k_o)^2 + (k_o^1)^3\, k_o$$

$$\left(\frac{d^4k}{d\ell^4}\right)_{\ell=0} = \frac{d^4k}{d\ell_p^4}\left(\frac{d\ell}{d\ell}p\right)^4 + 3\frac{d^3k}{d\ell_p^3}\frac{dk}{d\ell_p}\left(\frac{d\ell}{d\ell}p\right)^3 + 4\frac{d^3k}{d\ell_p^3}\frac{dk}{d\ell_p}\left(\frac{d\ell}{d\ell}p\right)^3 +$$

$$4\left(\frac{d^2k}{d\ell_p^2}\right)^2\left(\frac{d\ell}{d\ell}p\right)^3 + 8\frac{d^2k}{d\ell_p^2}\left(\frac{dk}{d\ell_p}\right)^2\left(\frac{d\ell}{d\ell}p\right)^2 + 3\frac{d^2k}{d\ell_p^2}\left(\frac{dk}{d\ell_p}\right)^2\left(\frac{d\ell}{d\ell}p\right)^2 +$$

$$\left(\frac{dk}{d\ell_p}\right)^4 \frac{d\ell}{d\ell}p \Bigg]_{\ell_p=0}$$

$$= k_o^4(k_o)^4 + 7k_o^3\, k_o^1(k_o)^3 + 4(k_o^2)^2(k_o)^3 + 11k_o^2(k_o^1)^2(k_o)^2 + (k_o^1)^4\, k_o.$$

The combination of the above six equations gives

$$\ell_p = \int_0^\ell \left[k_o + k_o^1\, k_o \ell + \frac{1}{2}\left(k_o^2\,(k_o)^2 + (k_o^1)^2 k_o\right)\ell^2 + \frac{1}{6}K_1\ell^3 + \right.$$

$$\left. + \frac{1}{24}K_2\ell^4 \right] d\ell,$$

where

$$K_1 = k_o^3(k_o)^3 + 4k_o^2\, k_o^1(k_o)^2 + (k_o^1)^3 k_o$$

and

$$K_2 = k_o^4(k_o)^4 + 7k_o^3\, k_o^1(k_o)^3 + 4(k_o^2)^2(k_o)^3 + 11k_o^2(k_o^1)^2(k_o)^2 + (k_o^1)^4 k_o.$$

Integration gives

$$\ell_p = k_o\ell + \frac{1}{2}k_o^1\, k_o\ell^2 + \frac{1}{6}\left(k_o^2(k_o)^2 + (k_o^1)^2 k_o\right)\ell^3 + \frac{1}{24}K_1\,\ell^4 +$$

$$+ \frac{1}{120}K_2\ell^5 + \ldots \qquad 4.23.$$

Thus

$$k_\ell = \frac{\ell_p}{\ell} = k_o + \frac{1}{2}k_o^1 k_o\ell + \frac{1}{6}\left(k_o^2(k_o)^2 + (k_o^1)^2 k_o\right)\ell^2 + \frac{1}{24}K_1\,\ell^3 +$$

$$+ \frac{1}{120}K_2\,\ell^4 + \ldots \qquad 4.24.$$

Also,

$$\ell_p - \ell = (k_o - 1)\ell + \frac{1}{2} k_o^1 k_o \ell^2 + \frac{1}{6}\left(k_o^2 (k_o)^2 + (k_o^1)^2 k_o\right)\ell^3 + \frac{1}{24} K_1 \ell^4 + \frac{1}{120} K_2 \ell^5 \quad \ldots . \quad 4.25.$$

Notes :-

1. For the transverse Mercator projection of a sphere, k_o is given by

$$k_o \overset{(3.58)}{=} 1 + \frac{1}{2}\left(\frac{E}{R}\right)^2 + o\{\left(\frac{E}{R}\right)^4\} .$$

The order of magnitude of its derivatives with respect to s(i.e., ℓ_p) are defined by the following relations.

$$\left(\frac{dk}{d\ell_p}\right)_{\ell_p=0} = \left(\frac{\partial k}{\partial E}\ \frac{dE}{d\ell_p}\right)_{\ell_p=0} = \frac{E_1}{R^2}\left(1 + o\{\left(\frac{E}{R}\right)^2\}\right)\sin\beta.$$

$$\left(\frac{d^2k}{d\ell_p^2}\right)_{\ell_p=0} = \left[\frac{\partial}{\partial E}\left(\frac{dk}{d\ell_p}\right)\frac{dE}{d\ell_p} + \frac{\partial}{\partial\beta}\left(\frac{dk}{d\ell_p}\right)\frac{d\beta}{d\ell_p}\right]_{\ell_p=0},$$

which from equations4.11 and 4.21 becomes

$$\left(\frac{d^2k}{d\ell_p^2}\right)_{\ell_p=0} = \frac{1}{R^2}\left(1 + o\{\left(\frac{E}{R}\right)^2\}\right)\sin^2\beta - \frac{E_1^2}{R^4}\cos^2\beta = \frac{\sin^2\beta}{R^2}\left(1 + o\{\left(\frac{E}{R}\right)^2\}\right).$$

Similarly,

$$\left(\frac{d^3k}{d\ell_p}\right)_{\ell_p=0} \simeq \frac{E}{R^4} \quad ; \quad \left(\frac{d^4k}{d\ell_p}\right)_{\ell_p=0} \simeq \frac{1}{R^4}.$$

The angle β in the above relations refers to the grid bearing of the projected geodesic at the initial terminal of the line.

For 100 km lines,

$$k_o - 1 \simeq 10^{-3} \quad ; \quad \frac{1}{2}k_o^1\ k_o \simeq 5\times10^{-4} \quad ; \quad \frac{1}{6}(k_o^1)^2\ell^2 \simeq 10^{-7};$$

the only other terms of magnitude 10^{-8} or larger (i.e., $(\ell/R)^6$ or greater) are

$$k_o^2\ \ k_o^1(k_o)^2 \simeq \frac{E_1\ell^3}{R^4} \quad ; \quad (k_o^2)^3\,(k_o)^3 \simeq \frac{\ell^4}{R^4}.$$

On ignoring terms smaller than 10^{-6} in the case of a <u>spherical</u> approximation of the Earth, the use of the above relations in equation 4.2.4 gives

$$k_\ell = 1 + \frac{1}{2}\left(\frac{E_1}{R}\right)^2 + \frac{1}{2}\left(1 + \frac{1}{2}\left(\frac{E_1}{R}\right)^2\right)\frac{E_1\ \ell\sin\beta}{R^2} + \frac{\ell^2\sin^2\beta}{6R^2} + o\{5\times10^{-7}\}$$

$$= 1 + \frac{3E_1^2 + 3E_1\,\ell\sin\beta + \ell^2\sin^2\beta}{6R^2}$$

$$= 1 + \frac{E_1 + E_1(E_1 + \ell\sin\beta) + (E_1 + \ell\sin\beta)^2}{6R^2} .$$

Thus

$$k_\ell = 1 + \frac{1}{6R^2}\left(E_1^2 + E_1E_2 + E_2^2 \right).$$

The correction $(k_\ell - 1)\ell$, commonly called the (s-S) correction, is given by

$$(k_\ell - 1)\ell = (s - S) = \frac{\ell}{6R^2}\left(E_1^2 + E_1E_2 + E_2^2 \right) \quad \ldots\ldots 4.26.$$

A final working formula on the transverse Mercator with six degree zones, for the line scale factor, and correct to $(\ell/R)^4$, is

$$k_\ell = k_o + \frac{1}{2}k_o^1k_o\ell + \frac{1}{6}\left(k_o^2(k_o)^2 + (k_o^1)^2\right)\ell^2 + \frac{1}{24}\left(k_o^3 + 4k_o^2\,k_o^1\right)\ell^3 + \frac{1}{120}\left(k_o^4 + 4(k_o^2)^2\right)\ell^4 + o\left\{\frac{E^4\ell^2}{R^6}\right\}$$

$$\ldots\ldots \quad 4.27.$$

2. It should be noted that the quantity E in the above equations, refers to the true distance along the orthogonal great circle system (i.e., Cassini-Soldner co-ordinates) and not T.M. co-ordinates. For all practical purposes,

$$E_t = E + o\{E\times4\times10^{-4}\} \quad ;$$

$$\frac{1}{6}\left(\frac{E_t}{R}\right)^2 = \frac{1}{6}\left(\frac{E}{R}\right)^2 + o\left\{\frac{4}{3}\left(\frac{E}{R}\right)^2\times10^{-4}(\simeq10^{-7})\right\} \quad \ldots\ldots \quad 4.28.$$

Thus no significant error is obtained on using transverse Mercator co-ordinates in lieu of E when evaluating equation 4.26.

4.4 The chord to arc correction

Also commonly known as

i. the arc to chord correction

and ii. the (t-T) correction

but with reversed signs.

The properties of smooth curves with limited length, can be studied with relative ease by considering their departure from the tangent P_1t at the initial terminal P_1 as a function of the length s along the projected curve, to the variable point P on it. If s is measured from P_1 along P_1P_2, consider the element of length ds of the curve at P and its relation to the tangent (t) - normal (n) system of rectangular co-ordinates with origin at P_1. The angle ξ between the element of length ds and the associated changes dt in t and dn in n, are related by the equations

$$\tan\xi = \frac{dn}{dt} = \frac{dn}{ds}\,\frac{ds}{dt} = \frac{n^1}{t^1}\,.$$

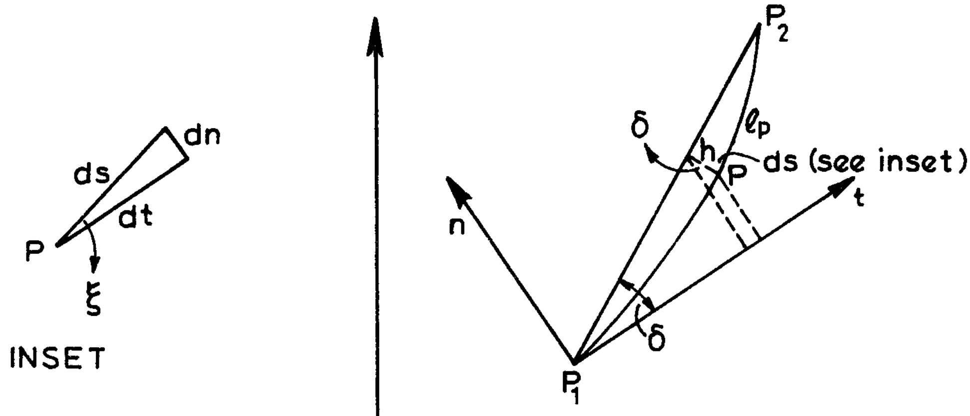


FIG.(4·4)

The pojected geodesic on the tangent normal system

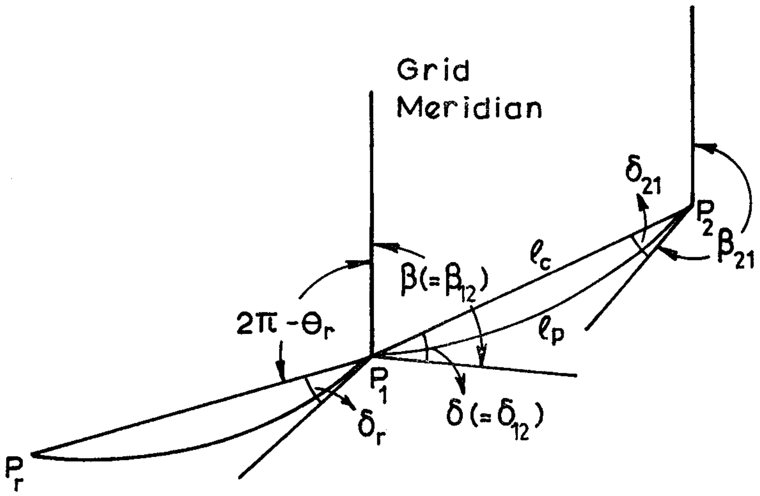


FIG.(4·5)

Sequential computations on the projection plane

The basic relations are illustrated in figure 4.4. The curvature

$$\sigma = \frac{d\xi}{ds}$$

is obtained by differentiating the above relation with respect to s, when

$$\sec^2\xi \frac{d\xi}{ds} = \sec^2\xi\ \sigma = -\frac{n^1}{(t^1)^2}\ t^2 + \frac{n^2}{t^1} = \frac{n^2t^1 - n^1t^2}{(t^1)^2}.$$

As

$$\cos^2\xi = (t^1)^2;$$

$$\sigma = n^2\ t^1 - n^1\ t^2 \qquad \ldots\ldots \qquad 4.29$$

and

$$(n^1)^2 + (t^1)^2 = 1 \qquad \ldots\ldots \qquad 4.30$$

(the notation is explained in the Guide to Notation).

The t and n co-ordinates can be expressed as a Maclaurin's series of the form

$$t = \sum_{i=1}^{k} \frac{s^i}{i!}\ t_o^i \qquad\qquad n = \sum_{i=1}^{k} \frac{s^i}{i!}\ n_o^i \ldots \quad 4.31,$$

where

$$x_o^i = \left(\frac{d^ix}{ds^i}\right)_{s=0} \quad ; \quad (x_o^i)^k = \left[\left(\frac{d^ix}{ds^i}\right)_{s=0}\right]^k \ldots\ldots \qquad 4.32.$$

The successive differential coefficients are obtained by differentiating equations 4.29 and 4.30 when these quantities are obtained in terms of the derivatives σ^i of the curvature with respect to s, where

$$\sigma^i = \frac{d^i\sigma}{ds^i} \quad .$$

Operation on equation 4.29 gives

$$\begin{aligned} \sigma^1 &= n^3t^1 + n^2t^2 - n^2t^2 - n^1t^3 = n^3t^1 - n^1t^3 \\ \sigma^2 &= n^4t^1 + n^3t^2 - n^2t^3 - n^1t^4 \qquad \ldots\ldots \qquad 4.33. \\ \sigma^3 &= n^5t^1 + 2n^4t^2 - 2n^2t^4 - n^1t^5 \end{aligned}$$

The differentiation of equation 4.30 in a similar manner gives

$$\begin{aligned} &n^1n^2 + t^1t^2 = 0 \\ &n^1n^3 + (n^2)^2 + (t^2)^2 + t^1t^3 = 0 \\ &n^1n^4 + 3(n^2n^3 + t^2t^3) + t^1t^4 = 0 \qquad \ldots\ldots \qquad 4.34. \\ &n^1n^5 + 4(n^2n^4 + t^2t^4) + 3(n^3)^2 + 3(t^3)^2 + t^1t^5 = 0 \end{aligned}$$

The quantities to be substituted in equations44.31 are obtained by evaluating equations 4.32 and 4.33 at $s = 0$ when $\sigma = \sigma_o$ is the curvature of the projected geodesic at the initial terminal P_1 , σ_o^i

being the derivatives of the curvature with respect to s and evaluated at P_1.

When $s \to 0$,

the curve coincides with the tangent, and

$$t_o^1 = 1 \quad ; \qquad n_o^1 = 0.$$

Substitution in the first equation of the set 4.34 gives

$$t_o^2 = 0.$$

The substitution of all known quantities into equation 4.29 gives

$$n_o^2 = \sigma_o.$$

Further substitution into the second equation of set 4.34 gives

$$t_o^3 = -(\sigma_o)^3 .$$

Continued substiution in to the first equation of set 4.33 and repetition of the procedure gives

$$n_o^3 = \sigma_o^1 \quad ;$$

$$t_o^4 = -3\sigma_o^1\sigma_o \quad ; \quad n_o^4 = \sigma_o^2 - (\sigma_o)^3 \quad ;$$

$$t_o^5 = 4(\sigma_o)^4 - \sigma_o^2\sigma_o - 3(\sigma_o^1)^2 - 3(\sigma_o)^4; \quad n_o^5 = \sigma_o^3 - 6\sigma_o^1(\sigma_o)^2$$

$$= (\sigma_o)^4 - 4\sigma_o^2\,\sigma_o - 3(\sigma_o^1)^2 \qquad \ldots\ldots \qquad 4.35.$$

The substitution of these values in equations 4.31 gives

$$t = s - \frac{1}{6}(\sigma_o)^2 s^3 - \frac{1}{8}\sigma_o^1\sigma_o s^4 - \frac{1}{120}\left(4\sigma_o^2\sigma_o + 3(\sigma_o^1)^2 - (\sigma_o)^4\right) s^5 \ldots \quad 4.36$$

$$n = \frac{1}{2}\sigma_o s^2 + \frac{1}{6}\sigma_o^1 s^3 + \frac{1}{24}\left(\sigma_o^2 - (\sigma_o)^3\right)s^4 + \frac{1}{120}\left(\sigma_o^3 - 6\,\sigma_o^1(\sigma_o)^2\right)s^5 \quad \ldots\ldots \quad 4.37.$$

The chord to arc correction δ is obtained by the evaluation of equations 4.36 and 4.37 when $s = \ell_p$, using the equation

$$\tan\delta = \frac{n}{t} = \tfrac{1}{2}\sigma_o\ell_p$$

For mapping projections of the transverse Mercator type, $\sigma_o \simeq 10^{-5}\ \mathrm{km}^{-1}$. Consequently, $\delta \simeq 100$ arcsec or less even for lines of length 100 km. Thus, for all practical purposes,

$$\tan\delta = \delta + \frac{\delta^3}{3} + \ldots = \delta + o\{10^{-10}\} .$$

As

$$\sigma^1 = \frac{d\sigma}{ds} = \frac{\partial\sigma}{\partial x_1}\frac{dx_1}{ds} + \frac{\partial\sigma}{\partial x_2}\frac{dx_2}{ds} + \frac{\partial\sigma}{\partial\beta}\frac{d\beta}{ds} ,$$

σ being given by equation 4.21 for the transverse Mercator, it follows that

$$\sigma^1 = \frac{1}{R^2}\cos\beta\,\sin\beta\left(1 + o\{(\tfrac{E}{R})^2\}\right) \quad \ldots\ldots \quad 4.38,$$

while

$$\sigma^2 \simeq \frac{E}{R^4} \quad \ldots\ldots \quad 4.39a,$$

as variations of magnitude $(\ell/R)^4$ can be considered to be negligible in the context of a spherical approximation of the Earth. Further,

$$\sigma^3 \simeq \frac{1}{R^4} \quad \ldots\ldots \quad 4.39b$$

for the transverse Mercator with six degree zone widths and geodetic lines which do not exceed 100 km in length, when the following are maximum orders of magnitude.

$$\sigma\ell_p \simeq 10^{-3} \;;\; \sigma^1\ell_p^2 \simeq 10^{-4} \;;\; (\sigma)^2\ell_p^2 \simeq 10^{-6} \;;\; \sigma^1\sigma\ell_p^3 \simeq 10^{-7};$$

$$(\sigma)^3\ell_p^3 \simeq 10^{-9} \;;\; \sigma^2\ell_p^3 \simeq 10^{-7} \;;\; \sigma^3\ell_p^4 \simeq 10^{-8}.$$

Adopting the criterion that values of δ should be correct to 0".001, the terms required in the evaluation of $\tan\delta$ are

$$\tan\delta = \left\{\frac{1}{2}\sigma_o\ell_p + \frac{1}{6}\sigma_o^1\ell_p^2 + \frac{1}{24}\left(\sigma_o^2 - (\sigma_o)^3\right)\ell_p^3 + \frac{1}{120}\left(\sigma_o^3 - 6\sigma_o^1(\sigma_o)^2\right)\ell_p^4\right\} \times$$

$$\left\{1 - \frac{1}{6}(\sigma_o)^2\ell_p^2 - \frac{1}{8}\sigma_o^1\sigma_o\ell_p^3\right\}^{-1}$$

$$= \frac{1}{2}\sigma_o\ell_p\left\{1 + \frac{1}{3}\frac{\sigma_o^1}{\sigma_o}\ell_p + \frac{1}{12\sigma_o}\left(\sigma_o^2 - (\sigma_o)^3\right)\ell_p^2 + \frac{1}{60\,\sigma_o}\left(\sigma_o^3 - 6\sigma_o^1(\sigma_o)^2\right)\ell_p^3\right\} \times$$

$$\left\{1 + \frac{1}{6}(\sigma_o)^2\ell_p^2 + \frac{1}{8}\sigma_o^1\sigma_o\ell_p^3\right\}.$$

On considering terms to order $(\sigma)^3\ell^3$ or greater, $\tan\delta$ is given by

$$\tan^3\delta = \frac{1}{8}(\sigma_o)^3\ell_p^3\left(1 + \frac{\sigma_o^1}{\sigma_o}\ell_p\right).$$

Thus

$$\delta = \tan\delta - \frac{1}{24}(\sigma_o)^3\ell_p^3\left(1 + \frac{\sigma_o^1}{\sigma_o}\ell_p\right)$$

$$= \frac{1}{2}\sigma_o\ell_p + \frac{1}{6}\sigma_o^1\ell_p^2 + \frac{1}{24}\left(\sigma_o^2 - 2(\sigma_o)^3 + 2(\sigma_o)^3\right)\ell_p^3 + \frac{1}{720}\left(6\sigma_o^3 + \right.$$

$$\left.\sigma_o^1(\sigma_o)^2\{-36+20+45-30\}\right)\ell_p^4$$

$$= \frac{1}{2}\sigma_o\ell_p + \frac{1}{6}\sigma_o^1\ell_p^2 + \frac{1}{24}\sigma_o^2\ell_p^3 + \frac{1}{720}\left(6\sigma_o^3 - \sigma_o^1(\sigma_o)^2\right)\ell_p^4 + o\{10^{-11}\}$$

$$\ldots\ldots \quad 4.40.$$

Equation 4.40 gives the chord to arc correction in terms of the curvature at the initial terminal of the line.

Notes:-

1. *For the transverse Mercator - spherical case*

The use of equations 4.21 and 4.38 in equation 4.40 gives

$$\delta = \frac{1}{2}\frac{E_1}{R^2}\left[1 + o\{(\tfrac{E}{R})^2\}\right] \ell_p \cos\beta + \frac{1}{6R^2}\left[1 + o\{(\tfrac{E}{R})^2\}\right]\ell_p^2 \cos\beta \sin\beta$$

$$= \frac{1}{2R^2}\left(E_1 + \frac{1}{3}\ell_p \sin\beta\right) \ell_p \cos\beta + o\{\frac{E^3\ell}{R^4}(\simeq 10^{-6})\} \quad .$$

If (N_i, E_i) are the co-ordinates of P_i $(i=1,2)$, as $\delta \simeq 100$ sec, the above formula can, for most practical purposes, be written as follows.

$$\ell_p \cos\beta \doteqdot N_2 - N_1 = \Delta N \quad ; \quad \ell_p \sin\beta \doteqdot E_2 - E_1$$

and

$$\delta = \frac{1}{6R^2}(2E_1 + E_2)(N_2 - N_1) \quad \ldots\ldots \quad 4.41.$$

2. The parametric equations of the projected curve can also be used to give

i. The linear arc to chord correction $(\ell_p - \ell_c)$ by the relation

$$\ell_c = (n^2 + t^2)^{\frac{1}{2}},$$

when (Thomas 1952, p.80)

$$\ell_p - \ell_c = \frac{1}{24}(\sigma_o)^2\ell_p^3 + \frac{1}{24}\sigma_o^1\sigma_o\ell_p^4 + \frac{1}{5760}\left(72\sigma_o^2\sigma_o + 64(\sigma_o^1)^2 - 3(\sigma_o)^4\right)\ell_p^5$$

$$\ldots\ldots \quad 4.42 \; ;$$

ii. the normal displacement h of the projected geodesic from the chord at the general point P(n,t) at a distance s from the initial terminal P_1, as measured along the curve whose projected length between the terminals $P_1(0,0)$ and $P_2(n_1,t_1)$ is ℓ_p. A consideration of figure 4.4 shows that

$$\tan\delta = \frac{n + h\cos\delta}{t - h\sin\delta}$$

or

$$h\{\cos^2\delta + \sin^2\delta\} = t\sin\delta - n\cos\delta.$$

The use of equations 4.36 and 4.37, together with equation 4.40, and the approximations

$$\cos\delta = 1 - \tfrac{1}{2}\delta^2 \quad ; \quad \sin\delta = \delta + \frac{1}{3!}\delta^3 \quad ,$$

gives (ibid, p.82)

$$h = \frac{1}{2}(\ell_p - s)\sigma_o s + \frac{1}{6}\sigma_o^1(\ell_p^2 - s^2)s + \frac{1}{24}\sigma_o^2(\ell_p^3 - s^3)s +$$

$$\frac{1}{48}(\sigma_o)^3\left(3\ell_p^2 s^2 + 2s^4 - s\ell_p^3 - 4s^3\ell_p\right) \quad \ldots\ldots \quad 4.43.$$

For $s = \frac{1}{2}\ell_p$, in the spherical case of the transverse Mercator, the maximum separation h_m is given by

$$h_m = \frac{1}{8}\sigma_o\ell_p^2 + \frac{1}{16}\sigma_o^1\ell_p^3 + o\{\ell\times 10^{-7}\} \quad \ldots\ldots \quad 4.44.$$

The use of equations 4.21 and 4.38 gives

$$h_m = \frac{1}{8}\ \frac{\ell_p \cos\beta}{R^2}\left(E_1 + \frac{1}{2}\ell_p \sin\beta\right) + o\{\ell\times\frac{E^3\ell}{R^4}\}$$

$$= \frac{\ell_p}{16R^2}\ (N_2 - N_1)(E_1 + E_2) + o\{\ell\times\frac{E^3\ell}{R^4}\} \quad \ldots\ 4.45.$$

For a 100 km line, $h_m \simeq 10$ metres, the term neglected having a magnitude approaching 1 cm.

4.5 The set of formulae for computations in plane rectangular co-ordinates

The results obtained in sections 4.2 to 4.4 can be summarised to obtain the following set of formulae for the computation of plane rectangular co-ordinates using plane grid bearings θ and lengths ℓ_c (= ℓ_p + $o\{10^{-8}\ell\}$) . The field observations made at a point P_1, whose projection co-ordinates (N_1, E_1) are known, are usually available in one of two forms.

i. An azimuth α and a length ℓ of a curve on the reference surface between P_1 and an adjacent point P_2, whose projection co-ordinates (N_2, E_2) are to be established. The required set of formulae in this case are as follows. The equation numbers refer to positions in the preceding text, all relations being for the transverse Mercator projection.

$$\phi_f = \frac{N_1}{R} + \phi_o \quad \ldots\ldots \quad 3.65,$$

where ϕ_o is the latitude of the origin

$$\gamma = \tan\phi_f\ \frac{E_1}{R}\left[1 - \frac{1}{3}(1+\tan^2\phi_f)\left(\frac{E_1}{R}\right)^2 + \frac{1}{15}(2+5\tan^2\phi_f+3\tan^4\phi_f)\left(\frac{E_1}{R}\right)^4\right] \quad \ldots\ldots \quad 3.64.$$

$$\beta = \alpha - \gamma$$

$$N_2 = N_1 + \ell\cos\beta + o\{\ell\times10^{-4}\}\ .$$

$$E_2 = E_1 + \ell\sin\beta + o\{\ell\times10^{-4}\}$$

$$\ell_c\ \left(= \ell_p + o\{10^{-8}\ell\}\right) = \ell\left(1 + \frac{1}{6R^2}\left(E_1^2 + E_1E_2 + E_2^2\right)\right) \quad \ldots 4.26.$$

$$\delta = \frac{1}{6R^2}(N_2 - N_1)(2E_1 + E_2) + o\{\frac{E^3\ell}{R^4}\} \quad \ldots\ldots \quad 4.41.$$

$$\theta = \beta - \delta \quad \ldots\ldots \quad 4.3.$$

$$N_2 = N_1 + \ell_c\cos\theta \quad ; \quad E_2 = E_1 + \ell_c\sin\theta \ \ldots\ 4.6.$$

ii. The second form in which the field data may be available is encountered in the representation of orientation in the local horizon. The alternative is a reverse grid bearing θ_r and a measured horizontal angle B instead of the azimuth α. A study of figure 4.5 shows that

$$\theta \neq \theta_r + B$$

even though the projection is conformal as the straight on the projection plane has no direct relation to the reference surface. The angle between the *tangents* to the projected geodesics however, is true on a conformal projection. Hence,

$$\theta = \theta_r + \delta_r + B - \delta \qquad \ldots\ldots \qquad 4.46,$$

where δ_r is the chord to arc correction for the line P_1P_r. Note that the chord to arc corrections at the two ends of a line are not equal in the case of the transverse Mercator projection as the signficnace of equation 4.41 is strictly algebraic.

4.6 Computation of projection co-ordinates from the grid bearing of the projected geodesic and tne measured length

It is also possible to obtain the transverse Mercator (TM) grid co-ordinates of the point $P_1(N_1,E_1)$ from those of $P_2(N_2,E_2)$ and the set of observed quantities which could either be of the form

a. ℓ, α

or else

b. ℓ, β_r, B,

all quantities having the same significance as in the previous section, β_r being the reverse grid bearing of a projected geodesic used as a reference direction from which the angle B has been measured to P_1 from P_2. Case a can be made identical with case b by the use of equations 3.64 and 4.3 in the case when the grid meridian is made tangential to the reference geodesic and

$$B = \beta .$$

Thus, for a sequential distribution of points P_i, it is common for observed data to be in couplets of the form (B,ℓ) and hence it is necessary not only to compute the grid co-ordinates of the fore point P_2 but also a reverse grid bearing (β_{21}) for the projected geodesic at P_2. On the TM which is a conformal projection,

$$\beta_{12} = \beta_r + B \qquad \ldots\ldots \qquad 4.47,$$

which is less complex than equation 4.46, though not by much. The problem can therefore be summarised as follows.

Given :- N_1, E_1, ℓ, (β_r, B) or β_{12} ;
Determine :- ΔN (hence N_2), ΔE (hence E_2), β_{21}.

From equations 4.3, 4.6, 4.26 and 4.41,

$$\Delta N = \ell_c \cos\theta \qquad ; \qquad \theta = \beta - \delta \quad .$$

Thus

$$N_2 - N_1 = (\ell + d\ell)\cos(\beta-\delta) = (\ell + d\ell)\left(\cos\beta\cos\delta + \sin\beta\sin\delta\right)$$

where

$$d\ell = \frac{\ell}{6R^2}\left(E_1^2 + E_1E_2 + E_2^2\right).$$

As $\delta \simeq 5\times10^{-3}$; $d\ell \simeq 10^{-3}\ell$ and $d\ell\times\delta \simeq 5\times10^{-7}\ell$, on ignoring terms smaller than 10^{-6};

$$N_2 - N_1 = (\ell + d\ell)\left(\cos\beta + \delta\sin\beta\right)$$

$$= \ell\cos\beta + d\ell\cos\beta + \delta\,\ell\sin\beta + o\{5\times10^{-7}\ell\}$$

$$= \ell\cos\beta\left[1 + \frac{E_1^2 + E_1E_2 + E_2^2}{6R^2} + \frac{2E_1 + E_2}{6R^2}\,\ell\sin\beta\right],$$

as

$$N_2 - N_1 = \ell\cos\beta$$

with sufficient precision in small order terms. A slightly simpler formula may be obtained by making the substitution

$$E_1 = E_2 - \ell\sin\beta$$

in these small order terms, when

$$N_2 - N_1 = \ell\cos\beta\Big(1 + \frac{1}{6R^2}\big((E_2 - \ell\sin\beta)^2 + (E_2 - \ell\sin\beta)E_2 + E_2^2 +$$

$$\{2(E_2 - \ell\sin\beta) + E_2\}\ell\sin\beta\big)\Big)$$

$$= \ell\cos\beta\Big[1 + \frac{1}{6R^2}\big(3E_2^2 - 3E_2\ell\sin\beta + \ell^2\sin^2\beta + 3E_2\ell\sin\beta$$

$$- 2\ell^2\sin^2\beta\big)\Big].$$

Thus

$$N_2 - N_1 = \ell\cos\beta\left[1 + \frac{E_2^2}{2R^2} - \frac{1}{6R^2}\ell^2\sin^2\beta\right] \quad \ldots\ldots \quad 4.49.$$

Similarly,

$$E_2 - E_1 = \ell_c\sin\theta = (\ell + d\ell)\sin(\beta-\delta) = (\ell + d\ell)\left(\sin\beta - \delta\cos\beta + o\{10^{-8}\}\right)$$

$$= \ell\sin\beta + d\ell\sin\beta - \delta\,\ell\cos\beta$$

$$= \ell\sin\beta + \frac{\ell\sin\beta}{6R^2}\left(E_1^2 + E_1E_2 + E_2^2\right) - \frac{\ell\cos\beta}{6R^2}(N_2 - N_1)(2E_1 + E_2).$$

As

$$\ell\sin\beta = E_2 - E_1 + o\{\ell\times10^{-3}\}$$

and

$$\ell\cos\beta = N_2 - N_1 + o\{\ell\times10^{-3}\},$$

$$E_2 - E_1 = \ell \sin \beta + \frac{1}{6R^2}\left[(E_1^2 + E_1E_1 + E_2^2)(E_2 - E_1) - \ell^2\cos^2\beta(2E_1 + E_2)\right]$$

Effecting the substitution

$$E_2 = E_1 + \ell \sin \beta + o\{10^{-3}\ell\},$$

the above equation becomes

$$E_2 - E_1 = \ell \sin \beta + \frac{1}{6R^2}\left(E_2^3 - E_1^3 - \ell^2\cos^2\beta(3E_1 + \ell \sin \beta)\right)$$

$$= \ell \sin \beta + \frac{E_2^3 - E_1^3}{6R^2} - \frac{E_1\ell^2\cos^2\beta}{2R^2} - \frac{1}{6R^2}\ell^3\sin \beta \cos^2\beta$$

$$\ldots\ldots \qquad 4.50.$$

Let δ_{12} and δ_{21} be the chord to arc corrections at P_1 and P_2, as illustrated in figure 4.5. Then

$$\beta_{12} = \theta + \delta_{12} = \beta$$

and

$$\beta_{21} = \pi + \theta + \delta_{21},$$

where

$$\delta_{12} = \frac{1}{6R^2}(N_2 - N_1)(2E_1 + E_2) = \frac{\ell \cos \beta}{6R^2}(2E_1 + E_2)$$

and

$$\delta_{21} = \frac{1}{6R^2}(N_1 - N_2)(2E_2 + E_1) = -\frac{\ell \cos \beta}{6R^2}(2E_2 + E_1).$$

Note:- δ_{12} and δ_{21} are of opposite sign, as illustrated correctly in figure 4.5, in the numerical sense, but the above equations are consistent with the fundamental algebraic definitions.

Therefore,

$$\theta = \beta_{12} - \delta_{12} = \beta_{21} - \pi - \delta_{21},$$

$$\beta_{21} = \beta_{12} + \pi + \delta_{21} - \delta_{12}$$

$$= \beta_{12} + \pi - \frac{\ell \cos \beta}{2R^2}(E_1 + E_2) \qquad \ldots\ldots \qquad 4.51.$$

Thus the required sequence of computations is

$$E_2 = E_1 + \ell \sin \beta + o\{10^{-3}\ell\}$$

$$E_2 = E_1 + \ell \sin \beta + \frac{E_2^3 - E_1^3}{6R^2} - \frac{E_1\ell^2\cos \beta}{2R^2} - \frac{1}{6R^2}\ell^3\sin \beta \cos^2\beta .$$

$$\ldots\ldots \qquad 4.52$$

$$N_2 = N_1 + \ell \cos \beta + \frac{E_2^2}{2R^2}\ell \cos \beta - \frac{1}{6R^2}\ell^3\cos \beta \sin^2\beta .$$

$$\beta_{21} = \beta_{12} + \pi - \frac{\ell \cos \beta}{2R^2}(E_1 + E_2) .$$

Notes :-

1. The two groups of formulae set out in sections 4.5 and 4.6 are only marginally different as regards convenience for hand computations and can be applied to the ellipsoid provided R^2 is replaced by the local mean radius of curvature in the normal section, as developed in appendix f.

2. The formulae developed in section 4.5 are in common usage currently as the plane grid bearings θ and the plane distance ℓ_c are quantities used directly in plane computations.

3. The central scale factor k_o is applied to the equations in sections 4.5 and 4.6 on the basis that its usage is equivalent to computing on a sphere of radius k_oR , as discussed in section 3.6.3.*1.* This requires that all measured distances ℓ are scaled down to $k_o\ell$, resulting in a uniform reduction in length of 4 parts in ten thousand. This consequently reduces all co-ordinates (E,N) to (k_oE,k_oN). Hence when computing ratios of the type $(E/R,\ \Delta N/R)$, where the projection co-ordinates are based on a central scale factor, the corresponding terms in the expression become

$$\frac{E}{k_oR} \quad , \quad \frac{\Delta N}{k_oR} .$$

It follows that *when the central scale factor k_o is used,* the equations in section 4.5 become

$$\phi_f = \frac{N}{k_oR} + \phi_o \quad ..... \quad 4.53.$$

$$\gamma = \tan\phi_f \frac{E_1}{k_oR}\left(1 - \frac{1}{3}(1+\tan^2\phi_f)\left(\frac{E_1}{k_oR}\right)^2 + \frac{1}{15}(2+5\tan^2\phi_f+3\tan^4\phi_f)\left(\frac{E_1}{k_oR}\right)^4\right) \quad .... \quad 4.54.$$

$$\beta = \alpha - \gamma \quad ..... \quad 4.3.$$

$$N_2 - N_1 = \Delta N = k_o\ell\cos\beta + o\{\ell\times10^{-4}\} \quad ..... \quad 4.55.$$

$$E_2 - E_1 = \Delta E = k_o\ell\sin\beta + o\{\ell\times10^{-4}\} \quad ...... \quad 4.56.$$

$$\ell_c = k_o\ell\left[1 + \frac{1}{6k_o^2R^2}\left(E_1^2 + E_1E_2 + E_2^2\right)\right] \quad ...... \quad 4.57.$$

$$\delta = \frac{1}{6k_o^2R^2}(2E_1 + E_2)(N_2 - N_1) \quad ...... \quad 4.58.$$

$$\theta = \beta - \delta \quad ..... \quad 4.1.$$

$$\Delta N = \ell_c\cos\theta \quad ; \quad \Delta E = \ell_c\sin\theta \quad ..... \quad 4.6.$$

The co-ordinates (E,N) in equations 4.53 to 4.58 refer to the transverse Mercator projection where the central scale factor k_o has been applied.

Similar expressions can be deduced for the equations at 4.52.

5. THE CONFORMAL MAPPING OF A SURFACE ONTO A PLANE

5.1 The ellipsoid of revolution as a reference surface

The development in sections 1 to 5 is based on a spherical reference surface. The reference figure which best fits the geoid is an ellipsoid of revolution, the current estimates of whose parameters are

$$a = 6{,}378{,}160 \text{ metres}$$
$$f^{-1} = 298.25 \qquad \ldots\ldots \qquad 5.1$$

where a is the equatorial radius and f the flattening, as described in appendix a.

The magnitude of the deviations of the geoid from a sphere of best fit are of the order of f(i.e., 3×10^{-3}) and hence all previous formulae would only be correct to this order of accuracy. It should be borne in mind however, that in all cases, relative angular displacements between identical points on both the ellipsoid and the equivalent sphere are in agreement to a much higher order of accuracy. In the case of surface displacements of geodetic magnitude (i.e., less than 100 km), these are smaller than 10^{-6}. The discrepancies in co-ordinates arise in the conversion of measured distances to angular displacements. In several cases, adequate formulae may be obtained by replacing the spherical radius R by the appropriate radius of curvature on the ellipsoid as given in appendix d. The comparisons will be drawn at the appropriate points in the subsequent development.

Angular observations on the ellipsoid are made in planes containing the local ellipsoid normal. Further, all electro-magnetic distance measurement (EDM) is also reduced in the normal section. Thus ellipsoidal curvatures in the normal section are of importance. The set of curvilinear surface co-ordinates (ϕ,λ) have parametric curves with relatively simple geometry. The rotation axis is normal to all parallels which are circles, the equator having a radius a. The meridians, as shown in appendix a , are orthogonal ellipses.

The perspective projection of the ellipsoid can be effected without difficulty as in the spherical case, once the nature of its geometry has been allowed for. The quasi perspective conical projections could be said to have some geometrical significance in the ellipsoidal case, too, but are best treated as a consistent set of mathematical rules.

Consider the following example.

Projections on a tangent cone - equidistant case

Refer section 3.2.1.

Rule 1. The scale is true along the standard parallel of co-latitude θ_o. As the radius of the parallel of co-latitude θ on the ellipsoid is $\nu \sin \theta$, it follows that

$$r_o k_c = \nu_o \sin \theta_o \qquad \ldots\ldots \qquad 5.2,$$

where ν has the same significance as in section 1.1.

Rule 2. All parallels are concentric circular arcs, the standard parallel plotting as one of radius r_o, given by

$$r_o = \nu_o \tan \theta_o .$$

Hence

$$k_c = \cos \theta_o .$$

Rule 3. All other parallels are circular arcs of radii r, given by

$$r = \nu_o \tan \theta_o - \int_{\theta}^{\theta_o} \rho \, d\theta \qquad \ldots\ldots \qquad 5.3,$$

where

$$\int_{\theta}^{\theta_o} \rho \, d\theta$$

is the meridian distance on the ellipsoid between the parallel of co-latitude θ and the standard parallel.

Such concepts are not complex in some of the direct cases but cause problems in oblique and transverse cases. These difficulties can be surmounted in the case of conformal projections by utilising the fact that the scale at a point is independent of the orientation of the element considered. In concept, the reference surface is mapped, element by element, onto the projection plane, using this condition, the process being known as *conformal mapping*.

5.2 The Caucny Riemann equations for conformal mapping

It is required to map the ω system of points P comprising the reference surface, illustrated in figure 5.1, onto the projection plane where they will be represented by the z system of points P*. Let the position of points on the reference surface which is assumed to to continuous and non-singular (e.g., Jeffreys & Jeffreys 1962,p.355), be defined by two surface parameters (u_1,u_2), similar in concept though not equal to the (ϕ,λ) system on the ellipsoid. In the subsequent development, it is assumed that the (u_1,u_2) system is an isometric one, as described in section 3.5.3.*1*. The general point P on the reference surface will be uniquely represented by the

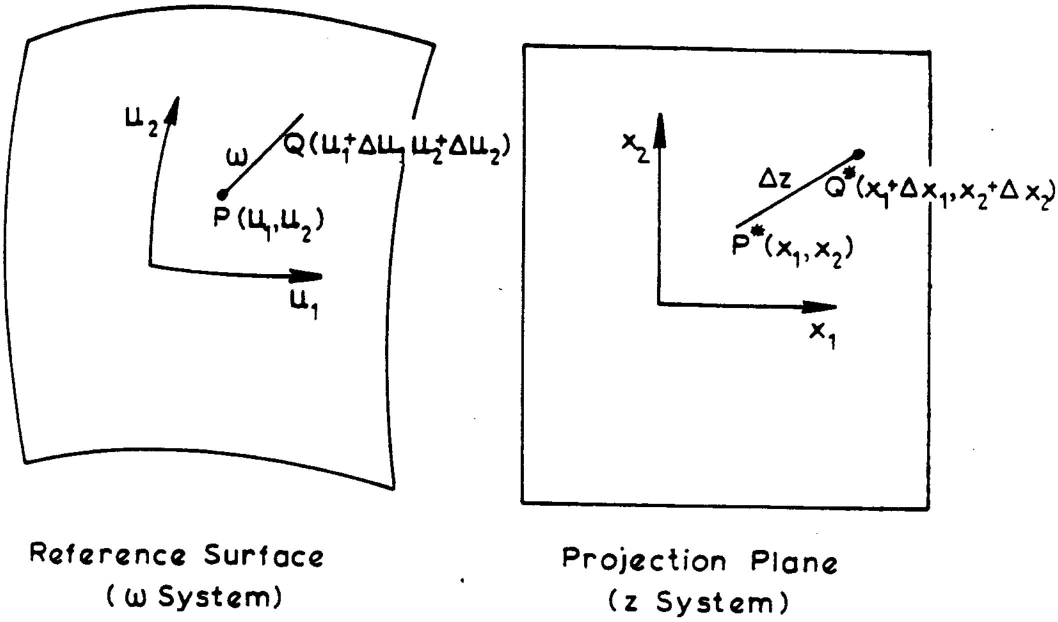


FIG.(5·1)

The elements for conformal mapping.

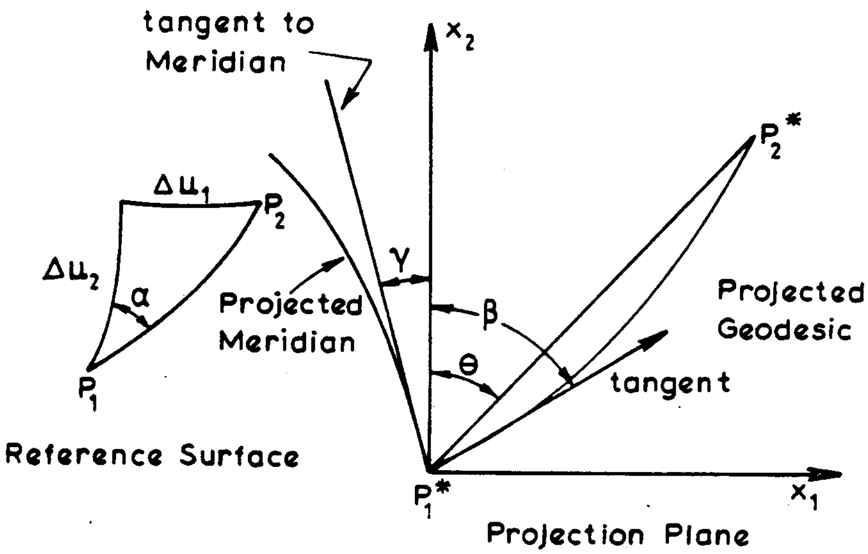


FIG.(5·2)

Convergence on conformal projections.

complex number

$$\omega = u_1 + iu_2 \qquad \ldots\ldots \qquad 5.4,$$

where

$$i = \sqrt{(-1)}$$

and (u_1,u_2) are a set of surface co-ordinates on the reference system which are isometric. Let ω be a single valued function in that any specified pair of values for u_1 and u_2, within prescribed limits, defines a unique point P on the reference surface. Let the point P be represented on the projection plane by P*, the latter being defined by the complex number

$$z = x_1 + ix_2 \qquad \ldots\ldots \qquad 5.5,$$

where (x_1,x_2) are the plane rectangular co-ordinates of P* on the projection plane. This representation is called the Argand diagram in pure mathematics (e.g., ibid,p.337 et seq), where it is primarily used for the illustration of the basic concepts of complex numbers. The essence of the above equations is that position on analytic two dimensional frames can be uniquely represented by a single complex number, in that two real numbers equivalent to the two surface co-ordinates, comprise the complex number. These complex numbers satisfy the general rules of algebra and calculus, provided the relevant functions are continuous.

Let the set of points P (u_1,u_2) on the reference surface be mapped onto the set of points P* (x_1,x_2) by the *mapping equation*

$$z = f(\omega) \qquad \ldots\ldots \qquad 5.6.$$

or

$$x_1 + ix_2 = f(u_1 + iu_2)$$

If the real and imaginary parts of the expression on the right of the equality at 5.6 are separated and equated with equivalent parts on the left, two relations of the form

$$x_i = x_i(u_1,u_2) \quad , \ i=1,2 \qquad \ldots\ldots \qquad 5.7$$

will result. If the mapping function $f(\omega)$ is analytic, i.e., it is continuous and its derivatives exist,

$$\frac{dz}{d\omega} = f'(\omega) \qquad \ldots\ldots \qquad 5.8.$$

Consider the point $Q(u_i+\Delta u_i, i=1,2)$, adjacent to P on the reference surface, and which maps as the point $Q^*(x_i+\Delta x_i, i=1,2)$. The vectors PQ and P*Q* represent changes $\Delta\omega$ in ω and Δz in z respectively. Thus

$$f'(\omega) = \lim_{\Delta\omega\to 0} \frac{\Delta z}{\Delta\omega} = \frac{dz}{d\omega} \qquad \ldots\ldots \qquad 5.9.$$

From equation 5.7, changes Δx_i in $x_i (i=1,2)$, due to changes Δu_i in u_i $(i=1,2)$ are given by

$$\Delta x_i = \frac{\partial x_i}{\partial u_1} \Delta u_1 + \frac{\partial x_i}{\partial u_2} \Delta u_2 = \sum_{k=1}^{2} \frac{\partial x_i}{\partial u_k} \Delta u_k, \; i=1,2.$$

Further,

$$\Delta z = \sum_{i=1}^{2} \left(\frac{\partial x_1}{\partial u_i} \Delta u_i + i \frac{\partial x_2}{\partial u_i} \Delta ui \right) \quad \ldots\ldots \quad 5.10.$$

Similarly, the consideration of equation 5.4 gives

$$\Delta \omega = \Delta u_1 + i \Delta u_2 \quad \ldots\ldots \quad 5.11.$$

Thus ,

$$f'(\omega) = \lim_{\Delta u_1, \Delta u_2 \to 0} \left(\frac{\left(\frac{\partial x_1}{\partial u_1} + i \frac{\partial x_2}{\partial u_1}\right) \Delta u_1 + \left(\frac{\partial x_1}{\partial u_2} + i \frac{\partial x_2}{\partial u_2}\right) \Delta u_2}{\Delta u_1 + i \Delta u_2} \right).$$

If the projection is conformal, the value of $f'(\omega)$ at P must be independent of the the orientation of the element PQ on the reference surface. When $\Delta u_1 \neq 0$,

$$f'(\omega) = \lim_{\Delta u_1, \Delta u_2 \to 0} \left(\frac{\frac{\partial x_1}{\partial u_1} + i \frac{\partial x_2}{\partial u_1} + \left(\frac{\partial x_1}{\partial u_2} + i \frac{\partial x_2}{\partial u_2}\right) \frac{\Delta u_2}{\Delta u_1}}{1 + i \frac{\Delta u_2}{\Delta u_1}} \right)$$

Evaluation in the limit when $\Delta u_2 = 0$, gives

$$f'(\omega) = \frac{\partial x_1}{\partial u_1} + i \frac{\partial x_2}{\partial u_1} \quad \ldots\ldots \quad 5.12.$$

Similarly, when $\Delta u_2 \neq 0$,

$$f'(\omega) = \lim_{\Delta u_1, \Delta u_2 \to 0} \left(\frac{\left(\frac{\partial x_1}{\partial u_1} + i \frac{\partial x_2}{\partial u_1}\right) \frac{\Delta u_1}{\Delta u_2} + \frac{\partial x_1}{\partial u_2} + i \frac{\partial x_2}{\partial u_2}}{\frac{\Delta u_1}{\Delta u_2} + i} \right).$$

Evaluation in the limit when $\Delta u_1 = 0$ gives

$$f'(\omega) = - i \frac{\partial x_1}{\partial u_2} + \frac{\partial x_2}{\partial u_2} \quad \ldots\ldots \quad 5.13.$$

For a conformal projection, $f'(\omega)$ must have the same value through both equations 5.11 and 5.12, as it is independent of the combination of values taken by Δu_1 and Δu_2. Thus

$$\frac{\partial x_1}{\partial u_1} + i\,\frac{\partial x_2}{\partial u_1} = \frac{\partial x_2}{\partial u_2} - i\,\frac{\partial x_1}{\partial u_2}.$$

The equation of real and imaginary parts gives

$$\frac{\partial x_1}{\partial u_1} = \frac{\partial x_2}{\partial u_2} \quad ; \quad \frac{\partial x_2}{\partial u_1} = -\frac{\partial x_1}{\partial u_2} \quad \ldots\ldots \quad 5.14.$$

These relations are called the *Cauchy Riemann equations for conformal mapping.*

Notes:-

1. The reverse case is also true. The derivation of equation 5.14 was a general one, as no specific properties were implied for either the (u_1,u_2) system or the (x_1,x_2) system, except that of isometry and that the surface to be mapped was continuous and non-singular.

The projection plane can therefore be mapped onto the ellipsoid (the reverse case) by the following equations, using the same developments as before.

$$\omega = f(z)$$
$$\frac{d\omega}{dz} = f'(z) \quad \ldots\ldots \quad 5.15a;$$

$$\frac{\partial u_1}{\partial x_1} = \frac{\partial u_2}{\partial x_2} \quad ; \quad \frac{\partial u_2}{\partial x_1} = -\frac{\partial u_1}{\partial x_2} \quad \ldots\ldots \quad 5.15b.$$

5.3 The point scale factor in conformal mapping

Consider the two adjacent points $P(u_1,u_2)$ and $Q(u_1+\Delta u_2, u_2+\Delta u_2)$ on the surface to be mapped in figure 5.1, and the equivalent points $P^*(x_1,x_2)$ and $Q^*(x_1+\Delta x_1,x_2+\Delta x_2)$ on conformal projection onto the plane. The positional displacement can be represented by the vector $\Delta\omega$ while the equivalent change on the projection plane is the vector Δz. Each vector can be represented by its modulus and direction (e.g., Jeffreys & Jeffreys 1962,p.63). As elemental changes are conformally mapped without change of orientation, the modulus $d\ell$ on the surface is mapped into the modulus $d\ell_p$ on the projection plane. If the parametric curves

$$u_i = \text{constant} \quad , \; i=1,2$$

form an orthogonal isometric system on the surface,

$$d\ell_p = \left((\Delta x_1)^2 + (\Delta x_2)^2\right)^{\frac{1}{2}}$$

and

$$d\ell = m\left((\Delta u_1)^2 + (\Delta u_2)^2\right)^{\frac{1}{2}} \qquad \ldots\ldots \qquad 5.17,$$

where m is the linearisation factor for converting the curvilinear co-ordinates u_1 and u_2 to their linear equivalents. From section 1.3.3, the point scale factor k is given by

$$k = \frac{d\ell_p}{d\ell} = \lim_{\Delta u_1, \Delta u_2 \to 0}\left(\frac{\left((\Delta x_1)^2 + (\Delta x_2)^2\right)^{\frac{1}{2}}}{m\left((\Delta u_1)^2 + (\Delta u_2)^2\right)^{\frac{1}{2}}}\right)$$

If the mapping equation is given by equations 5.6 and 5.7,

$$\Delta x_i = \lim_{\Delta u_1, \Delta u_2 \to 0}\left(\frac{\partial x_i}{\partial u_1}\Delta u_1 + \frac{\partial x_i}{\partial u_2}\Delta u_2\right) = \lim_{\Delta u_1, \Delta u_2 \to 0}\left(\sum_{j=1}^{2}\frac{\partial x_i}{\partial u_j}\Delta u_j\right) \quad i=1,2$$

$$\ldots\ldots \qquad 5.19.$$

Thus

$$\sum_{i=1}^{2}(\Delta x_i)^2 = \left[\left(\left(\frac{\partial x_1}{\partial u_1}\right)^2 + \left(\frac{\partial x_2}{\partial u_1}\right)^2\right)(\Delta u_1)^2 + 2\left(\frac{\partial x_1}{\partial u_1}\frac{\partial x_1}{\partial u_2} + \frac{\partial x_2}{\partial u_1}\frac{\partial x_2}{\partial u_2}\right)\Delta u_1 \Delta u_2 + \right.$$

$$\left.\left(\left(\frac{\partial x_1}{\partial u_2}\right)^2 + \left(\frac{\partial x_2}{\partial u_2}\right)^2\right)(\Delta u_2)^2\right] = \sum_{i=1}^{2}\sum_{j=1}^{2}\left\{\left(\frac{\partial x_j}{\partial u_i}\right)^2(\Delta u_i)^2 + \frac{\partial x_j}{\partial u_1}\frac{\partial x_j}{\partial u_2}\Delta u_1 \Delta u_2\right\}.$$

If $\Delta u_1 \neq 0$, divide both numerator and denominator of equation 5.18 by Δu_1 when

$$k = \lim_{\Delta u_1, \Delta u_2 \to 0}\left(\frac{\left\{\sum_{j=1}^{2}\left(\left(\frac{\partial x_j}{\partial u_1}\right)^2 + \left(\frac{\partial x_j}{\partial u_2}\right)^2\left(\frac{\Delta u_2}{\Delta u_1}\right)^2 + 2\frac{\partial x_j}{\partial u_1}\frac{\partial x_j}{\partial u_2}\frac{\Delta u_2}{\Delta u_1}\right)\right\}^{\frac{1}{2}}}{m\left\{1 + \left(\frac{\Delta u_2}{\Delta u_1}\right)^2\right\}^{\frac{1}{2}}}\right).$$

As k is a function of position only and not of orientation in conformal mapping, the value of k is independent of those assigned to the increments Δu_1 in u_1 and Δu_2 in u_2. It is therefore possible to select a convenient set of values for u_1 and u_2 which will provide a unique evaluation of k at the point in question. Selecting $\Delta u_1 \neq 0$ and $\Delta u_2 = 0$,

$$\Delta u_2/\Delta u_1 = 0.$$

Thus

$$k = \frac{1}{m}\left(\left(\frac{\partial x_1}{\partial u_1}\right)^2 + \left(\frac{\partial x_2}{\partial u_2}\right)^2\right)^{\frac{1}{2}}.$$

The use of the Cauchy Riemann conditions from equation 5.14 gives

$$k = \frac{1}{m}\left(\left(\frac{\partial x_1}{\partial u_1}\right)^2 + \left(\frac{\partial x_2}{\partial u_2}\right)^2 = \frac{1}{m}\left(\left(\frac{\partial x_2}{\partial u_2}\right)^2 + \left(\frac{\partial x_1}{\partial u_2}\right)^2\right)^{\frac{1}{2}} \quad \ldots \quad 5.21$$

Notes:-

1. The second equality could also have been obtained from equation 5.20 by selecting the set of values

$$\Delta u_2 \neq 0 \; ; \qquad \Delta u_1 = 0$$

for the evaluation of k.

2. *The linearisation factor m*

It should be noted that u_1 and u_2, which constitute an isometric system of curvilinear surface co-ordinates , have been converted to their linear equivalents by the use of the linearisation factor m, which is a constant at a given point, but is a function of position. In the case of an ellipsoidal reference surface, the element of length $d\ell$ equivalent to changes $d\lambda, d\phi$ in the surface co-ordinates is given in appendix c as

$$d\ell^2 = \nu^2\cos^2\phi \, d\lambda^2 + \rho^2 d\phi^2 \qquad \ldots\ldots \qquad 5.22$$

where the (λ,ϕ) system of surface co-ordinates is not an isometric one, as discussed in section 3.5.3.*1*. The required isometric system of surface co-ordinates is afforded by the parameters (λ, u_2), where

$$du_2 = \frac{\rho}{\nu \cos\phi} d\phi \qquad \ldots\ldots \qquad 5.23.$$

Thus

$$d\ell = \nu\cos\phi (d\lambda^2 + du_2^2)^{\frac{1}{2}}$$

which is of the form

$$m\left(du_1^2 + du_2^2\right)^{\frac{1}{2}}$$

given in equation 5.17. The linearisation factor for the (λ,u_2) system is given by

$$m = \nu \cos\phi \qquad \ldots\ldots \qquad 5.24$$

and is a position dependent function. The parameter u_2, given by

$$u_2 = \int \frac{\rho}{\nu\cos\phi} d\phi \qquad \ldots\ldots \qquad 5.25,$$

is called the *isometric latitude.*

3. *The Jacobian functional determinant*

A convenient means of representing the alternate possibilities in equation 5.21, which are

$$\left(\frac{\partial x_1}{\partial u_1}\right)^2 + \left(\frac{\partial x_2}{\partial u_1}\right)^2 = \left(\frac{\partial x_1}{\partial u_2}\right)^2 + \left(\frac{\partial x_2}{\partial u_2}\right)^2 \quad \ldots\ldots \qquad 5.26,$$

is afforded by the use of the determinant

$$\frac{\partial(x_1,x_2)}{\partial(u_1,u_2)} \left(\equiv J\left(\frac{x_1,x_2}{u_1,u_2}\right) \right) = \begin{vmatrix} \frac{\partial x_1}{\partial u_1} & \frac{\partial x_2}{\partial u_1} \\ \frac{\partial x_1}{\partial u_2} & \frac{\partial x_2}{\partial u_2} \end{vmatrix} \quad \ldots.5.27,$$

also called the *Jacobian* (e.g., Jeffreys & Jeffreys 1962,p.183) or the *Jacobian functional determinant* (e.g., Thomas 1952,p.21) and represented by either of the notations at the left of equation 5.27. The application of the Cauchy Riemann conditions to the right hand side give both terms in the equality numbered 5.26. Thus

$$k = \frac{1}{m}\left(J\left(\frac{x_1,x_2}{u_1,u_2}\right) \right)^{\frac{1}{2}} \quad \ldots\ldots \quad 5.28.$$

4. Equation 5.28 is of significance in the definition of the point scale factor only if expressions are available for the projection co-ordinates (x_i, i=1,2) in terms of the surface co-ordinates (u_i,i=1,2) through equations of the type at 5.7. If, on the other hand, the reverse relations were available for the definition of the surface parameters in terms of the projection co-ordinates, by equations of the type

$$u_i = u_i(x_1,x_2) \quad , \quad i=1,2 \quad \ldots\ldots \quad 5.29,$$

the point scale factor can be expressed through the following equations obtained on the same principles as used in the derivation of equations 5.19 to 5.21.

Using the same co-ordinate increments as before,

$$\Delta u_i = \sum_{j=1}^{2} \frac{\partial u_i}{\partial x_j} \Delta x_j \quad , \ i=1,2$$

$$\sum_{i=1}^{2} (\Delta u_i)^2 = \sum_{j=1}^{2} \sum_{k=1}^{2} \left\{ \left(\frac{\partial u_k}{\partial x_j}\right)^2 \Delta x_j^2 + \frac{\partial u_k}{\partial x_1}\frac{\partial u_k}{\partial x_2} \Delta x_1 \Delta x_2 \right\} .$$

If $\Delta x_1 \neq 0$,

$$\frac{1}{k} = m\left\{ \lim_{\Delta x_1,\Delta x_2 \to 0} \left(\frac{\sum_{k=1}^{2}\left[\left(\frac{\partial u_k}{\partial x_1}\right)^2 + \left(\frac{\partial u_k}{\partial x_2}\right)^2\left(\frac{\Delta x_2}{\Delta x_1}\right)^2 + 2\frac{\partial u_k}{\partial x_1}\frac{\partial u_k}{\partial x_2}\frac{\Delta x_2}{\Delta x_1}\right]}{1 + \left(\frac{\Delta x_2}{\Delta x_1}\right)^2} \right) \right\}^{\frac{1}{2}} .$$

As k^{-1} is independent of the values assigned to the set $(\Delta x_1, \Delta x_2)$ on a conformal projection, evaluation for the specific case

$$\Delta x_2 = 0 \quad ; \quad \Delta x_1 \neq 0$$

gives

$$\frac{1}{k} = m \left\{ \sum_{k=1}^{2} \left(\frac{\partial u_k}{\partial x_1}\right)^2 \right\}^{\frac{1}{2}} = m\left\{ \left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 \right\}^{\frac{1}{2}} .$$

The use of the Cauchy Riemann conditions for the inverse case, given in equation 5.16, provides the final form for k as

$$\frac{1}{k} = m\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2\right]^{\frac{1}{2}} = m\left[\left(\frac{\partial u_2}{\partial x_2}\right)^2 + \left(\frac{\partial u_1}{\partial x_2}\right)^2\right]^{\frac{1}{2}} = m\left[J\left(\frac{u_1, u_2}{x_1, x_2}\right)\right]^{\frac{1}{2}} \quad \ldots\ldots \quad 5.30.$$

5. *Conclusion*

Given expressions for either the projection co-ordinates in terms of the surface co-ordinates or vice versa, it is possible to derive expressions for the point scale factor k by partial differentiation and manipulation of the results. This is of considerable significance in deriving formulae for point-to-point computations on the ellipsoid, as a knowledge of the expression for k enables the curvature of the projected geodesic to be computed by the use of equation 4.21. This, in turn, provides expressions for all related quantities through the equations set out in sections 4.4 to 4.6.

5.4 Convergence on conformal projections

Consider the general case of the geodesic $P_1^*P_2^*$ on a conformal projection and let its grid bearing at P_1^* be β. Let the plane bearing of the chord $P_1^*P_2^*$ be θ. Elementary trigonometry gives

$$\tan\theta = \frac{\Delta x_1}{\Delta x_2} \quad \ldots\ldots \quad 5.31,$$

where

$$\Delta x_i = x_{i2} - x_{i1},\ i=1,2 \quad \ldots\ldots \quad 5.32.$$

Let the corresponding changes in the parameters on the reference surface be $\Delta u_i (i=1,2)$, where

$$\Delta u_i = u_{i2} - u_{i1},\ i=1,2 \quad \ldots\ldots \quad 5.33.$$

If α is the angle between the geodesic and the parametric curve

$$u_1 = \text{constant}$$

at P_1 and if the parametric curves on the reference surface form an orthogonal system,

$$\tan\alpha = \lim_{\Delta u_1, \Delta u_2 \to 0}\left(\frac{\Delta u_1}{\Delta u_2}\right) \quad \ldots\ldots \quad 5.34$$

As the projection co-ordinates are related to the surface parameters by equation 5.6,

$$x_i = x_i(u_1, u_2) \quad \ldots\ldots \quad 5.6$$

the quantities in equation 5.32 are related to those in equation 5.33 by the equation

$$\Delta x_i = \frac{\partial x_i}{\partial u_1}\Delta u_1 + \frac{\partial x_i}{\partial u_2}\Delta u_2,\ i=1,2.$$

Thus,

$$\tan\theta \overset{(5.31)}{=} \lim_{\Delta u_1,\Delta u_2\to 0} \frac{\sum_{k=1}^{2} \frac{\partial x_1}{\partial u_k}\Delta u_k}{\sum_{k=1}^{2} \frac{\partial x_2}{\partial u_k}\Delta u_k} .$$

If $\Delta u_2 \neq 0$, the use of equation 5.34 gives

$$\tan\theta = \frac{\frac{\partial x_1}{\partial u_1}\frac{\Delta u_1}{\Delta u_2} + \frac{\partial x_1}{\partial u_2}}{\frac{\partial x_2}{\partial u_1}\frac{\Delta u_1}{\Delta u_2} + \frac{\partial x_2}{\Delta u_2}} = \frac{\frac{\partial x_1}{\partial u_1}\tan\alpha + \frac{\partial x_1}{\partial u_2}}{\frac{\partial x_2}{\partial u_1}\tan\alpha + \frac{\partial x_2}{\partial u_2}} \quad \text{as} \quad \Delta u_1,\ \Delta u_2 \to 0.$$

Note:-

If $\Delta u_2 = 0$ and $\Delta u_1 \neq 0$, $\alpha = \frac{1}{2}\pi$ or $\frac{3}{2}\pi$.

In this case

$$\tan\theta = \lim_{\Delta u_1\to 0} \frac{(\partial x_1/\partial u_1)}{(\partial x_2/\partial u_1)} \quad \ldots\ldots \quad 5.35.$$

When $\alpha = 0$, $\theta = 2\pi - \gamma$ and

$$\tan(2\pi-\gamma) = \frac{(\partial x_1/\partial u_2)}{(\partial x_2/\partial u_2)}$$

or

$$\tan\gamma = -\frac{(\partial x_1/\partial u_2)}{(\partial x_2/\partial u_2)} \quad \ldots\ldots \quad 5.36.$$

Alternately, as the projection is conformal, the use of the Cauchy Riemann equations gives

$$\tan\gamma = \frac{(\partial x_2/\partial u_1)}{(\partial x_1/\partial u_1)} \quad \ldots\ldots \quad 5.37.$$

Equation 5.37 can also be obtained directly from equation 5.35, as the angle between the prime vertical and the grid meridian is $\frac{1}{2}\pi - \gamma$.

Notes:-

1. Equations 5.36 and 5.37 are convenient expressions for the evaluation of the convergence from *surface co-ordinates*. Relations for the reverse case can be obtained either as shown in section 1.3.2 or by the use of the partial derivatives of the equations at 5.29 in equation 5.34 when

$$\Delta u_i = \lim_{\Delta x_1,\Delta x_2\to 0} \sum_{k=1}^{2} \frac{\partial u_i}{\partial x_k}\Delta x_k , \quad i=1,2.$$

If $\Delta x_2 \neq 0$,

$$\tan\alpha = \lim_{\Delta x_1,\Delta x_2\to 0}\left(\frac{\frac{\partial u_1}{\partial x_1}\frac{\Delta x_1}{\Delta x_2} + \frac{\partial u_1}{\partial x_2}}{\frac{\partial u_2}{\partial x_1}\frac{\Delta x_1}{\Delta x_2} + \frac{\partial u_2}{\partial x_2}}\right) = \frac{\frac{\partial u_1}{\partial x_1}\tan\theta + \frac{\partial u_1}{\partial x_2}}{\frac{\partial u_2}{\partial x_1}\tan\theta + \frac{\partial u_2}{\partial x_2}}$$

When $\alpha = 0, \quad \theta = 2\pi - \gamma,$ whence

$$\tan(2\pi-\gamma) = -\frac{(\partial u_1/\partial x_2)}{(\partial u_1/\partial x_1)} \quad \text{or} \quad \tan\gamma = \frac{(\partial u_1/\partial x_2)}{(\partial u_1/\partial x_1)} \quad ..5.38.$$

The use of the Cauchy Riemann equations as set out at 5.16, gives an alternate expression for tan γ as

$$\tan\gamma = -\frac{(\partial u_2/\partial x_1)}{(\partial u_2/\partial x_2)} \quad ...... \quad 5.39.$$

It should be noted that the use of the Cauchy Riemann conditions makes other expressions possible.

2. *The relation between scale factor and convergence*

Convenient expressions relating the point scale factor to the convergence are also of interest in the study of those conformal projections with limited variations in one of the surface parameters u_1 (and hence x_1). In these cases, the expressions at 5.6 and 5.29 are rapidly converging power series in either Δu_1 or x_1 and consequently, the partial derivatives

$$\left(\frac{\partial x_1}{\partial u_1}, \frac{\partial x_2}{\partial u_1}\right) \quad \text{and} \quad \left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_1}\right)$$

are easy to evaluate. Tan γ can be defined in terms of these derivatives only through either equation 5.37 or by the relation

$$\tan\gamma = -\frac{(\partial u_2/\partial x_1)}{(\partial u_1/\partial x_1)} \quad ...... \quad 5.40,$$

which is obtained by the use of the Cauchy Riemann conditions and equation 5.39. The combination of equations 5.37 and 5.21 gives

$$k = \frac{1}{m}\frac{\partial x_1}{\partial u_1}\left(1 + \tan^2\gamma\right)^{\frac{1}{2}} \quad ...... \quad 5.41.$$

Similarly, equations 5.30 and 5.40 give

$$\frac{1}{k} = m\frac{\partial u_1}{\partial x_1}\left(1 + \tan^2\gamma\right)^{\frac{1}{2}} \quad ...... \quad 5.42.$$

5.5 Conclusion and summary

If position on the reference surface is defined by an orthogonal isometric system of parameters (u_1,u_2), the surface can be conformally mapped onto the (x_1,x_2) projection plane by the use of the mapping equation

$$x_1 + i\, x_2 = f(u_1 + i\, u_2) \qquad \ldots\ldots \qquad 5.6.$$

The equation of real and imaginary parts of this equation give the required expressions for projection co-ordinates in terms of surface parameters, in the form

$$x_i = x_i(u_1, u_2) \qquad \ldots\ldots \qquad 5.7.$$

Direct partial differentiation of the equations at 5.7, give expressions for tan γ through equation 5.37, and hence for k through equation 5.41.

Similar equations can also be defined for the mapping of the projection plane onto the ellipsoid, and called the *reverse case.* The formulae necessary for the sequential calculations which arise in the case of the establishment of a geodetic network, are obtained by the use of equations 4.20, 4.40 and 4.25 which are all capable of evaluation once the expression for k has been defined.

Three distinct cases can be outlined in practice.

Case 1. The conversion of surface co-ordinates (u_1,u_2) to projection co-ordinates (x_1,x_2).

The required equations are

a. the mapping equations

$$x_1 + i\, x_2 = f(u_1 + i\, u_2)$$

and hence

b. $$x_i = x_i(u_1, u_2), \quad i=1,2.$$

The sorting of terms after partial differentiation gives

c.

$$\gamma = \tan^{-1}\left(\frac{(\partial x_2/\partial u_1)}{(\partial x_1/\partial u_1)}\right) = \tan^{-1}\left(\frac{(\partial x_1/\partial u_2)}{(\partial x_2/\partial u_2)}\right) = \gamma(u_1, u_2)$$

and

d. $$k = \frac{1}{m}\,\frac{\partial x_1}{\partial u_1}\left(1 + \tan^2\gamma\right)^{\frac{1}{2}} = k(u_1, u_2) \quad \ldots\ldots \quad 5.41.$$

Case 2. Conversion of projection co-ordinates (x_1,x_2) to surface co-ordinates (u_1,u_2)

The required equations for the reverse case are

a. the mapping equation

$$u_i + iu_2 = f(x_1 + ix_2)$$

and hence

b. $u_i = u_i(x_1, x_2)$, $i=1,2$.

Partial differentiation of the above equations, gives

c. $$\gamma = \tan^{-1}\left(- \frac{(\partial u_2/\partial x_1)}{(\partial u_1/\partial x_1)} \right) = \gamma(x_1, x_2) \; \ldots\ldots \qquad 5.40$$

or any other convenient version which can be obtained by the use of the Cauchy Riemann conditions;
and

d. $$k = \frac{1}{m}\left[\frac{\partial u_1}{\partial x_1}\left(1 + \tan^2\gamma\right)^{\frac{1}{2}}\right]^{-1} = k(x_1, x_2) \ldots. \qquad 5.42.$$

Case 3. *Point-to-point computations :-*
Given $P_1(x_{11}, x_{21})$, ℓ and β, to determine x_1, x_2 and hence $P_2(x_{12}, x_{22})$

The curvature of the projected geodesic σ can be obtained from equation 5.42 through equation 4.20 as

$$\sigma = \frac{1}{k}\left(\frac{\partial k}{\partial x_1}\cos\beta - \frac{\partial k}{\partial x_2}\sin\beta\right) = (x_{11}, x_{21}, \beta, \ell) \ldots 4.20,$$

where β is obtained from the measured azimuth α through equation 4.3 as

$$\beta = \alpha - \gamma$$

The length ℓ_p of the projected geodesic is obtained by the use of equation 4.25 as

$$\ell_p = k_o\ell + \frac{1}{2}k_o^1 k_o \ell^2 + \frac{1}{6}\left(k_o^2(k_o)^2 - (k_o^1)^2 k_o\right)\ell^3 + \ldots. \qquad 4.24,$$

where the expressions for k and its derivatives are evaluated at P_1. Thus

$$\ell_p = \ell_p(x_{11}, x_{21}, \beta, \ell);$$

The chord to arc correction δ is given by equation 4.40 as

$$\delta = \frac{1}{2}\sigma_o\ell_p + \frac{1}{6}\sigma_o^1\ell_p^2 + \frac{1}{24}\sigma_o^2\ell_p^3 + \ldots = \delta(x_{11}, x_{21}, \beta, \ell) \;..\; 4.40;$$

the final equations giving the co-ordinates at P_1 being numbers 4.4 and 4.6, which are

$$x_{i2} = x_{i1} + \Delta x_i, \quad i=1,2 \; \ldots\ldots \qquad 4.3,$$

where

$$\Delta x_1 = \ell_c \cos\theta \quad ; \qquad \Delta x_2 = \ell_c \sin\theta \quad \ldots 4.6,$$

θ and ℓ_c being given by

$$\theta = \beta - \delta \qquad \text{and} \qquad \ell_c = \ell_p + o\{10^{-8}\}.$$

Notes:-

1. *Initialisation or definition of the mapping equation*

The mapping equation is merely the set of rules to be used in effecting the representation of the reference surface on the projection plane. For example, in the case of the Mercator projection of the ellipsoid, the equator plots as a straight line, true to scale. Hence

a. $u_2 = 0$ when $x_2 = 0$

and

b. $x_1 = f(u_1) = a\lambda$,

where λ is the difference in longitude from the reference meridian which plots as a straight line whose equation is

$$x_1 = 0.$$

Hence

$$x_1 + ix_2 = f(u_1 + iu_2) = a(\lambda + iu_2).$$

The equation of real and imaginary parts, together with the use of equation 5.25, gives

$$x_1 = a\lambda$$

and $\ldots\ldots$ 5.43.

$$x_2 = a\,u_2 = a\int_0^{\phi} \frac{\rho}{\nu\cos\phi}\, d\phi$$

The integration of the second equation at 5.43 gives

$$x_2 = a\int_0^{\phi} \frac{(1-e^2)\,d\phi}{(1-e^2\sin^2\phi)\cos\phi} = \int_0^{\phi} d\phi\left(\frac{e}{2}\frac{-e\cos\phi}{1-e\sin\phi} - \frac{e}{2}\frac{e\cos\phi}{1+e\sin\phi} + \right.$$

$$\left. + \frac{1}{\cos\phi}\right) \overset{(3.44)}{=} a\log\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}\right] \quad \ldots 5.44.$$

Thus u_2 for an ellipsoid, is given by the equation

$$u_2 = \log\left[\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\left(\frac{1-e\sin\phi}{1+e\sin\phi}\right)^{\frac{e}{2}}\right] \quad \ldots\ldots \quad 5.45$$

As $\partial x_2/\partial u_1 = 0$, the use of the procedures outlined above will give

$$\gamma = 0 \; ; \qquad k = \frac{1}{m}a = \frac{a}{\nu\cos\phi} \quad \ldots\ldots \quad 5.45.$$

The mapping functions for other projections are more complex. The projection of most interest is the ellipsoidal version of the transverse Mercator. The relevant formulae are derived in section 6.

6. THE TRANSVERSE MERCATOR PROJECTION OF THE ELLIPSOID IN ZONES

6.1 Introduction

The basic characteristics of the transverse Mercator projection have already been dealt with in sections 3.6 and 4. This projection has been adopted for affording a consistent global coverage using six degree zones about a central meridian in what is titled the Universal Transverse Mercator (UTM) system of co-ordinates, and described in section 3.6.3.3. Such a reference frame incorporates a central scale factor which deviates from unity by half the maximum scale error on an equivalent transverse Mercator projection. Consequently, the purely positive scale errors on the latter, with a maximum scale error $\simeq 10^{-3}$, are converted to both positive and negative errors with a maximum magnitude of 5×10^{-4}, on the former. The reduction in the magnitude of the scale errors so achieved makes the projection suitable for large scale cadastral work.

The transverse Mercator projection was strictly the *transverse case* of the Mercator projection in the spherical case. The conformal condition was introduced by varying the east co-ordinate distance, which is measured along the great circle orthogonal to the central meridian, as described in section 3.6. An exact "transverse case" of the Mercator projection is not possible on the ellipsoid. In fact, the meridians and parallels form the only geodesic isometric system of parametric curves on the ellipsoid (Thomas 1952,p.69).

It is possible to derive expressions for the projection of an ellipsoid by appropriate amendment of the cylindrical equidistant projection (Cassini-Soldner) where the great circle, orthogonal to the central meridian, and constituting the east axis on the projection plane in the spherical case, is replaced by the equivalent geodesic on the projection (Jordan-Eggert 1962, vol III second half,section 27). The resulting projection can be made conformal by adopting the technique used in section 3.6.3. The end product is slightly different from that obtained by the use of the technique outlined in section 5.4. The system of co-ordinates obtained by the latter technique are also known as *Gauss-Kruger* co-ordinates (ibid, section 32).

Two cases will be considered.

a. Mapping the ellipsoid onto the projection plane

In this case, the geographical co-ordinates of the general point $P(\phi,\Delta\lambda)$, where $\Delta\lambda$ is the difference in longitude from the central meridian, positive if P is east of it, are defined, along with the latitude ϕ_o of the origin P_o. It is required to find (x_1,x_2,γ) at P through the intermediate isometric ellipsoidal parameters $(\Delta\lambda,u_2)$, where u_2 is the isometric latitude, given by

$$u_2 = \int_0^\phi \frac{\rho}{\nu \cos\phi}\, d\phi \qquad \ldots\ldots \qquad 5.25.$$

b. The conversion of transverse Mercator co-ordinates to ellipsoidal co-ordinates

This problem requires the determination of $(\phi,\Delta\lambda,\gamma)$ in terms of the defined projection co-ordinates (x_1,x_2), using the intermediate isometric parameters $(\Delta\lambda,u_2)$.

A third problem is described as case 3 in section 5.4.

6.2 Conversion of ellipsoidal co-ordinates to projection co-ordinates

Problem:-

Stated as case a in section 6.1.
Given $(\Delta\lambda,\phi)$ at the general point P on an ellipsoid, and the origin $P_o(0,\phi_o)$ on the central meridian;
required to find the TM projection co-ordinates $(x_1,x_2) \equiv (E,N)$ of the equivalent point P* on the projection plane, as shown in figure 6.1, and the convergence γ in terms of the geographical co-ordinates.

The general mapping equation is

$$x_1 + ix_2 = f(u_1 + iu_2)$$

where $u_1 = \Delta\lambda$ and u_2 is the isometric latitude, given by equation 5.25. The characteristic properties of the transverse Mercator projection are

a. the central meridian plots as the straight line $x_1 = 0$;

Thus, when $u_1 = \Delta\lambda = 0$,

$x_1 = 0$ and $ix_2 \overset{(5.6)}{=} f(iu_2)$.

and b. the central meridian plots true to scale.

Hence when $x_1 = 0,\ x_2 = \int_{\phi_o}^{\phi} \rho\, d\phi \qquad \ldots\ldots \qquad 6.1.$

The right hand side of equation 5.6 can be expanded as a

Taylor series, $f(u_1+ iu_2)$ being an analytic function whose derivatives exist. Such an expansion is a power series in $\Delta\lambda$ and is convergent as

$$\Delta\lambda \not> 3^{o} (\simeq 5\times 10^{-2})$$

In such a case,

$$x_1 + ix_2 = f(\Delta\lambda + iu_2) = f(iu_2) + \sum_{i=1}^{5} \frac{(\Delta\lambda)^i}{i!} \frac{d^i}{d(iu_2)^i}\left(f(iu_2)\right) + o\{10^{-10}\} \quad \ldots . \quad 6.2.$$

As $f(iu_2)$ is the value of $f(u_1+ iu_2)$ when $u_1= 0$, i.e., on the central meridian, it follows from b above that

$$f(iu_2) = ix_2 = i\int_{\phi_o}^{\phi} \rho \, d\phi \quad \ldots . \quad 6.3.$$

From equation 5.25,

$$\frac{d}{d(iu_2)}\left(f(iu_2)\right) = \frac{\partial}{\partial\phi}\left(f(iu_2)\right)\frac{d\phi}{du_2}\cdot\frac{du_2}{d(iu_2)}$$

$$= i\rho \; \frac{\nu\cos\phi}{\rho}\cdot\frac{1}{i} = \nu\cos\phi \quad \ldots \quad 6.4.$$

From equation 18 in appendix c,

$$\frac{d^2}{d(iu_2)^2}\left(f(iu_2)\right) = \frac{\partial}{\partial\phi}(\nu\cos\phi)\cdot\frac{d\phi}{du_2}\cdot\frac{du_2}{d(iu_2)} = -\rho\sin\phi\cdot\frac{\nu\cos\phi}{\rho}\cdot\frac{1}{i}$$

$$= i\nu\cos\phi\sin\phi \quad \ldots\ldots \quad 6.5.$$

Further differentiation in a similar manner gives

$$\frac{d^3}{d(iu_2)^3}\left(f(iu_2)\right) = i\,\frac{\partial}{\partial\phi}(\nu\cos\phi\sin\phi)\;\frac{\nu\cos\phi}{\rho}\;\frac{1}{i}$$

$$= \left(-\rho\sin^2\phi + \nu\cos^2\phi\right)\;\frac{\nu\cos\phi}{\rho} = \cos^2\phi\left(\frac{\nu}{\rho} - \tan^2\phi\right)\nu\cos\phi$$

$$\ldots\ldots \quad 6.6.$$

As the derivative of ν/ρ with respect to ϕ is given by equation 30 in appendix e as

$$\frac{\partial}{\partial\phi}\left(\frac{\nu}{\rho}\right) = -2\tan\phi\left(\frac{\nu}{\rho} - 1\right) \quad \ldots\ldots \quad 6.7,$$

$$\frac{d^4}{d(iu_2)^4}\left(f(iu_2)\right) = \frac{\partial}{\partial\phi}\left(\left(\frac{\nu}{\rho} - \tan^2\phi\right)\nu\cos^3\phi\right)\frac{\nu\cos\phi}{\rho}\;\frac{1}{i}$$

$$= -i\,\frac{\nu\cos\phi}{\rho}\Bigg[\left(\frac{\nu}{\rho} - \tan^2\phi\right)\left(2\cos\phi(-\sin\phi)\nu\cos\phi - \rho\sin\phi\cos^2\phi\right) + $$

$$+ \nu\cos^3\phi\left(-2\tan\phi\left(\frac{\nu}{\rho} - 1\right) - 2\tan\phi\sec^2\phi\right)\Bigg]$$

$$= i\,\frac{\nu\cos\phi}{\rho}\;\rho\cos^2\phi\sin\phi\left[\left(2\frac{\nu}{\rho} + 1\right)\left(\frac{\nu}{\rho} - \tan^2\phi\right) + 2\frac{\nu}{\rho}\left(\frac{\nu}{\rho} - 1+1+\tan^2\phi\right)\right]$$

$$= i\nu\cos^3\phi\sin\phi\left(\frac{\nu}{\rho} - \tan^2\phi + 4\left(\frac{\nu}{\rho}\right)^2\right) \quad \ldots\ldots \quad 6.8.$$

In computing $\frac{d^5}{d(iu_2)^5}\left(f(iu_2)\right)$, it should be borne in mind that $(\Delta\lambda)^5/5! \simeq 10^{-9}$. Also,

$$r = \frac{\nu}{\rho} = \frac{1 - e^2\sin^2\phi}{1 - e^2} = \frac{1 - e^2 + e^2(1 - \sin^2\phi)}{1 - e^2} = 1 + \varepsilon\cos^2\phi$$

$$= 1 + o\{6\times10^{-3}\} \qquad \ldots\ldots \qquad 6.9.$$

Thus,

$$\frac{d^5}{d(iu_2)^5}\left(f(iu_2)\right) = i\frac{\partial}{\partial\phi}\left(\nu\cos^3\phi\sin\phi(r - \tan^2\phi + 4r^2)\right)\frac{\nu\cos\phi}{\rho}\frac{1}{i}$$

$$= \frac{\nu\cos\phi}{\rho}\Big[(r - \tan^2\phi + 4r^2)\big(-\rho\sin\phi\cos^2\phi\sin\phi -$$

$$\nu\cos\phi\sin\phi(2\cos\phi\sin\phi) + \nu\cos^3\phi\cos\phi\big) + \nu\cos^3\phi\sin\phi\times$$

$$\big(-2\tan\phi(r - 1) - 2\tan'\phi\sec^2\phi + 8r\{-2\tan\phi(r - 1)\}\big)\Big]$$

The above expression is simplified by grouping the factor $\rho\cos^4\phi$ and replacing $\sec^2$ by $(1+\tan^2\ )$, when

$$\frac{d^5}{d(iu_2)^5}\left(f(iu_2)\right) = -\nu\cos^5\phi\Big[(r - \tan^2\phi + 4r^2)\big(\tan^2\phi + 2r\tan^2\phi - r\big) +$$

$$r\tan^2\phi\big((r - 1)(2 + 16r) + 2 + 2\tan^2\phi\big)\Big]$$

$$= -\nu\cos^5\phi\Big[\tan^4\phi\big(-1 + r(2 - 2)\big) + \tan^2\phi\big(r(1+1) +$$

$$r^2(2+4-16+2) + r^3(+8+16)\big) - r^2 - 4r^3\Big]$$

$$= \nu\cos^5\phi\left[\tan^4\phi - \tan^2\phi(2r - 8r^2 + 24r^3) + r^2 + 4r^3\right]$$

$$\ldots\ldots 6.10.$$

Note :-

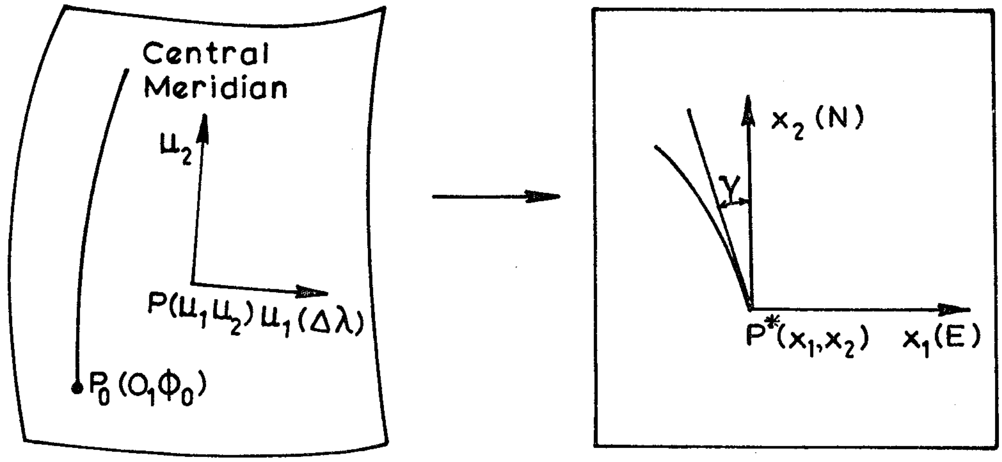
The above forms can be reduced to any of the common expressions (e.g., those given by Jordan-Eggert and/or Clark) by making the appropriate substitutions, bearing in mind that

$$r = \frac{\nu}{\rho} = 1 + o\{6\times10^{-3}\} = 1 + \frac{e^2}{1-e^2}\cos^2\phi = 1 + \varepsilon\cos^2\phi$$

$$= 1 + \eta^2 ,$$

where η^2 is the abbreviation used for the term of order e^2 by Jordan-Eggert. From a computational point of view, the difference of r from unity can be considered to be negligible when evaluating terms of small order.

A study of equations 6.3 to 6.10 shows that the terms obtained for even values of i in equation 6.2, along with $f(iu_2)$

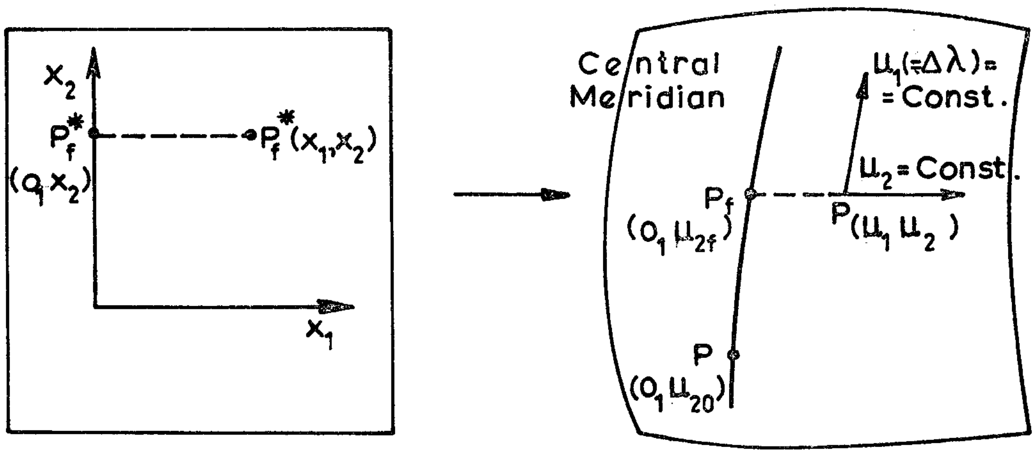


Reference Ellipsoid

Projection Plane

FIG (6·1)

The transverse Mercator projection of the reference ellipsoid the direct case



Projection Plane

Reference Ellipsoid

FIG. (6·2)

The transverse Mercator projection of the reference ellipsoid the reverse case are imaginary while those for odd values of i are real. On equating real and imaginary parts of this latter equation, the required expressions for (x_1,x_2) in terms of $(\phi,\Delta\lambda)$ are given as

$$x_1 = \nu \cos\phi\,\Delta\lambda + \frac{1}{6}\nu\cos^3\phi(r-\tan^2\phi)(\Delta\lambda)^3 + \frac{1}{120}\nu\cos^5\phi\big(\tan^4\phi - 2r(1-4r+12r^2)\tan^2\phi + r^2+4r^3\big)(\Delta\lambda)^5 + o\{10^{-10}\} \quad \ldots.6.11$$

and

$$x_2 = \int_{\phi_o}^{\phi}\rho\,d\phi + \frac{1}{2}\nu\cos\phi\sin\phi(\Delta\lambda)^2 + \frac{1}{24}\nu\cos^3\phi\sin\phi(r-\tan^2\phi+4r^2)(\Delta\lambda)^4 + o\{10^{-10}\} \quad \ldots\ldots \quad 6.12.$$

Notes:-

1. Equations 6.11 and 6.12 can easily be transformed into the forms given by Jordan-Eggert(1962, III Second Half, section 32)or Thomas(1952,p.2) by replacing r by $(1+\eta^2)$. The current form has the advantage of having less terms and being easier to differentiate. It is also no more difficult to evaluate on an electronic computer. In the present form, it is possible to perform a differentiation to any order with a knowledge of only two special derivatives,

i.e., $\frac{\partial}{\partial\phi}(\nu\cos\phi)$ and $\frac{\partial}{\partial\phi}(r)$.

2. For the equivalent spherical expressions, see equations 3.46 and 3.57. The discrepancies are of order

$(\Delta\lambda)^3\times 6\times 10^{-3}$,

provided the correct radius of curvature is used.

6.3 Grid convergence in terms of ellipsoidal co-ordinates

On referring to section 5.4, it can be seen that the simplest expressions for convergence are those obtained from equations 6.11 and 6.12, on partial differentiation with respect to λ. From equation 5.37,

$$\tan = \frac{(\partial x_2/\partial\lambda)}{(\partial x_1/\partial\lambda)} \quad \ldots\ldots \quad 6.13.$$

An expression for γ which will have an accuracy of 0".01 under all circumstances for six degree zones, can be obtained only if all terms of order larger than $(\Delta\lambda)^6$ are considered. It will therefore be necessary to consider the term arising when i=6 in equation 6.2. This is obtained by differentiating equation 6.10 and adopting the abbreviations

$$t = \tan\phi \quad ; \qquad \frac{\partial}{\partial\phi}(r) \overset{(6.7)}{=} -2t(r-1) \qquad \ldots\ldots \quad 6.14,$$

when the use of equation 6.10 and the relation

$$\sec^2\phi = 1 + t^2$$

gives

$$\frac{d^6}{d(iu_2)^6}\left(f(iu_2)\right) = \frac{\partial}{\partial\phi}\left[\nu\cos^5\phi\left(t^4 - t^2(2r - 8r^2 + 24r^3) + r^2 + 4r^3\right)\right]\frac{\nu\cos\phi}{\rho}\;\frac{1}{i}$$

$$= -i\,\frac{\nu\cos\phi}{\rho}\Big[\left(t^4 - t^2(2r - 8r^2 + 24r^3) + r^2 + 4r^3\right)\left(-\rho\sin\phi\cos^4\phi - 4\nu\cos\phi\cos^3\phi\sin\phi\right) + \nu\cos^5\phi\left(4t^3(1+t^2) - 4t\,r(1+t^2)(1-4r+12r^2) - 2t^2(1-8r+36r^2)\{-2t(r-1)\} + (2r+12r^2)\{-2t(r-1)\}\right)\Big]$$

$$= i\,\frac{\nu\cos\phi}{\rho}\,\rho\sin\phi\cos^4\phi\Big[(1+4r)\left(t^4 - t^2(2r-8r^2+24r^3) + r^2 + 4r^3\right) + r\left(4r(1+t^2)(1-4r+12r^2) - 4t^2(1-8r+36r^2)(r-1) + 4r(1+6r)(r-1) - 4t^2(1+t^2)\right)\Big]$$

$$= i\nu\cos^5\phi\sin\phi\Big[t^4\left(1 + r(4-4)\right) + t^2\left(r(-2-4+4) + r^2(+8-8+4-36) + r^3(-24+32-16+176) + r^4(-96+48-144)\right) + r^2(1+4-4) + r^3(4+4-16-20) + r^4(16+48+24)\Big]\ .$$

Thus,

$$\frac{d^6}{d(iu_2)^6}\left(f(iu_2)\right) = i\nu\cos^5\phi\sin\phi\left(t^4 + t^2(-2r-32r^2+168r^3-192r^4) + r^2 - 28r^3 + 88r^4\right)\ .$$

The appropriate expression for x_2 which is of adequate precision prior to differentiation is

$$x_2 = \int_{\phi_o}^{\phi}\rho\,d\phi + \frac{1}{2}\nu\cos\phi\sin\phi(\Delta\lambda)^2 + \frac{1}{24}\nu\cos^3\phi\sin\phi(r-t^2+4r^2)(\Delta\lambda)^4 + \frac{1}{720}\nu\cos^5\phi\sin\phi\left(t^4 - 2t^2r(1+16r-84r^2+96r^3) + r^2 - 28r^3 + 88r^4\right)(\Delta\lambda)^6$$

$$\ldots\ldots \qquad 6.15.$$

The partial differentiation of equations 6.15 and 6.11 with respect to λ gives

$$\frac{\partial x_2}{\partial\lambda} = \nu \cos\phi \sin\phi\, \Delta\lambda\Big[1+ \tfrac{1}{6}\cos^2\phi(r-t^2+4r^2)(\Delta\lambda)^2 + \tfrac{1}{120}\cos^4\phi\big(t^4-$$

$$2t^2r(1+16r-84r^2+96r^3)+ r^2- 28r^3+ 88r^4\big)(\Delta\lambda)^4 \quad + o\{(\Delta\lambda)^6\}\Big]$$

and 6.16

$$\frac{\partial x_1}{\partial\lambda} = \nu \cos\phi\Big[1+ \tfrac{1}{2}\cos^2\phi(r-t^2)(\Delta\lambda)^2 + \tfrac{1}{24}\cos^4\phi\big(t^4-2t^2r(1-4r+12r^2) +$$

$$r^2+ 4r^3\big)(\Delta\lambda)^4 + o\{(\Delta\lambda)^6\}\Big] \quad \text{...... 6.17.}$$

The substitution of equations 6.16 and 6.17 in 6.13, and the use of the binomial expansion on the denominator, gives the following relations. As

$$\Big[1 + \tfrac{1}{2}\cos^2\phi(r-t^2)(\Delta\lambda)^2+ \tfrac{1}{24}\cos^4\phi\big(t^4-2t^2r(1-4r+12r^2)+ r^2+ 4r^3\big)(\Delta\lambda)^4\Big]^{-1}$$

$$= 1 - \tfrac{1}{2}\cos^2\phi(r-t^2)(\Delta\lambda)^2- \tfrac{1}{24}\cos^4\phi\big(t^4(1-6)-2t^2r(1-6-4r+12r^2)+ r^2(1-6) +$$

$$4r^3\big)(\Delta\lambda)^4 + o\{(\Delta\lambda)^6\} ,$$

therefore,

$$\tan\gamma = \Delta\lambda \sin\phi\Big[1 + \tfrac{1}{6}\cos^2\phi(r-t^2+4r^2)(\Delta\lambda)^2 + \tfrac{1}{120}\cos^4\phi\big(t^4- 2t^2r(1+$$

$$16r-84r^2+96r^3)+ r^2- 28r^3+ 88r^4\big)(\Delta\lambda)^4\Big]\Big[1 - \tfrac{1}{2}\cos^2\phi(r-t^2)(\Delta\lambda)^2+$$

$$\tfrac{1}{24}\cos^4\phi\big(5t^4- 2t^2r(5+4r-12r^2)+ 5r^2- 4r^3\big)(\Delta\lambda)^4\Big]$$

$$= \Delta\lambda \sin\phi\Big[1 + \tfrac{1}{6}\cos^2\phi\big(r(1-3)+ 4r^2- t^2(1-3)\big)(\Delta\lambda)^2 +$$

$$\tfrac{1}{120}\cos^4\phi\big(t^4(1+25-10)+ t^2r\{(-2-50+20)+ r(-32-40+40) +$$

$$r^2(168+120)-192r^3\}+ r^2(1+25-10)- r^3(28+40+20)+ 88r^4\big)(\Delta\lambda)^4\Big].$$

Completion of the additions and slight simplification gives

$$\tan\gamma = \Delta\lambda \sin\phi\Big[1 + \tfrac{1}{3}\cos^2\phi(2r^2-r+t^2)(\Delta\lambda)^2 + \tfrac{1}{15}\cos^4\phi\ \big(2t^4-$$

$$4t^2r(1+r-9r^2+6r^3)+ 2r^2- 11r^3+ 11r^4\big)(\Delta\lambda)^4\Big]+o\{(\Delta\lambda)^7\} \quad \text{...6.18.}$$

As

$$\gamma = \tan\gamma - \frac{\tan^3\gamma}{3} + \frac{\tan^5\gamma}{5} ,$$

$$-\frac{\tan^3\gamma}{3} = - \tfrac{1}{3}\sin^3\phi(\Delta\lambda)^3\Big[1 + \tfrac{1}{3}\cos^2\phi(2r^2-r+t^2)(\Delta\lambda)^2\Big]^3$$

$$= \tfrac{1}{3}\sin\phi \cos^2\phi(\Delta\lambda)^3\big(-t^2- \cos^2\phi\ t^2(2r^2-r+t^2)(\Delta\lambda)^2\big)+ o\{(\Delta\lambda)^7\}$$

and

$$\frac{\tan^5\gamma}{5} = \frac{1}{5}\sin^5\phi\ (\Delta\lambda)^5 = \frac{1}{15}\cos^4\phi\ \sin\phi(\Delta\lambda)^5\left(3t^4\right) + o\{(\Delta\lambda)^7\}.$$

Thus

$$\gamma = \Delta\lambda\ \sin\phi\Big[1 + \frac{1}{3}\cos^2\phi(2r^2-r+t^2-t^2)(\Delta\lambda)^2 + \frac{1}{15}\cos^4\phi\big(t^4(2-5+3) - t^2r\left(4-5+r(4+10)-36r^2+24r^3\right) + 2r^2 - 11r^3 + 11r^4\big)(\Delta\lambda)^4\Big].$$

Thus the final expression for γ is

$$\gamma = \Delta\lambda\ \sin\phi\Big[1 + \frac{1}{3}\cos^2\phi(2r^2-r)(\Delta\lambda)^2 + \frac{1}{15}\cos^4\phi\left(t^2r(1-14r+36r^2-24r^3) + 2r^2 - 11r^3 + 11r^4\right)(\Delta\lambda)^4 + o\{(\Delta\lambda)^6\}\Big] \quad \ldots\ldots \quad 6.19.$$

Notes:-

1. For most practical purposes, the term in $(\Delta\lambda)^5$ can be simplified by putting

$$r = 1 + o\{6\times10^{-3}\},$$

when

$$\frac{(\Delta\lambda)^5}{15}\times o\{6\times10^{-3}\} \simeq 10^{-10}.$$

Thus, for $|\phi| \ngtr \frac{1}{4}\pi$,

$$\gamma = \Delta\lambda\ \sin\phi\Big[1 + \frac{1}{3}\cos^2\phi(2r^2-r)(\Delta\lambda)^2 + \frac{1}{15}\cos^4\phi\left(2-t^2\right)(\Delta\lambda)^4\Big] \quad \ldots\ldots \quad 6.20$$

with adequate precision.

2. Also see equation 3.53 for the spherical case. The difference in magnitude between the spherical and ellipsoidal expressions have an order of magnitude

$$(\Delta\lambda)^3\times6\times10^{-3} \simeq 10^{-6} < 1 \text{ arc sec for } 6^{\circ} \text{ zones.}$$

6.4 Conversion of projection co-ordinates to ellipsoidal co-ordinates

Problem:-

Stated as case b in section 6.1. Given the TM projection co-ordinates (x_1, x_2) of the general point P* on the projection plane, and the geographical co-ordinates $(0,\phi_o)$ of the origin.

Required to find the surface parameters $\Delta\lambda$ and u_2 (and hence ϕ), along with the convergence in terms of the projection co-ordinates.

The problem is apparently complicated by the fact that all the small order correction terms are latitude dependent. As in section 3.6.3.4, use is made of the fact that for any given value of x_2, there is a unique latitude ϕ_f, called the foot point latitude, which is the latitude of the point P^*_f in figure 6.2, on the central meridian and with the same x_2 value as the general point P*. The difference between the latitude ϕ of P and ϕ_f can be seen to be of order

$$\tfrac{1}{2}(\Delta\lambda)^2 \approx 10^{-3} \text{ rad}$$

from the development preceding equation 3.66. Thus x_2 can be uniquely defined by the equation

$$x_2 = (x_2)_{x_1=0} = \int_{\phi_o}^{\phi_f} \rho \, d\phi \qquad \ldots\ldots \qquad 6.21.$$

The required mapping equation in terms of the orthogonal isometric system of co-ordinates on the surface of the ellipsoid is

$$\Delta\lambda + iu_2 = f(x_1 + ix_2) \qquad \ldots\ldots 6.22,$$

where u_2 is the isometric latitude, given by equation 5.25. As the mapping function is analytic, the application of Taylor's theorem to equation 6.22 gives

$$\Delta\lambda + iu_2 = f(ix_2) + \sum_{i=1}^{n} \frac{x^i}{i!} \left(\frac{d^i}{d(ix_2)^i}\left(f(ix_2)\right)\right)_{x_1=0}$$

$$\ldots\ldots \qquad 6.23,$$

where the derivatives are evaluated at $x_1 = 0$, and hence equation 6.21 applies. Initialisation *on the projection plane* gives

$$\Delta\lambda = 0 \qquad \text{when} \qquad x_1 = 0.$$

Thus, on the x_2 axis itself,

$$iu_{2f} = f(ix_2) \overset{(5.25)}{=} i \int_0^{\phi_f} \frac{\rho}{\nu \cos\phi} \phi \quad \ldots \ 6.24,$$

where ϕ_f is the foot point latitude, as described above. Thus,

$$\frac{d}{d(ix_2)}\left(f(ix_2)\right) = \left(\frac{\partial}{\partial\phi}\{f(ix_2)\} \cdot \frac{d\phi}{dx_2} \cdot \frac{dx_2}{d(ix_2)} \right)_{x_1=0}$$

$$= \left(i \frac{\rho}{\nu \cos\phi} \ \frac{1}{\rho} \ \frac{1}{i} \right)_{\phi=\phi_f} = \frac{1}{\nu_f \cos\phi_f} \quad ..6.25a,$$

where the subscript $_f$ refers to evaluation at the foot point latitude. Further differentiation on the same lines using equation 18 in appendix c gives

$$\frac{d^2}{d(ix_2)^2}\left(f(ix_2)\right) = \left(\frac{\partial}{\partial\phi}\left(\frac{1}{\nu \cos\phi}\right) \frac{1}{\rho} \ \frac{1}{i} \right)_{x_1=0} = -\left(\frac{(-\rho \sin\phi)}{(\nu \cos\phi)^2} \ \frac{1}{i\rho} \right)_{\phi=\phi_f}$$

$$= -i\left(\frac{1}{\nu^2 \cos\phi} \tan\phi \right)_{\phi=\phi_f} \qquad \ldots\ldots \qquad 6.25b,$$

and

$$\frac{d^3}{d(ix_2)^3}\left(f(ix_2)\right) = \left[\frac{\partial}{\partial\phi}\left(-i\frac{\sin\phi}{(\nu\cos\phi)^2}\right)\cdot\frac{1}{\rho}\cdot\frac{1}{i}\right]_{x_1=0}$$

$$= \left[-\frac{1}{\rho}\left[-\frac{2\sin\phi}{(\nu\cos\phi)^3}(-\rho\sin\phi) + \frac{\cos\phi}{(\nu\cos\phi)^2}\right]\right]_{x_1=0}$$

$$= -\left[\left(2\sin^2\phi + \frac{\nu}{\rho}\cos^2\phi\right)\frac{1}{\nu^3\cos^3\phi}\right]_{x_1=0}$$

$$\overset{(6.9)}{=} -\left(\frac{1}{\nu^3\cos\phi}\{2\tan^2\phi + r\}\right)_{\phi=\phi_f} \qquad \ldots\ldots \qquad 6.25c.$$

When $i = 6$ in equation 6.23, the order of magnitude of the term is controlled by the expression

$$\frac{1}{720}\left(\frac{x_1}{\nu}\right)^6 \simeq 10^{-10}$$

Hence truncation of the series at $i = 5$ will be adequate for most geodetic purposes. Further differentiation, using the abbreviated forms defined at equations 6.14, gives

$$\frac{d^4}{d(ix_2)^4}\left(f(ix_2)\right) = \left[\frac{\partial}{\partial\phi}\left[-\left(2\sin^2\phi + r\cos^2\phi\right)\frac{1}{\nu^3\cos^3\phi}\right]\cdot\frac{1}{\rho}\cdot\frac{1}{i}\right]_{x_1=0}$$

$$= i\left[\frac{1}{\rho}\left(\frac{3\rho\sin\phi}{\nu^4\cos^4\phi}\left(2\sin^2\phi + r\cos^2\phi\right) + \frac{1}{\nu^3\cos^3\phi}\left(4\sin\phi\cos\phi + \cos^2\phi\{-2t(r-1)\} - 2r\cos\phi\sin\phi\right)\right)\right]_{x_1=0}.$$

Thus

$$\frac{d^4}{d(ix_2)^4}\left(f(ix_2)\right) = i\left[\frac{1}{\nu^4\cos^4\phi}\left[3\sin\phi\left(2\sin^2\phi + r\cos^2\phi\right) + r\cos^2\phi\sin\phi\left(4-2(r-1)-2r\right)\right]\right]_{x_1=0}$$

$$= i\left[\frac{1}{\nu^4\cos\phi}\left(t\{3(2t^2+r) + r(6-4r)\}\right)\right]_{x_1=0}$$

$$= i\left[\frac{1}{\nu^4\cos\phi}\,t\left(6t^2 + 9r - 4r^2\right)\right]_{\phi=\phi_f} \qquad \ldots\ldots \qquad 6.25d.$$

$$\frac{d^5}{d(ix_2)^5}\left(f(ix_2)\right) = \left[\frac{\partial}{\partial\phi}\left(\frac{1}{\nu^4\cos^4\phi}\sin\phi\left(6\sin^2\phi + 9r\cos^2\phi - 4r^2\cos^2\phi\right)\right)\frac{1}{\rho}\;\frac{1}{i}\right]_{x_1=0}$$

$$\frac{d^5}{d(ix_2)^5}\left(f(ix_2)\right) = \left[\frac{1}{\rho}\left[\left(6\sin^2\phi + 9r\cos^2\phi - 4r^2\cos^2\phi\right)\left(\frac{4\rho\sin^2\phi}{\nu^5\cos^5\phi} + \right.\right.\right.$$

$$\left.\frac{\cos\phi}{\nu^4\cos^4\phi}\right) + \frac{\sin\phi}{\nu^4\cos^4\phi}\left(12\sin\phi\cos\phi - 2\cos\phi\sin\phi(9r-4r^2) + \right.$$

$$\left.\left.\left.\cos^2\phi(9-8r)\{-2t(r-1)\}\right)\right]\right]_{x_1=0}$$

$$= \left[\frac{1}{\rho}\ \frac{\rho}{\nu^5\cos^5\phi}\cos^4\phi\left[\left(6t^2+9r-4r^2\right)(4t^2+r) + r\left(12t^2 - \right.\right.\right.$$

$$\left.\left.\left.2t^2(9r-4r^2) - 2t^2(r-1)(9-8r)\right)\right]\right]_{x_1=0}$$

$$= \left[\frac{1}{\nu^5\cos\phi}\left[\left(6t^2+9r-4r^2\right)(4t^2+r) + r\left(12t^2-2t^2(26r-12r^2-9)\right)\right]\right]_{x_1=0}$$

$$= \left[\frac{1}{\nu^5\cos\phi}\left(24t^4 + t^2\left(r(36+6+12+18) + r^2(-16-52) + 24r^3\right) + \right.\right.$$

$$\left.\left.9r^2 - 4r^3\right)\right]_{x_1=0}.$$

Thus

$$\frac{d^5}{d(ix_2)^5}\left(f(ix_2)\right) = \left[\frac{1}{\nu^5\cos\phi}\left(24t^4 + t^2(72r-68r^2+24r^3) + 9r^2 - 4r^3\right)\right]_{\phi=\phi_f}$$

..... 6.25e.

The substitution of equations 6.25 a to e, in equation 6.23, together with the equation of real and imaginary parts, gives the expressions for $\Delta\lambda$ and u_2 as

Real parts

$$\Delta\lambda = \left[\frac{1}{\cos\phi}\left[\left(\frac{x_1}{\nu}\right) - \frac{1}{6}(r+2t^2)\left(\frac{x_1}{\nu}\right)^3 + \frac{1}{120}\left(9r^2-4r^3+24t^4+\right.\right.\right.$$

$$\left.\left.\left.4t^2(18r-17r^2+6r^3)\right)\left(\frac{x_1}{\nu}\right)^5\right]\right]_{\phi=\phi_f} + o\{10^{-10}\}.. \qquad 6.26.$$

Imaginary parts

As set out in equation 6.24,

$$u_2 = \left(u_2 - \frac{t}{\cos\phi}\left[\frac{1}{2}\left(\frac{x_1}{\nu}\right)^2 - \frac{1}{24}(6t^2+9r-4r^2)\left(\frac{x_1}{\nu}\right)^4\right]\right]_{\phi=\phi_f} + o\{10^{-10}\}$$

...... 6.27,

where

$$t = \tan\phi \quad ; \quad r = \frac{\rho}{\nu} \qquad \text{......} \qquad 6.28.$$

The difference $\Delta\phi$ between the latitude ϕ of P and the foot point latitude ϕ_f, given by

$$\Delta\phi = \phi - \phi_f$$

is related to the difference Δu_2 between the equivalent isometric latitudes, given by

$$\Delta u_2 = u_2 - u_{2f}$$

through a Taylor's series of the form

$$\Delta\phi = \Delta u_2 \frac{d\phi}{du_2} + \frac{1}{2}(\Delta u_2)^2 \frac{d^2\phi}{du_2^2} + o\{10^{-10}\} ,$$

the differential coefficients being evaluated at $x_1 = 0$ at the foot point. From equation 5.25,

$$\frac{d\phi}{du_2} = \frac{\nu \cos\phi}{\rho} \qquad \ldots\ldots \qquad 6.29.$$

The use of equation 6.7 gives

$$\frac{d}{du_2}\left(\frac{d\phi}{du_2}\right) = \frac{\partial}{\partial\phi}\left(\frac{d\phi}{du_2}\right)\frac{d\phi}{du_2} = \left(\cos\phi \frac{\partial}{\partial\phi}\left(\frac{\nu}{\rho}\right) - \frac{\nu}{\rho}\sin\phi\right)\frac{\nu\cos\phi}{\rho}$$

$$= -\left(2\sin\phi(r-1) + r\sin\phi\right) r\cos\phi$$

$$= r\sin\phi\cos\phi(2-3r) \qquad \ldots\ldots \qquad 6.30$$

As

$$(\Delta u_2)^2 \overset{(6.27)}{=} \left(\frac{1}{4}\ \frac{t^2}{\cos^2\phi}\left(\frac{x_1}{\nu}\right)^4 + o\{10^{-10}\}\right)_{\phi=\phi_f} ,$$

evaluation at the foot point latitude using equations 6.29 and 6.30 gives

$$\Delta\phi = \left(\Delta u_2 r\cos\phi + \frac{1}{2}(\Delta u_2)^2(2-3r)\,r\,t\cos^2\phi\right)_{\phi=\phi_f}$$

The use of equation 6.27 gives

$$\Delta\phi = \phi - \phi_f = \left(-r\,t\left[\frac{1}{2}\left(\frac{x_1}{\nu}\right)^2 - \frac{1}{24}\left(t^2(6+6-9r) + 9r - 4r^2\right)\left(\frac{x_1}{\nu}\right)^4\right]\right)_{\phi=\phi_f}.$$

Thus

$$\phi = \left(\phi - \frac{1}{2}\frac{x_1^2}{\rho\nu}t + \frac{1}{24}t\left(t^2(12-9r) + 9r - 4r^2\right)\frac{x_1^4}{\rho\nu^3}\right)_{\phi=\phi_f} + o\{10^{-10}\} \quad \ldots 6.31.$$

Equations 6.26 and 6.31 define the conversion of projection co-ordinates to geographical co-ordinates.

6.5 Grid convergence from projection co-ordinates

The grid convergence in terms of projection co-ordinates is obtained by the use of equation 5.40

$$\tan\ \gamma = -\frac{(\partial u_{2}/\partial x_{1})}{(\partial u_{1}/\partial x_{1})}\ .$$

As in section 6.3, it is necessary to express u_2 in terms of x_1 to order $(x_1/\nu)^6$ so that the final expression may contain all the relevant terms whose contributions are in excess of 0.″01. The term not included in equation 6.27 is

$$\frac{x_{1}^{6}}{6!}\ \frac{d^{6}}{d(ix_{2})^{6}}\{f(ix_{2})\}$$

From equation 6.25e,

$$\frac{d^{5}}{d(ix_{2})^{5}}\{f(ix_{2})\} = \left.\frac{\cos^{4}\phi}{(\nu\ \cos\ \phi)^{5}}\left(24t^{4}+\ t^{2}(72r-68r^{2}+24r^{3})+\ 9r^{2}-\ 4r^{3}\right)\right|_{x_1=0}\ .$$

The differentiation of this relation by parts, using equation 6.14 and the relation

$$\frac{d}{d\phi}(t) = \sec^{2}\phi = 1 + t^{2},$$

gives

$$\frac{d^{6}}{d(ix_{2})^{6}}\{f(ix_{2})\} = \left.\left\{\frac{\partial}{\partial\phi}\left(\frac{d^{5}}{d(ix_{2})^{5}}\{f(ix_{2})\}\right)\cdot\frac{1}{\rho}\cdot\frac{1}{i}\right\}\right|_{x_1=0}$$

$$= \left\{-i\ \frac{1}{\rho}\left[\left(\frac{5\rho}{(\nu\ \cos\ \phi)^{6}}\sin\ \phi\ \cos^{4}\phi - \frac{4\cos^{3}\phi\ \sin\ \phi}{(\nu\ \cos\ \phi)^{5}}\right)\left(24t^{4}+\ t^{2}(72r-68r^{2}+24r^{3})+\ 9r^{2}-\ 4r^{3}\right) + \frac{1}{\nu^{5}\cos\ \phi}\left(96t^{3}(1+t^{2})(72r-68r^{2}+24r^{3})\ 2t(r-1)\{t^{2}(72-136r+72r^{2})+\ 18r-\ 12r^{2}\}\right)\right]\right\}_{x_1=0}$$

$$= \left\{-\ \frac{1}{i}\ \frac{t}{\nu^{6}\cos\ \phi}\left[(5-4r)\left(24t^{4}+\ t^{2}(72r-68r^{2}+24r^{3})+\ 9r^{2}-\ 4r^{3}\right) + r\left(96t^{4}+\ t^{2}\{96+144+r(144-416)+r^{2}(-136+416)+r^{3}(48-144)\}+ r(144+36)+\ r^{2}(-136-60)+\ r^{3}(48+24)\right)\right]\right\}_{x_1=0}$$

$$\frac{d^6}{d(ix_2)^6}\big(f(ix_2)\big) = \Big(-i\frac{t}{\nu^6\cos\phi}\big(t^4\{120+r(-96+96) + t^2\{r(360+240)+r^2(-340-288-272)$$

$$+r^3(120+272+280)+r^4(-96-96)\}+ r^2\{45+180\}+ r^3\{-20-36-196\}+$$

$$r^4\{16+72\}\big)\Big)_{x_1=0}.$$

Thus

$$\frac{d^6}{d(ix_2)^6}\big(f(ix_2)\big) = \Big(-i\,\frac{t}{\nu^6\cos\phi}\,\big(120t^4+ r\,t^2(600-900r+672r^2-192r^3)+$$

$$225r^2- 252r^3+ 88r^4\big)\Big)_{\phi=\phi_f} \qquad \ldots\ldots \qquad 6.32.$$

The combination of equation 6.32 with equation 6.27 gives

$$u_2 = \Big(u_2 - \frac{t}{\cos\phi}\,\Big(\frac{1}{2}\Big(\frac{x_1}{\nu}\Big)^2- \frac{1}{24}\big(6t^2+ 9r- 4r^2\big)\Big(\frac{x_1}{\nu}\Big)^4+ \frac{1}{720}\big(120t^4+$$

$$t^2r(600-900r+672r^2-192r^3)+ r^2(225-252r+ 88r^2)\big)\Big(\frac{x_1}{\nu}\Big)^6\Big)\Big)_{\phi=\phi_f},$$

which can be written as

$$u_2= \Big(u_2 - \frac{t}{\cos\phi}\,\Big(\frac{1}{2}\Big(\frac{x_1}{\nu}\Big)^2- \frac{1}{24}A\Big(\frac{x_1}{\nu}\Big)^4+ \frac{1}{720}B\Big(\frac{x_1}{\nu}\Big)^6\Big)\Big)_{\phi=\phi_f}.$$

Differentiation with respect to x_1 gives

$$-\frac{\partial u_2}{\partial x_1} = \Big(\frac{t}{\nu\cos\phi}\Big(\frac{x_1}{\nu}\Big)\Big(1 - \frac{1}{6}A\Big(\frac{x_1}{\nu}\Big) + \frac{1}{120}B\Big(\frac{x_1}{\nu}\Big)\Big)\Big)_{\phi=\phi_f}$$

If equation 6.26 is written as

$$\Delta\lambda = \Big(\frac{1}{\cos\phi}\Big(\Big(\frac{x_1}{\nu}\Big)- \frac{1}{6}C\Big(\frac{x_1}{\nu}\Big)^3+ \frac{1}{120}D\Big(\frac{x_1}{\nu}\Big)^5\Big)\Big)_{\phi=\phi_f},$$

$$\frac{\partial\lambda}{\partial x_1} =\Big(\frac{1}{\nu\cos\phi}\Big(1 - \frac{1}{2}C\Big(\frac{x_1}{\nu}\Big)^2+ \frac{1}{24}D\Big(\frac{x_1}{\nu}\Big)^4\Big)\Big)_{\phi=\phi_f} \qquad \ldots\ldots \qquad 6.33.$$

$$\tan\gamma = \Big(t\Big(\frac{x_1}{\nu}\Big)\Big(1 - \frac{1}{6}A\Big(\frac{x_1}{\nu}\Big)^2+ \frac{1}{120}B\Big(\frac{x_1}{\nu}\Big)^4\Big)\Big(1 - \frac{1}{2}C\Big(\frac{x_1}{\nu}\Big)^2+ \frac{1}{24}D\Big(\frac{x_1}{\nu}\Big)^4\Big)^{-1}\Big)_{\phi=\phi_f}$$

$$= \Big(t\Big(\frac{x_1}{\nu}\Big)\Big(1 - \frac{1}{6}A\Big(\frac{x_1}{\nu}\Big)^2+ \frac{1}{120}B\Big(\frac{x_1}{\nu}\Big)^4\Big)\Big(1 + \frac{1}{2}C\Big(\frac{x_1}{\nu}\Big)^2+ \frac{1}{24}(6C -D)\Big(\frac{x_1}{\nu}\Big)^4\Big)\Big)_{\phi=\phi_f}$$

$$=\Big(t\,\Big(\frac{x_1}{\nu}\Big)\Big(1 + \frac{1}{6}(3C-A)\Big(\frac{x_1}{\nu}\Big)^2+ \frac{1}{120}(B-5D+30C -10AC)\Big(\frac{x_1}{\nu}\Big)^4\Big)\Big)_{\phi=\phi_f}.$$

On making the appropriate substitutions,

$$\tan\gamma = \left[t\left(\frac{x_1}{\nu}\right)\left[1 + \frac{1}{6}\left(-6r+ 4r^2\right)\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{120}\left(t^4(120-120+120-120)+ t^2\{r(600-360+120-60-180)+r^2(-900+340+80)+r^3(672-120)-192r^4\}+ r^2(225-45+30-90)+ r^3(-252+20+40)+ 88r^4\right)\left(\frac{x_1}{\nu}\right)^4\right]\right]_{\phi=\phi_f}$$

$$= \left[t\left(\frac{x_1}{\nu}\right)\left[1 - \frac{1}{3}\left(3r- 2r^2\right)\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{120}\left(t^2(120r-480r^2+552r^3-192r^4)+ 120r^2- 192r^3+ 88r^4\right)\left(\frac{x_1}{\nu}\right)^4\right]\right]_{\phi=\phi_f} \quad \ldots\ldots \quad 6.34.$$

As

$$\gamma = \tan\gamma - \frac{1}{3}\tan^3\gamma + \frac{1}{5}\tan^5\gamma - \ldots\ldots,$$

the small order terms are evaluated from equation 6.34 as

$$-\frac{1}{3}\tan^3\gamma \doteq \left[-\frac{1}{3}t^3\left(\frac{x_1}{\nu}\right)^3\left[1 - \left(3r- 2r^2\right)\left(\frac{x_1}{\nu}\right)^2\right]\right]_{\phi=\phi_f} + o\{10^{-10}\}$$

and

$$\frac{1}{5}\tan^5\gamma = \frac{1}{5}t_f^5\left(\frac{x_1}{\nu_f}\right)^5 + o\{10^{-10}\}.$$

Thus,

$$\gamma = \left[t\left(\frac{x_1}{\nu}\right)\left[1 - \frac{1}{3}\left(t^2+ 3r- 2r^2\right)\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{15}\left(3t^4+ t^2r\{15+15+ r(-60-10)+ 69r^2- 24r^3\}+ 15r^2 - 24r^3 + 11r^4\right)\left(\frac{x_1}{\nu}\right)^4\right]\right]_{\phi=\phi_f}.$$

The final expression for the grid convergence in terms of projection co-ordinates is therefore given by

$$\gamma = \left[t\left(\frac{x_1}{\nu}\right)\left[1 - \frac{1}{3}\left(t^2+ 3r- 2r^2\right)\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{15}\left(3t^4+ t^2r(30-70r+69r^2-24r^3)+ 15r^2- 24r^3+ 11r^4\right)\left(\frac{x_1}{\nu}\right)^4\right]\right]_{\phi=\phi_f} + o\{10^{-10}\}\ldots 6.35.$$

Notes :-

1. Equation 6.35 reduces to the expression obtained for the spherical case in section 3.6.3.*4* if all expressions which are functions of the eccentricity, are put equal to zero. This is equivalent to assigning the value unity for r.

2. As stated earlier, the foot point latitude ϕ_f is a defined value for every value of x_2 (i.e., the north co-ordinate).

6.6 The point scale factor from projection co-ordinates

An expression for the point scale factor in terms of projection co-ordinates is obtained with the minimum effort from equation 5.42 on putting $u_1 = \Delta\lambda$, when

$$\frac{1}{k} = m \frac{\partial\lambda}{\partial x_1}\left(1 + \tan^2\gamma\right)^{\frac{1}{2}} \qquad \ldots\ldots \qquad 5.42,$$

where m is the linearisation factor, given by equation 5.24 for the ellipsoid as

$$m = \nu \cos\phi \ .$$

As $(\partial\lambda/\partial x_1)$, given by equation 6.33, and $\tan\gamma$, given by equation 6.34, are in terms of the foot point latitude ϕ_f. The latter is a known quantity once x_2 is defined. The latitude ϕ is however, an unknown quantity which has to be determined. It therefore becomes desirable to express m too, in terms of ϕ_f.

This is achieved by expressing the variation of m with ϕ in terms of a Taylor series in relation to the value m_f of m at the foot point latitude and given by

$$m_f = \nu_f \cos\phi_f \ .$$

Thus,

$$m = m_f + \frac{dm}{d\phi}\Delta\phi + \frac{1}{2}\frac{d^2m}{d\phi^2}(\Delta\phi)^2 + \ldots\ldots,$$

where $\Delta\phi$ is the difference $(\phi-\phi_f)$, given by equation 6.31. Thus,

$$m = \left\{\nu\cos\phi + \rho\sin\phi\, r\, t\left[\frac{1}{2}\left(\frac{x_1}{\nu}\right)^2 - \frac{1}{24}\left(t^2(12-9r) + 9r - 4r^2\right)\left(\frac{x_1}{\nu}\right)^4\right] + \frac{1}{2}\frac{\partial}{\partial\phi}\left(-\rho\sin\phi\right)\left[\frac{1}{4}r^2t^2\left(\frac{x_1}{\nu}\right)^4\right]\right\}_{\phi=\phi_f} .$$

As

$$\rho\sin\phi = \frac{\rho}{\nu}(\nu\cos\phi)t = \frac{1}{r}(\nu\cos\phi)t\ ,$$

$$\frac{\partial}{\partial\phi}\left(\rho\sin\phi\right) = \frac{1}{r^2}\left(2t(r-1)\right)\nu\cos\phi\, t - \frac{1}{r}\rho\sin\phi\, t + \frac{1}{r}\nu\cos\phi(1+t^2)$$

$$= \frac{1}{r}\rho\cos\phi\left[\frac{1}{r}(r-1)2t^2r - t^2 + r(1+t^2)\right].$$

Thus,

$$\frac{\partial}{\partial\phi}\left(-\rho\sin\phi\right) = -\frac{1}{r}\rho\cos\phi\left(3t^2(r-1) + r\right),$$

and evaluation at $\phi = \phi_f$ prior to substitution in the earlier equation gives

$$m = \left\{\nu \cos\phi + \rho \sin\phi \, rt \left(\frac{1}{2}\left(\frac{x_1}{\nu}\right)^2 - \frac{1}{24}\left(t^2(12-9r) + 9r - 4r^2\right)\left(\frac{x_1}{\nu}\right)^4\right) - \frac{1}{8}\frac{\rho \sin\phi}{r} \; \frac{1}{t} r^2 t^2 \left(3t^2(r-1) + r\right)\left(\frac{x_1}{\nu}\right)^4\right\}_{\phi=\phi_f}$$

$$= \left\{\nu \cos\phi + \rho \sin\phi \, rt \left(\frac{1}{2}\left(\frac{x_1}{\nu}\right)^2 - \frac{1}{24}\left(t^2\{12-9+r(-9+9)\} + 12r - 4r^2\right)\left(\frac{x_1}{\nu}\right)^4\right)\right\}_{\phi=\phi_f}$$

$$= \left\{\nu \cos\phi \left[1 + t^2\left(\frac{1}{2}\left(\frac{x_1}{\nu}\right)^2 - \frac{1}{24}\left(3t^2 + 12r - 4r^2\right)\left(\frac{x_1}{\nu}\right)^4\right)\right]\right\}_{\phi=\phi_f}$$

$$= \left\{\nu \cos\phi \left[1 + \frac{1}{2}t^2\left(\frac{x_1}{\nu}\right)^2\left(1 - \frac{1}{12}A_1\left(\frac{x_1}{\nu}\right)^2\right)\right]\right\}_{\phi=\phi_f} \quad \ldots\ldots \quad 6.36.$$

From equation 6.34,

$$\left(1 + \tan^2\gamma\right)^{\frac{1}{2}} = 1 + \frac{1}{2}\tan^2\gamma - \frac{1}{8}\tan^4\gamma + o\{10^{-10}\} \quad .$$

As

$$\frac{1}{2}\tan^2\gamma = \left\{\frac{1}{2}t^2\left(\frac{x_1}{\nu}\right)^2\left(1 - \frac{2}{3}\left(3r-2r^2\right)\left(\frac{x_1}{\nu}\right)^2\right)\right\}_{\phi=\phi_f} + o\{10^{-10}\}$$

and

$$-\frac{1}{8}\tan^4\gamma = -\frac{1}{8}t_f^4\left(\frac{x_1}{\nu_f}\right)^4 ,$$

$$\left(1 + \tan^2\gamma\right)^{\frac{1}{2}} = \left\{1 + \frac{1}{2}t^2\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{24}\left(t^2\{-24r + 16r^2\} - 3t^4\right)\left(\frac{x_1}{\nu}\right)^4\right\}_{\phi=\phi_f}$$

$$= \left\{1 + \frac{1}{2}t^2\left(\frac{x_1}{\nu}\right)^2\left(1 - \frac{1}{12}A_2\left(\frac{x_1}{\nu}\right)^2\right)\right\}_{\phi=\phi_f} \quad \ldots\ldots \quad 6.37,$$

where

$$A_2 = 24r_f - 16r_f^2 + 3t_f^2.$$

The evaluation of equation 5.42, using equations 6.33, 6.36 and 6.37 gives

$$\frac{1}{k} = \left\{\nu \cos\phi \left[1 + \frac{1}{2}t^2\left(\frac{x_1}{\nu}\right)^2\left(1 - \frac{1}{12}A_1\left(\frac{x_1}{\nu}\right)^2\right)\right]\left[1 + \frac{1}{2}t^2\left(\frac{x_1}{\nu}\right)^2\left(1 - \frac{1}{12}A_2\left(\frac{x_1}{\nu}\right)^2\right)\right]\frac{1}{\nu \cos\phi}\left(1 - \frac{1}{2}C\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{24}D\left(\frac{x_1}{\nu}\right)^4\right)\right\}_{\phi=\phi_f}$$

$$\frac{1}{k} = \left(1 + \frac{1}{2}(2t^2 - C)\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{24}(D - A_1t^2 - A_2t^2 + 6t^4 - 12Ct^2)\left(\frac{x_1}{\nu}\right)^4\right)_{\phi=\phi_f}$$

$$= \left(1 - \frac{1}{2}r\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{24}(9r^2 - 4r^3 + t^4(24-3-3+6-24) + t^2\{r(72-24-12-12) + r^2(-68+16+4) + 24r^3\})\left(\frac{x_1}{\nu}\right)^4\right)_{\phi=\phi_f}.$$

Thus,

$$\frac{1}{k} = \left(1 - \frac{1}{2}r\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{24}(9r^2 - 4r^3 + t^2(24r - 48r^2 + 24r^3))\left(\frac{x_1}{\nu}\right)^4\right)_{\phi=\phi_f} + o\{10^{-10}\} \quad \ldots\ldots \quad 6.38.$$

and k is obtained by inverting the above equation, when

$$k = \left(1 + \frac{1}{2}r\left(\frac{x_1}{\nu}\right)^2 - \frac{1}{24}(3r^2 - 4r^3 + t^2(24r - 48r^2 + 24r^3))\left(\frac{x_1}{\nu}\right)^4\right)_{\phi=\phi_f} + o\{10^{-10}\} \quad \ldots\ldots \quad 6.39.$$

Notes :-

1. The final expressions in sections 6.2 to 6.6 can all be converted to the form given by Thomas (1952,pp.92-106) on replacing r by $(1+\eta^2)$, as noted in section 6.2.*1*.

6.7 Formulae for computations using plane rectangular co-ordinates

The procedures for obtaining these formulae have already been outlined in sections 4.2 to 4.4. The curvature σ of the projected geodesic is obtained from equations 4.20, 6.38 & 6.39, when

$$\sigma = \frac{1}{k}\frac{\partial k}{\partial x_1}\cos\beta \quad ; \qquad \frac{\partial k}{\partial x_2} = 0.$$

$$\sigma = \left(\left(1 - \frac{1}{2}r\left(\frac{x_1}{\nu}\right)^2 + \frac{1}{24}(9r^2 - 4r^3 + t^2(24r - 48r^2 + 24r^3))\left(\frac{x_1}{\nu}\right)^4\right) \times \left(\frac{r}{\nu^2}x_1\cos\beta\left(1 - \frac{1}{6}(3r^2 - 4r^3 + t^2(24r - 48r^2 + 24r^3))\left(\frac{x_1}{\nu}\right)^2\right)\right)\right)_{\phi=\phi_f}$$

$$= \left(\left(\frac{x_1}{\nu}\right)\frac{r\cos\beta}{\nu}\left(1 - \frac{1}{6}(3r + 3r^2 - 4r^3 + t^2(24r - 48r^2 + 24r^3))\left(\frac{x_1}{\nu}\right)^2\right)\right)_{\phi=\phi_f}$$

$$= \left(\left(\frac{x_1}{\nu}\right)\frac{r\cos\beta}{\nu}\left(1 - \frac{1}{6}r\,B_1\left(\frac{x_1}{\nu}\right)^2\right)\right)_{\phi=\phi_f} + o\{2\times10^{-11}\text{km}^{-1}\}\ldots \quad 6.40,$$

where

$$B_1 = 3 + 3r_f - 4r_f^2 + t_f^2(24-48r_f+24r_f^2).$$

6.7.1 The chord to arc correction

The *chord to arc correction* δ for the line P_1P_2 in figure 4.5, is obtained by the use of equation 4.40, given by

$$\delta = \frac{1}{2}\sigma_o \ell_p + \frac{1}{6}\sigma_o^1 \ell_p^2 + \frac{1}{24}\sigma_o^2 \ell_p^3 + \frac{1}{720}\left(6\sigma_o^3 - \sigma_o^1(\sigma_o)^2\right)\ell_p^4 + o\{10^{-11}\} \quad ..(4.40),$$

where σ_o, σ_o^1 and σ_o^2 are the curvature and its derivatives of the projected geodesic (length ℓ_p) at P (x_{11}, x_{21}). The derivatives (σ_o^1, σ_o^2) of the curvature with respect to length on the projection plane, are obtained by the differentiation of equation 6.40, noting that

$$\left(\partial\sigma/\partial x_2\right) = 0,$$

when

$$\sigma^1 = \frac{\partial\sigma}{\partial x_1}\frac{dx_1}{d\ell_p} + \frac{\partial\sigma}{\partial\beta}\frac{d\beta}{d\ell_p} = \left[\frac{r\cos\beta\sin\beta}{\nu^2}\left(1 - \frac{1}{2}B_1\frac{x_1^2}{\rho\nu}\right) + \right.$$

$$\left.\left(-\frac{x_1}{\nu}\frac{\sin\beta}{\rho}\left[1 - \frac{1}{6}rB_1\left(\frac{x_1}{\nu}\right)^2\right]\right)(-\sigma)\right]_{\phi=\phi_f}$$

$$\overset{6.40}{=} \left[\frac{\cos\beta\sin\beta}{\rho\nu}\left(1 - \frac{1}{2}B_1\frac{x_1^2}{\rho\nu}\right) + \left(-\frac{x_1\sin\beta}{\rho\nu}\right)\left(-\frac{x_1\cos\beta}{\rho\nu}\right)\right]_{\phi=\phi_f} + o\{\frac{x_1^4\ell^2}{\nu^6}\}.$$

Thus ,

$$\sigma^1 = \left[\frac{\sin\beta\cos\beta}{R^2}\left[1 - \left(\frac{1}{2}B_1 - 1\right)\left(\frac{x_1}{R}\right)^2\right]\right]_{\phi=\phi_f} \quad ......6.41,$$

where

$$r\frac{1}{\nu^2} = \frac{1}{\rho\nu} = \frac{1}{R^2} \quad ..... 6.42.$$

Similarly,

$$\sigma^2 = \frac{\partial\sigma^1}{\partial x_1}\frac{dx_1}{d\ell_p} + \frac{\partial\sigma^1}{\partial\beta}\frac{d\beta}{d\ell_p} = \left[-\frac{\cos\beta\sin^2\beta}{R^4}2x_1\left(\frac{1}{2}B_1 - 1\right) + \right.$$

$$\left.\left(\frac{\cos^2\beta - \sin^2\beta}{R^2}\right)\left(-\frac{x_1}{\nu}\frac{\cos\beta}{\rho}\right)\right]_{\phi=\phi_f} + o\{\frac{x_1^3\ell^3}{R^6}\}$$

$$= \left(-\frac{\cos\beta}{R^4}x_1\left(\sin^2\beta\,(B_1-3) + \cos^2\beta\right)\right)_{\phi=\phi_f} + o\{\frac{x_1^3\ell^3}{R^6}\} \quad ....6.42a$$

and

$$\sigma^3 = \frac{\partial\sigma^2}{\partial x_1}\frac{dx_1}{d\ell_p} + \frac{\partial\sigma^2}{\partial\beta}\frac{d\beta}{d\ell_p} = \left(-\frac{\cos\beta\sin\beta}{R^4}\left(\sin^2\beta(B_1-3)+\cos^2\beta\right)\right)_{\phi=\phi_f} + o\{\frac{x_1^2\ell^4}{R^6}\sigma^3\} \qquad \ldots\ldots 6.42b.$$

As the term $\sigma^1(\sigma)^2$ is of the order of magnitude of terms neglected, the use of equations 6.41 and 6.42a-b along with equation 6.40 in 4.40 gives, on evaluation at P_1,

$$\delta_{12} = \delta = \left(\frac{\ell_p\cos\beta\, x_{11}}{2R^2}\left(1-\frac{1}{6}B_1\left(\frac{x_{11}}{R}\right)^2\right) + \frac{\ell_p\cos\beta\,\ell_p\sin\beta}{6R^2}\left(1+\left(1-\frac{1}{2}B_1\right)\left(\frac{x_{11}}{R}\right)^2\right) - \frac{\ell_p\cos\beta}{24R^4}x_{11}\left(\ell_p^2\sin^2\beta(B_1-3)+\ell_p^2\cos^2\beta\right) - \frac{\ell_p\cos\beta\,\ell_p\sin\beta}{120\,R^4}\left(\ell_p^2\cos^2\beta+(B_1-3)\ell_p^2\sin^2\beta\right)\right)_{\phi=\phi_f} + o\{\frac{x_1^4\ell^2}{R^6}\} .$$

On replacing B_1 by 2 in the smallest term,

$$\delta = \frac{\ell_p\cos\beta}{6R^2}\left(3x_{11}+\ell_p\sin\beta\right) - \frac{\ell_p\cos\beta}{24R^4}x_{11}\left(B_1(2x_{11}^2+2x_{11}\ell_p\sin\beta+\ell_p^2\sin^2\beta) - \ell_p\sin\beta(4x_{11}-3\ell_p\sin\beta)+\ell_p^2\cos^2\beta\right) - \frac{\ell_p\cos\beta\,\ell_p\sin\beta}{120R^4}\left(\ell_p^2\cos^2\beta-\ell_p^2\sin^2\beta\right)\Bigg)_{\phi=\phi_f} + o\{10^{-9}\}.$$

As the order of magnitude of the first term is 10^{-4}, and that of the second 10^{-6},

$$\delta = \left(\frac{\ell_p\cos\beta}{6R^2}\left(3x_{11}+\ell_p\sin\beta\right)+o\{10^{-6}\}\right)_{\phi=\phi_f} .$$

The plane grid bearing θ of the line P_1P_2 on the projection is related to β by equation 4.1 as

$$\beta = \theta + \delta .$$

Thus

$$\cos\beta = \cos\theta - \delta\sin\theta + o\{10^{-8}\}$$

and

$$\sin\beta = \sin\theta + \delta\cos\theta + o\{10^{-8}\}.$$

The replacement of β by θ in the expression for δ does not affect the equation except in the first term which is three orders larger than the other terms. The first term can be replaced by the expression

$$\frac{\ell_p \cos\beta}{6R^2}\left(3x_{11} + \ell_p \sin\beta\right) = \frac{\ell_p(\cos\theta - \delta\sin\theta)}{6R^2}\left(3x_{11} + \ell_p(\sin\theta + \delta\cos\theta)\right)$$

$$= \frac{\ell_p \cos\theta}{6R^2}\left(3x_{11} + \ell_p \sin\theta\right) - \frac{\delta}{6R^2}\left(\ell_p \sin\theta\,(3x_{11} + \ell_p\sin\theta) - \ell_p^2\cos^2\theta\right) .$$

The second term above is four orders smaller than the first. The use of the exact relations (subject to equation 4.22)

$$\Delta x_2 = x_{22} - x_{21} = \ell_p \cos\theta \ ; \qquad \Delta x_1 = x_{12} - x_{11} = \ell_p \sin\theta \ ,$$

noting that

$$B_1 = 2 + o\{10^{-2}\},$$

$$\delta_{12} = \Big(\frac{\Delta x_2}{6R^2}\left(2x_{11} + x_{12}\right) - \frac{\Delta x_2}{24R^4}\, x_{11}\big(B_1(x_{11}^2 + x_{12}^2) - 4(x_{12}-x_{11})x_{11} +$$
$$3(x_{12} - x_{11})^2 + (\Delta x_2)^2\big) - \frac{\Delta x_2 (x_{12} - x_{11})}{120\, R^4}\left((\Delta x_2)^2 - (x_{12} - x_{11})^2\right)$$
$$- \frac{\Delta x_2}{36R^4}(2x_{11} + x_{12})\ \left(3x_{11}(x_{12} - x_{11}) + (x_{12} - x_{11})^2 - (\Delta x_2)^2\right)\Big)_{\phi=\phi_f}$$

This equation can also be written as

$$\delta_{12} = \Big(\frac{\Delta x_2}{6R^2}\ (2x_{11} + x_{12})\Big[\ 1 - \frac{1}{6R^2}\left(3x_{11}\,\Delta x_1 + (\Delta x_1)^2 - (\Delta x_2)^2\right)\Big] -$$
$$\frac{\Delta x_2}{120\ R^4}\ \left(5x_{11}\{4x_{11}^2 + 5(\Delta x_1)^2 + (\Delta x_2)^2\} + \Delta x_1\{(\Delta x_2)^2 - (\Delta x_1)^2\}\right)\Big)_{\phi=\phi_f}$$

...... 6.43 ,

on noting that

$$B_1(x_{11}^2 + x_{12}^2) - 4(x_{12} - x_{11})x_{11} + 3(x_{12} - x_{11})^2$$
$$= 9x_{11}^2 - 10\, x_{11}x_{12} + 5\, x_{12}^2 = 4x_{11}^2 + 5(\Delta x_1)^2 .$$

Notes :-

1. Note that all values of ϕ in equation 6.43 refer to the foot point latitude, corresponding to the x_2 co-ordinate of the initial terminal P_1(i.e., x_{21}) and *not* to either the latitude at the mid point of the line or one third way along it.

2. The accuracy of equation 6.43 is 0.0001 arc sec over a 60 km line for a six degree zone.

3. A formula correct to 0.01 arc sec is obtained under these same circumstances is obtained by ignoring all the differential terms in equation 6.43, which then reduces to

$$\delta = \Big(\frac{\Delta x_2}{6R^2}\left(2x_{11} + x_{12} - \frac{1}{R^2}\, x_{11}^3\ \right) \Big)_{\phi=\phi_f} \qquad \ldots\ldots \qquad 6.44$$

The contribution of the third term over the extreme test line given by Bomford (1962, table 1) between P_1 (25°N, 4°E) and P_2 (26°N, 5°E), is approximately 0.2 arcsec, δ being approximately 127 sec. The resulting error is less than 0.02 arc sec over a line whose length is approximately 150 km and situated at the extremity of a six degree zone.

6.7.2 The line scale factor

The line scale factor k_ℓ for the line P_1P_2 is given by equation 4.27 as

$$k_\ell = \frac{\ell}{\ell p} = k_o + \frac{1}{2} k_o^1 k_o \ell + \frac{1}{6}\left(k_o^2 (k_o)^2 + (k_o^1)^2 k_o\right)\ell^2 +$$

$$\frac{1}{24}\left(k_o^3 + 4k_o^2 k_o^1\right)\ell^3 + \frac{1}{120}\left(k_o^4 + 4(k_o^2)^2\right)\ell^4 + o\{\frac{x_1^3 \ell^3}{R^6}\} \quad .. \ 4.27,$$

k_o, k_o^1, k_o^2, k_o^3 and k_o^4 are the values of the point scale factor and its derivatives with respect to ℓ_p at P_1. From equation 6.39, it can be seen that

$$(\partial k/\partial x_2) = 0.$$

Thus

$$k_o^1 = \left(\frac{\partial k}{\partial x_1}\frac{dx_1}{d\ell_p}\right)_{\ell_p=0} = \left[\frac{\sin\beta}{R^2}\left(x_{11} + \frac{1}{6} B_2 \frac{x_{11}^3}{\nu^2}\right)\right]_{\phi=\phi_f} \quad \ldots\ldots \ 6.45,$$

where

$$B_2 = -\left(3r^2 - 4r^3 + t^2(24r - 48r^2 + 24r^3)\right) = 1 + o\{10^{-2}\} \quad \ldots\ldots 6.46$$

and

Further differentiation gives

$$k_o^2 = \left(\frac{\partial k^1}{\partial x_1}\frac{dx_1}{d\ell_p} + \frac{\partial k^1}{\partial \beta}\frac{d\beta}{d\ell_p}\right)_{\ell_p=0} = \left(\frac{\sin^2\beta}{R^2}\left[1 + \frac{1}{2}B_2\left(\frac{x_{11}}{\nu}\right)^2\right] + \frac{\cos\beta}{R^2} x_{11}(-\sigma)\right)_{\phi=\phi_f}$$

$$\overset{(6.40)}{=} \left(\frac{\sin^2\beta}{R^2}\left[1 + \frac{1}{2}B_2\left(\frac{x_{11}}{\nu}\right)^2\right] + \frac{\cos\beta}{R^2} x_{11}\left(-\frac{\cos\beta}{R^2} x_{11}\right)\right)_{\phi=\phi_f} + o\{\frac{x_{11}^4}{R^6}\}$$

$$= \left(\frac{\sin^2\beta}{R^2}\left(1 + \frac{1}{2}B_2\left(\frac{x_{11}}{\nu}\right)^2\right) - \frac{\cos^2\beta}{R^4} x_{11}^2\right)_{\phi=\phi_f} + o\{\frac{x_{11}^4}{R^6}\} \quad \ldots\ldots \ 6.47,$$

$$k_o^3 = \left(\frac{\partial k^2}{\partial x_1}\frac{dx_1}{d\ell_p} + \frac{\partial k^2}{\partial \beta}\frac{d\beta}{d\ell_p}\right)_{\ell_p=0} = \left(\frac{\sin^3\beta}{R^2\nu^2} B_2 x_{11} - \frac{2\cos^2\beta\sin\beta}{R^4} x_{11} + \right.$$

$$\left.\frac{2\sin\beta\cos\beta}{R^2}\left(-\frac{\cos\beta}{R^2}x_{11}\right)\right)_{\phi=\phi_f} + o\{\frac{x_{11}^3}{R^6}\}$$

$$k_o^3 = \left\{\frac{\sin^3\beta}{R^2\nu^2}B_2\, x_{11} - 4\,\frac{\cos^2\beta\,\sin\beta}{R^4}x_{11} + o\{\frac{x_{11}^3}{R^6}\}\right\}_{\phi=\phi_f} \quad ..6.48,$$

and

$$k_o^4 = \left(\frac{\partial k^3}{\partial x_1}\frac{dx_1}{d\ell_p}\right)_{\ell_p=0} + o\{\frac{x_{11}}{R^6}\} = \left(\left(\frac{\sin^2\beta}{R^2\nu^2}B_2 - 4\,\frac{\cos^2\beta}{R^4}\right)\sin^2\beta\right)_{\phi=\phi_f} + o\{\frac{x_{11}^2}{R^6}\}\ .$$

...... 6.49.

Thus the point scale factor k_ℓ is given by substitutiong the results of equations 6.39, 6.45 and 6.47-6.49 in equation 4.27, when

$$k_\ell = \left[1 + \frac{1}{2R^2}x_{11}^2 + \frac{1}{24}B_2\frac{x_{11}^4}{R^2\nu^2} + \frac{1}{2}\left(1+\frac{x_{11}^2}{2R^2}\right)\left(1+\frac{1}{6}B_2\left(\frac{x_{11}}{\nu}\right)^2\right)\frac{\ell\sin\beta}{R^2}x_{11} + \right.$$

$$\frac{\ell^2}{6R^2}\left[\sin^2\beta\left(1+\frac{1}{2}B_2\left(\frac{x_{11}}{\nu}\right)^2\right)\left(1+\frac{x_{11}^2}{R^2}\right) - \frac{\cos^2\beta}{R^2}x_{11}^2 + \frac{\sin^2\beta}{R^2}x_{11}^2\right] +$$

$$\frac{\ell^3}{24}\left(\frac{\sin\beta}{R^2}x_{11}\left(\frac{\sin^2\beta}{\nu^2}B_2 - 4\frac{\cos^2\beta}{R^2}\right) + 4\,\frac{\sin^2\beta}{R^2}\;\frac{\sin\beta}{R^2}x_{11}\right) +$$

$$\left.\frac{\ell^4}{120}\left[\left(\frac{\sin^2\beta}{R^2\nu^2}B_2 - 4\frac{\cos^2\beta}{R^4}\right)\sin^2\beta + 4\,\frac{\sin^4\beta}{R^4}\right]\right]_{\phi=\phi_f} + o\{\frac{x_{11}^6}{R^6}\} \quad .. \; 6.50.$$

The major contribution to k_ℓ is from terms of the type

$$\left(\frac{x_1}{R}\right)^2, \qquad \frac{x_1\,\Delta x_1}{R^2} \quad \text{and} \quad \left(\frac{\Delta x_i}{R}\right)^2.$$

The appropriate sorting of terms gives

$$k_\ell = \frac{\ell}{\ell}p = \left[1 + \frac{1}{6R^2}\left(3x_{11}^2 + 3x_{11}\ell\sin\beta + \ell^2\sin^2\beta\right) \quad + \right.$$

$$+ \frac{B_2}{120\,R^2\nu^2}\left(5x_{11}^4 + 10x_{11}^3\ell\sin\beta + 10x_{11}^2\ell^2\sin^2\beta + 5x_{11}\ell^3\sin^3\beta + \ell^4\sin^4\beta\right) +$$

$$+\frac{1}{120\,R^4}\left(30x_{11}^3\ell\sin\beta + 40x_{11}^2\ell^2\sin^2\beta - 20x_{11}^2\ell^2\cos^2\beta - \right.$$

$$20x_{11}\ell^3\cos^2\beta\sin\beta + 20x_{11}\ell^3\sin^3\beta - 4\ell^4\cos^2\beta\sin^2\beta +$$

$$\left.\left.4\ell^4\sin^4\beta\right)\right]_{\phi=\phi_f} + o\{10^{-10}\} \qquad ...... \quad 6.51.$$

The above expression can be defined in terms of projection co-ordinates alone, to this same degree of accuracy. As ℓ is the distance on ellipsoid and β the grid bearing of the projected curve, the relevant variables required in equation 6.51 are ℓ_p and θ. The latter can, in turn, be directly related to the projection co-ordinates of the terminals by plane trigonometry. From equations 4.4, 4.27, 6.44 and 6.51, it follows that β can be replaced by θ in all terms except the first, in the last equation. The discrepancy between β and θ in the large order term is allowed for as follows.

$$\text{As} \quad \beta = \theta + \delta \qquad \text{and} \qquad \ell = \ell_p - d\ell,$$

where

$$d\ell = (k_\ell - 1)\ell \quad ,$$

$$\sin\beta = \sin\theta + \delta\cos\theta = \sin\theta\,(1 + \delta\cot\theta + o\{10^{-8}\}).$$

Thus,

$$\ell \sin\beta = (\ell_p - d\ell)\sin\theta(1 + \delta\cot\theta)$$
$$= \ell_p \sin\theta\left(1 - \frac{d\ell}{\ell} + \delta\cot\theta + o\{10^{-8}\}\right) .$$

Further,

$$\ell^2 \sin^2\beta = \ell_p^2 \sin^2\theta\left(1 - 2\frac{d\ell}{\ell} + 2\delta\cot\theta + o\{10^{-8}\}\right).$$

The contribution of the larger order terms in equation 6.51 can be expressed as

$$\frac{1}{6R^2}(3x_{11}^2 + 3x_{11}\ell\sin\beta + \ell^2\sin^2\beta) = \frac{1}{6R^2}\Big(3x_{11}^2 + 3x_{11}\ell_p\sin\theta + \ell_p^2\sin^2\theta +$$
$$3x_{11}\ell_p\sin\theta\{\delta\cot\theta - \frac{d\ell}{\ell}\} + 2\ell_p^2\sin^2\theta\{\delta\cot\theta - \frac{d\ell}{\ell}\}\Big)$$

$$= \frac{1}{6R^2}\Big(x_{11}^2 + x_{11}(x_{11} + \ell_p\sin\theta) + \{x_{11}^2 + 2x_{11}\ell_p\sin\theta + \ell_p^2\sin^2\theta\} +$$
$$3x_{11}\ell_p\sin\theta\{\delta\cot\theta - \frac{d\ell}{\ell}\} + 2\ell_p^2\sin^2\theta\{\delta\cot\theta - \frac{d\ell}{\ell}\}\Big) \quad \ldots .6.52.$$

The first three terms are three orders larger than the rest of the expression and reduce to

$$x_{11}^2 + x_{11}x_{12} + x_{12}^2 = x_{1\mu}^2. \qquad \ldots\ldots 6.53.$$

It would suffice if δ and $d\ell/\ell$ were replaced by the expressions

$$\delta = \frac{\Delta x_2}{6R^2}(2x_{11} + x_{12}) \qquad \ldots\ldots 6.54$$

and

$$\frac{d\ell}{\ell} = \frac{1}{6R^2}(x_{11}^2 + x_{11}x_{12} + x_{12}^2) = \frac{1}{6R^2}x_{1\mu}^2 \qquad \ldots\ldots 6.55$$

when using the results of equation 6.52 in equation 6.51. On doing so, noting that

$$\ell_p\cos\theta = \Delta x_2 \qquad \ldots\ldots 6.56,$$

$$k_\ell = \Big(1 + \frac{1}{6R^2} x_{1\mu}^2 + \frac{B_2}{120R^2\nu^2} \big(5x_{11}^4 + 10x_{11}^3(x_{12}-x_{11}) + 10x_{11}^2(x_{11}^2-2x_{11}x_{12}+x_{12}^2) +$$

$$+5x_{11}(x_{12}^3-3x_{11}x_{12}^2+3x_{11}^2x_{12}-x_{11}^3) + (x_{12}^4-4x_{11}x_{12}^3+6x_{11}^2x_{12}^2-4x_{11}^3x_{12}+x_{11}^4) +$$

$$+ \frac{1}{360R^4}\big(90x_{11}^3(x_{12}-x_{11}) + 120x_{11}^2(x_{12}^2-2x_{11}x_{12}+x_{11}^2) + 60x_{11}(x_{12}^3-3x_{11}x_{12}^2+$$

$$3x_{11}^2x_{12}-x_{11}^3) + (12x_{12}^4-48x_{11}x_{12}^3+72x_{11}^2x_{12}^2-48x_{11}^3x_{12}+12x_{11}^4) -$$

$$10(x_{11}^2+x_{11}x_{12}+x_{12}^2)(2x_{12}^2-x_{11}x_{12}-x_{11}^2) + \Delta x_2^2\big(-60x_{11}^2 - 60x_{11}(x_{12}-x_{11}) -$$

$$12(x_{12}^2-2x_{11}x_{12}+x_{11}^2) + 10(2x_{11}^2+5x_{11}x_{12}+2x_{12}^2)\big)\big)\Big)\Big|_{\phi=\phi_f}$$

as

$$3x_{11}\ell_p \sin\theta(\delta\cot\theta - \frac{d\ell}{\ell}) + 2\ell_p^2\sin^2\theta(\delta\cot\theta - \frac{d\ell}{\ell}) =$$

$$\ell_p\cos\theta\,(x_{11}+2x_{12}) - \frac{1}{6R^2}x_{1\mu}^2(x_{11}+2x_{12})(x_{12}-x_{11}).$$

Further sorting of terms gives

$$k_\ell = \Big(1 + \frac{1}{6R^2}x_{1u}^2 + \frac{B_2}{120R^2\nu^2}\big(x_{11}^4(5-10+10-5+1) + x_{11}^3x_{12}(10-20+15-4) +$$

$$x_{11}^2x_{12}^2(10-15+6) + x_{11}x_{12}^3(5-4) + x_{12}^4\big)$$

$$+ \frac{1}{360R^4}\big(x_{11}^4(-90+120-60+12+10) + x_{11}^3x_{12}(90-240+180-48+20) +$$

$$x_{11}^2x_{12}^2(120-180+72+10-10) + x_{11}x_{12}^3(60-48-10) + x_{12}^4(12-20) +$$

$$\Delta x_2^2\big(x_{11}^2(-60+60-12+20) + x_{11}x_{12}(-60+24+50) + x_{12}^2(-12+20)\big)\big)\Big)\Big|_{\phi=\phi_f}.$$

On completing the summations,

$$k_\ell = \Big(1 + \frac{1}{6R^2}x_{1\mu}^2 + \frac{B_2}{120R^2\nu^2}\big(x_{11}^4 + x_{11}^3x_{12} + x_{11}^2x_{12}^2 + x_{11}x_{12}^3 + x_{12}^4\big) -$$

$$\frac{1}{360R^4}\big((8x_{11}^4-2x_{11}^3x_{12}-12x_{11}^2x_{12}^2-2x_{11}x_{12}^3+8x_{12}^4) -$$

$$2\Delta x_2^2(4x_{11}^2+7x_{11}x_{12}+4x_{12}^2)\big)\Big)\Big|_{\phi=\phi_f} \qquad \ldots\ldots 6.57.$$

As

$$\frac{B_2}{\nu^2} = \frac{1}{R^2} + o\{\frac{f}{R^2}\} ,$$

equation 6.57 could be simplified slightly to give

$$k_\ell = \left[1 + \frac{1}{6R^2}x_{1\mu}^2 + \frac{1}{360R^4}\left(-5x_{11}^4 + 5x_{11}^3x_{12} + 15x_{11}^2x_{12}^2 + 5x_{11}x_{12}^3 - 5x_{12}^4 + \right.\right.$$

$$\left.\left. + 2\Delta x_2(4x_{1\mu}^2 + 3x_{11}x_{12})\right)\right]_{\phi=\phi_f}$$

$$= \left[1 + \frac{1}{6R^2} x_{1\mu}^2 + \frac{1}{360R^4}\left(5\{(x_{12}-x_{11})(x_{11}^3-x_{12}^3) + 3x_{11}^2x_{12}^2\} \right.\right.$$

$$\left.\left. + 2\,\Delta x_2(4x_{1\mu}^2+3x_{11}x_{12})\right)\right]_{\phi=\phi_f}$$

$$= \left[1 + \frac{1}{6R^2} x_{1\mu}^2 + \frac{1}{360R^4}\left(5\{3x_{11}^2x_{12}^2-(x_{12}-x_{11})^2x_{1\mu}^2\} + 2\Delta x_2(4x_{1\mu}^2+3x_{11}x_{12})\right)\right]_{\phi=\phi_f}$$

$$+o\{10^{-9}\} \quad \ldots\ldots 6.58.$$

Notes :-

1. An accuracy of 1 part in 10^6 is achieved by the use of the first two terms in equation 6.58. For exhaustive tests of various expressions which allow for the fact that the first two terms always *underestimate* k_ℓ (see Bomford 1962). He also suggests the use of the following convenient expression obtained by the numerical integration of the second term in equation 6.39 by Simpson's rule, together with the assumption that the third term has an insignificant variation over any line. Hence, if

$$x_{1m} = \frac{1}{2}(x_{11} + x_{12}),$$

and

$$x_{11}^4 \doteq x_{12}^4 \doteq \frac{1}{9} x_{1\mu}^4,$$

$$k_\ell = \left[1 + \frac{1}{6}\ \frac{1}{2R^2}\left(x_{11}^2 + 4x_{1m}^2 + x_{12}^2\right) + \frac{B_2}{24R^2\nu^2}\ \frac{1}{9}x_{1\mu}^4\right]_{\phi=\phi_f}$$

$$\doteq \left[1 + \frac{1}{6R^2}x_{1\mu}^2\left(1 + \frac{1}{36R^2}x_{1\mu}^2\right)\right]_{\phi=\phi_f} \quad \ldots\ldots \quad 6.59.$$

2. For most practical purposes however, the following set of formulae will enable computations to be carried out with an accuracy of 1 part in 10^6.

$$\theta = \alpha_{12} - \gamma - \delta_{12} \quad \ldots\ldots \quad 4.2,$$

where

$$\gamma = \left[\frac{x_{11}t}{\nu}\left(1 - \frac{1}{3}(t^2+3r-2r^2)\left(\frac{x_{11}}{\nu}\right)^2 + \frac{1}{15}(2+3t^4+5t^2)\left(\frac{x_{11}}{\nu}\right)^4\right)\right]_{\phi=\phi_f}$$

$$\ldots\ldots\ 6.35$$

and

$$\delta_{12} = \frac{\Delta x_2}{6R^2}\left(2x_{11} + x_{12}\right) \quad \ldots\ldots \quad 6.54.$$

Also,

$$\ell_p = \ell \left(1 + \frac{1}{6R^2}\left(x_{11}^2 + x_{11}x_{12} + x_{12}^2\right)\right)_{\phi=\phi_f} \quad ......6.55.$$

These quantities enable computations using plane rectangular formulae

$$x_{i2} = x_{i1} + \ell_p \cos \theta_i, \ i=1,2 \quad ...... \ 6.60,$$

where

$$\theta_1 = \theta \quad ; \qquad \theta_2 = \frac{\pi}{2} - \theta \quad ...... \ 4.6.$$

E N D

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A U S T R A L I A

A P P E N D I X

THE GEOMETRY OF THE ELLIPSOID OF REVOLUTION

A convenient mathematical model which best fits the geoid (e.g.,Bomford 1962,p.84) to an accuracy of 1 part in 10^5 is the ellipsoid of revolution, obtained by rotating an ellipse about its minor axis. The positions of points P on the Earth's surface can be uniquely related to a system of two dimensional system of parameters on the surface of the ellipsoid, on representing P by the equivalent point P_o on the latter, the surface normal at which passes through P. Thus points on the Earth's surface in three dimensional Earth space, which has the same rotational and galactic motion as the Earth, can be related to one of the following reference frames

i. two dimensional system of surface parameters (u_1,u_2), the surface being known;

ii. a general curvilinear co-ordinate system;

iii. a three dimensional rectangular Cartesian co-ordinate system (x_1,x_2,x_3).

The second system will not be considered in the present development.

As the ellipsoid is obtained by rotating an ellipse, called the *meridian ellipse*, about its minor axis, it is a necessary preliminary to study the geometry of the ellipse.

a. Geometry of the meridian ellipse

The relation between the general meridian ellipse, shown in figure i, and the ellipsoid of revolution, is illustrated in figure 1.1 on page 2. The equation of an ellipse in relation to the axes as shown in figure i, is well known to be

$$\frac{x^2}{a^2} + \frac{x_3^2}{b^2} = 1 \qquad \ldots\ldots \qquad 1,$$

where the lengths a of the semi-major and b of the semi-minor axes are related to the eccentricity e by the relation

$$b^2 = a^2\left(1 - e^2\right) \qquad \ldots\ldots \qquad 2.$$

It is customary, in geodetic literature, to refer to a and b as the equatorial and polar radii respectively. The flattening f

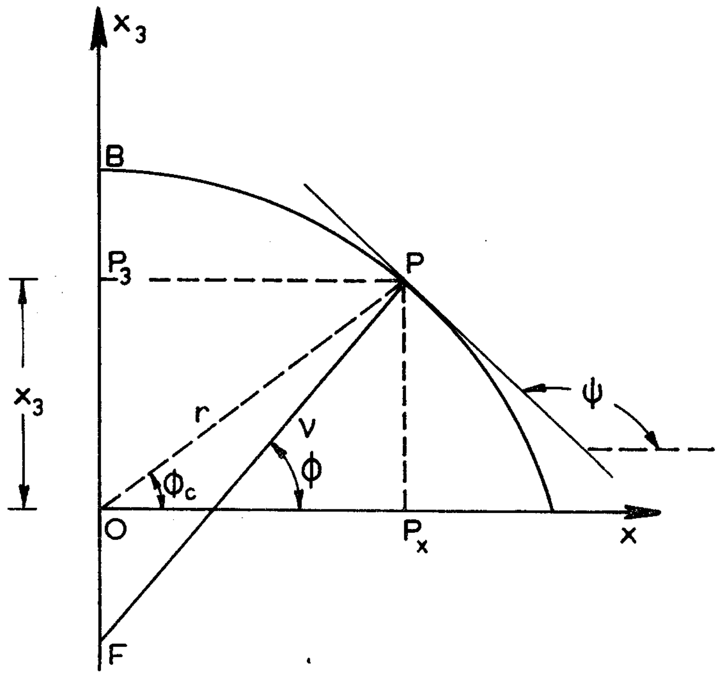

FIG. (i)

The meridian ellipse

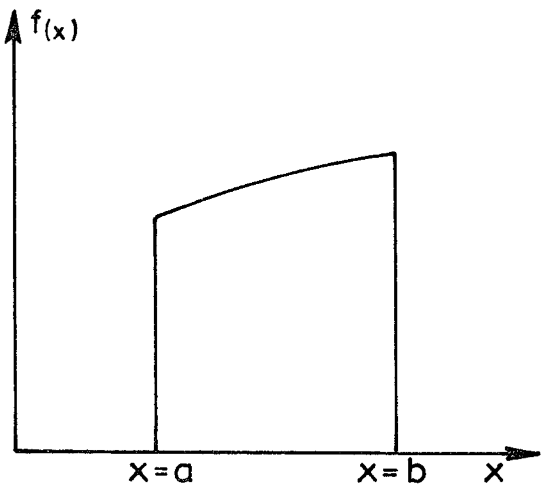

FIG. (ii)

The mean value of f(x)

of the meridian ellipse is given by

$$f = \frac{a - b}{a} = 1 - \frac{b}{a} \qquad \ldots\ldots \qquad 3.$$

From equations 2 and 3,

$$\frac{b^2}{a^2} = 1 - e^2 = (1 - f)^2$$

or

$$e^2 = 2f - f^2 \qquad \ldots\ldots 4.$$

Cartesian co-ordinates

If PF is the normal to the meridian ellipse at the general point P, and is of length ν, and the geocentric radius OP is of length r, the position of P can be defined by

either a pair of Cartesian co-ordinates (x, x_3),

or the latitude ϕ which is the angle the normal PF makes with the major axis Ox.

These quantities are related by the equations at 5 which follow from figure i.

$$x = \nu \cos\phi = r \cos\phi_c$$
$$x_3 = \nu \sin\phi - OF = r \sin\phi_c \qquad \ldots\ldots \qquad 5.$$

ϕ_c is the geocentric latitude.

The differentiation of equation 1 with respect to x, gives

$$\frac{x}{a^2} + \frac{x_3}{b^2} \frac{dx_3}{dx} = 0.$$

If ψ is the angle the tangent to the ellipse at P, makes with the x axis,

$$\tan\psi = \frac{dx_3}{dx} = -\frac{x}{x_3} \frac{b^2}{a^2} \qquad \ldots\ldots \qquad 6.$$

As the gradient of the normal is $\tan\phi$ and

$$\tan\phi \times \tan\psi = -1,$$

$$\tan\phi = -\frac{1}{\tan\psi} = \frac{x_3}{x} \frac{a^2}{b^2}.$$

The use of equation 2 gives

$$x_3 = x(1-e^2)\tan\phi \overset{(5)}{=} \nu(1-e^2)\sin\phi \qquad \ldots\ldots \qquad 7.$$

From equations 5 and 7,

$$OF = \nu e^2 \sin\phi \qquad \ldots\ldots \qquad 8.$$

The substitution of equations 5 and 7 in equation 1 gives

$$\frac{\nu^2\cos^2\phi}{a^2} + \frac{\nu^2(1-e^2)\sin^2\phi}{a^2(1-e^2)} = 1.$$

Hence

$$\nu^2(1 - e^2\sin^2\phi) = a^2,$$

or

$$\nu = \frac{a}{(1-e^2\sin^2\phi)^{\frac{1}{2}}} = \frac{a}{T^{\frac{1}{2}}} \qquad \ldots\ldots \qquad 9,$$

where

$$T = 1 - e^2\sin^2\phi \qquad \ldots\ldots \qquad 10.$$

Curvature in the meridian ellipse

If the radius of curvature in the plane of the meridian ellipse is ρ, the curvature $1/\rho$ is obtained from equation 6 as follows.

$$\frac{1}{\rho} = -\frac{d\psi}{ds} = -\frac{d}{ds}\left(\tan^{-1}\left(\frac{dx_3}{dx}\right)\right) = -\frac{1}{\left(1 + \left(\frac{dx_3}{dx}\right)^2\right)} \cdot \frac{d}{dx}\left(\frac{dx_3}{dx}\right) \cdot \frac{dx}{ds}$$

where $d\psi$ is the change in ψ over the element of length ds along the ellipse, corresponding to changes dx in x and dx_3 in x_3. Thus

$$\frac{dx}{ds} = \lim_{dx, dx_3 \to 0}\left(\frac{dx}{\left((dx)^2 + (dx_3)^2\right)^{\frac{1}{2}}}\right) = \frac{1}{\left(1 + \left(\frac{dx_3}{dx}\right)^2\right)^{\frac{1}{2}}}$$

and

$$\frac{1}{\rho} = -\frac{\frac{d}{dx}\left(\frac{dx_3}{dx}\right)}{\left(1 + \left(\frac{dx_3}{dx}\right)^2\right)^{3/2}} = -\frac{-\frac{b^2}{a^2}\left(\frac{1}{x_3} - \frac{x}{x_3^2}\frac{dx_3}{dx}\right)}{\left(1 + \left(\frac{dx_3}{dx}\right)^2\right)^{3/2}}$$

$$= -\frac{-\frac{b^2}{a^2}\cdot\frac{b^2}{x_3^3}\cdot\left(\frac{x_3^2}{b^2} + \frac{x^2}{a^2}\right)}{\left(1 + \frac{x^2}{a^4}\frac{b^4}{x_3^2}\right)^{3/2}} = \frac{b^4}{a^2x_3^3}\;\frac{a^6x_3^3}{\left(a^4x_3^2 + b^4x^2\right)^{3/2}}.$$

The use of equations 5 and 7 gives

$$\frac{1}{\rho} = \frac{a^8(1-e^2)^2}{a^6\left(\nu^2(1-e^2)^2\sin^2\phi + \nu^2(1-e^2)^2\cos^2\phi\right)^{3/2}} = \frac{a^2(1-e^2)^2}{\nu^3(1-e^2)^3}.$$

The use of equation 9 gives

$$\rho = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{3/2}} = \frac{a(1-e^2)}{T^{3/2}} \qquad \ldots\ldots \qquad 11.$$

b. The ellipsoid of revolution

Fundamental quantities

As the ellipsoid of revolution is obtained by rotating the meridian ellipse about its minor axis, the x_3 co-ordinate and the latitude are invariant on rotation. The exact location of the meridian plane can be defined uniquely by

either the angle λ between some reference plane x_3Ox_1 in figure 1.1 and the general meridian plane x_3Ox ;

or the co-ordinates x_1 and x_2 in a rectangular three dimensional Cartesian system $(x_1x_2x_3)$, with coincidence between the rotation and x_3 axes, where x_1 and x_2 are related to x defined in equation 5, and λ by the equations

$$x_1 = x \cos\lambda \quad ; \quad x_2 = x \sin\lambda \qquad \ldots\ldots \quad 12.$$

Thus position on the ellipsoid of revolution can be defined by

either the set of surface parameters (ϕ,λ)

or the set of three dimensional rectangular Cartesian co-ordinates (x_1,x_2,x_3).

A study of equations 5, 7 and 11 shows that these two systems are related by the following set of equations

$$\begin{aligned} x_1 &= \nu \cos\phi \cos\lambda \\ x_2 &= \nu \cos\phi \sin\lambda \\ x_3 &= \nu(1-e^2)\sin\phi \end{aligned} \qquad \ldots\ldots \quad 13,$$

which is of the form

$$x_i = x_i(\phi,\lambda) \qquad \ldots\ldots \quad 14.$$

The direction cosines $\ell_i (i=1,3)$ of the normal to the ellipsoid at P are given by

$$\begin{aligned} \ell_1 &= \frac{x_1}{PF} = \cos\phi\cos\lambda \\ \ell_2 &= \frac{x_2}{PF} = \cos\phi\sin\lambda \\ \ell_3 &= \frac{P_3F}{PF} = \sin\phi \end{aligned} \qquad \ldots\ldots \quad 15.$$

c. The element of length on an ellipsoid

The general element of length ds on the surface of the reference ellipsoid can be defined in terms of the equivalent changes $d\phi$ and $d\lambda$

in the surface parameters, or in terms of the changes dx_i (i=1,3) in the Cartesian co-ordinates. Obviously,

$$ds^2 = \sum_{i=1}^{3} (dx_i)^2 \qquad \ldots\ldots \qquad 16,$$

which, from equation 13, can be expressed in the form

$$ds^2 = \sum_{i=1}^{3}\left[\frac{\partial x_i}{\partial\phi} d\phi + \frac{\partial x_i}{\partial\lambda} d\lambda\right]^2$$

$$= (d\phi)^2 \sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial\phi}\right)^2 + 2\, d\phi\, d\lambda \sum_{i=1}^{3}\frac{\partial x_i}{\partial\phi}\frac{\partial x_i}{\partial\lambda} + (d\lambda)^2 \sum_{i=1}^{3}\left(\frac{\partial x_i}{\partial\lambda}\right)^2 \ldots 17.$$

Two special derivatives are required to evaluate equation 17 for the ellipsoid. The first is

$$\frac{\partial}{\partial\phi}(\nu \cos\phi) \overset{(9)}{=} a \frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{(1-e^2\sin^2\phi)^{\frac{1}{2}}}\right)$$

$$= a\left(\frac{-\sin\phi}{T^{\frac{1}{2}}} + \left(-\frac{1}{2}\right)\frac{\cos\phi}{T^{3/2}}\left(-2e^2\sin\phi\cos\phi\right)\right)$$

$$= -\frac{a\sin\phi}{T^{3/2}}\left(1 - e^2\sin^2\phi - e^2\cos^2\phi\right) = -\frac{a(1-e^2)}{T^{3/2}}\sin\phi$$

$$\overset{(11)}{=} -\rho\sin\phi \qquad \ldots\ldots \qquad 18.$$

Also,

$$\frac{\partial x_3}{\partial\phi} = a(1-e^2)\frac{\partial}{\partial\phi}\left(\frac{\sin\phi}{T^{\frac{1}{2}}}\right) = a(1-e^2)\left(\frac{\cos\phi}{T^{\frac{1}{2}}} + \frac{\sin\phi}{T^{3/2}}e^2\sin\phi\cos\phi\right)$$

$$= \frac{a(1-e^2)}{T^{3/2}}\cos\phi\left(1-e^2\sin^2\phi+e^2\sin^2\phi\right) = \rho\cos\phi \quad \ldots \qquad 19.$$

The differentiation of equations at 13, using equations 18 and 19 gives

$$\frac{\partial x}{\partial\phi} = -\rho\sin\phi\cos\lambda \qquad\qquad \frac{\partial x}{\partial\lambda} = -\nu\cos\phi\sin\lambda$$

$$\frac{\partial x_2}{\partial\phi} = -\rho\sin\phi\sin\lambda \quad \ldots 20. \qquad \frac{\partial x}{\partial\lambda} = \nu\cos\phi\cos\lambda \ \ldots 21.$$

$$\frac{\partial x_3}{\partial\phi} = \rho\cos\phi \qquad\qquad \frac{\partial x}{\partial\lambda} = 0$$

Thus

$$ds^2 = \left(\rho^2\sin^2\phi(\cos^2\lambda + \sin^2\lambda) + \rho^2\cos^2\phi\right)(d\lambda)^2 + \left(\rho\nu\sin\phi\cos\phi \times \sin\lambda\cos\lambda - \rho\nu\sin\phi\cos\phi\sin\lambda\cos\lambda\right)d\phi\, d\lambda + \nu^2\cos^2\phi(\sin^2\lambda + \cos^2\lambda)(d\lambda)^2 .$$

Simplification gives

$$ds^2 = \rho^2(d\phi)^2 + \nu^2\cos^2\phi(d\lambda)^2 \qquad \ldots\ldots \qquad 22.$$

As $d\phi$ is a change entirely in the meridian ellipse whose curvature is ρ, equation 22 can be interpreted, in the limit, as a plane right angled triangle, as shown in figure 3.9, where the length $\nu \cos \phi \, d\lambda$ is orthogonal to the local meridian.

d. Curvature in the normal section

The curvature σ of the ellipsoid in the normal section at any azimuth α can be considered to have three components σ_i (i=1,3) along each of the Cartesian axes. These components are given by

$$\sigma_i = \frac{d\ell_i}{ds} = \frac{\partial \ell_i}{\partial \phi}\frac{d\phi}{ds} + \frac{\partial \ell_i}{\partial \lambda}\frac{d\lambda}{ds} \quad , \; i=1,3,$$

where ℓ_i (i=1,3) are the direction cosines of the normal.

The radius of curvature R_α in the normal section is given by

$$\frac{1}{R_\alpha^2} = \sigma^2 = \sum_{i=1}^{3} (\sigma_i)^2 \qquad \ldots\ldots \qquad 23.$$

As

$$\sum_{i=1}^{3} (\ell_i)^2 = 1$$

(e.g., Eisenhart 1960,p.5), differentiation with respect to s gives

$$\sum_{i=1}^{3} \ell_i \sigma_i = 0.$$

Thus the components σ_i are proportional to the direction cosines of a line in the tangent plane to the ellipsoid of revolution, corresponding to the element of length ds, and which are equal to dx_i/ds. Thus

$$\sigma_i \propto \frac{dx_i}{ds} .$$

But

$$1 \overset{(23)}{=} R_\alpha^2 \sum_{i=1}^{3} (\sigma_i)^2 \qquad \ldots\ldots \qquad 24,$$

if ds lies in the normal section whose radius of curvature is R_α. Therefore

$$\frac{dx_i}{ds} = R_\alpha \sigma_i \quad , \; i=1,3 \qquad \ldots\ldots \qquad 25.$$

Substitution of the result from 25 into 24 gives

$$\frac{1}{R_\alpha} = \sum_{i=1}^{3} \sigma_i \frac{dx_i}{ds} = \sum_{i=1}^{3} \left(\frac{\partial \ell_i}{\partial \phi} \frac{d\phi}{ds} + \frac{\partial \ell_i}{\partial \lambda} \frac{d\lambda}{ds}\right)\left(\frac{\partial x_i}{\partial \phi} \frac{d\phi}{ds} + \frac{\partial x_i}{\partial \lambda} \frac{d\lambda}{ds} \right)$$

$$= \left(\frac{d\phi}{ds}\right)^2 \sum_{i=1}^{3} \left(\frac{\partial \ell_i}{\partial \phi} \frac{\partial x_i}{\partial \phi}\right) + \frac{d\phi}{ds}\frac{d\lambda}{ds} \sum_{i=1}^{3} \left(\frac{\partial \ell_i}{\partial \phi} \frac{\partial x_i}{\partial \lambda} + \frac{\partial \ell_i}{\partial \lambda} \frac{\partial x_i}{\partial \phi} \right) +$$

$$\left(\frac{d\lambda}{ds}\right)^2 \sum_{i=1}^{3} \left(\frac{\partial \ell_i}{\partial \lambda} \frac{\partial x_i}{\partial \lambda}\right) \qquad \ldots\ldots \qquad 26.$$

The differentiation of equation 15 gives

$$\frac{\partial \ell_1}{\partial \phi} = - \sin \phi \cos \lambda \qquad \frac{\partial \ell_1}{\partial \lambda} = -\cos \phi \sin \lambda$$

$$\frac{\partial \ell_2}{\partial \phi} = - \sin \phi \sin \lambda \qquad \frac{\partial \ell_2}{\partial \lambda} = \cos \phi \cos \lambda$$

$$\frac{\partial \ell_3}{\partial \phi} = \cos \phi \qquad \frac{\partial \ell_3}{\partial \lambda} = 0$$

$$\ldots\ldots \qquad 27.$$

Substitution from equations 20,21 and 27 in equation 26 gives

$$\frac{1}{R_\alpha} = \left(\frac{d\phi}{ds}\right)^2 \left(\rho \sin^2\phi (\cos^2\lambda + \sin^2\lambda) + \rho \cos^2\phi\right) +$$

$$\frac{d\phi}{ds}\frac{d\lambda}{ds}\begin{pmatrix} +\nu \cos\phi \sin\phi \cos\lambda \sin\lambda - \nu \cos\phi \sin\phi \cos\lambda \sin\lambda + 0 \\ +\rho \sin\phi \cos\phi \cos\lambda \cos\lambda - \rho \sin\phi \cos\phi \sin\lambda \cos\lambda + 0 \end{pmatrix} +$$

$$\left(\frac{d\lambda}{ds}\right)^2 \left(\nu \cos^2\phi (\sin^2\lambda + \cos^2\lambda)\right).$$

Hence

$$\frac{1}{R_\alpha} = \rho \left(\frac{d\phi}{ds}\right)^2 + \nu \cos^2\phi \left(\frac{d\lambda}{ds}\right)^2 \qquad \ldots\ldots \qquad 28.$$

On applying figure 3.9 to the ellipsoid, as

$$\lim_{ds\to 0}\{ds \cos \alpha\} = \rho \; d\phi \qquad \text{and} \qquad \lim_{ds\to 0}\{ds \sin \alpha\} = \nu \cos \phi \; d\lambda ,$$

where α is the azimuth of the normal section,

$$\frac{1}{R_\alpha} = \frac{\cos^2\alpha}{\rho} + \frac{\sin^2\alpha}{\nu} \qquad \ldots\ldots \qquad 29.$$

This result is known as *Euler's theorem*.

When $\alpha = 0$, the normal section coincides with the meridian and $R_\alpha = \rho$,

which is in agreement with equation 11.

Similarly, when

$$\alpha = \tfrac{1}{2}\pi,$$

$$R_\alpha = \nu .$$

Thus ν is the radius of curvature in the normal section orthogonal to the meridian. This section is commonly known as the prime vertical section and ν is known as the radius of curvature in the prime vertical.

e. The ratio ν/ρ

An expression requiring frequent evaluation in conformal projection theory is the differential of the ratio ν/ρ with respect to ϕ. From equations 9, 10 and 11,

$$\frac{\nu}{\rho} = \frac{1 - e^2\sin^2\phi}{1 - e^2} .$$

Differentiation with respect to ϕ gives

$$\frac{\partial}{\partial\phi}\left(\frac{\nu}{\rho}\right) = \frac{2e^2\sin\phi\cos\phi}{1 - e^2} = - 2\frac{\sin\phi}{\cos\phi}\quad\frac{e^2\cos^2\phi}{1 - e^2}$$

$$= -2\tan\phi\left(\frac{1-e^2\sin^2\phi}{1-e^2} + \frac{e^2-1}{1-e^2}\right)$$

$$= -2\tan\phi\left(\frac{\nu}{\rho} - 1\right) = -2t(r-1) \qquad \ldots\ldots \qquad 30.$$

f. The mean radius of curvature at a point on the ellipsoid

The mean value $M\{f(x)\}$ of the function $f(x)$ between the limits $x=a$ and $x=b$, is

$$M\{f(x)\} = \frac{\sum_{i=1}^{n} f(x_i)}{n} ,$$

where the n values of $f(x)$ are sampled at regular intervals between the abscissae x_{i-1} and x_i. As

$$n = \frac{b-a}{dx} ,$$

$$M\{f(x)\} = \lim_{dx\to 0}\left(\frac{1}{b-a}\sum_{i=1}^{n} f(x_i)\ dx\right) = \frac{1}{b-a}\int_a^b f(x)\ dx \qquad \ldots\ldots \qquad 31.$$

The mean radius of curvature R at a point on the ellipsoid is obtained by the application of equation 31 to equation 29, when

$$R = M\{R_\alpha\} = \frac{1}{2\pi - 0}\int_0^{2\pi} \frac{\rho\nu}{\rho\,\sin^2\alpha + \nu\,\cos^2\alpha}\,d\alpha.$$

As the function to be integrated undergoes the same changes in each quadrant, the trigonometric functions being independent of sign,

$$R = \frac{4}{2\pi}\,\rho \int_0^{\frac{1}{2}\pi} \frac{\sec^2\alpha\,d\alpha}{1 + \frac{\rho}{\nu}\tan^2\alpha} = \frac{2}{\pi}\,\rho\,\left(\frac{\nu}{\rho}\right)^{\frac{1}{2}}\int_0^{\infty} \frac{d\left(\sqrt{\left(\frac{\rho}{\nu}\right)}\tan\alpha\right)}{1 + \frac{\rho}{\nu}\tan^2\alpha}$$

$$= \frac{2}{\pi}\sqrt{(\rho\nu)}\,\left[\tan^{-1}\left(\sqrt{\left(\frac{\rho}{\nu}\right)}\,\tan\alpha\right)\right]_0^{\frac{\pi}{2}} = \frac{2}{\pi}\sqrt{(\rho\nu)}\left(\frac{\pi}{2} - 0\right)$$

$$= \sqrt{(\rho\nu)} \qquad \ldots\ldots \qquad 32.$$

g. The geodesic on the ellipsoid

An infinite number of curves can be drawn between two points *on the surface* of an ellipsoid. The curve of minimum length is called the *geodesic*. By Meusnier's theorem (e.g., Eisenhart 1960, p.118) which states that

the radius of curvature R_α in the normal section between two adjacent points on a surface, is related to that of an oblique section (R_O)

a. between the same points;

and b. whose principal normal is inclined at an angle θ to the surface normal,

by the relation

$$R_O = R_\alpha \cos\theta \qquad \ldots\ldots \qquad 33,$$

it follows that the shortest distance between two adjacent points on a surface, in the limit, lies in the normal section between them.

If the geodesic on the ellipsoid is defined by the parametric equations

$$x_i = x_i(s) \qquad \ldots\ldots \qquad 34,$$

where s is the distance along the geodesic from some reference point on it, then x_i^1, given by

$$x_i^1 = \frac{dx_i}{ds}\ ,\ i=1,3 \qquad \ldots\ldots \qquad 35,$$

are the direction cosines of the tangent to the geodesic as

$$\sum_{i=1}^{3} (x_i^1)^2 = 1 \qquad \ldots\ldots \qquad 36.$$

The differentiation of equation 36 with respect to s gives

$$\sum_{i=1}^{3} x_i^1 \, x_i^2 = 0.$$

As $x_i^1 (i=1,3)$ are the direction cosines of the tangent to the geodesic, and as the sum of the products of like direction cosines of mutually perpendicular lines are zero, it follows that

$$x_i^2 \propto \ell_i \quad , \; i=1,3,$$

where $\ell_i (i=1,3)$ are the direction cosines of the principal normal to the curve which, being a geodesic, has the normal to the surface as its principal normal as it lies entirely in the normal section. Hence ℓ_i for the geodesic on the ellipsoid are given by equation 15. It follows that

$$\frac{x_1^2}{\ell_1} = \frac{x_2^2}{\ell_2}$$

The use of equation 15 gives

$$\frac{x_1^2}{x_1} = \frac{x_2^2}{x_2}$$

or

$$x_1 x_2^2 - x_2 x_1^2 = 0,$$

where x_1 and x_2 are obtained from equation 13. Integration with respect to s gives

$$x_1 x_2^1 - x_2 x_1^1 = \text{Constant} = C,$$

where x_1^1, x_2^1 are given by equations 17,20 and 21. Appropriate substitution gives

$$\nu \cos\phi \cos\lambda\left(-\rho \sin\phi \sin\lambda \frac{d\phi}{ds} + \nu \cos\phi \cos\lambda \frac{d\lambda}{ds}\right) -$$
$$\nu \cos\phi \sin\lambda\left(-\rho \sin\phi \cos\lambda \frac{d\phi}{ds} - \nu \cos\phi \sin\lambda \frac{d\lambda}{ds}\right) = C.$$

Simplification gives

$$\nu^2 \cos^2\phi \, \frac{d\lambda}{ds} = C.$$

Consideration of the elemental triangle in figure 3.9, as adapted to the ellipsoid, gives

$$\frac{d\lambda}{ds} = \frac{\sin\alpha}{\nu \cos\phi}.$$

The use of this relation gives the final form of the equation which is satisfied by all geodesics on the ellipsoid, as

$$\nu \cos \phi \sin \alpha = C \qquad \ldots\ldots \qquad 37,$$

where C is a constant for a given geodesic.

E N D

R E F E R E N C E S

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E N D

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ADDENDUM

The expressions for δ_{12} on P 112 and k_ℓ on P 117 do not take into account the ellipticity factorwhich arises as a consequence of $\rho\nu$ having a variation with x_2. If $\rho\nu = R$, then

$$\left(\frac{\partial R}{\partial x_2}\right)_{x_1=0} = \left(\frac{\partial}{\partial\phi}\left(\frac{(1-e^2)}{(1-e^2\sin^2\phi)^2}\right)\frac{d\phi}{dx_2}\right)_{\phi=\phi_f}$$

$$= \left(\frac{-2a^2(1-e^2)[-2e^2\sin\phi\cos\phi]}{(1-e^2\sin^2\phi)^3}\frac{1}{\rho}\right)_{\phi=\phi_f} = \left(\frac{2R}{\nu}\frac{e^2}{1-e^2}\sin 2\phi\right)_{\phi=\phi_f}$$

$$= \frac{2R}{\nu}\varepsilon^2 \sin 2\phi \qquad \text{....(1)},$$

where $\varepsilon^2 = \dfrac{e^2}{1-e^2}$.

The contribution to σ on P 109 is

$$\sigma_c = -\frac{1}{k}\frac{\partial k}{\partial x_2}\sin\beta = -\frac{\partial}{\partial R}\left(\frac{x_1^2}{2R^2}\right)\frac{\partial R}{\partial x_2}\sin\beta = \left(\frac{x_1^2}{2R^2}\frac{2R}{\nu}\varepsilon^2\sin 2\phi\sin\beta\right)_{\phi=\phi_f}$$

$$= \left(\frac{x}{\rho\nu}\sin\beta\left(\frac{x_1}{\nu}\varepsilon^2\sin 2\phi\right)\right)_{\phi=\phi_f} \qquad \text{....(2)},$$

whose order of magnitude is one smaller than that of order$\{x_1/\nu\}^4$ as $\varepsilon^2 \approx 6\times10^{-3}$. Similarly, the contributions of significance σ_c^1 to σ^1 and σ_c^2 to σ^2 are given by

$$\sigma_c^1 = \frac{\partial\sigma}{\partial x_2}\frac{dx_2}{d\ell_p} = \frac{\partial}{\partial R}\left(\frac{x_1}{R}\cos\beta\right)\frac{\partial R}{\partial x_2}\cos\beta = \left(\frac{\cos^2\beta}{R}\left(\frac{x_1}{\nu}2\varepsilon^2\sin 2\phi\right)\right)_{\phi=\phi_f} \qquad \text{...(3)}$$

and

$$\sigma_c^2 = \frac{\partial\sigma^1}{\partial x_2}\frac{dx}{d\ell_p} = \left(\frac{-\cos^2\beta\sin\beta}{R}\left(\frac{2}{\nu}\varepsilon^2\sin 2\phi\right)\right)_{\phi=\phi_f} \qquad \text{....(4)}.$$

The correction δ_c to δ_{12} is given from equation 4.40 as

$$\delta_c = \tfrac{1}{2}\sigma_{co}\ell_p + \frac{1}{6}\sigma^1_{co}\ell_p^2 + \frac{1}{24}\sigma^2_{co}\ell_p^3$$ to the required order of precision.

Thus the term

$$\delta_c = \left(\frac{1}{12\rho\nu^2}\varepsilon^2\sin 2\phi\left(6x_{11}^2\Delta x_1 - \Delta x_2^2(3x_{11}+x_{12})\right)\right)_{\phi=\phi_f} \qquad \text{....(5)}$$

should be added to equation 6.43 on page 112, where

$$\Delta x_1 = x_{12} - x_{11} \quad \text{and} \quad \Delta x_2 = x_{22} - x_{21} \qquad \text{....(6)}.$$

Similarly, the ellipticity factor ffects k_ℓ through the terms

$$\tfrac{1}{2}k_o^1\ell + \frac{1}{6}k_o^2\ell^2 + \frac{1}{24}k_o^3\ell^3$$ in equation 4.27.

Similar consideration of equations 6.39, 6.45 and 6.47 give the correction term $k_{\ell c}$ to k_ℓ in equation 6.58 as

$$k_{\ell c} = \left(-\frac{\varepsilon^2\sin 2\phi}{12\rho\nu^2}\Delta x_2(3x_{11}^2 + 2x_{11}x_{12} + x_{12}^2)\right)_{\phi=\phi_f} \qquad \text{....(7)}$$

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