SURVEY COMPUTATIONS

M. Maughan

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Table of Contents

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1. Introduction

These lecture notes have been prepared for the use of students taking the degree course in Surveying at the University of New South Wales, with the aim of showing them the best methods of carrying out survey computations using tables of natural trigonometrical functions and hand operated or non-programmable calculating machines. Whilst logarithms may occasionally enter into the calculations, this method of calculation is today almost entirely superseded and it will not in consequence be dealt with. The programming of calculations for electronic computers may involve different problems and procedures, and whilst this is probably the method of calculation which will be most used in the future, it also will not be dealt with in these notes.

1.1. Accuracy of Calculation

It is useless to carry out a calculation unless this is done to the accuracy required by the problem or as limited by the accuracy of the original data. A sufficient number of significant figures must be used to ensure this accuracy and in general, in order to reduce the effect of rounding off errors, one more significant figure is usually employed in the main body of the calculation than is required for the final result. On the other hand it is inefficient to carry too many figures in the calculation and here common sense must prevail. For instance, if as part of a problem, the area is required of a rectangular plot, the sides of which have been measured as 120.63 and 275.91, a straightforward multiplication would give the area as 33283.0233 square units. This however would give a false idea of the accuracy of the areas as all we know is that the length of one side lies between 120.625 and 120.635 and that of the other between 275.905 and 275.915, which gives limits of 33281 and 33285 approximately for the area. The decimal part of the area is thus of little significance and in this case an answer to the nearest whole number would suffice.

1.2 Checking of Calculations

As accuracy is of paramount importance and since everyone is liable to make mistakes, it is essential that the calculation should be checked. In checking, the checker should not take the original calculation and check it step by step as there is always a great danger of accepting as correct a figure already written down even when it is wrong, and checks should be carried out as far as possible in two ways:

(a) an independent calculation by a second computer, the results of the calculations being subsequently compared, any discrepancies being investigated and rectified.

(b) a second calculation by the same computer but using a different mathematical process so that the tendency to repeat an error in the original calculation is eliminated.

The best methods of calculation are thus those with the most built-in checks but it is not always easy to devise these in a way which does not involve a lot of extra calculation. When only a partial checking system is used, it is essential for the computer to realise which parts of the computation are unchecked, so that he can treat these parts with special care.

No checking system will deal with inaccurate data and so the computer must ensure that his data has been extracted correctly. Whilst this may seem obvious, experience has shown that in a fair percentage of student calculations the data has been incorrectly transferred and the subsequent effort has been completely wasted.

1.3 Setting-out of the Calculations

Opinions will probably differ on this question. Whilst the use of standard forms may be suitable for computations by technicians, their use in a University course is to be deplored as they give the students a type of calculation which does not involve any thought or understanding. When left to their own devices, however, students generally cover masses of paper with computations, set down in an untidy fashion and difficult for others to follow.

Calculations should in general be set-out in tabular form in a neat and logical way, with the minimum of explanatory matter. It is open to question how much of the calculation should be committed to paper and how much carried on the machine. Some people consider that as much as possible should be carried on the machine owing to the possibility of transferring figures incorrectly to paper but whilst this is probably true in the case of expert computers, it is felt that with students learning to carry out calculations, as much as possible of the calculation should be written down, in order that any errors may be easily traced.

In the examples to be given later in these notes the calculations will be set out in tabular form. These are given merely as examples of neat and logical methods of setting out the calculations and it is not suggested that they are the only methods or that they are necessarily the best ones.

2. Elementary Formulae

Most survey computations involve a knowledge of plane trigonometry, which students will already have acquired at school, but a list of the more important formulae is given below for reference purposes.

2.1 General Trigonometrical Relationships

$$\cos^2 A + \sin^2 A = 1 \qquad 2.1.1$$

$$\sec^2 A = 1 + \tan^2 A \qquad 2.1.2$$

$$\operatorname{cosec}^2 A = 1 + \cot^2 A \qquad 2.1.3$$

$$\sin (A \pm B) = \sin A \cos B \pm \cos A \sin B \qquad 2.1.4$$

$$\cos (A \pm B) = \cos A \cos B \mp \sin A \sin B \qquad 2.1.5$$

$$\tan (A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A \tan B} \qquad 2.1.6$$

$$\sin 2A = 2 \sin A \cos A \qquad 2.1.7$$

$$\cos 2A = \cos^2 A - \sin^2 A = 2 \cos^2 A - 1 = 1 - 2 \sin^2 A \qquad 2.1.8$$

$$\tan 2A = \frac{2 \tan A}{1 - \tan^2 A} \qquad 2.1.9$$

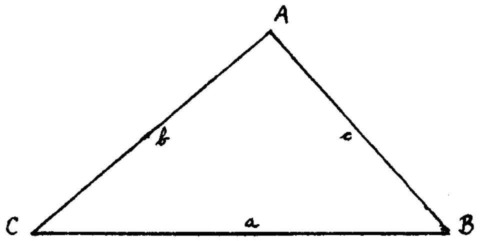
$$\sin X + \sin Y = 2 \sin \frac{(X + Y)}{2} \cos \frac{(X - Y)}{2} \qquad 2.1.10$$

$$\sin X - \sin Y = 2 \cos \frac{(X + Y)}{2} \sin \frac{(X - Y)}{2} \qquad 2.1.11$$

$$\cos X + \cos Y = 2 \cos \frac{(X + Y)}{2} \cos \frac{(X - Y)}{2} \qquad 2.1.12$$

$$\cos X - \cos Y = 2 \sin \frac{(X + Y)}{2} \sin \frac{(Y - X)}{2} \qquad 2.1.13$$

2.2 Formulae Relating Elements of the Triangle



$$a^2 = b^2 + c^2 - 2\,bc \cos A \qquad 2.2.1$$

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} \qquad 2.2.2$$

$$\tan \frac{(A - B)}{2} = \frac{a - b}{a + b} \cot \frac{C}{2} \qquad 2.2.3$$

$$\sin \frac{A}{2} = \sqrt{\frac{(s - b)(s - c)}{bc}} \quad \text{where } 2s = a + b + c \qquad 2.2.4$$

$$\cos \frac{A}{2} = \sqrt{\frac{s(s - a)}{bc}} \qquad 2.2.5$$

$$\text{Area of Triangle} = \tfrac{1}{2}\, bc \sin A \qquad 2.2.6$$

$$= \frac{1}{2} \frac{a^2 \sin B \sin C}{\sin A} \qquad 2.2.7$$

$$= \sqrt{s(s - a)(s - b)(s - c)} \qquad 2.2.8$$

3. Co-ordinate Systems and Bearings

Most of the subsequent calculations will involve co-ordinates and these co-ordinates will be denoted by N (northings positive) and E (eastings positive) whilst bearings will be measured by the clockwise angle from the North direction from 0° to 360°. The quadrantal form of expressing bearings has become increasingly obsolete, and N and E will be employed for the co-ordinates as it is quite obvious what they stand for, unlike X and Y which have different meanings in different parts of the world. The use of N and E also does away with any confusion with the school mathematical system of x and y with angles measured anti-clockwise from the x (or East) direction. When co-ordinates are quoted in these notes, the Easting co-ordinate will be given first and the Northing co-ordinate second. This is inconsistent with the general method of quoting Latitude before Longitude but it is the method adopted in New South Wales where the majority of our students will eventually be operating.

3.1 Calculations Involving a Line of Given Length and Bearing

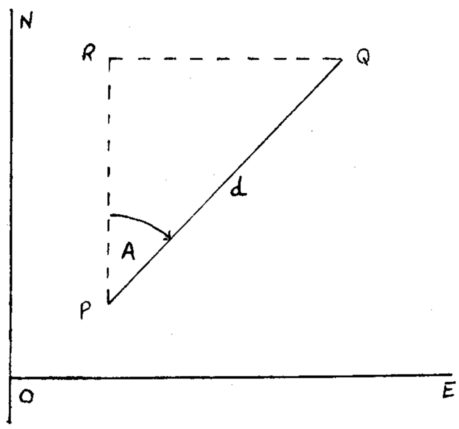
When dealing with computations involving a surveyed line PQ, the triangle used in the calculation is the right angled triangle which the line makes with the northing line through P and the easting line through Q. The treatment will differ slightly according to the quadrant in which the bearing of the line PQ lies. The length of the line is always taken as positive but the trigonometrical functions of the bearing may be either positive or negative.

Let the length of the line be d and the bearing A

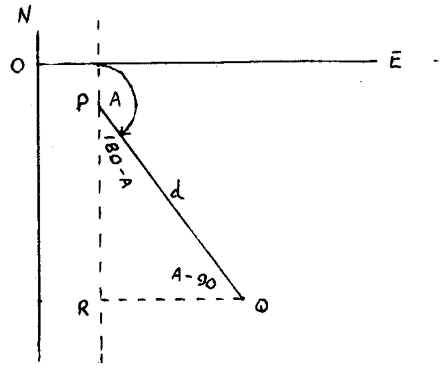
(a) 1st quadrant $0^\circ < A < 90^\circ$

Difference in Northings $\Delta N = N_Q - N_P = d \cos A$ 3.1.1a

Difference in Eastings $\Delta E = E_Q - E_P = d \sin A$



From the diagram it is obvious that the ΔN and ΔE are both positive and hence in the first quadrant cos A and sin A are also both positive.

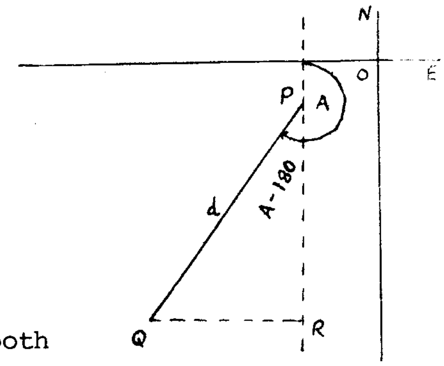


(b) <u>2nd quadrant</u> $90^\circ < A < 180^\circ$

Here we will have

$$\left.\begin{array}{l}\Delta N = d \cos A = -d \cos (180-A) = -d \sin (A-90) \\ \Delta E = d \sin A = d \sin (180-A) = d \cos (A-90)\end{array}\right\} \quad 3.1.1b$$

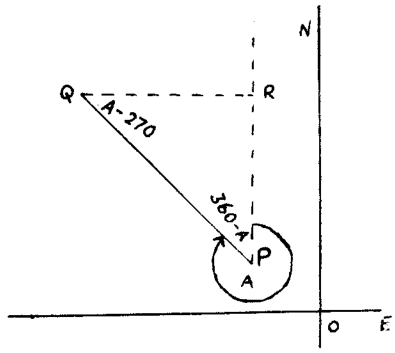
In this quadrant cos A will be negative and sin A positive.



(c) <u>3rd quadrant</u> $180^\circ < A < 270^\circ$

$$\left.\begin{array}{l}\Delta N = d \cos A = -d \cos (A-180) \\ \Delta E = d \sin A = -d \sin (A-180)\end{array}\right\} \quad 3.1.1c$$

In this quadrant sin A and cos A will be both negative.



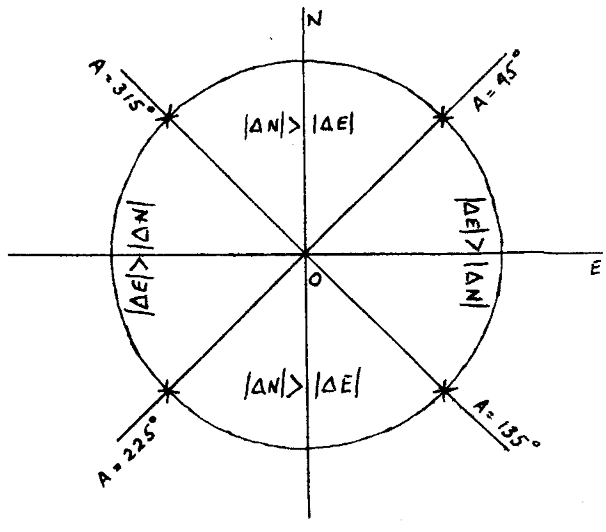
(d) <u>4th quadrant</u> $270^\circ < A < 360^\circ$

$$\left.\begin{array}{l}\Delta N = d \cos A = d \cos (360-A) = d \sin (A-270) \\ \Delta E = d \sin A = -d \sin (360-A) = -d \cos (A-270)\end{array}\right\} \quad 3.1.1d$$

In this quadrant, cos A will be positive and sin A will be negative.

It will be noticed that in the second and fourth quadrants, two expressions, besides d cos A and d sin A were given for ΔN and ΔE. This is done to facilitate the looking up of the trigonometrical functions in the tables as it is easier to subtract 90° or 270° from an angle than to subtract the angle from 180° or 360°. It should be stressed however that this facility is sometimes a source of error as students are liable to forget to look up the sine instead of the cosine or v.v. with the result that the numerical values of ΔN and ΔE are reversed.

Drawing a simple diagram showing the line in its approximate bearing would safeguard against this error as it would show whether ΔN or ΔE was <u>numerically</u> the greater.



3.2 Basic Calculations

There are three basic computations in problems involving co-ordinate systems in surveying. They are

(a) Given the co-ordinates (E_P, N_P) of a point P and the bearing A and distance d of a line PQ to find the co-ordinates of Q

From section 3.1 the co-ordinates of Q are obviously given by

$$\left.\begin{aligned} E_Q &= E_P + d \sin A \\ N_Q &= N_P + d \cos A \end{aligned}\right) \qquad 3.2.1$$

(b) The reverse problem - Given the co-ordinates (E_P, N_P) and (E_Q, N_Q) of two points P & Q to find the bearing and distance of the line between them.

As before

$$d \sin A = E_Q - E_P$$

$$d \cos A = N_Q - N_P$$

$$\tan A = \frac{E_Q - E_P}{N_Q - N_P} \text{ or } A = \tan^{-1}\left(\frac{E_Q - E_P}{N_Q - N_P}\right) \qquad 3.2.2$$

and

$$d = \frac{E_Q - E_P}{\sin A} = \frac{N_Q - N_P}{\cos A} \quad \text{(angle A having been found)} \qquad 3.2.3$$

It should be noted that as tan X = tan (180 + X) equation (3.2.2) will give two values for A, i.e. it will give the bearing PQ and the bearing QP. A simple diagram will show which is which.

The length d could also be calculated from the formula $d = \sqrt{(E_Q - E_P)^2 + (N_Q - N_P)^2}$ but this is a more complicated procedure and does not provide the check given by formula (3.2.3)

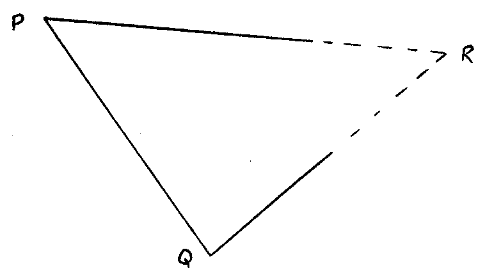
(c) Given the length of one side of a triangle and two angles of the triangle to find the lengths of the other two sides.

Let PQ be the known side and P and Q be the known angles.

Then R = 180 - P - Q

and formula (2.2.2) gives

$$\frac{PQ}{\sin (P + Q)} = \frac{PR}{\sin Q} = \frac{QR}{\sin P}$$



These three calculations can all be illustrated by the calculation of the co-ordinates of an intersected point.

Example 1.

At two known stations P & Q, the co-ordinates of which are (37928.3, 42398.7) and (43527.5, 37814.3) respectively, the angles $\lfloor QPR$ and $\lfloor PQR$ have been measured to a point R which is east of the line PQ. These angles are 45°43' 19" and 67°19'28" respectively. Find the co-ordinates of R

(1) See section 3.2(b)

Station	E	N
Q	43527.5	37814.3
P	37928.3	42398.7
	5599.2	-4584.4

$$\text{Bearing } \vec{PQ} = \tan^{-1} \frac{5599.2}{-4584.4} = \tan^{-1} (-1.221359) = 129^\circ\ 18'\ 33''$$

$$\text{Distance } PQ = \frac{-4584.4}{\cos 129^\circ 18' 33''} = \frac{-4584.4}{-.633506} = 7236.55$$

$$= \frac{5599.2}{\sin 129^\circ 18' 33''} = \frac{5599.2}{.773738} = 7236.56$$

Note that here in the calculation of the distance we are carrying one more decimal than is justified by the data. The two values for the distance should be the same but may differ by 1 or 2 in the last place of decimals in which case the value worked from whichever is the larger of the sine or cosine is accepted i.e. in this case 7236.56 as sin A >cos A

(2) See section 3.2(c)

Station	Angle	Sine	Side Length	Side
P	45° 43' 19"	.715961	5630.47	QR
Q	67 19 28	.922702	7256.33	RP
R	66 57 13	.920188	7236.56	PQ
	180 00 00			

Angles P & Q are known and angle R is obtained by making the three angles add up to 180°. PQ is the known side and is shown on the same line as angle R which is opposite to it in the triangle. Then QR and PR are calculated from $\frac{PQ \sin P}{\sin R}$ and $\frac{PQ \sin Q}{\sin R}$

(3) See section 3.2(a)

Bearing	PQ	129° 18' 33"		QP	309° 18' 33"	
	− ∠P	45 43 19		+ ∠Q	67 19 28	
Bearing	PR	83 35 14	dist. 7256.33	QR	16 38 01	5630.47
		sin + .993743	cos +.111691		+ .286251	+ .958155
Co-ords	P	37928.3	42398.7	Q	43527.5	37814.3
ΔE and ΔN		+7210.9	+ 810.5		+1611.7	+ 5394.9
	R	45139.2	43209.2		45139.2	43209.2

Note that in the final calculation the figures have been cut down to one decimal again as the co-ordinates of R cannot be quoted to a greater accuracy than those of P & Q

The first part of this section consists of finding the bearings of PR and QR from those of PQ and QP and the known angles. The bearings PQ and QP will obviously differ by 180°.

3.3 Methods of Checking these Computations

In the numerical example given in the last section the agreement of the co-ordinates of R, when finally computed from P and from Q, checks that all three parts of the calculation have been carried out correctly but the situation is rather different when each part is considered separately.

In part (1) the agreement of the two values for the length PQ, checks the calculation provided that the distances ΔN and ΔE have been correctly obtained. There has been no automatic check on the values for ΔN and ΔE, so these must be checked or the whole part checked by a different method of calculation. One possible method of checking would be to do the subtraction first in your head and then check on the machine.

Part (2) is completely unchecked except for making sure that all the angles add up to 180° exactly, but as this calculation is seldom made except as a preliminary to a subsequent calculation of co-ordinates, check calculations are not normally carried out.

Part (3) is checked only because the calculations of the co-ordinates of R have been carried out from two separate points. If they had been calculated from one point P only, then the results would be unchecked and a checking method would have to be devised.

For parts (1) and (3) of the calculation, an independent checking method, called the "45° check" has been devised, based on the facts that $\sin 45^\circ = \cos 45^\circ = \frac{1}{\sqrt{2}}$ and $\tan 45^\circ = 1$

As regard calculations of the type shown in part (1) it is only necessary to check the calculation of the bearing as the subsequent calculation of the distance is checked by the double calculation in equations (3.2.3). The bearing is checked by considering two auxiliary points P' and Q' such that $N_P' = N_P - E_P$ and $E_P' = N_P + E_P$ and similarly for Q.

Then if the bearing of the line P'Q' is A'

$$\tan A' = \frac{\Delta E'}{\Delta N'} = \frac{\Delta N + \Delta E}{\Delta N - \Delta E} = \frac{1 + \frac{\Delta E}{\Delta N}}{1 - \frac{\Delta E}{\Delta N}} = \frac{1 + \tan A}{1 - \tan A} = \tan (45^{\circ} + A) \qquad 3.3.1$$

$$A' = 45^{\circ} + A$$

Example 2 - same data as Example 1

Station	E	N	E' = N + E	N' = N - E
Q	43527.5	37814.3	81341.8	- 5713.2
P	37928.3	42398.7	80327.0	+ 4470.6
	5599.2	-4584.4	1014.8	-10183.6

$$A' = \tan^{-1} \frac{1014.8}{-10183.6} = \tan^{-1} (-.099650) = 174^{\circ}\ 18'\ 33''$$

= A + 45 (from example 1) - thus checking the calculation. Note that $\Delta N'$ and $\Delta E'$ should first be obtained from the N' and E ' for P' and Q' and then checked from ΔN and ΔE.

There are two forms of the check for the calculations of the co-ordinates of Q from the known co-ordinates of P and the known bearing and distance between them. In both of these checks an auxiliary bearing of (A + 45°) is used, in the first case with an auxiliary distance of $d\sqrt{2}$ and in the second case with an auxiliary distance of $d/\sqrt{2}$

Case 1 $\qquad A' = A + 45^{\circ} \qquad d' = d\sqrt{2}$

$$d' \sin A' = d\sqrt{2} \sin (A + 45) = d\sqrt{2} \left(\frac{1}{\sqrt{2}} \cos A + \frac{1}{\sqrt{2}} \sin A \right)$$

$$= d \cos A + d \sin A$$

$$= \Delta N + \Delta E \qquad 3.3.2$$

Example 3 same data as example 1 for line PR

1.414214 d	A + 45	d' sin A'	ΔN + ΔE (from example 1)
10262.0	128° 35' 14"	8021.4	8021.4

<u>Case 11</u> $A' = A + 45 \qquad d' = d/\sqrt{2}$

Then $C = d' \cos A' = \frac{d}{\sqrt{2}} \cos (A + 45^{\circ}) = \frac{d}{2} (\cos A - \sin A)$

$S = d' \sin A' = \frac{d}{\sqrt{2}} \sin (A + 45^{\circ}) = \frac{d}{2} (\cos A + \sin A)$

$\Delta N = d \cos A = C + S$ and $\Delta E = \sin A = S - C$ 3.3.3

<u>Example 4</u> same data as example 1 for line QR

.707107 d	A + 45	C	S	C + S	S - C
3981.3	61° 38' 01"	1891.5	3503.3	5394.8	1611.8

When the values of (C + S) and (S - C) are compared with ΔN and ΔE from example 1, it will be seen that there is a discrepancy in each case of 1 in the last figure. This is due to rounding off errors and is quite acceptable.

Of these two methods the first will obviously appeal to many computers as it involves less work but it should be pointed out that its use is not without danger as the computer could have transposed his ΔN and ΔE and the check would apparently show the calculation to be right when in fact it is wrong. This type of mistake is quite likely to occur when A is in the second or fourth quadrants and the angles (A - 90) and (A - 270) are used without transposing the trigonometrical functions (see section 3.1) Theoretically the check should still work under these conditions but experience shows that the student often tends to concentrate on the numerical value of $(\Delta N + \Delta E)$ and to ignore its sign. The second method is therefore preferable. Quite small errors can be detected by this check as the maximum discrepancy due to rounding off should not exceed 2 in the last place of decimals.

3.4 Traverse Calculations

Traverse calculations consist of a series of calculations of the type 3.2(a) together with adjustments designed to eliminate as far as possible the errors in the angular and linear observations. There are a number of different methods of adjustment but the only one which will be dealt with here is the normal form of the Bowditch adjustment. The method is best explained by a numerical example.

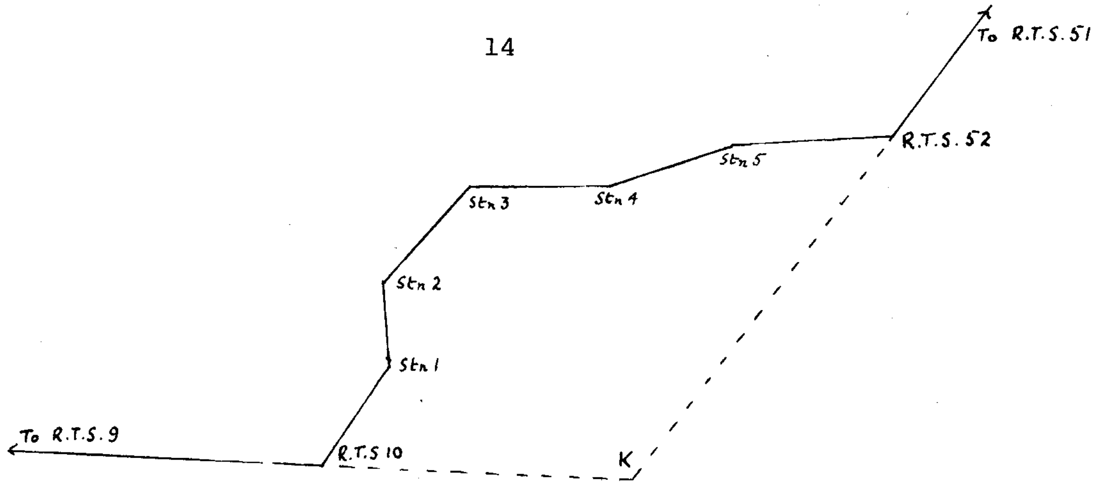
Example 5

Consider a traverse run between two control stations, R.T.S. 10 and R.T.S. 52 for which the following data is given.

Station	E	N	Datum Bearing to Trig
R.T.S. 10	8916.37	7854.72	279° 43' 29"
R.T.S. 52	10406.73	9424.95	53 30 46

and for which the following reduced angular and linear observations have been obtained.

At Station	Clockwise Angle From Point Above to Point Below	Distance To Point Below
R.T.S. 9		
R.T.S. 10	95° 12' 13"	579.82
STN 1	164 58 42	284.48
STN. 2	207 15 40	400.45
STN. 3	250 19 33	292.54
STN. 4	145 30 06	438.03
STN. 5	183 26 41	521.73
R.T.S. 52	167 05 02	
R.T.S. 51		



The first calculation to be carried out is the adjustment of the angular observations and this can be done by making corrections either to the bearings direct or initially to the observed angles. The latter method gives an additional check on the work and has been adopted in the specimen computation. In this computation the station names are entered in the first column, the observed angles in the second and the datum opening and closing bearings are entered on the appropriate lines of the fourth column. It should be noted that these bearings are from the point above to the point below and hence the opening datum bearing differs by by 180° from that given in data. The traverse is now changed for angular purposes only into a loop traverse by adding underneath the observed angles an additional angle which will be found by subtracting the last datum bearing from the first datum bearing. This angle will differ from the external angle at point K in the diagram by 180°. The sum of all the external (or internal) angles of a polyon is some integral multiple of 180° and hence so must be the sum of the angles now written in the second column, but on summing these angles a total of 1260° 00' 40" is obtained, indicating an angular misclosure of 40". In order to make the angles total 1260° i.e. 7 x 180° exactly, a correction of -40" must be made, in total, to the observed angles and since we are not dealing with decimals of a second, 5 angles will receive a correction of -6" and 2 a correction of -5", these two being spaced evenly through the traverse. It should be noted that no correction is made to the additional angle as this is not an observed angle but is fixed by the data.

The corrections are entered in column 3 and after correction of the angles in column 2, the bearings in column 4 are completed by adding the adjusted angle to the bearing on the line above and adding or subtracting 180°. As a final important check, the last observed angle should be added to the previous bearing, altered by 180°, to see whether the closing bearing is obtained. If it is, this will check that all the adjustments and the bearing calculations have been carried out correctly. This check is not obtained if the bearings are calculated from the unadjusted angles and then adjusted by an accumulative process i.e., the first bearing would have a correction of -6", the second a correction of -11" and so on. If properly carried out, both methods will give the same result but in the bearing adjustment, a small error of 1' say in one of the calculations could change a small positive misclosure to a small negative one or v.v. and could pass unnoticed. In the angle correction method, this error would be shown up by the final check.

The observed distances are now entered and, if thought necessary, the sines and cosines of the adjusted bearings, and the difference Northings and Eastings are then calculated and entered in the appropriate columns. At this stage, the 45° check computations should be made as shown in the lower part of the worked example as it is a waste of time to proceed to calculate final co-ordinates if the differences Northings or Eastings are incorrect. When they have been checked, the total difference Northings and Eastings should be compared with the differences between the corresponding co-ordinates of the opening and closing stations to give the misclosures in Northings and Eastings.

In the Bowditch correction method, these misclosures are adjusted on each traverse line in proportion to the ratio of the length of that line to the total distance traversed, e.g. the correction to the Northing difference for the first line would be $\frac{-.238 \times 579.82}{2517.05} = -.055$

The misclosures are most easily calculated by dividing the total Northing or Easting misclosure by the total distance and then multiplying this figure in turn by the length of each traverse line. The differences in Northings and Eastings and the corrections have been calculated to one extra place of decimals to avoid rounding off errors. After correction the differences should be added to the appropriate co-ordinates to give the co-ordinates of the new stations but these, though calculated to 3 places of decimals (in this example) should only be recorded to two places in conformity with the data co-ordinates. A check should be obtained on the final datum co-ordinates.

The adjustments could have been applied on an accumulative principle direct to the co-ordinates but for a reason similar to that given for the angle adjustment, it has been considered preferable to apply them to the difference Northings and Eastings.

For all traverses the angular and linear misclosures should be shown to ensure that these conform to local regulations or to the requirements of a particular contract. They are normally shown both in total and as a ratio. The total linear misclosure is of course the hypotenuse of the triangle formed by the northings and eastings misclosures.

It should be noted that the bearings and distances will not be consistent with the final co-ordinates but this is not of great importance as, for the type of work for which a Bowditch adjustment is appropriate, the co-ordinates are only required for plotting purposes. The adjusted bearings shown in the calculation will probably be more accurate than those computed from co-ordinates, as they are unaffected by errors in the distances and in the co-ordinates of the control points.

STATION	OBSERVED ANGLE	CORR.	BEARING	DISTANCE	SINE COSINE	ΔE	CORR.	ΔN	CORR.	E	N	STATION
R.T.S 9			99 43 29									
R.T.S.10	95 12 ~~13~~ 07	-6	14 55 36	579·82	+·257582 +·966256	+ 149·~~351~~ 392	+041	+ 560·~~255~~ 200	-055	8916·37	7854·72	R.T.S.10
STN 1	164 58 ~~42~~ 37	-5	359 54 13	284·48	-·001682 +·999999	− ·~~478~~ 458	+020	+ 284·~~480~~ 453	-027	9065·76	8414·92	STN 1
STN 2	207 15 ~~40~~ 34	-6	27 09 47	400·45	+·456524 +·889711	+ 182·~~815~~ 843	+028	+ 356·~~285~~ 247	-038	9065·30	8699·37	STN 2
STN 3	250 19 ~~33~~ 27	-6	97 29 14	292·54	+·991474 -·130305	+ 290·~~046~~ 068	+021	− 38·~~119~~ 147	-028	9248·15	9055·62	STN 3
STN 4	145 30 ~~06~~ 00	-6	62 59 14	438·03	+·890905 +·454189	+ 390·~~243~~ 274	+031	+ 198·~~948~~ 907	-041	9538·22	9017·47	STN 4
STN 5	183 26 ~~41~~ 36	-5	66 25 50	521·73	+·916576 +·399860	+ 478·~~205~~ 242	+037	+ 208·~~619~~ 570	-049	9928·49	9216·38	STN 5
R.T.S.52	167 ~~05~~ 04 ~~02~~ 56	-6	53 30 46	2517·05		+1490·182		+1570·468		10406·73	9424·95	R.T.S.52
R.T.S.51	46 12 43									+ 1490·36	+ 1570·23	
	1260 00 40	-40								1490·182	1570·468	
										+ ·178	− ·238	

Miscloses

Angular 40" or 5.7" per angle

Linear ·297 in 2517·05 or 1 in 8475 approx

CHECK COMPUTATION

BEARING + 45°	·707107 x DISTANCE		S	C	S−C = ΔE	S+C = ΔN
59 55 36	409·995	+·865384 +·501108	+354·803	+205·452	+149·351 ✓	+560·255 ✓
44 54 13	201·158	+·705416 +·708296	+142·001	+142·479	− ·478 ✓	+284·480 ✓
72 09 47	283·161	+·951933 +·306309	+269·550	+ 86·735	+182·815 ✓	+356·285 ✓
142 29 14	206·857	+·608938 −·793218	+125·963	−164·083	+290·046 ✓	− 38·120 ✓
107 59 14	309·734	+·951125 −·308805	+294·596	− 95·647	+390·243 ✓	+198·949 ✓
111 25 50	368·919	+·930861 −·365373	+343·412	−134·793	+478·205 ✓	+208·619 ✓

3.5 Missing Data Problems

Traverse computations are frequently used in Australia to solve problems involving missing data. They are generally computed in the form of a loop traverse, i.e. a traverse opening and closing on the same point, but the principle is equally applicable to traverses opening on one co-ordinated point and closing on another. The assumption is made that the known observations are accurate and that there is no misclosure in the traverse. There will thus be two equations, one equating the sum of the difference Northings to the difference between the Northings of the closing and opening stations (zero for a loop traverse) and the other doing the same thing for the Eastings. From this it will be seen that, in general, two unknown elements can be found.

As examples of this type of problem the following cases will be dealt with:

(1) bearing and distance of one line unknown

(2) two distances unknown

(a) For the first type let us assume that in Example 5, the bearing and distance for the line from Station 3 to Station 4 was unknown but all the other observations and data were as given in the example. Then if d and A are the required distance and bearing

$$1608.587 + d \cos A = 1570.23 \qquad d \cos A = -38.357$$

$$1200.136 + d \sin A = 1490.36 \qquad d \sin A = 290.224$$

$$\cot A = \frac{-38.357}{290.224} = -.132163 \qquad A = 97^{\circ}\ 31'\ 44''$$

$$d = \frac{-38.357}{.131024} = \frac{290.224}{.991379} = 292.75$$

If there is a slight discrepancy between the two values of d so obtained, the value arising from the formula which uses the larger numbers should be accepted.

(b) For the second type, taking the same Example 5 again with the same data and observations except that the distances Stn.2 - Stn. 3 and Stn. 4 - Stn. 5 are unknown, then if these distances are denoted by d_1 and d_2 respectively

$$.889711\ d_1 + .454189\ d_2 + 1015.235 = 1570.23$$

$$.456524\ d_1 + .890905\ d_2 + 917.124 = 1490.36$$

and the two equations of this nature can always be solved for d_1 and d_2 except when the two lines, the distances of which are unknown are parallel (e.g. opposite sides of a road reserve).

When the traverse is a loop traverse, the solution can be simplified by a preliminary rotation of the axes so that either d_1 or d_2 takes up a cardinal bearing before calculation of the difference Northings and Eastings. The side lengths calculated in Example 1(2) could be found by this procedure as follows:-

Line	Bearing	Distance	Transformed Bearing	ΔE	ΔN
PQ	129°18'33"	7236.56	135° 43' 19"	+5052.140	-5181.089
QR	16 38 01	d_1	23 02 47	$+.391477\ d_1$	$+.920188\ d_1$
RP	263 35 14	d_2	270 00 00	$-d_2$	0

The transformed bearings are obtained by adding 6°24'46" to all the original bearings making the line RP due West.

This gives

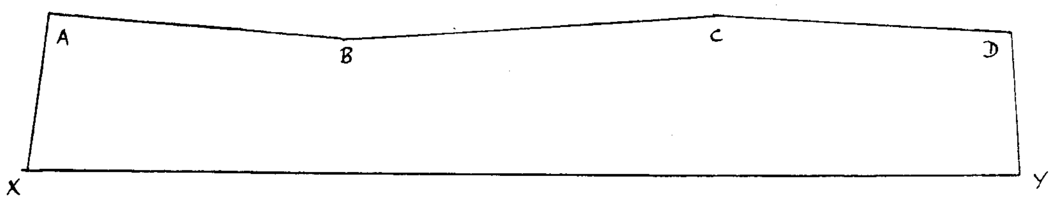
$$-5181.089 + .920188\ d_1 = 0 \qquad d_1 = 5630.47$$

$$d_2 = 5052.140 + .391477\ d_1 = 7256.34$$

which agree with the values previously obtained.

3.6 Offset Distance Calculations

A further use of traverse computations is in the calculation of offset distances to a given line, generally a boundary. Consider the problem of emplacing points on line between two known points X and Y, over which it is not possible to set up an instrument. The general procedure is to set up within a few feet of X, observe to X and then run a traverse as nearly parallel to the line XY as possible, until a point in the vicinity of Y is reached and at this point observe to Y.



The unclosed traverse is then computed on an assumed bearing, a slight reduction in calculation being obtained if one line, say the first long traverse line, AB, is taken on a cardinal bearing. From this computation the total differences Northing and Easting between X and Y will be obtained and from these the bearing XY can be calculated. This bearing should not in general be much different from that of AB and the system should now be swung so that XY lies along the cardinal direction (say West-East) Recomputation, using the amended bearings, of the Northing differences only will, by accumulation give the required offset distances. A 45^{o} check need not be made in this case if a zero offset is obtained at Y on the second calculation.

STATION	OBSERVED ANGLE	CORR.	ASSUMED BEARING	DISTANCE	SINE / COSINE	ΔE	CORR.	ΔN	CORR.	E	N	STATION
X												
			1 44 50	2.53	+.030490 / +.999535	+ .077		+ 2.529				
A	268 15 10				1							
			90 00 00	740.25	0	+ 740.250		0				
B	180 07 40											
			90 07 40	1263.71	+.999998 / −.002230	+ 1263.707		− 2.818				
C	179 57 40											
			90 05 20	1499.95	+.999999 / −.001551	+ 1499.949		− 2.326				
D	268 29 30											
			178 34 50	3.27	+.024771 / −.999693	+ .081		− 3.269				
Y												
						+ 3504.064		− 5.884				

Bearing XY = $\cot^{-1}\left(\frac{-5.884}{3504.064}\right)$

$= \cot^{-1}(-.001679)$

$= 90^\circ\ 05'\ 46''$

Deduct 05′ 46″ from each Bearing

TRANSFORMED BEARING	DISTANCE	SINE / COSINE	ΔN	OFFSET DISTANCE	STATION
1 39 04	2.53	+.999585	+2.529	+ 2.53	A
89 54 14	740.25	+.001677	+1.241	+ 3.77	B
90 01 54	1263.71	−.000553	− .699	+ 3.07	C
89 59 34	1499.95	+.000126	+ .189	+ 3.26	D
178 29 04	3.27	−.999650	−3.269	− .01	

CHECK COMPUTATION

BEARING + 45°	.707107 × DISTANCE		S	C	S − C = ΔE	S + C = ΔN

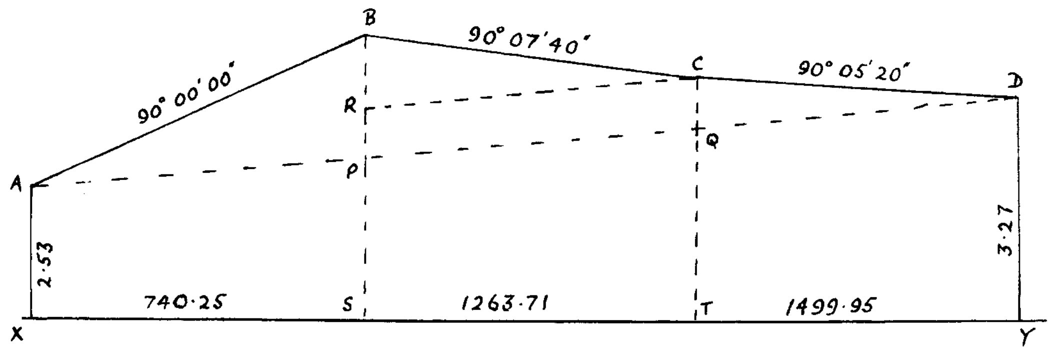
Example 6

Station	Observed Angle	Distance
X		
A	268^{O} 15' 10"	2.53
B	180 07 40	740.25
C	179 57 40	1263.71
D	268 29 30	1499.95
Y		3.27

The actual final offset distance obtained in this worked example (-.01) is due to the facts that the cosines of angles close to 90^{O} are changing rapidly and the bearing XY was taken to the nearest second.

Alternative Method

In cases where the main part of the traverse is practically a straight line and the radiations to the end boundary marks are almost at right angles to the boundary, an approximate method of calculation can be carried out which will involve little use of tables and which will give results almost identical with those obtained from the accurate calculation. The example previously dealt with provides a suitable case to illustrate this method.



In the diagram given above, the angles have been exaggerated to make the diagram clearer. AX and DY have been taken as perpendicular to XY and also equal in length to the measured distances 2.53 and 3.27. BRPS and CQT are perpendiculars to XY cutting AD in P and Q whilst CR is parallel to DA.

The method depends on the use of the following approximations for small angles

$\cos A \simeq 1 \qquad \sin A \simeq \tan A \simeq A$ (in radians)

Then $\angle BAD$ = (offset distance of D from AB produced)/distance AD

$$\frac{740.25 \times 0 + 1263.71 \times 460 + 1499.95 \times 320}{740.25 + 1263.71 + 1499.95} \quad \text{(working in seconds)}$$

$$= 303 \text{ seconds} = 5' \, 03''$$

Now by similar triangles

$$PS = 2.53 + \frac{740.25}{3503.91}(3.27 - 2.53) = 2.686$$

$$QT = 2.53 + \frac{2003.96}{3503.91}(3.27 - 2.53) = 2.953$$

and

$$BP = 740.25 \times .0014689 \; (5' \, 03'' \text{ in radians}) = 1.087$$

$$CQ = 1499.95 \times .0000824 \; (17'' \text{ in radians}) = .124$$

$$BR = 1263.71 \times .0007612 \; (2' \, 37'' \text{ in radians}) = .962$$

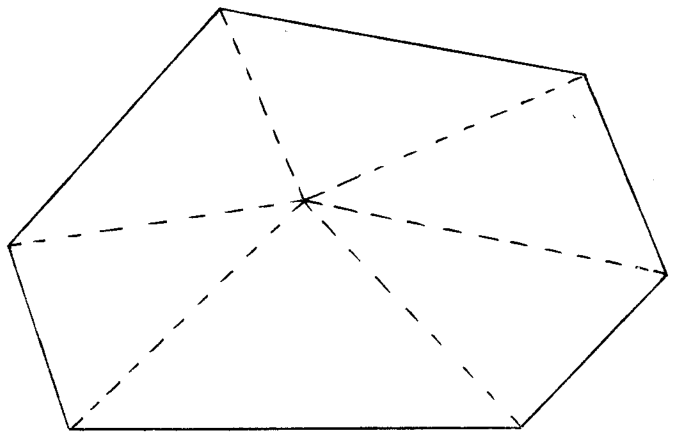
Offset distance at B = PS + BP = 3.77

Offset distance at C = CQ + QT = 3.08

and a check is given by the fact that BP = QC + BR

4. Areas

4.1 Plane Figures with straight sides.

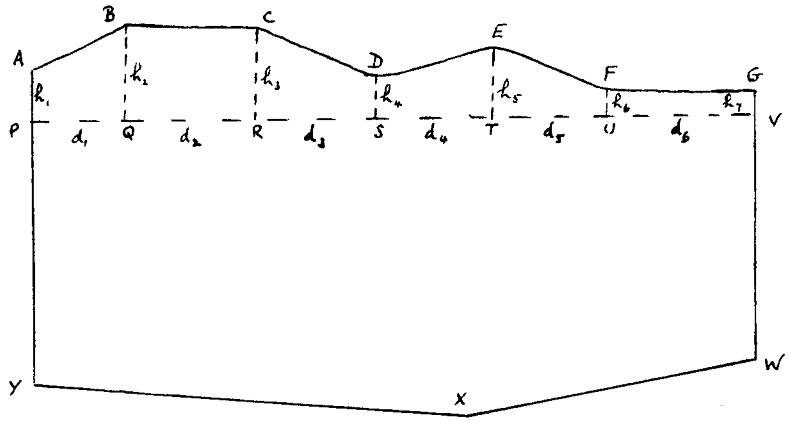


Any plane figure with straight sides can be divided up into a series of triangles and so its area can be computed using the various formulae for the area of a triangle.

For chain surveys, formula 2.2.8 can be used or the formula $A = \frac{1}{2}$ base x height, using measured distances where possible or scaled distances if measurements are not available.

For theodolite surveys formulae 2.2.6 and 2.2.7 may be used.

4.2 Plots with a curvilinear boundary.

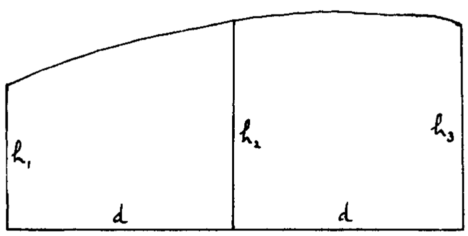


If a plot has one or more of its boundaries irregular, the irregular boundary is usually surveyed by chainage and offset distance, and the area of the irregular portion of the plot between the boundary and the traverse line is then calculated by one of two methods.

(a) $\text{Area} = \frac{1}{2}\left\{ d_1(h_1+h_2) + d_2\ (h_2+h_3) + d_3\ (h_3+h_4) + \dots + d_n(h_n+h_{n+1}) \right\}$

The offset distances are measured at unequal intervals as above when the irregular boundary can be divided with sufficient accuracy, into sections which can be regarded as straight lines. When this is not the case the offset distances are measured at equal distances d along the chainage line and the formula then becomes $A = \frac{d}{2}\left\{ h_1+2h_2+2h_3+ \dots \qquad + 2h_n+h_{n+1} \right\}$ 4.1

(b) In the other method, offset distances are again measured at equal intervals d along the chainage line and the assumption is made that the irregular boundary going through the ends of three neighbouring offsets can be represented with sufficient accuracy by a second or third order polynomial



If this assumption is justifiable the area of this section will be given by

$$A = \frac{d}{3}\ (h_1 + 4h_2 + h_3)$$

and combining a number of such sections gives the general formula

$$A = \frac{d}{3}\left\{ h_1+4h_2+2h_3+4h_4+2h_5+\dots \qquad +2h_{2n-1} + 4h_{2n} + h_{2n+1} \right\} \qquad 4.2$$

This formula is generally accepted to be more accurate than formula (4.1) which assumes straight line sections for the irregular boundary but it suffers from a slight disadvantage in that it needs an odd number of offset distances. This disadvantage can be got over however, if an even number of offsets have been observed, by calculating the area of one of the end sections by formula (4.1) and the remainder by formula (4.2)

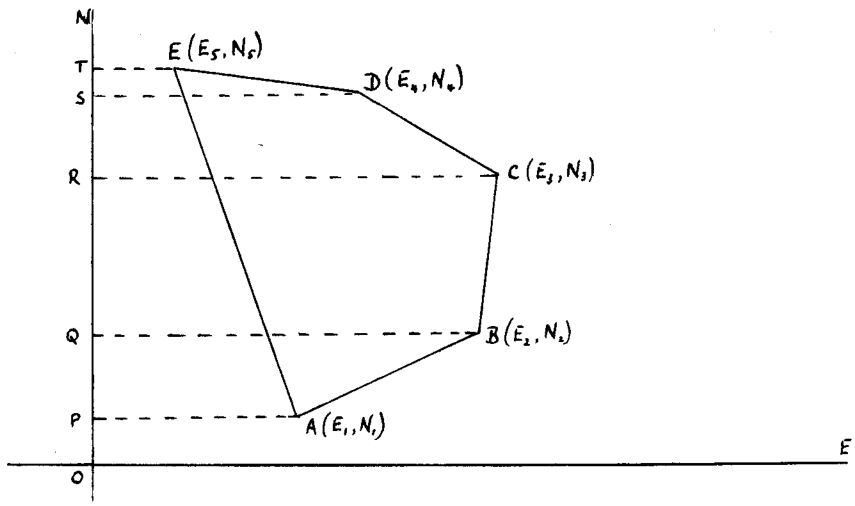
It should be noted that these formulae can also be used for the calculations of volumes, the offset distances h_1 h_2 etc. being replaced by the areas A_1 A_2 A_3 etc. of parallel cross sections at distances d apart. The formulae then are

$$V = \frac{d}{2}\left\{A_1+2A_2+2A_3+\ldots\quad+2A_n+A_{n+1}\right\} \qquad 4.1a$$

and

$$V = \frac{d}{3}\left\{A_1+4A_2+2A_3+4A_4+2A_5+\ldots\quad+2A_{2n-1}+4A_{2n}+A_{2n+1}\right\} \qquad 4.2a$$

4.2 Areas from Co-ordinates



Consider a plane figure ABCDE where the co-ordinates of the points A, B, C, D & E are known. A is (E_1,N_1) etc. as shown on the diagram. Drop perpendiculars AP, BQ etc. on to the N-S axis. Then

$$\text{Area ABCD} = \text{Area (ABQP + BCRQ + CDSR + DETS - AETP)}$$

$$= \tfrac{1}{2}\left\{(N_2-N_1)(E_2+E_1)+(N_3-N_2)(E_3+E_2)+(N_4-N_3)(E_4+E_3)+(N_5-N_4)(E_5+E_4)+(N_1-N_5)(E_1+E_5)\right\} \qquad 4.3a$$

This is known as the Double Longitude formula. It can be transformed by rearrangement of the terms into other forms as follows.

$$\text{Area} = \tfrac{1}{2}\left\{(E_1-E_2)(N_1+N_2)+(E_2-E_3)(N_2+N_3)+(E_3-E_4)(N_3+N_4)+(E_4-E_5)(N_4+N_5)+(E_5-E_1)(N_5+N_1)\right\} \qquad 4.3b$$

$$= \tfrac{1}{2}\left\{N_1(E_5-E_2)+N_2(E_1-E_3)+N_3(E_2-E_4)+N_4(E_3-E_5)+N_5(E_4-E_1)\right\} \qquad 4.3c$$

$$= \tfrac{1}{2}\left\{E_1(N_2-N_5)+E_2(N_3-N_1)+E_3(N_4-N_2)+E_4(N_5-N_3)+E_5(N_1-N_4)\right\} \qquad 4.3d$$

For a plot bounded by n sides, equations 4.3a and 4.3c become

$$\text{Area} = \tfrac{1}{2}\sum_{i=1}^{n} (N_{i+1} - N_i)(E_{i+1} + E_i) = \tfrac{1}{2}\sum_{i=1}^{n} N_i(E_{i-1} - E_{i+1})$$

where if $(i-1)<1$ add n and if $(i+1)> n$ subtract n.

Formulae 4.3a and 4.3b are commonly used in Australia as full co-ordinates are generally not available and only the ΔN and ΔE for the various lines have been calculated.

For a properly co-ordinated system, however, formulae 4.3c and 4.3d are the most suitable for calculations on a hand operated calculating machine.

4.3 Numerical Examples

4.3.1 Double Latitude and Double Longitude Method.

Line	ΔE	ΔN	DLONG	DLAT	ΔNxDLONG	ΔExDLAT
AB	+169	-298	+169	-298	- 50362	- 50362
BC	+362	-151	+700	-747	-105700	-270414
CD	+383	+630	+1445	-268	+910350	-102644
DE	-560	+301	+1268	+663	+381668	-371280
EA	-354	-482	+ 354	+482	-170628	-170628
					965328	-965328

Area = ½ x 965328 = 482664 square units

The DLONG column is formed by adding to the ΔE on the same line, twice the sum of the preceding ΔE's. It is checked by the numerical agreement (but with the opposite sign) of the last entry with the last entry in the ΔE column. The DLAT column is similarly formed and checked from the ΔN column. The individual (ΔN x DLONG) and (ΔE x DLAT) would not be calculated and entered in practice, only the total being accumulated on the machine.

4.3.2 Zigzag Method from Co-ordinates.

In order to keep positive values throughout, let A be given the co-ordinates (500, 0) Then from the ΔN and ΔE given above the co-ordinates of the whole system will be

Station	E	N
A	0	500
B	169	202
C	531	51
D	914	681
E	354	982

It should be obvious that a change of origin will not affect the area of the plot and so any constant can be added to or subtracted from the N co-ordinate (or the E co-ordinate) in order to make the numbers smaller and easier to handle. The stations are then written down in cyclic order on successive lines, twice, and the first two stations are then repeated a third time as shown below.

Station	N	E
A	500	
B		169
C	51	
D		914
E	982	
A		0
B	202	
C		531
D	681	
E		354
A	500	
B		169

The Northing co-ordinate is entered on the first line, the Easting on the second, the Northing on the third and so on alternately. The calculation is then carried out as an accumulative process as follows. Zero is put on the Setting Register and multiplied by 500 giving a total zero in the Produce Register. Without clearing the Multiplying or Produce Registers, 169 is put on the Setting Register and the 500 at present on the Multiplying Register is changed to 51. This gives a total of 169(51-500) in the Product Register. 914 is now put on the Setting Register and the 51 on the Multiplying Register changed to 982. This gives a total of {169(51-500) + 914(982-51)} on the Product Register and the process is continued until the list has been finished when the final total will be twice the area, giving Area = ½ x 965328 = 482664 as before.

The calculation is checked by carrying out the same process from the bottom up i.e. 0 on the Setting Register multiplied by 169, then 500 on the Setting Register and the 169 changed to 354 etc.

It will be seen from the above that ignoring the repetition of the first two lines, the Northing co-ordinate and the Easting co-ordinate of each station appears once. This would not be the case if there were an even number of stations in the plot and in this case an amended procedure has to be adopted, by repeating the last station each time it appears in order to ensure that each co-ordinate is listed at least once. For instance if we required the area of the plot ABCDA, then the co-ordinates would be set out as follows.

Station	N	E
A	500	
B		169
C	51	
D		914
D	681	
A		O
B	202	
C		531
D	681	
D		914
A	500	
B		169

The area ABCDA would then be computed as

$\frac{1}{2}\{169(51-500)+914(681-51)+0(202-681)+531(681-200)+914(500-681)\}$

Formula 4.3d for four stations

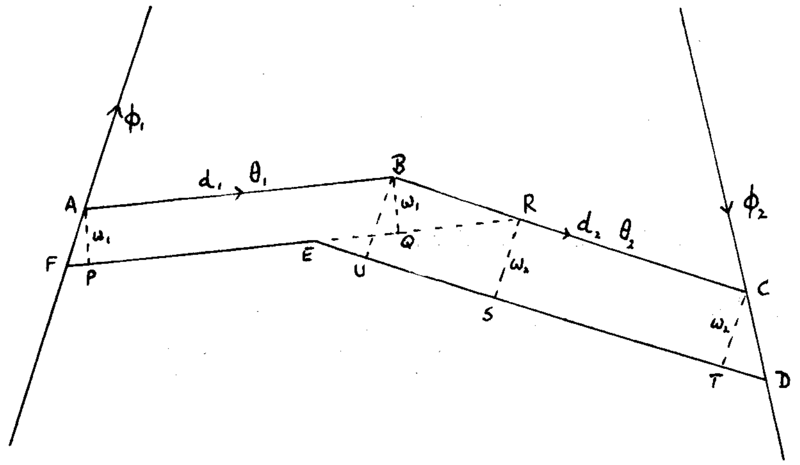
$\frac{1}{2}\{E_1(N_2-N_4)+E_2(N_3-N_1)+E_3(N_4-N_2)+E_4(N_1-N_3)\}$ gives

$\frac{1}{2}\{0(202-681)+169(51-500)+531(681-202)+914(500-51)\}$

and the formula used in practice gives the same result since

$N_1-N_3 = 500-51 = (500-681) + (681-51) = (N_1-N_4) + (N_4-N_3)$

5. Road Intersection Calculations



These are the calculations required for the setting out of a road through one or more lots and the subsequent calculations of the areas to be resumed. They involve only elementary trigonometry.

Example.

One side of a road reserve ABC through a lot has been set out and the distances and bearings of the lines AB & BC have been measured as d_1 & θ_1 and d_2 & θ_2 respectively. The bearings of the lot boundaries are ϕ_1 & ϕ_2 as shown on the diagram, all bearings being those in the directions of the arrows. The width of the road at A is w_1 and that of the road at C is w_2, these widths remaining constant, i.e. FE is parallel to AB and ED is parallel to BC.

The problem is to set out the points F, E & D and to calculate the area resumed.

Produce FE to cut BC in R. AP & BQ are perpendicular to FR and BU, RS & CT are perpendicular to ED. Then

$\underline{/AFP} = \theta_1 - \phi_1$ AF $= w_1 \text{cosec } (\theta_1 - \phi_1)$ FP $= w_1 \cot(\theta_1 - \phi_1)$

$\underline{/CDT} = \phi_2 - \theta_2$ CD $= w_2 \text{cosec } (\phi_2 - \theta_2)$ TD $= w_2 \cot(\phi_2 - \theta_2)$

$\underline{/BRQ} = \underline{/RES} = (\theta_2 - \theta_1)$

$ER = w_2 \operatorname{cosec}(\theta_2 - \theta_1)$ $\qquad$ $QR = w_1 \cot(\theta_2 - \theta_1)$

$ES = w_2 \cot(\theta_2 - \theta_1)$ $\qquad$ $US = BR = w_1 \operatorname{cosec}(\theta_2 - \theta_1)$

If the setting out of E is to be done from B, then

$$\underline{/ABE} = \tan^{-1}(w_1/EQ) = \tan^{-1}\left(\frac{w_1}{w_2 \operatorname{cosec}(\theta_2-\theta_1) - w_1 \cot(\theta_2-\theta_1)}\right)$$

and distance $BE = w_1 \operatorname{cosec} \underline{/ABE}$ or can be computed by Pythagoras

$$FE = FP + PQ + QR - ER$$

$$= w_1 \cot(\theta_1 - \phi_1) + d_1 + w_1 \cot(\theta_2 - \theta_1) - w_2 \operatorname{cosec}(\theta_2 - \theta_1)$$

$$ED = DT + TU + ES - US$$

$$= w_2 \cot(\phi_2 - \theta_2) + d_2 + w_2 \cot(\theta_2 - \theta_1) - w_1 \operatorname{cosec}(\theta_2 - \theta_1)$$

Area to be resumed $= \frac{w_1}{2}(AB+FE) + \frac{w_2}{2}(BC+ED)$

$$= \frac{w_1}{2}\left(2d_1 + w_1\cot(\theta_1 - \phi_1) + w_1\cot(\theta_2-\theta_1) - w_2\operatorname{cosec}(\theta_2-\theta_1)\right) + \frac{w_2}{2}\left(2d_2 + w_2\cot(\phi_2-\theta_2) + w_2\cot(\theta_2-\theta_1) - w_1\operatorname{cosec}(\theta_2-\theta_1)\right)$$

6. Curve Setting Out Computations.

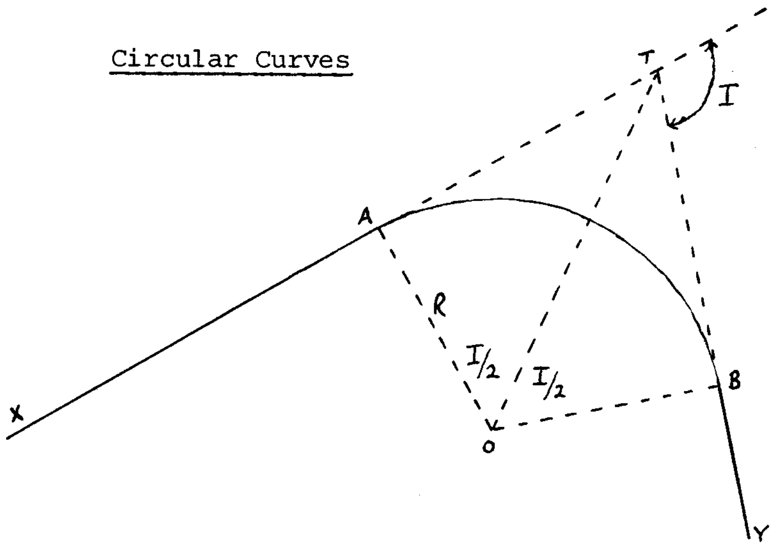
6.1 The curves used in setting out road or railway alignments are of two main types:

(i) curves of constant radius, i.e. circles, and straight sections can be regarded as circles of infinite radius

(ii) curves of gradually changing radius - generally referred to as transition curves.

Any particular alignment can be made up of a combination of different curves and the curvature of any particular section may be in the same or opposite direction to that of the neighbouring sections, the only restraints being that two neighbouring sections must have a common tangent at their junction point, and that a transition curve must have the same radius at its ends as the curve to which it is joined.

6.2 Circular Curves



Circular curves are commonly defined by the radius R, though in British practice it has been defined by the degree D which is the angle (in degrees) subtended at the centre by a chord 100 ft. long. This method of defining a curve is now becoming obsolete but, if used, the radius can be found from the relationship $R = 50/\sin {}^{D}/_{2}$

In the diagram two straight sections XA and BY are to be connected by a circular curve of radius R which will be tangential to the straight sections at A & B respectively. XA & YB produced meet at T, and I is the angle at T between the forward directions of the two straights.

It follows that $\angle AOT = \angle BOT = I/2$

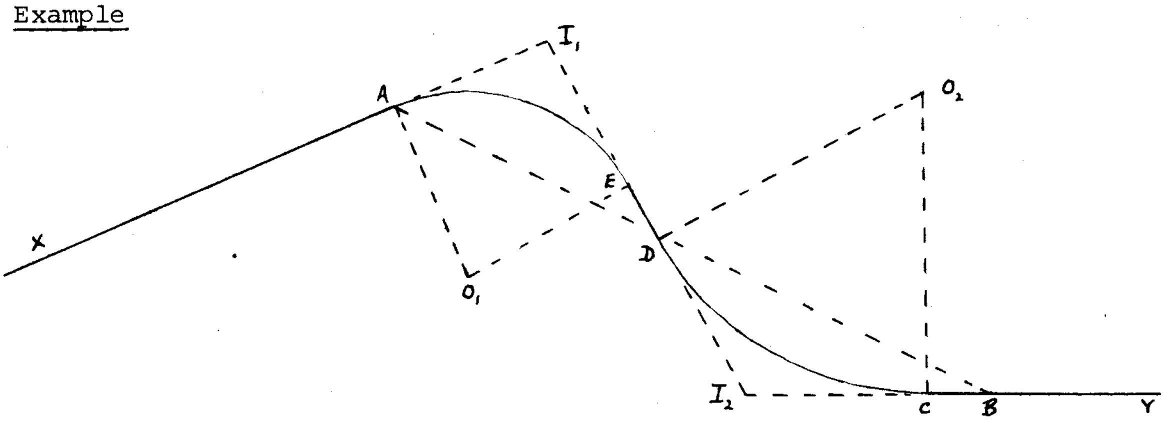
Tangent Lengths AT = BT = R tan I/2

TO = R sec I/2

Arc length AB = R.I (in radian measure)

and problems involving combinations of circular curves can be dealt with using the above relationships, by a number of techniques of which perhaps the most commonly employed is the "Traverse with Missing Data" method dealt with in Section 3.5

Example



Two straight sections XA on a bearing of $67^{\circ}15'$ and BY on a bearing of 90° are to be connected by a circular curve AE of radius 600 followed by a straight ED of length 200 and a reverse circular curve DC of radius 1000, C being an unknown point on YB (produced if necessary). The points A & B have been connected by survey and the bearing of AB is $116^{\circ}14'$ and the distance 2156.6

To find the bearing of the straight section ED, the length CB and the arc lengths AE and DC

Let the bearing of $\vec{ED}$ be A. Then the bearings of O_1E and DO_2 will be $(A-90^\circ)$

Consider the traverse AO_1EDO_2CB and let the distance of CB = d

Then $600(\sin 157^\circ 15' + \sin(A-90^\circ)) + 200 \sin A + 1000 \sin(A-90) + d$

$= 2156.6 \sin 116^\circ 14'$ (1)

$600(\cos 157^\circ 15' + \cos(A-90)) + 200 \cos A + 1000 \cos(A-90) - 1000$

$= 2156.6 \cos 116^\circ 14'$ (2)

Equation (2) reduces to

$$200 \cos A + 1600 \sin A = -953.28 + 1000 + 553.32 = 600.04 \quad (3)$$

This type of equation can be solved in a number of ways but all of them introduce ambiguities which can generally be resolved by looking at the diagram.

Let $K = \sqrt{200^2 + 1600^2}$ and $\cos B = 200/K$, $\sin B = 1600/K$. Then on dividing equation (3) by K

$$\cos(A-B) = .37212891 = \cos 68^\circ 09' 11''$$

Therefore $A = \pm 68^\circ 09' 11'' + B = \pm 68^\circ 09' 11'' + \cos^{-1}(.124035)$

$= \pm 68^\circ 09' 11'' + 82^\circ 52' 30''$

The ambiguity in this case arises because cos X = cos(-X) but inspection of the diagram shows that the positive sign must be accepted and $A = 151^\circ 01' 41''$.

Note that if we had put sin B = 200/K, an ambiguity would still arise owing to the fact that sin X = sin (180-x)

Equation (1) now gives

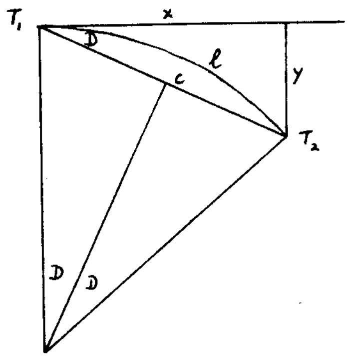
BC = d = 1934.47 - 232.03 + 1600 cos A - 200 sin A = 205.8

Arc length AE = 600 x 1.462203 = 877.3

Arc length DC = 1000 x 1.065141= 1065.1

The chainages along the curve can then be determined and also the positions of the tangent intersection points I_1 & I_2 if these are required.

6.3 Setting out Circular Curves



Setting out can be done in several ways

(i) by distances along the tangent at any point and offset distances from it. This involves calculation of the distances x and y in the diagram.

(ii) by chord distances from some point and deflection angles from the tangent at that point to the chords. For this method the distance c and the deflection angle D have to be calculated

(iii) when two points on the curve have been established by either of the methods given above, intermediary points if required can be fixed by traverse along the chord and offset distances from it.

The process can always be restarted from any point which has already been fixed on the curve if the offset distances become too long for practical purposes or if the line of sight determined by the deflection angle is obscured.

If ℓ is the length of the arc T_1T_2 $\ell = 2RD$

$x = R\sin 2D$ $y = R(1 - \cos 2D)$ $c = 2R\sin D$

or $$\frac{x}{\ell} = \frac{\sin 2D}{2D} = 1 - \frac{2D^2}{3} + \frac{2D^4}{15} - \qquad 6.1$$

$$\frac{y}{\ell} = \frac{1 - \cos 2D}{2D} = D - \frac{D^3}{3} + \frac{2D^5}{45} - \qquad 6.2$$

$$\frac{c}{\ell} = \frac{\sin D}{D} = 1 - \frac{D^2}{6} + \frac{D^4}{120} - \qquad 6.3$$

Tables of these functions, a specimen of which is shown overleaf, can easily be prepared and these may be of use when only hand operated calculating machines are available but if electronic machines with trigonometric functions are being used it is easier to work from the original formulae and of course the results will not be subject to the slight errors of linear interpolation. It should be noted that the column x/ℓ is not necessary as the factors in this column are the same as those in the c/ℓ column for twice the angle and interpolation from this column would give a better result because of the smaller differences. In both cases the Deflection Angle D in radian measure has first to be calculated from the known radius R of the curve and the distance ℓ, along the curve, at which it is intended to emplace a mark.

TABLES FOR SETTING OUT CIRCULAR CURVES

θ	X/L		Y/L		C/L		DEFLECTION ANGLE
0.01	0.999933	-200	0.009999	9999	0.999983	-49	0 34 23
0.02	0.999733	-333	0.019997	9994	0.999933	-83	1 8 45
0.03	0.999400	-466	0.029991	9988	0.999850	-117	1 43 8
0.04	0.998934	-600	0.039979	9980	0.999733	-150	2 17 31
0.05	0.998334	-732	0.049958	9970	0.999583	-183	2 51 53
0.06	0.997602	-865	0.059928	9958	0.999400	-217	3 26 16
0.07	0.996737	-998	0.069836	9944	0.999184	-250	4 0 39
0.08	0.995739	-1130	0.079829	9928	0.998934	-283	4 35 1
0.09	0.994609	-1262	0.089757	9910	0.998651	-316	5 9 24
0.10	0.993347	-1394	0.099667	9890	0.998334	-350	5 43 46
0.11	0.991953	-1525	0.109557	9868	0.997985	-383	6 18 9
0.12	0.990428	-1656	0.119425	9844	0.997602	-416	6 52 32
0.13	0.988771	-1787	0.129269	9818	0.997186	-449	7 26 54
0.14	0.986984	-1917	0.139088	9791	0.996737	-482	8 1 17
0.15	0.985067	-2047	0.148878	9761	0.996254	-515	8 35 40
0.16	0.983021	-2176	0.158639	9729	0.995739	-549	9 10 2
0.17	0.980844	-2305	0.168369	9696	0.995190	-582	9 44 25
0.18	0.978540	-2433	0.178064	9660	0.994609	-615	10 18 48
0.19	0.976106	-2561	0.187725	9623	0.993994	-648	10 53 10
0.20	0.973546		0.197348		0.993347		11 27 33

Numerical Example

Two straights, making a deflection angle of 75° intersect at I (chainage 2853.24) & are to be joined by a circular curve of radius 800 to be set out at intervals of 50 units of chainage.

Tangent distance = 800 tan $37^{\circ}30'$ = 613.86

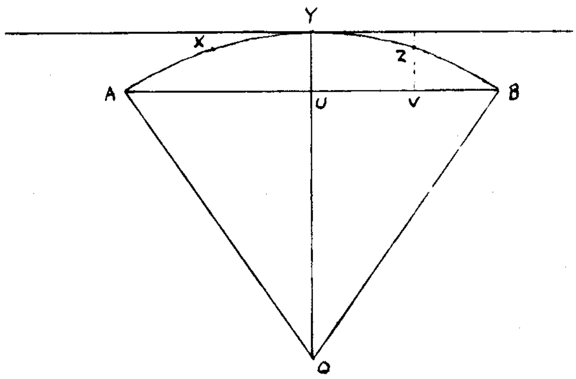
Chainage of first tangent point T_1 = 2853.24 - 613.86 = 2239.38

Arc length = 800 x 1.30900 = 1047.20

Chainage of second tangent point T_2 = 3286.58

Setting out data

ℓ	D	x	y	c	Deflection Angle
10.62	.0066375	10.62	.07	10.62	0° 22' 49"
60.62	.0378875	60.56	2.30	60.61	2° 10' 15"
.	.	.	.	.	.
.	.	.	.	.	.



The same formulae or tables can be used if it is required to survey in additional points by traverse along a chord and offset from it. Suppose it is required to emplace 3 extra marks X, Y & Z at equal intervals along the curve between A & B, the two points which have been fixed by the survey data given above.

Then U the mid point of AB can be easily fixed

$\angle AOB = 2(.0378875 - .0066375) = .0625$ radians

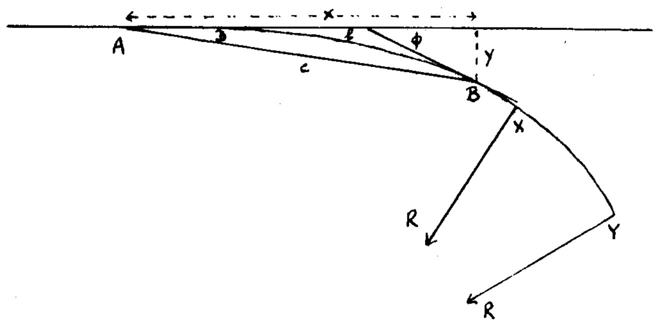
Offset distance YU = offset distance of A (or B) from the tangent at Y

$= 800\ (1 - \cos .03125) = .39$

Offset distance ZV $= 800\ (1 - \cos .03125) - 800\ (1 - \cos .015625) = .29$

and distance UV $= 800 \sin .015625 = 12.50$

6.4 Transition Curves



Transition curves in road and rail alignments are placed between straights and circular curves or between circular curves of different radii in order to cushion the effect of what would otherwise be a sudden change of direction. The most suitable transition curve is the clothoid which has a constant rate of change of curvature with respect to arc. A number of other mathematical curves have been used in the past as transition curves, mainly due to the complicated formulae for computation of the clothoid but in these days of electronic computation there is little justification for the use of such curves.

In the diagram B is any point on a clothoid AX joining a straight to a circular curve XY of radius R. The tangent at B makes an angle ϕ with the tangent at A and x, y, ℓ, c & D have the same meanings as in Section 6.3 The equation of the clothoid is given by $\frac{d^2\phi}{d\ell^2} = K$ (a constant) and so by integration $\frac{d\phi}{d\ell} = K\ell$ and $\phi = \frac{1}{2}K\ell^2$ since both are zero when $\ell = 0$

At X, the junction with the circular curve, if L and Φ are the arc lengths and tangential angle at that point

$$\frac{1}{R} = \frac{d\phi}{d\ell} = KL \qquad \Phi = \frac{1}{2}KL^2 = \frac{L}{2R} \qquad \phi = \frac{\ell^2}{2LR}$$

$$\text{Now} \quad x = \int \cos\phi \, d\ell = \int \cos\left(\frac{\ell^2}{2LR}\right) d\ell$$

$$= \int \left(1 - \frac{1}{2}\left(\frac{\ell^2}{2LR}\right)^2 + \frac{1}{24}\left(\frac{\ell^2}{2LR}\right)^4 - \dots \right) d\ell$$

$$= \ell \left(1 - \frac{1}{10}\left(\frac{\ell^2}{2LR}\right)^2 + \frac{1}{216}\left(\frac{\ell^2}{2LR}\right)^4 - \dots \right)$$

$$= \ell \left(1 - \frac{\phi^2}{10} + \frac{\phi^4}{216} - \quad \right) \tag{6.4}$$

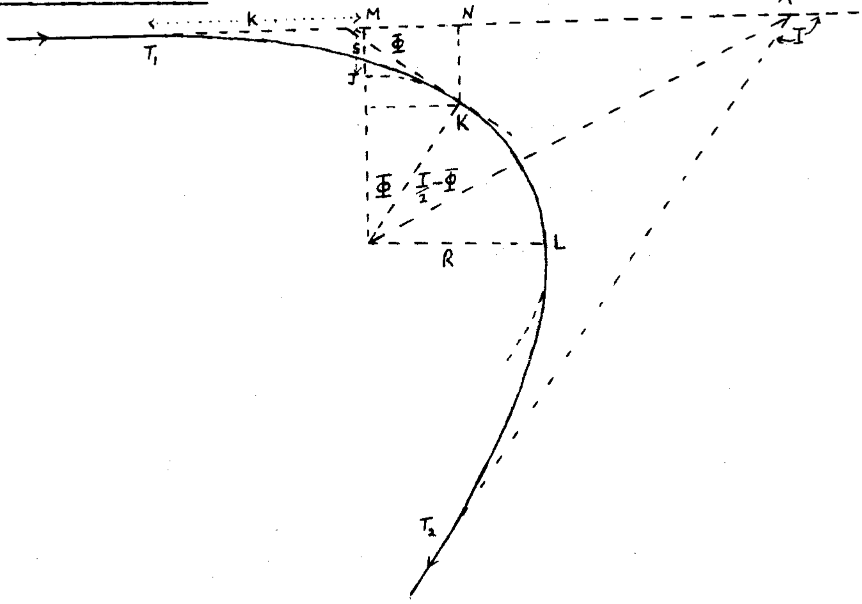
Similarly it can be shown that

$$y = \ell \left(\frac{\phi}{3} - \frac{\phi^3}{42} + \frac{\phi^5}{1320} - \dots \right) \tag{6.5}$$

$$c = \sqrt{x^2 + y^2} = \ell \left(1 - \frac{2\phi^2}{45} + \frac{2\phi^4}{2835} - \right) \tag{6.6}$$

$$D = \tan^{-1}(y/x) = \frac{\phi}{3} - \frac{216}{2835}\left(\frac{\phi}{3}\right)^3 - \frac{7776}{467775}\left(\frac{\phi}{3}\right)^5 \dots \tag{6.7}$$

6.5 Setting out transition curves



Consider the case of two straights T_1A & T_2A meeting at A with an intersection angle I, which are to be connected by a circular curve of radius R and two identical transition curves of length L. The circular curve will have to be offset from the straights as shown in the diagram and if OM, the perpendicular from the centre of the circle on to T_1A cuts the circular curve (produced) at J, then MJ, which will be denoted by s, is known as the shift. The distance T_1M will be denoted by k

The data will be the radius R, the length L of the transition curve, both of which will be determined from practical mechanical considerations and the observed intersection angle I between the two straights.

If the tangential angle at the point K where the transition curve joins the circular curve is Φ then $\Phi = L/2R$ and X & Y the distances along the tangent T_1A and perpendicular to it are obtained by substituting L and Φ in equations 6.4 and 6.5

$$R + s = R\cos\Phi + Y$$

$$k = X - R\sin\Phi$$

The length of the circular curve $KL = R(I - 2\Phi) = RI - L$ and the total length of the combined curves $T_1KLT_2 = RI + L$

From these formulae and those of Section 6.4 all the setting out details can be computed. In this case since the formulae are more complicated the use of prepared tables is of great advantage and a specimen of these tables is given.

Numerical Example

The data will be the same as that of the example in Section 6.3 except that each of the straights will be joined to the circular curve by a transition curve of length 300.

Tangential angle at junction of transition and circular curves

$$= \frac{300}{1600} = .1875 \text{ radians}$$

Entering the tables with this value for D gives

$$s = 300 \times .015605 = 4.68$$

$$k = 300 \times .499414 = 149.82$$

Distance $T_1A = k + (R + s)\tan 37^\circ 30' = 149.82 + 804.68 \times .767327$

$$= 767.27$$

Chainage of tangent point $T_1 = 2853.24 - 767.27 = 2085.97$

Length of combined curve $= RI + L = 800 \times 1.308997 + 300$

$$= 1347.20$$

Chainage of tangent point $T_2 = 3433.17$

TABLES FOR SETTING OUT TRANSITION CURVES

D	X/L		Y/L		C/L		DEFLECTION ANGLE	S/L		K/L	
0.01	[illegible]	-3[illegible]	0.003333	3333	0.999995	-13	0 11 28	0.000833	833	0.499998	-5
0.02	0.999960	-5[illegible]	0.006666	3333	0.999982	-22	0 22 55	0.001667	833	0.499993	-8
[illegible]	[illegible]	-70	[illegible]	3332	0.999960	-3[illegible]	0 34 [illegible]	0.002500	833	0.499985	-12
0.04	[illegible]	-90	0.013332	3332	0.999920	-40	0 45 5[illegible]	0.003333	833	0.499973	-15
0.05	0.99975[illegible]	-110	0.016664	3331	0.999880	-46	0 57 18	0.004166	833	0.499958	-18
0.06	0.99964[illegible]	-130	0.019995	3330	0.999840	-58	1 8 45	0.004999	833	0.499940	-22
[illegible]	[illegible]	-150	0.023325	[illegible]	0.999782	-67	1 20 [illegible]	0.005333	833	0.499918	-25
0.08	[illegible]	-170	0.026654	3328	0.999715	-76	1 31 4[illegible]	0.006665	833	0.499893	-28
0.09	[illegible]	-190	0.029983	3327	0.999640	-84	1 43 8	0.007498	833	0.499865	-32
0.10	[illegible]	[illegible]	[illegible]	3325	0.999556	-92	1 54 35	0.008330	832	0.499833	-35
[illegible]	[illegible]	-230	[illegible]	3324	[illegible]	-102	2 6 [illegible]	0.009163	832	0.499798	-38
0.12	[illegible]	-250	[illegible]	3322	[illegible]	-111	2 17 [illegible]	0.009995	832	0.499760	-42
0.13	[illegible]	-27[illegible]	[illegible]	[illegible]	0.999250	-12[illegible]	2 28 57	0.010827	832	0.499718	-45
0.14	[illegible]	[illegible]	[illegible]	[illegible]	0.999120	-129	2 40 24	0.011659	831	0.499674	-48
[illegible]	[illegible]	[illegible]	[illegible]	3316	[illegible]	[illegible]	2 51 51	0.012490	831	0.499625	-52
0.16	[illegible]	-32[illegible]	[illegible]	3314	0.998963	-147	3 3 1[illegible]	0.013321	831	0.499574	-55
0.17	[illegible]	[illegible]	[illegible]	3311	[illegible]	-155	3 14 45	0.014152	831	0.499519	-58
0.18	[illegible]	[illegible]	[illegible]	3309	[illegible]	-164	3 26 [illegible]	0.014983	830	0.499460	-62
[illegible]	[illegible]	[illegible]	[illegible]	3306	[illegible]	-173	3 37 [illegible]	0.015813	830	0.499399	-65
[illegible]	[illegible]		[illegible]		[illegible]		3 49 6	0.016643		0.499334	

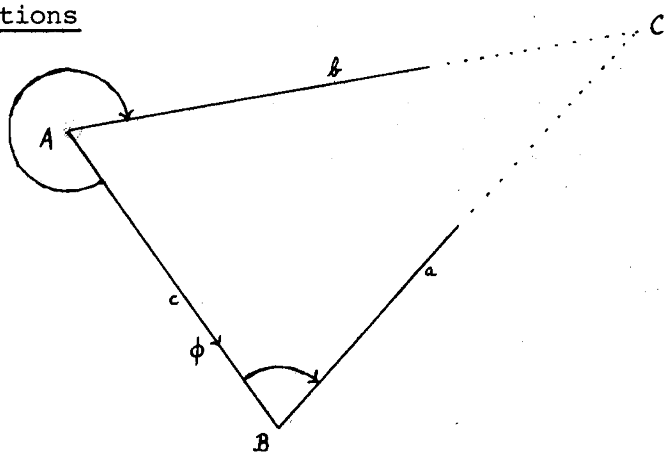
Setting out data.

ℓ	D	x	y	c	Deflection Angle
14.03	.008769	14.03	.04	14.03	$0^{o}10'03''$
64.03	.040019	64.02	.85	64.03	$0^{o}45'51''$
.	.	.	.	.	.
.	.	.	.	.	.
.	.	.	.	.	.

7. Intersections and Resections ; Satellite Stations.

7.1 Fixation by Angular Observations

7.1(a) Intersections



A & B are two established trig stations whose co-ordinates (E_A, N_A) and (E_B, N_B) and also the distance and bearing AB are known from the trig data sheets.

7.1(a)1. To fix a third point C a minimum of two observations is required and in this case the two observations are the angles at A & B between the other known station and the new station C, and to overcome the ambiguity of on which side of the line AB, C lies ,the measured angles will be taken as the clockwise angle from the known station to the new station as shown in the diagram.

Then angle C = 180-B-(360-A) = A-B-180

and $E_C = E_A + b \sin (\phi + A)$

$$= E_A + \frac{c \sin B}{\sin C} (\sin \phi \cos A + \cos \phi \sin A)$$

$$= E_A - \frac{\sin B}{\sin (A-B)} \{(E_B - E_A) \cos A + (N_B - N_A) \sin A\}$$

$$= E_A - \frac{\{(E_B - E_A) \sin B \cos A + (N_B - N_A) \sin B \sin A\}}{\sin A \cos B - \cos A \sin B}$$

On expansion

$$= \frac{E_A \cot B - E_B \cot A - (N_B - N_A)}{\cot B - \cot A} \qquad 7.1.1$$

and by a similar derivation

$$N_C = \frac{N_A \cot B - N_B \cot A + (E_B - E_A)}{\cot B - \text{Cot } A} \qquad 7.1.2$$

These formulae give an alternative method of calculating the co-ordinates of an intersected point to that given in Example 1 of Section 3.2 and the data of that example can be used for a numerical illustration of the present method

Data P (37928.3, 42398.7) Q (43527.5, 37814.3)

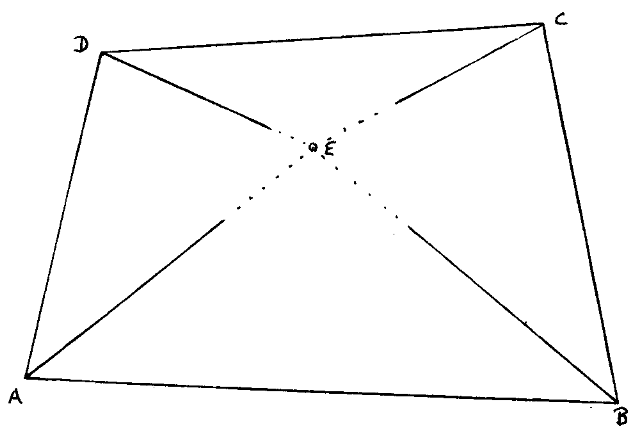
Angle P = 314° 16' 41" Angle Q = 67° 19' 28"

$E_P \cot Q$	15846.71	E_P	37928.3	cot Q	.417807	N_P	42398.7	$N_P \cot Q$	17714.47
$E_Q \cot P$	-42444.14	E_Q	43527.5	cot P	-.975111	N_Q	37814.3	$N_Q \cot P$	-36873.14
	58290.85	$E_Q - E_P$	5599.2		1.392918	$N_Q - N_P$	-4584.4		54587.61
	+ 4584.40								+5599.20
	62875.25	E_C	45139.2			N_C	43209.2		60186.81

This method of calculation however does not contain any built in checks.

7. 1(a)2. Semigraphic Method of dealing with redundant data

It is good survey practice however to take more than the minimum number of observations in order to provide a check on the final results and it will be assumed that these additional observations have taken the form of more angular observations at established trig stations.



In this example angular observations have been taken at four established trig stations A, B, C & D to fix a new station E. These four rays can be combined in pairs to give 6 separate values for the co-ordinates of E, if calculated by the method given above, and the problem is to decide on the best values for the co-ordinates, without going through the full calculations. This is done by a combination of calculation and graphical techniques and is best illustrated by an example.

<u>Data</u>	Station	E	N	Bearing to E
	A	2589.399	11717.848	72° 26' 58"
	B	9307.040	8423.634	338° 09' 25"
	C	14697.340	17233.067	241° 20' 16"
	D	4949.507	14603.003	119° 25' 06"

The bearing from A to E given above is obtained by adding the observed angle <u>/DAE</u> to the known bearing AD and by subtracting the observed angle <u>/EAB</u> from the known bearing AB and taking the mean. Similarly for the other bearings.

The procedure is as follows

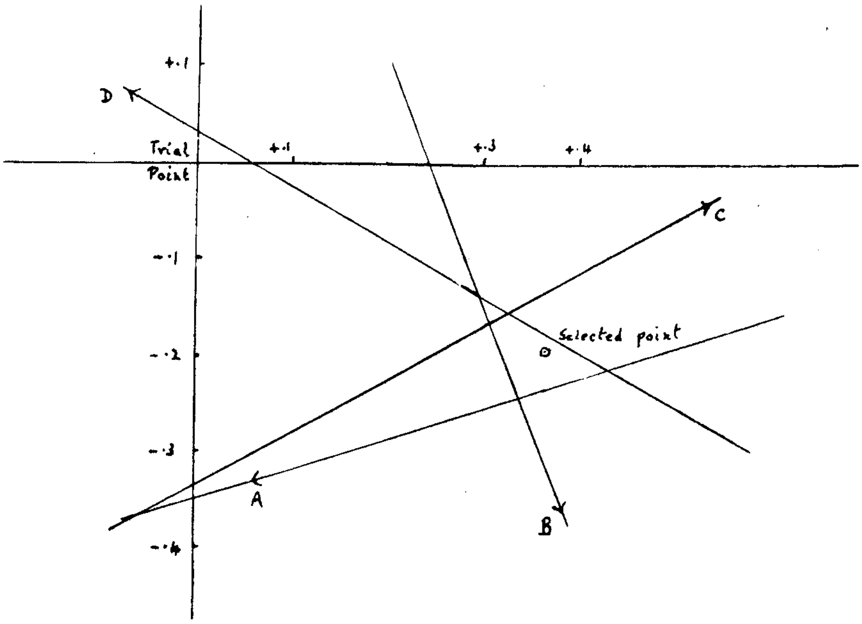
(a) by graphical means or by accepting the observations from two of the stations and computing, approximate co-ordinates are obtained for E. It will be assumed that this has been done and that the trial co-ordinates for E are

(7379.000, 13233.000)

(b) The positions of the incoming rays in the vicinity of this trial point are then investigated by computing the cuts on either the N-S line or the E-W line through the trial point. In order to deal with the smaller of the two cuts and so to be able to draw the subsequent diagram at the largest scale, it is normal, if the ray is mainly N-S to work from the difference Northings to a cut on the E-W line and if the ray is mainly E-W to work from the difference Eastings to a cut on the N-S line. At the same time approximate distances from the trigs to the new station are found. The calculations could be set out as follows.

	Station A	Station B	Station C	Station D
Bearing to E	72° 26' 58"	338° 09' 25"	241° 20' 16"	119° 25' 06"
Trial Pt N/E	7379.000	13233.000	7379.000	7379.000
Trig N/E	2589.399	8423.634	14697.340	4949.507
ΔN/ΔE	+ 4789.601	+ 4809.366	- 7318.340	+ 2429.493
tan/cot Bg	+ .316269	- .400843	+ .546627	- .563893
ΔE/ΔN	+ 1514.802	- 1927.801	- 4000.402	- 1369.974
Trig E/N	11717.848	9307.040	17233.067	14603.003
Cut E/N	13232.650	7379.239	13232.665	13233.029
Cut	S .350	E .239	S .335	N .029
Distance	5020	5180	8340	2790

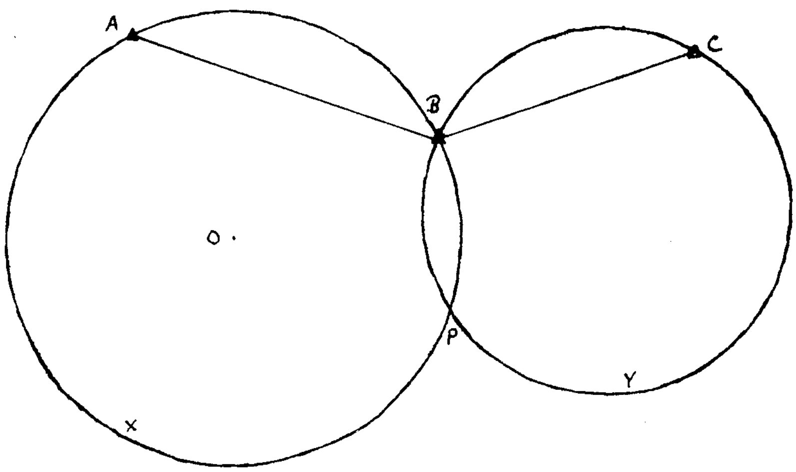
(c) These cuts on the axes through the trial point are then plotted on as large a scale as is convenient and rays are drawn through them on the appropriate bearings. The final point is then determined by inspection of the error figure with the aim of making the corrections to the observations as small as possible. The observations in this problem are angular ones and since in general there will be no indication that one bearing is more in error than another, the angular corrections will be treated as as nearly equal as possible which means that the distances of the selected point from the individual rays should be proportional to the distance from the trial point to the trigs. For any group of 3 rays a unique point can be found by this method but if there are more than 3 rays it is most unlikely that the condition can be satisfied for all rays. In this case most weight should be given to the rays from the 3 nearest trigs as the same offset distance means a larger angular correction to the observations at a near trig than to those at more distant ones. The diagram and selected point for this particular example are shown below



The selected point has co-ordinate (+.355 - .200) relative to the trial point and hence the co-ordinates of Station E will be taken as (7379.355, 13232.800)

(d) This method of determining the co-ordinates of an intersected point has the advantage that a large error in any observation will immediately be seen in the diagram. Before rejecting any observations however checks should be placed on that part of the computation to ensure that the plotted ray has not been put in the wrong position owing to a computing error.

7.1(b) Resections



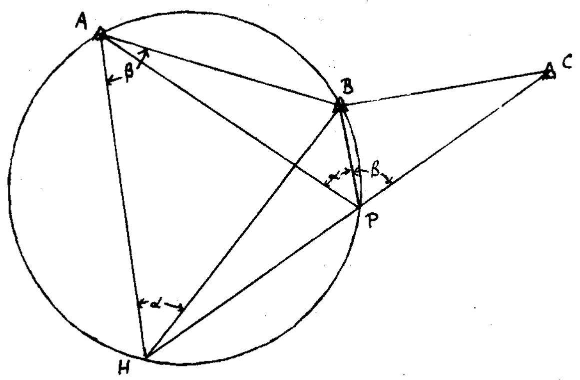
Resection consists of the determining of the co-ordinates of a new station from the observation of angles at it, between a number of previously established stations.

If the angle is observed at a new station P between two old trig stations A & B all that we know from this observation is that P lies somewhere on the arc AXB of a circle centre O such that /AOB = twice the measured angle. If however the angle between Stations B & C is also measured then P must lie on the arc BYC of another circle and where these two arcs cut will be the position of Station P. For resections therefore a minimum of three rays must be observed as against two for intersections. This is because there is one other unknown to be determined, the orienting factor to convert the observed directions into bearings.

The trig stations to be observed from P should be carefully selected as if A, B, C & P all lie on the same circle there would only be one circle and not two, whilst if they were nearly on the same circle, the cut would be a poor one and the fixation of P would be relatively inaccurate.

7.1(b).1 Calculation of a 3 ray Resection.

There are a large number of methods of calculating a three ray resection but probably the simplest to understand and the one with the most built in checks is the Collins Point Resection Method which will now be described.



A, B & C are the three old trig stations whose co-ordinates are known and theodolite observations at P have produced an angle α between A & B and β between B & C

A circle through A, B & P is drawn and CP is joined to cut this circle again at H. Then by ordinary geometry $\underline{/AHB} = \alpha$ & $\underline{/ HAB} = \beta$

The steps in the calculation are as follows.

1. From the co-ordinates $(E_A N_A)$ of A & $(E_B N_B)$ of B calculate the bearing $\phi_{\vec{AB}}$ and distance d_{AB} of AB

2. From this information and the known angles of ΔABH, calculate the bearings AH & BH and also by use of the sin rule, their distances.

3. Calculate the co-ordinates of H from A and from B as a check

4. Calculate the bearing of the line HPC from the co-ordinates of H & C

5. Using the bearing PC and the observed angles find the bearings PA and PB and hence the angles of the triangle PAB

6. Calculate the distances PA & PB

7. Calculate the co-ordinates of P from A and from B as a check

Numerical Example

Station	Eastings	Northings	Theodolite Reading at P
A	13761.69	23056.19	313° 07' 30"
B	15022.76	21116.83	5° 52' 53"
C	17099.81	20388.26	64° 59' 30"

1.	Station B	15022.76	21116.83	Bg $\vec{AB}$ = 146° 57' 58"
	Station A	13761.69	23056.19	Distance AB = 2313.313
		+1261.07	-1939.36	

2.	A	59 06 37	.858157	2493.736	$\vec{AB}$ 146° 57' 58"	$\vec{BA}$ 326° 57' 58"
	B	68 08 00	.928053	2696.848	59 06 37	68 08 00
	H	52 45 23	.796069	2313.313	$\vec{AH}$ 206° 04' 35"	$\vec{BH}$ 258° 49' 58"
		180 00 00				

3.	AH 206° 04' 35"	2696.848		BH 258° 49' 58"	2493.736	
	A 13761.690	23056.190		B 15022.760	21116.830	
	- 1185.451	-2422.333		-2446.520	- 482.970	
	H 12576.239	20633.857		H 12576.240	20633.860	
4.				C 17099.810	20388.260	
	Bearing PC 93° 06' 28"			C - H +4523.570	- 245.600	
5.	Bearing PB 33° 59' 51"					
	Bearing PA 341° 14' 28"					

6.	A	14° 16' 30"	.246576	716.530
	B	112° 58' 07"	.920719	2675.536
	P	52° 45' 23"	.796069	2313.313
		180 00 00		

7.	AP 161° 14' 28"	2675.536	BP 213° 59' 51"	716.530
	A 13761.690	23056.190	B 15022.760	21116.830
	860.416	-2533.412	- 400.653	- 594.048
	P 14622.106	20522.778	P 14622.107	20522.782

Accept P (14622.11 20522.78)

It should be noted that the only part of this calculation which is not checked is stage 4 and this could easily be done by a 45° check. If the stage 4 calculation is incorrect the bearings at P and its final co-ordinates will be wrong.

There are a number of other methods of calculating a three ray resection but details of these will not be given as this type of computation is becoming obsolete since it contains no check on observational errors.

7.1(b) 2. Semigraphic treatment of Resections with redundant data.

In order to provide a check on the accuracy of the field observations it is normal to observe directions to at least 4 established trig stations. The semigraphic treatment of these observations is very similar in parts to that of intersection observations but there are important differences between the two cases.

(i) Co-ordinates of a trial point are first found either by computing using the directions to three known stations, or more simply by a tracing paper resection.

(ii) Using these trial co-ordinates the angular observations are oriented by computing a bearing from the trial point to the most distant trig station and using the observations to compute the other bearings. The trial co-ordinates, if obtained by a tracing paper resection, may be in error by a considerable amount and the bearings obtained from them will be inaccurate. The rays to the trigs will not now all pass through the same point for two reasons

(a) the assumed bearings are inaccurate and

(b) there are observational errors in the recorded directions.

If the trial point is significantly inaccurate the first source of error will swamp the second and the error due to bad orientation must be removed first.

(iii) Cuts are computed and rays plotted in the vicinity of the trial point as in the semigraphic intersection example but the diagram is then treated slightly differently. The directions of the distant trigs must be shown on the diagram and rays are then drawn parallel to the plotted rays at distances from them which are proportional to the distance to the appropriate trig station. These parallel rays must all lie on the same side (either right or left) of the original ray when looking towards the distant trigs. For every pair of stations a line is drawn joining the intersection of the two rays to the intersection of their two parallel rays, giving a series of points through which pass three of these last group of lines. These points will be the positions obtained by computing from the observations to any group of 3 stations and if the mean position is accepted it can be assumed that the orientation error has been eliminated.

(iv) Using this second trial point the procedure is repeated as for a semigraphic intersection. If the original trial point had been obtained by computation the semigraphic intersection method could have been used straight away.

Numerical Example

Observed Station	Eastings	Northings	Theodolite Reading at E
A	12589.399	31717.848	10° 39' 10"
B	16140.580	28012.682	83° 08' 11"
C	19.962	6511.864	209° 18' 12"
D	10060.660	29232.227	315° 09' 05"

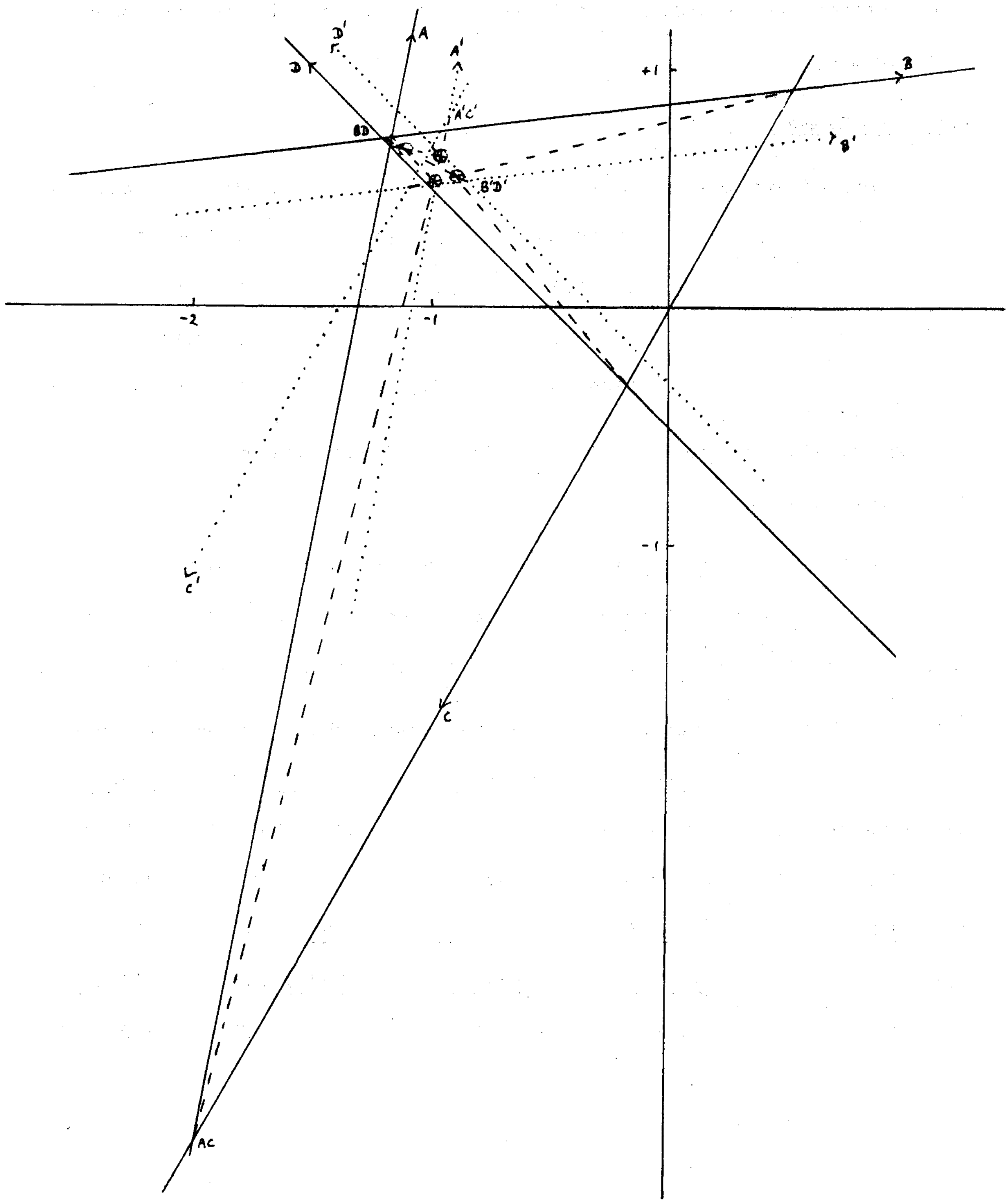
A moderately poor trial point E 11795 N 27489 has purposely been chosen for this calculation.

The most distant station is obviously C and the bearing to C calculated from its co-ordinates and those of the trial point is 209° 18' 24". An adjustment of + 12" has therefore to be made to all the theodolite readings.

The calculation of the cuts is then made using these approximate bearings.

	Station A	Station B	Station C	Station D
Approx.Bg from E	10 39 22	83 08 23	209 18 24	315 09 17
Trial Pt. N/E	27489.000	11795.000	27489.000	27489.000
Trig. N/E	31717.848	16140.580	6511.864	29232.227
ΔN/ΔE	-4228.848	-4345.580	20977.136	-1743.227
tan/cot Bg	.188159	.120310	.561327	-.994614

	Station A	Station B	Station C	Station D
ΔE/ΔN	-795.696	-522.817	11775.033	+1733.838
Trig E/N	12589.399	28012.682	19.962	10060.660
Cut E/N	11793.703	27489.865	11794.995	11794.498
Cut	W 1.297	N .865	W .005	W .502
Approx.Dist.	4300	4380	24060	2460



Co-ordinates relative to the Trial Point of the 4 points through which 3 rays pass are

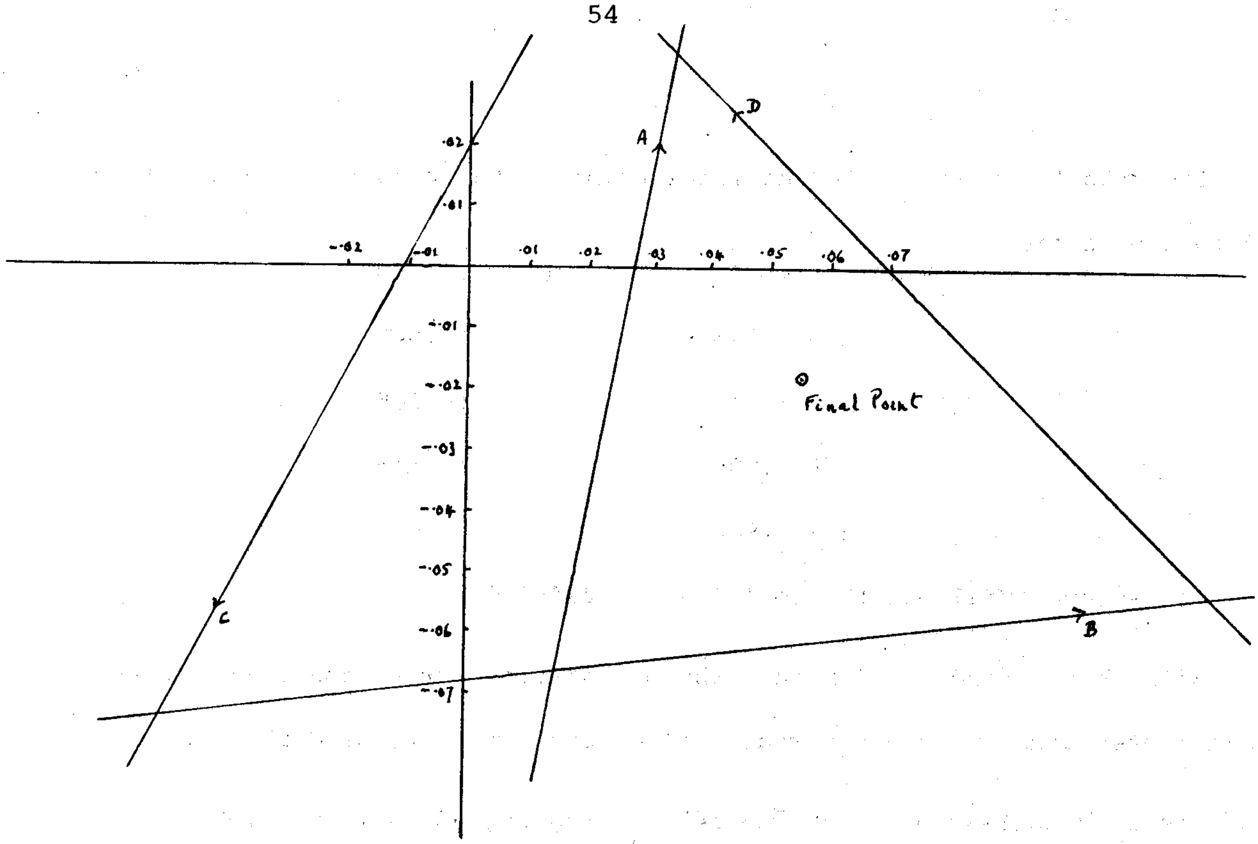
	E		N
	- 1.100		+ .660
	- .975		+ .640
	- .915		+ .565
	- .995		+ .545
Mean	- .996		+ .603

For second trial point take 11794.0 27489.6

With this second trial point, the orientation error should have been eliminated and only the observational errors remain to be dealt with.

New bearing to Station C 209°18'14" Bearing adjustment - 10"

	Station A	Station B	Station C	Station D
Bearing from E	10 39 12	83 08 13	209 18 14	315 09 07
Trial Pt N/E	27489.600	11794.000	27489.600	27489.600
Trig N/E	31717.848	16140.580	6511.864	29232.227
ΔN/ΔE	-4228.248	-4346.580	20977.736	-1742.627
Tan/cot Bg	.188109	.120359	.561263	-.994710
ΔE/ΔN	- 795.372	- 523.150	11774.027	1733.409
Trig E/N	12589.399	28012.682	19.962	10060.660
Cut E/N	11794.027	27489.532	11793.989	11794.069
Cut	E .027	S .068	W .011	E .069
Approx.Dist.	4300	4380	24060	2460



Accepted Co-ordinates for Resected Point E 11794.055 27489.581

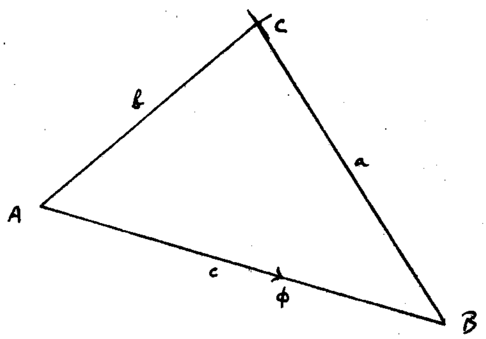
Computed Bearings to the Trig Stations and corrections

Station A	10°	39'	10"	-2"
Station B	83	08	15	+2"
Station C	209	18	14	0
Station D	315	09	09	+2"

7.2 Fixation by Observation of Distances

7.2.1 In this case there is no difference between resection and intersection as it is immaterial from which end of the line the distance is measured and to fix a third point from two known stations, two distance observations are needed

Given The co-ordinates (E_A, N_A) , (E_B, N_B) of the known points A and B, the corresponding distance **c** and bearing ϕ of the line AB and the two measured distances a & b



Then $E_C = E_A + b \sin (\phi_{AB} - A)$

$$= E_A + b\ (\sin\phi \cos A - \cos\phi \sin A)$$

$$= E_A + \frac{b}{c} \left\{ (E_B - E_A) \cos A - (N_B - N_A) \sin A \right\} \qquad 7.2.1a$$

and the angle A can be obtained from $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$

This can be checked by computing from B which gives

$$E_C = E_B - \frac{a}{c} \left\{ (E_B - E_A) \cos B + (N_B - N_A) \sin B \right\} \qquad 7.2.1b$$

If tables or electronic calculators giving trig functions are not available these formulae can be amended by using the relationships

$$\cos A = 2\cos^2 \frac{A}{2} - 1 = \frac{2s(s-a)}{bc} - 1$$

$$\sin A = 2\cos \frac{A}{2} \sin \frac{A}{2} = \frac{2\sqrt{s(s-a)(s-b)(s-c)}}{bc}$$

Making these substitutions in the two previous formulae and taking the mean gives a nearly symmetrical formula

$$E_C = \frac{1}{2c^2} \left\{ E_A (a^2-b^2+c^2) + E_B\ (-a^2+b^2+c^2) - 4\ (N_B-N_A) \sqrt{s(s-c)(s-b)(s-c)} \right\} \qquad 7.2.1c$$

These formulae assume that the angle at A from B to C is anticlockwise. If the reverse is the case, the sign of the (N_B-N_A) coefficient must be changed to +

The corresponding formulae for the northing co-ordinate are

$$N_C = N_A + \frac{b}{c} \left\{ (N_B - N_A) \cos A + (E_B-E_A) \sin A \right\} \qquad 7.2.2a$$

$$= N_B - \frac{a}{c} \left\{ (N_B - N_A) \cos B - (E_B-E_A) \sin B \right\} \qquad 7.2.2b$$

$$= \frac{1}{2c^2} \left\{ N_A\ (a^2-b^2+c^2) + N_B\ (-a^2+b^2+c^2) + 4\ (E_B-E_A) \sqrt{s(s-a)(s-b)(s-c)} \right\} \qquad 7.2.2c$$

with the same proviso for the sign of the (E_B-E_A) term.

It should be noted that in the last of the three formulae the coefficients of the three terms are numerically the same for both E_C and N_C and that the coefficient of the second term = (1 - the coefficient of the first term). This latter fact can be used either for checking purposes or to effect a slight reduction of the computational work by putting the formulae in the form

$$E_C = E_A + \frac{(b^2+c^2-a^2)(E_B-E_A) - 4\,(N_B-N_A)\,\sqrt{s(s-a)(s-b)(s-c)}}{2c^2} \qquad 7.2.3$$

$$N_C = N_A + \frac{(b^2+c^2-a^2)(N_B-N_A) + 4\,(E_B-E_A)\sqrt{s(s-a)(s-b)(s-c)}}{2c^2}$$

Numerical Example

Given

E_B =	6788.67	N_B =	8328.27	a =	5191.05	s-a =	4522.735
E_A =	1240.22	N_A =	2628.80	b =	6282.32	s-b =	3431.465
$E_B - E_A$ =	5548.45	N_B-N_A =	5699.47	c =	7954.20	s-c =	1759.585
				2s =	19427.57	Sum =	9713.785
				s =	9713.785		

$\dfrac{b^2+c^2-a^2}{2c^2}$ = .5989464 (p) $\qquad \dfrac{2\sqrt{s(s-a)(s-b)(s-c)}}{c^2}$ = .5148452 (q)

E_A =	1240.220	N_A =	2628.800
$p(E_B-E_A)$ =	3323.224	$p(N_B-N_A)$ =	3413.677
$-q(N_B-N_A)$ =	-2934.345	$q(E_B-E_A)$ =	2856.593
E_C =	1629.10	N_C =	8899.07

7.2.2 Semi-Graphic Methods

(a) With only the minimum number of distance observations

The procedure is as follows

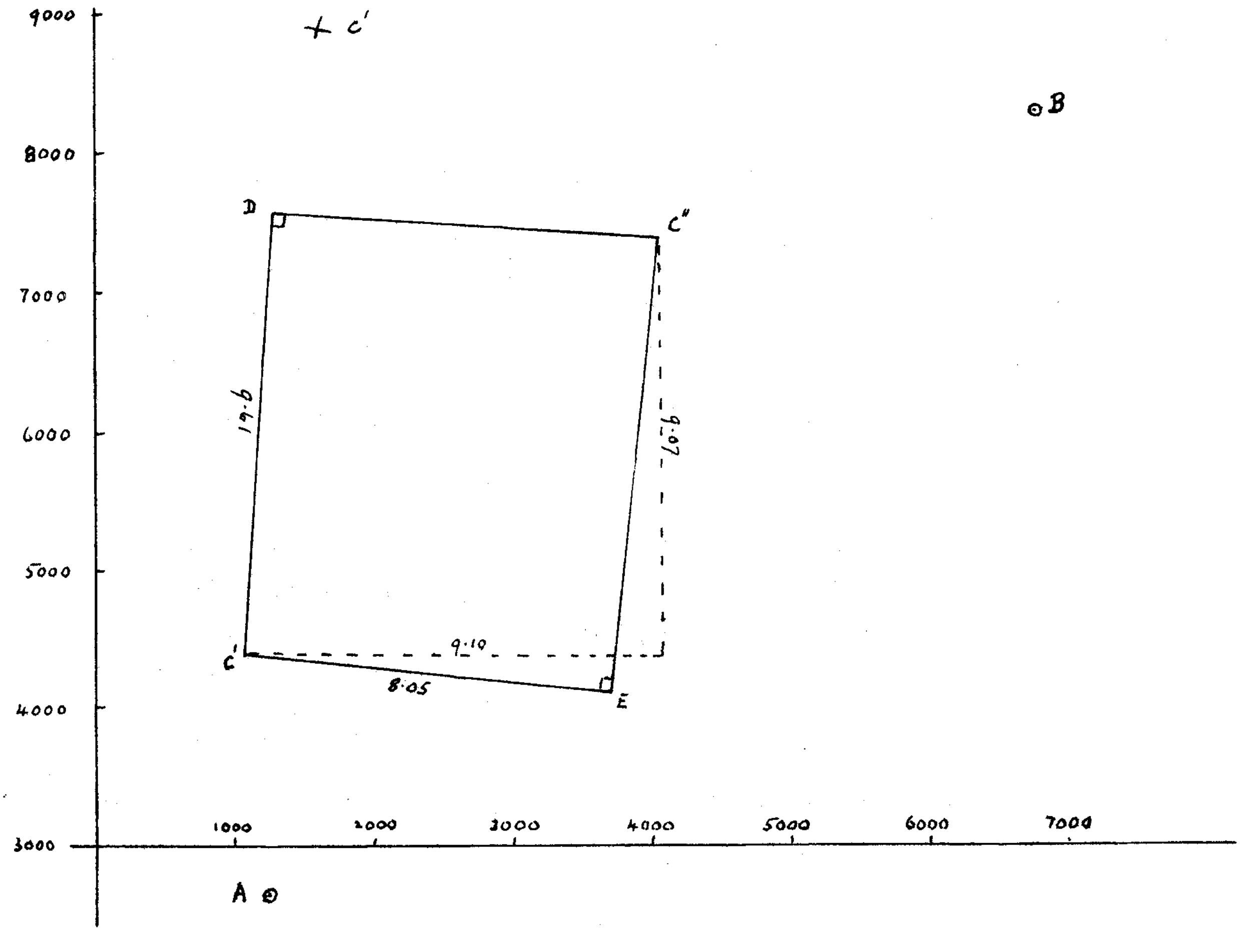
(i) On as large a scale as possible plot the positions of A & B and find an approximate position of C by the intersection of arcs equal to the lengths a & b

(ii) With these approximate co-ordinates for C calculate the distances CA & CB and find the adjustments required to give the observed distances.

(iii) On any convenient scale plot these adjustments in the directions CA & CB, turn off lines at right angles and where these lines intersect will give a second trial point.

(iv) Read off the ΔE & ΔN between the trial points and repeat the procedure with the new co-ordinates if necessary.

Using the same data as in the previous example the diagrams and calculations will be as follows



Point	Easting	Northing	Length	Point	Easting	Northing	Length
A	1240.22	2628.80	6282.32	B	6788.67	8328.27	5191.05
C'	1620.00	8890.00		C'	1620.00	8890.00	
Δ'	+379.78	+6261.20	6272.71	Δ'	-5168.67	+561.73	5199.10
	+ 9.10	+ 9.07	+ 9.61		+ 9.10	+ 9.07	- 8.05
	+388.88	+6270.27	6282.318		-5159.57	+570.80	5191.048
			+ .002				+ .002

In this case further plotting and calculation is unnecessary

C	1629.10	8899.07			1629.10	8899.07	

(b) With redundant Observations

When there are redundant observations, two lines giving a good intersection angle are selected and trial co-ordinates are computed from these two observations by either of the methods previously detailed. Using these trial co-ordinates shifts are computed for each of the other rays and tangents for each measured length are then plotted on the diagram and the final point is selected by inspection of the error figure.

Numerical Example

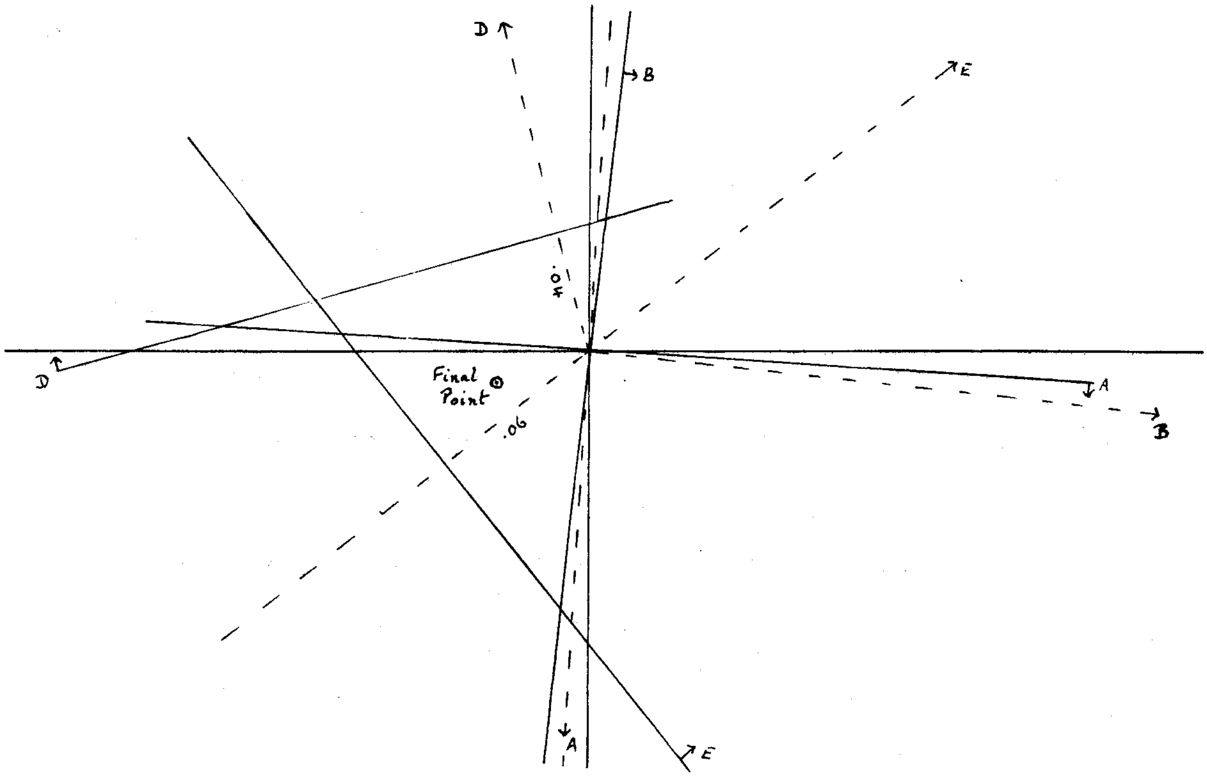
Suppose that in the previous example in addition to the lines CA & CB, distances were also measured from C to known stations D and E

Station D	Easting	44.25	Northing	14752.70	Distance	6064.34
Station E	Easting	6442.71	Northing	12778.96	Distance	6182.65

Take as trial co-ordinates 1629.10 8899.07 as obtained from A & B

	Station A		Station B		Station D		Station E	
Trig E/N	1240.22	2628.80	6788.67	8328.27	44.25	14752.70	6442.71	12778.96
T.P. E/N	1629.10	8899.07	1629.10	8899.07	1629.10	8899.07	1629.10	8899.07
Δ	-388.88	-6270.27	5159.57	-570.80	-1584.85	5853.63	4813.61	3879.89
Dist & Shift	6282.32	.00	5191.05	.00	6064.38	- .04	6182.59	+ .06
Δ/s	- .062	.998	+ .994	- .110	- .261	+ .965	+ .779	+ .628

The last line gives the direction of the trig from the trial point.



Final Position

-.03 -.01

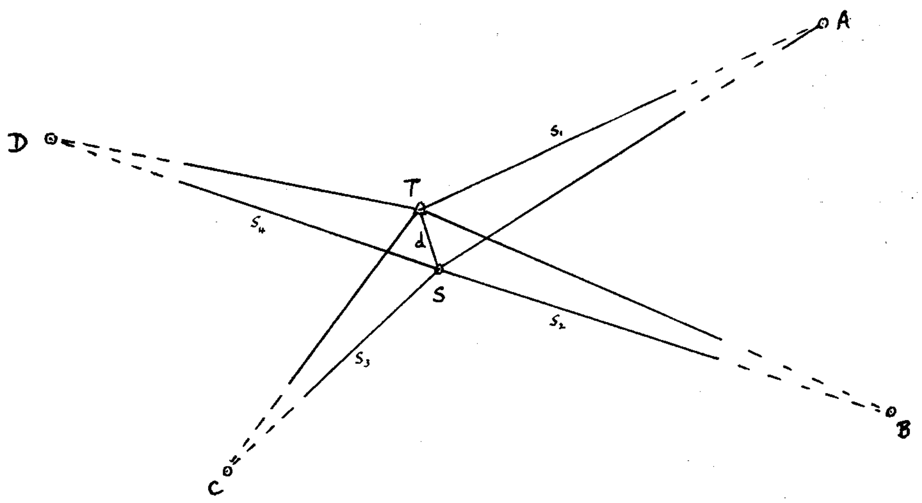
Final Co-ordinates of C

1629.07 8899.06

In selecting the final position it should be remembered that the variances of electromagnetic distance measurements are partly proportional to the distances.

7.3 Satellite Stations

There are occasions when observations are required at a trig station which is marked by a beacon supported by a large cairn of stones and it is not convenient to dismantle the cairn in order to set up the theodolite over the trig station. The procedure in this case is to set up the theodolite a short distance from the trig station and observe to the required points and to the neighbouring trig in addition. As this will be a very close sight it need only be recorded to the nearest minute of arc, but the distance must also be measured whilst the distances to the other points observed must be ascertainable approximately by scaling from a map or other means.



In the diagram T is the trig station, S the satellite station, A, B, C & D the other four points to which observations are required. The distance TS is d and s_i (i = 1,4) are the approximate distances from T to A, B, C & D

The required correction to the observed direction from S to A to give the direction T to A

$$= \text{Bearing TA} - \text{Bearing SA}$$

$$= \text{Bearing AT} - \text{Bearing AS} = \underline{\angle SAT}$$

The required correction is then given by the sine formula

$$\sin \underline{/ SAT} = \frac{d}{s_1} \sin \underline{/ AST}$$

It should be noted that this formula automatically gives the correct sign to the correction. The clockwise angle at the satellite station S, from the trig station T to the distant stations is, for A & B, in the first two quadrants for which the sine is positive whilst for C & D it is in the last two quadrants for which the sine is negative. Inspection of the diagram demonstrates that the corrections to the directions must have the corresponding sign.

Numerical Example

Observations taken at a satellite station near Gap Trig

Station Observed	Theodolite Reading	Distance	Clockwise Angle from Trig	Sin	Correction to Reading
Gap	36 34 00.0	2.68			
Bluff	102 04 11.5	18690 .0	65 30 11.5	+.909984	+26.9"
Corona	162 15 08.5	29120 .0	125 41 08.5	+.812229	+15.4"
Witeroc	249 00 47.2	8820 .0	212 26 47.2	-.536511	-33.6"
F.G.6	265 43 56.7	7560 .0	229 09 56.7	-.756603	-55.3"

8. Transformations

8.1 In this course only linear transformations will be considered although later, in photogrammetry you will deal with transformations which are non-linear.

Linear transformations are those in which the connection between the old co-ordinate system and the new one is expressed in a linear form, e.g.

$$E' = a_{11} E + a_{12} N + b_1 \qquad \text{or} \qquad \begin{bmatrix} E' \\ N' \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \times \begin{bmatrix} E \\ N \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$N' = a_{21} E + a_{22} N + b_2$$

where (E', N') are the co-ordinates in the new system corresponding to co-ordinates (E, N) in the old system.

The general form of the linear transformation is called an affine transformation but there is a special form, in which the scale change is the same in all directions, called a similarity transformation, which will be dealt with first.

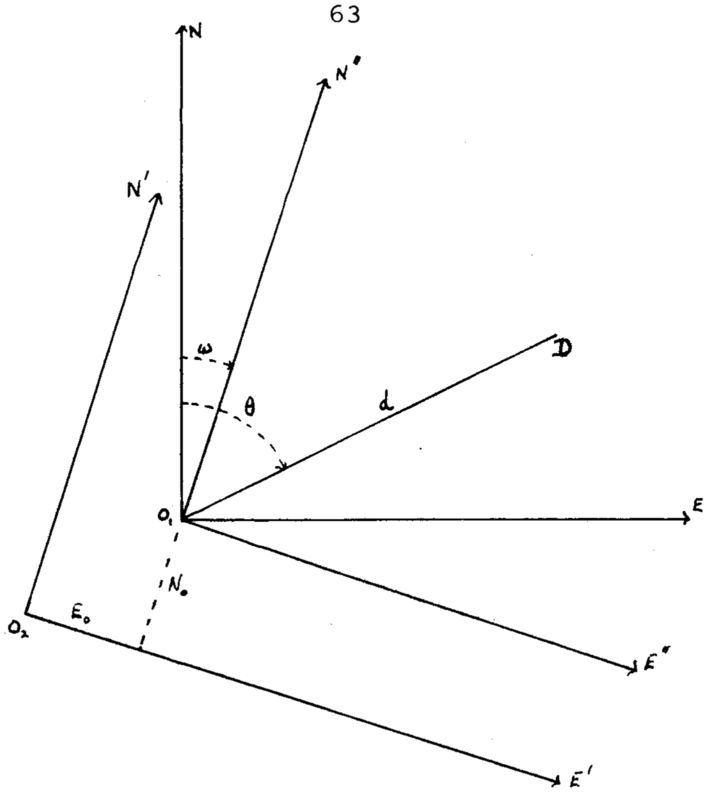
8.2 Similarity Transformation.

A similarity transformation is made up of three separate operations

(a) a rotation of the axes

(b) a uniform scale change in all directions, and

(c) a change of origin



In the diagram O_1E and O_1N are the old co-ordinate axes. O_2E' and O_2N' the new co-ordinate axes and O_1E'' and O_1N'' axes parallel to the new ones but through the old origin.

Consider a point D at a distance d from the origin and such that the bearing of O_1D in the old system is θ Dealing first with the rotation of the axes through a clockwise angle w, the new bearing of O_1D will be $(\theta-w)$ and the new co-ordinates will be

$$E' = d \sin(\theta-w) = d(\sin\theta\cos w - \cos\theta\sin w) = E\cos w - N\sin w$$

$$N' = d\cos(\theta-w) = d(\cos\theta\cos w + \sin\theta\sin w) = E\sin w + N\cos w$$

Next applying a uniform change of scale by a factor λ in all directions gives $E' = \lambda(E\cos w - N\sin w)$

$$N' = \lambda(E\sin w + N\cos w)$$

and finally the change of origin from O_1 to O_2 gives

$$E' = \lambda(E \cos w - N \sin w) + E'_o$$

$$N' = \lambda(E \sin w + N \cos w) + N'_o.$$

where (E'_o, N'_o) are the co-ordinates, in the new system, of the old origin.

These relationships between the two sets of co-ordinates can be expressed in a slightly shorter form as

$$E' = aE - bN + c_1$$

$$N' = bE + aN + c_2$$

where $a = \lambda \cos w$ and $b = \lambda \sin w$; $\lambda = \sqrt{a^2 + b^2}$ and $w = \tan^{-1}(b/a)$

For each point for which the co-ordinates in both systems are known, there will thus be two equations and since there are four constants a, b, c_1 and c_2 to be found, there must be two common points for which the co-ordinates in both systems are known before the transformation can be effected. In practice the constants c_1 and c_2 are eliminated by dealing with differences in co-ordinates, rather than with the co-ordinates themselves.

$$E'_Y - E'_X = a(E_Y - E_X) - b(N_Y - N_X)$$

$$N'_Y - N'_X = b(E_Y - E_X) + a(N_Y - N_X)$$

Before these formulae can be applied the constants a and b must be calculated from the co-ordinates in the two systems of the two common points P and Q.

Let the bearing and distance PQ in the old system be α and d and in the new system be α' and d'

Then $\lambda = \frac{d'}{d}$ and $w = \alpha - \alpha'$

$$a = \lambda \cos w = \frac{d'}{d}(\cos\alpha \cos\alpha' + \sin\alpha \sin\alpha') = \frac{dd'(\cos\alpha \cos\alpha' + \sin\alpha \sin\alpha')}{d^2}$$

$$= \frac{(N_Q - N_P)(N'_Q - N'_P) + (E_Q - E_P)(E'_Q - E'_P)}{d^2}$$

and similarly $b = \lambda \sin w$

$$= \frac{(E_Q - E_P)(N'_Q - N'_P) - (E'_Q - E'_P)(N_Q - N_P)}{d^2}$$

Numerical Example

Data	Station	Old System Co-ordinates E	Old System Co-ordinates N	New System Co-ordinates E'	New System Co-ordinates N'
	A	21.13	22.25	17.21	64.71
	B	31.02	77.71	18.11	8.34
	C	13.06	36.93		
	D	14.81	52.31		

To find the co-ordinates in the new system of C & D and the scale change

Calculation $d^2 = (55.46)^2 + (9.89)^2 = 3173.6237$

$a = \{(55.46)(-56.37) + (9.89)(.90)\} / d^2 = -.982278$

$b = \{-(.90)(55.46) + (9.89)(-56.37)\}/d^2 = -.191394$

Scale change $= \lambda = \sqrt{a^2 + b^2} = 1.000751$

Old System		Station	New System	
E_i	N_i	i	E'_i	N'_i
$E_j - E_i$	$N_j - N_i$		$a(E_j - E_i)$	$a(N_j - N_i)$
			$-b(N_j - N_i)$	$b(E_j - E_i)$
E_j	N_j	j	E'_j	N'_j
21.13	22.25	A	17.21	64.71
-8.07	+14.68		+7.927	-14.420
			+2.810	+ 1.545
13.06	36.93	C	27.947	51.835
+1.75	+15.38		- 1.719	-15.107
			+ 2.944	- .335
14.81	52.31	D	29.172	36.393
+16.21	+25.40		-15.923	-24.950
			+ 4.861	- 3.102
31.02	77.71	B	18.110	8.341

The calculation is carried out by working with differences of co-ordinates, starting from one common point and finishing on the other as a check on the whole of the calculation. One extra place of decimals is carried throughout the calculation and the new co-ordinates are subsequently rounded off to conform to the data.

8.3 Affine Transformation

In this general case the formulae connecting the two sets of co-ordinates

$$E' = a_1E + b_1N + C_1$$

$$N' = a_2E + b_2N + C_2$$

involve six constants a_1, a_2, b_1, b_2, c_1 & c_2 and hence the co-ordinates in both systems for three common points must be known. If these common points are P, Q and R then dealing with differences of co-ordinates

$$E'_Q - E'_P = a_1(E_Q - E_P) + b_1(N_Q - N_P)$$

$$(E'_R - E'_Q) = a_1(E_R - E_Q) + b_1(N_R - N_Q)$$

giving

$$a_1 = \frac{(E'_Q - E'_P)(N_R - N_Q) - (E'_R - E'_Q)(N_Q - N_P)}{(E_Q - E_P)(N_R - N_Q) - (E_R - E_Q)(N_Q - N_P)}$$

$$b_1 = \frac{(E'_R - E'_Q)(E_Q - E_P) - (E'_Q - E'_P)(E_R - E_Q)}{(E_Q - E_P)(N_R - N_Q) - (E_R - E_Q)(N_Q - N_P)}$$

and similarly from

$$N'_Q - N'_P = a_2(E_Q - E_P) + b_2(N_Q - N_P)$$

$$N'_R - N'_Q = a_2(E_R - E_Q) + b_2(N_R - N_Q)$$

we get

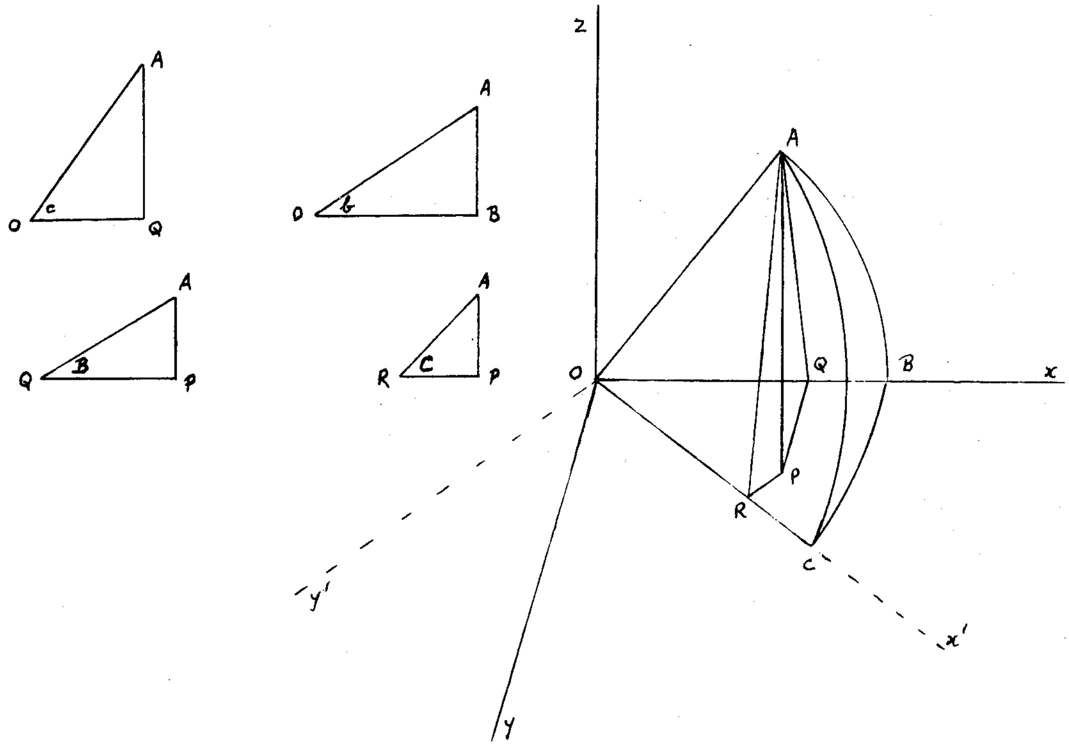
$$a_2 = \frac{(N'_Q - N'_P)(N_R - N_Q) - (N'_R - N'_Q)(N_Q - N_P)}{(E_Q - E_P)(N_R - N_Q) - (E_R - E_Q)(N_Q - N_P)}$$

$$b_2 = \frac{(N'_R - N'_Q)(E_Q - E_P) - (N'_Q - N'_P)(E_R - E_Q)}{(E_Q - E_P)(N_R - N_Q) - (E_R - E_Q)(N_Q - N_P)}$$

Note that for all four constants the denominator is the same.

8.4 Other Uses of Transformations.

Transformations can also be used to derive formulae in other branches of mathematics. Consider a spherical triangle ABC on a sphere of unit radius (see Section 9 for definitions of the sides and angles of a spherical triangle)



P is the foot of the perpendicular from A on to the BOC plane, PQ is perpendicular to OB and PR is perpendicular to OC. Then consider co-ordinate axes through O the centre of the sphere such that Ox lies along OB and the xOy plane coincides with the BOC plane. The co-ordinates of A will then be (OQ, QP, PA) or (cos c, sin c cos B, sin c sin B) since by definition $\angle AOB$ = c and $\angle AQP$ = B.

Now rotate these axes through an angle a about the OZ axis so that the new axis Ox' lies along OC.

Relative to this system the co-ordinates of A will be (OR, RP, PA) or (cos b, -sin b cos C, sin b sin C) and the two systems will be connected by the transformation

$$\begin{bmatrix} \cos b \\ -\sin b \cos C \\ \sin b \sin C \end{bmatrix} = \begin{bmatrix} \cos a & \sin a & 0 \\ -\sin a & \cos a & 0 \\ 0 & 0 & 1 \end{bmatrix} \times \begin{bmatrix} \cos c \\ \sin c \cos B \\ \sin c \sin B \end{bmatrix}$$

giving

$$\cos b = \cos a \cos c + \sin a \sin c \cos B$$

$$-\sin b \cos C = -\sin a \cos c + \cos a \sin c \cos B$$

$$\sin b \sin C = \sin c \sin B$$

The first of these formulae is a variant of formula 9.1, the second is a variant of 9.5 whilst the last is part of 9.3

Spherical Trigonometry

9.1 Introduction

Spherical Trigonometry is a very important branch of Mathematics in survey work. Its most important application is in astronomical surveying but it can also be used to solve a number of three dimensional problems. In geodesy it is used to derive the formulae for the calculation of differences of Latitude and Longitude for short and medium length geodetic lines whilst in photogrammetry it can be used to find the magnitude of the errors involved in assuming that the angle between two lines on the ground is equal to the angle between the corresponding lines on the photograph.

Numerical Examples of some of the uses of spherical trigonometry will be given at the end of this section.

9.2 Definitions

Great Circle is a circle formed by the intersection of the sphere and any plane passing through the centre of the sphere. It will of course have the same centre and same radius as the sphere itself.

Small Circle is a circle formed by the intersection of the sphere and a plane which does not pass through the centre of the sphere. Its centre will be the foot of the perpendicular from the centre of the sphere to the plane and if this distance is p and the radius of the sphere is r, the radius of the small circle will be $\sqrt{r^2 - p^2}$

Side of a Spherical Triangle is an arc of a Great Circle. Its length will be quoted in circular measure by the angle it subtends at the centre of the sphere. The circular measure will be used in all subsequent formulae.

Angle of a Spherical Triangle is the angle between the tangents to the two great circles at the point of intersection. It is also the angle between the planes of the great circles of which the sides are arcs.

Spherical Excess denoted by E, is the amount by which the sum of the three angles of a spherical triangle exceeds 180°

Pole. The poles of a great circle are the ends of the diameter of the sphere which is at right angles to the plane of the great circle.

Polar Triangle. The Polar Triangle A'B'C' of a spherical triangle ABC is formed by the great circles connecting A', B' & C' which are the poles of sides BC, CA & AB respectively.

Right Angled Triangle. A Spherical Triangle with at least one of its angles a right angle. (Note that two or all three angles can be right angles)

9.3 Formulae

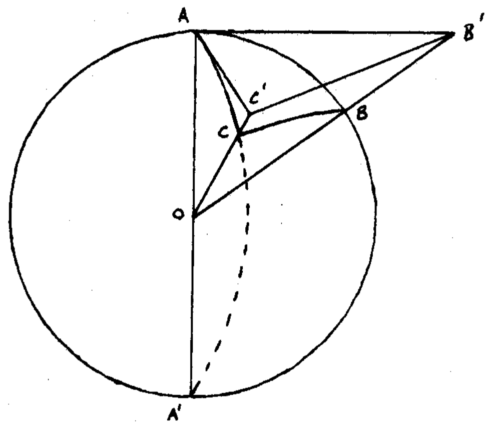


Figure 9.1

There is only one basic formula in spherical trigonometry and all the other formulae can be derived from it by mathematical operations.

Let ABC be a spherical triangle on a sphere centre O and let the radii OB, OC (produced) cut the tangent plane to the sphere at A in B' & C' respectively.

Then by definition

Spherical angle $\angle BAC$ = Plane angle $\angle B'AC' = A$

$\angle B'OC' = a$ $\quad\quad$ $\angle C'OA = b$ $\quad\quad$ $\angle AOB' = c$

Let the sphere have unit radius

Then $AB' = \tan c$ $\quad\quad$ $OB' = \sec c$

$AC' = \tan b$ $\quad\quad$ $OC' = \sec b$

giving from the $\Delta AB'C'$ $\quad$ $(B'C')^2 = \tan^2 c + \tan^2 b - 2\tan c \tan b \cos A$

and from the $\Delta OB'C'$ $\quad$ $(B'C')^2 = \sec^2 c + \sec^2 b - 2\sec c \sec b \cos a$

Equating these two values and remembering that $\sec^2\theta = 1 + \tan^2\theta$

$$\sec b \sec c \cos a = 1 + \tan b \tan c \cos A$$

or
$$\cos a = \cos b \cos c + \sin b \sin c \cos A \qquad 9.1$$

This is the basic formula of spherical trigonometry and there are of course, two other exactly similar formulae for cosb and cosc.

This proof, or demonstration rather, of the formula suffers from the defect that in order that OB and OC should cut the tangent plane at A, the sides b and c of the spherical triangle must both be less than 90°. The size of the side a is immaterial. However this is not an overriding difficulty as if two (or three) sides of the triangle are $>90^{\circ}$ we then consider the triangle formed by the other part of the lune - A'BC in the diagram given above. Then $A' = A$

$b' = A'C = \pi - b$ $\quad\quad$ $c' = A'B = \pi - c$ $\quad$ and a is common.

If b and c are both $> \frac{\pi}{2}$ b' & c' will both be $< \frac{\pi}{2}$

and hence from equation (8.1)

$$\cos a = \cos b' \cos c' + \sin b' \sin c' \cos A$$

$$= \cos(\pi - b)\cos(\pi - c) + \sin(\pi - b)\sin(\pi - c)\cos A$$

$$= \cos b \cos c + \sin b \sin c \cos A$$

Hence the equation is generally true irrespective of the size of the sides of the triangle.

From equation

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 A}{\sin^2 a} = \frac{\sin^2 b \sin^2 c - (\cos a - \cos b \cos c)^2}{\sin^2 a \sin^2 b \sin^2 c}$$

$$= \frac{(1-\cos^2 b)(1-\cos^2 c) - (\cos a - \cos b \cos c)^2}{\sin^2 a \sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 b - \cos^2 c + \cos^2 b \cos^2 c - (\cos^2 a + \cos^2 b \cos^2 c - 2\cos a \cos b \cos c)}{\sin^2 a \sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}$$

and since this last expression is symmetrical in a, b & c the same result would have been obtained if we had started from $\frac{\sin^2 B}{\sin^2 b}$ or from $\frac{\sin^2 C}{\sin^2 c}$

$$\therefore \frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c} \qquad (9.3)$$

Again from equation (9.1)

$$\sin b \cos A = \frac{\cos a - \cos b \cos c}{\sin c} = \frac{\cos a(\sin^2 c + \cos^2 c) - \cos b \cos c}{\sin c}$$

$$= \cos a \sin c - \cos c \frac{(\cos b - \cos a \cos c)}{\sin c}$$

$$= \cos a \sin c - \cos c \sin a \cos B \qquad (9.5)$$

These three formulae can all be proved by another construction as follows:

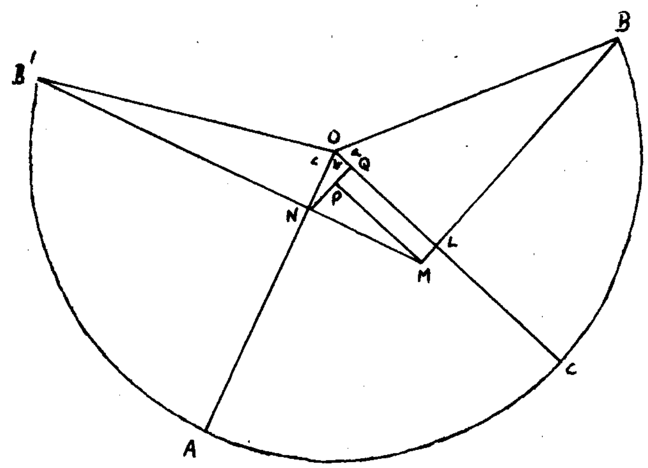


Figure 9.2

Cut out a sector of circle OBCAB'O after having marked off arc lengths BC, CA & AB' corresponding to the three sides a, b, c of a spherical triangle. Drop perpendiculars BL & B'N on OC & OA respectively and continue them to meet in M. Draw NQ perpendicular to OC and MP perpendicular to NQ

Now crease the cardboard along OA & OC and bend the two end sectors up until B and B' meet. Fasten in this position and join B to M by a peice of thread. We now have a representation of a spherical triangle and the planes joining it to the centre of the sphere.

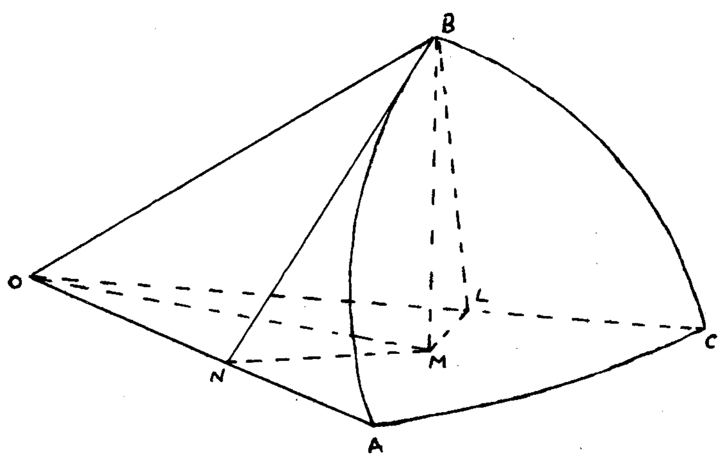


Figure 9.3

Now originally line BNM was perpendicular to OA and so after creasing and bending about OA, the plane BNM will be perpendicular to OA and $\angle BNM$ = angle between planes BOA & COA = A

Similarly plane BLM will be perpendicular to OC and $\angle BLM$ = C

Since planes BNM and BLM are perpendicular to OA and OC respectively, BM, the line in which these two planes cut, will be perpendicular to plane OAC and hence Δs BMN and BML are right angled at M

Take the radius of the circle as unity and then in terms of the sides and angles of the spherical triangle

(a) in Figure 9.2 obtain expressions for B'N, ON, BL, OL, NQ, OQ and finally QL = OL - OQ

(b) in Figure 9.3 obtain expressions for LM and NM. Also obtain expressions for BM from both ΔBLM and ΔBNM - this will give one of the formulae

(c) in Figure 9.2 from ΔNMP (note $\angle PNM$ = b) obtain expressions for NP and PM

(d) Equate ML = PQ = NQ - NP to give another formula

(e) Equate PM = QL to give a third formula

Answers: (a) B'N = sin c ON = cos c BL = sin a OL = cos a

NQ = cos c sin b OQ = cos c cos b OL = cos a - cos bcos c

(b) LM = sin a cos C NM = sin c Cos A

BM = sin c sin A = sin a sin C giving formula (9.3)

(c) NP = sin c cos A cos b PM = sin c cos A sin b

(d) sina cos C = cos c sin b - sin c cos b cos A - Formula (9.5)

(e) sin c sin b cos A = cos a - cos b cos c Formula (9.1)

<u>Half Angle Formulae</u>

$$\sin\frac{A}{2} = \sqrt{\frac{1-\cos A}{2}} = \sqrt{\frac{1-\dfrac{\cos a - \cos b \cos c}{\sin b \sin c}}{2}} = \sqrt{\frac{\cos b \cos c + \sin b \sin c - \cos a}{2 \sin b \sin c}}$$

$$= \sqrt{\frac{\cos(b-c)-\cos a}{2\sin b \sin c}} \qquad = \sqrt{\frac{2\sin\left(\frac{b-c+a}{2}\right)\sin\left(\frac{a+c-b}{2}\right)}{2\sin b\sin c}}$$

$$= \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}} \qquad \text{where } 2s = a+b+c \qquad (9.7)$$

Similarly $\cos\frac{A}{2} = \sqrt{\frac{1+\cos A}{2}} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$ (9.8)

$$\tan\frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin s \sin(s-a)}}$$

Also $\tan\frac{(B+C)}{2} = \dfrac{\tan\frac{B}{2}+\tan\frac{C}{2}}{1-\tan\frac{B}{2}\tan\frac{C}{2}} = \dfrac{\sqrt{\dfrac{\sin(s-a)}{\sin s}}\left(\sqrt{\dfrac{\sin(s-c)}{\sin(s-b)}}+\sqrt{\dfrac{\sin(s-b)}{\sin(s-c)}}\right)}{1-\dfrac{\sin(s-a)}{\sin s}}$

$$= \sqrt{\frac{\sin s \sin(s-a)}{\sin(s-b)\sin(s-c)}}\left(\frac{\sin(s-c)+\sin(s-b)}{\sin s - \sin(s-a)}\right)$$

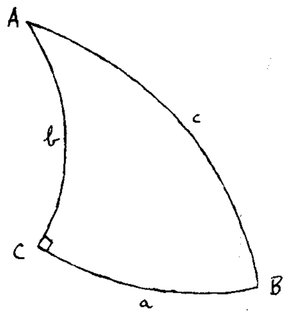
$$= \cot\frac{A}{2}\left(\frac{2\sin\left(\frac{2s-b-c}{2}\right)\cos\left(\frac{b-c}{2}\right)}{2\cos\left(\frac{2s-a}{2}\right)\sin\frac{a}{2}}\right)$$

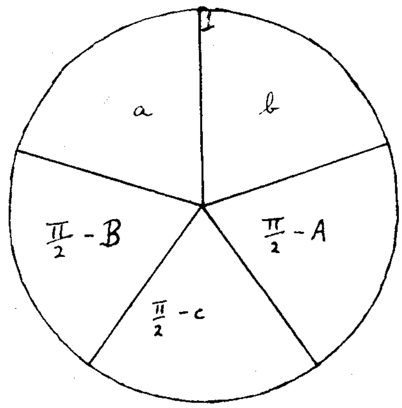
$$= \frac{\cos\left(\frac{b-c}{2}\right)}{\cos\left(\frac{b+c}{2}\right)}\cot\frac{A}{2} \qquad (9.11)$$

Similarly $\tan\frac{(B-C)}{2} = \dfrac{\sin\left(\frac{b-c}{2}\right)}{\sin\left(\frac{b+c}{2}\right)}\cot\frac{A}{2}$ (9.12)

Napiers Rules for Formulae for Right Angled Triangles

For Right Angled Triangles, Napier devised an easy memory aid





ABC is a spherical triangle right angled at C

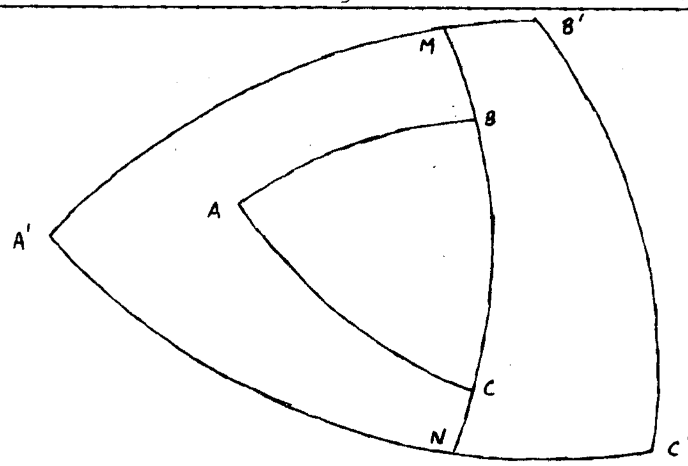
(a) Divide a circle into 5 segments and in these segments enter in cyclic order round the circle and triangle, the 5 elements of the triangle excluding the right angle C. The sides bordering on the right angle are entered in their normal forms and the other 3 elements by their complements.

(b) Any segment of the circle can be called the "Middle Part" The two segments adjacent to it are the "Adjacent Parts" and the remaining two are called the "Opposite Parts"

(c) Sine (Middle Part) = Product of Tangents of Adjacent Parts (9.15)

= Product of Cosines of Opposite Parts (9.16)

9.4 Relation between sides and angles of Polar & Primitive Triangles



Let A' B' & C' be the poles of the sides BC, CA & AB. Then A'B 'C' is the Polar triangle corresponding to the Primitive triangle ABC

Let the great circle arc BC, produced if necessary, cut the arcs A'B' and A'C' at M and N respectively

(a) Then since A' is the pole of BC arc $A'B = \pi/2$

since C' is the pole of AB arc $C'B = \pi/2$

∴ B is the pole of the great circle arc A'C' and similarly it can be shown that C and A are the poles of A'B' and B'C' respectively.

Hence if A'B'C' is the Polar triangle of ABC, ABC is also the polar triangle of A'B'C'

(b) As B is the pole of A'C' arc $BN = \pi/2$

Since C is the pole of A'B' arc $CM = \pi/2$

Also since A' is the pole of BC arc $MN = A'$

But MN = BN + CM - BC i.e.

Similarly

$$\left.\begin{aligned} A' &= \pi-a \\ B' &= \pi-b \\ C' &= \pi-c \\ A &= \pi-a' \\ B &= \pi-b' \\ C &= \pi-c' \end{aligned}\right\} \quad (9.17)$$

When these relationships are applied to formulae involving the elements of the spherical triangle, additional formulae will be obtained. Consider for example formula (9.8)

$$\cos\frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}} = \sqrt{\frac{\sin(\frac{a+b+c}{2}) \sin(\frac{b+c+a}{2})}{\sin b \sin c}}$$

Substituting from (8.17) gives

$$\cos\left(\frac{\pi-a'}{2}\right) = \sqrt{\frac{\sin\left(\frac{3\pi-A'-B'-C'}{2}\right)\sin\left(\frac{\pi+A'-B'-C'}{2}\right)}{\sin(\pi-B')\sin(\pi-C')}}$$

or $$\sin\frac{a'}{2} = \sqrt{\frac{-\cos S \cos(S-A')}{\sin B' \sin C'}} \quad \text{where } 2S = A'+B'+C'$$

This formula will refer to any triangle and so the dashes can be dropped and the formula written

$$\sin\frac{a}{2} = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}} \quad \text{where } 2S = A + B + C$$

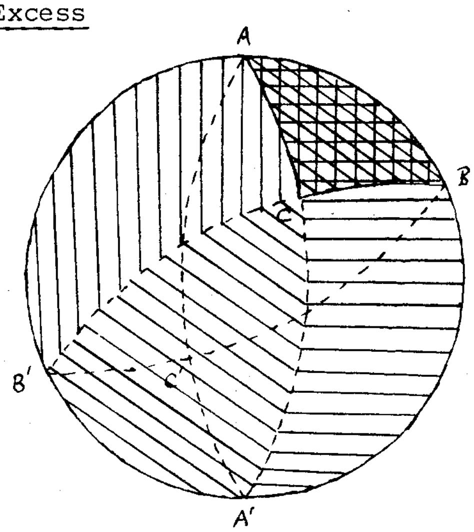
Formulae involving the determination of an angle by calculation of its sine should be avoided as far as possible since there will be an ambiguity as both A and (180-A) are solutions of sin A = x
There will be no ambiguity if sin A/2 is calculated.

A summary of the formulae most used in survey work follows.

Spherical Trigonometrical Formulae

Name	Form	Formulae	No. of Variants	
Cosine	Primitive	$\cos a = \cos b \cos c + \sin b \sin c \cos A$	3	9.1
	Polar	$\cos A = -\cos B \cos C + \sin B \sin C \cos a$	3	9.2
Sine	Primitive & Polar	$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}$	1	9.3
Four Part	Primitive & Polar	$\sin B \cot A = \cot a \sin c - \cos c \cos B$	6	9.4
Five Part	Primitive	$\sin b \cos A = \cos a \sin c - \sin a \cos c \cos B$	6	9.5
	Polar	$\sin B \cos a = \cos A \sin C + \sin A \cos C \cos b$	6	9.6
Half Angle	Primitive	$\sin \frac{A}{2} = \sqrt{\frac{\sin(s-b)\sin(s-c)}{\sin b \sin c}}$	3	9.7
		$\cos \frac{A}{2} = \sqrt{\frac{\sin s \sin(s-a)}{\sin b \sin c}}$	3	9.8
	Polar	$\cos \frac{a}{2} = \sqrt{\frac{\cos(S-B)\cos(S-C)}{\sin B \sin C}}$	3	9.9
		$\sin \frac{a}{2} = \sqrt{\frac{-\cos S \cos(S-A)}{\sin B \sin C}}$	3	9.10
Napier's Analogies	Primitive	$\tan \frac{(B+C)}{2} = \frac{\cos\left(\frac{b-c}{2}\right)}{\cos\left(\frac{b+c}{2}\right)} \cot \frac{A}{2}$	3	9.11
		$\tan \frac{(B-C)}{2} = \frac{\sin\left(\frac{b-c}{2}\right)}{\sin\left(\frac{b+c}{2}\right)} \cot \frac{A}{2}$	3	9.12
	Polar	$\tan \frac{(b+c)}{2} = \frac{\cos\left(\frac{B-C}{2}\right)}{\cos\left(\frac{B+C}{2}\right)} \tan \frac{a}{2}$	3	9.13
		$\tan \frac{(b-c)}{2} = \frac{\sin\frac{(B-C)}{2}}{\sin\left(\frac{B+C}{2}\right)} \tan \frac{a}{2}$	3	9.14

9.5 Spherical Excess



The surface area of a sphere is $4\pi R^2$ and the surface area of a lune enclosed between two planes through a common diameter and making an angle θ w ith each other will by symmetry be $\frac{\theta}{2} \cdot 4\pi R^2 = 2\theta R^2$

Consider a spherical triangle ABC and let the great circle arcs formed by the sides be continued right round the sphere to meet again in A', B' and C'

Then the area of lune ABA'CA is $2AR^2$ hatched ≡

the area of lune BAB'CB is $2BR^2$ hatched ||||

the area of lune CBC'AC is $2CR^2$

This last lune can by symmetry be taken as ΔABC + area CA'B'C and as such has been hatched \\\\

From the diagram it will be seen that the triangle ABC has been included three times and hence

Sum of areas of the three lunes = Area of hemisphere + 2 x area ABC

i.e. $2R^2(A + B + C) = 2\pi R^2 + 2\Delta ABC$

Area of $\Delta ABC = R^2(A + B + C - \pi) = ER^2$ (9.18)

where $E = A + B + C - \pi$ is the Spherical Excess

The spherical excess is very important in geodesy when a geodetic triangulation network has to be adjusted as the observed angles of a triangle have to be adjusted to sum to $(180^\circ + E)$ and not to 180°. Its value is determined by calculating the area of the triangle as if it were a plane triangle and dividing this by R^2 to obtain E in radian measure.

9.6 Legendre's Theorem

If the sides of a spherical triangle are small compared with the radius of the sphere, then each angle of the spherical triangle exceeds by (1/3rd spherical excess) the corresponding angles of a plane triangle, the sides of which are of the same length as those of the spherical triangle.

Consider a spherical triangle ABC with side lengths a, b, c in radian measure and a corresponding plane $\Delta A'B'C'$ with sides α, β & γ where α = ar etc.

Then $\cos A = \dfrac{\cos a - \cos b \cos c}{\sin b \sin c}$

$$= \frac{\left(1 - \frac{\alpha^2}{2r^2} + \frac{\alpha^4}{24r^4} - ..\right) - \left(1 - \frac{\beta^2}{2r^2} + \frac{\beta^4}{24r^4} - ..\right)\left(1 - \frac{\gamma^2}{2r^2} + \frac{\gamma^4}{24r^4} - ..\right)}{\frac{\beta}{r}\left(1 - \frac{\beta^2}{6r^2} + ..\right) \quad \frac{\gamma}{r}\left(1 - \frac{\gamma}{6r}2 + ..\right)}$$

$$= \frac{\dfrac{\beta^2 + \gamma^2 - \alpha^2}{2r^2} + \dfrac{\alpha^4 - \beta^4 - \gamma^4 - 6\beta^2\gamma^2}{24r^4}}{\dfrac{\beta\gamma}{r^2}\left(1 - \dfrac{\beta^2 + \gamma^2}{6r^2}\right)}$$ ignoring terms of order 4 in $(\frac{1}{r})$

$$= \frac{1}{2\beta\gamma}\left(\beta^2 + \gamma^2 - \alpha^2 + \frac{\alpha^4 - \beta^4 - \gamma^4 - 6\beta^2\gamma^2}{12r^2}\right)\left(1 + \frac{\beta^2 + \gamma^2}{6r^2}\right) + O(\frac{1}{r^4})$$

$$= \frac{\beta^2 + \gamma^2 - \alpha^2}{2\beta\gamma} - \frac{2\beta^2\gamma^2 + 2\gamma^2\alpha^2 + 2\alpha^2\beta^2 - \alpha^4 - \beta^4 - \gamma^4}{24\,\beta\gamma r^2} + O\,(\frac{1}{r^4})$$

$$= \cos A' - \sin^2 A' \frac{\beta\gamma}{6r^2}$$

if $A = A' + \theta$ where θ is small

$\cos A = \cos(A' + \theta) \qquad = \cos A' - \theta \sin A'$

in this case $\theta = \dfrac{\beta\gamma \sin A'}{6r^2} = \dfrac{\Delta A'B'C'}{3r^2} = \dfrac{\Delta ABC}{3r^2} = \dfrac{E}{3}$

$\therefore A = A' + \frac{E}{3}$ and similarly for B and C (9.19)

9.7 Numerical Examples

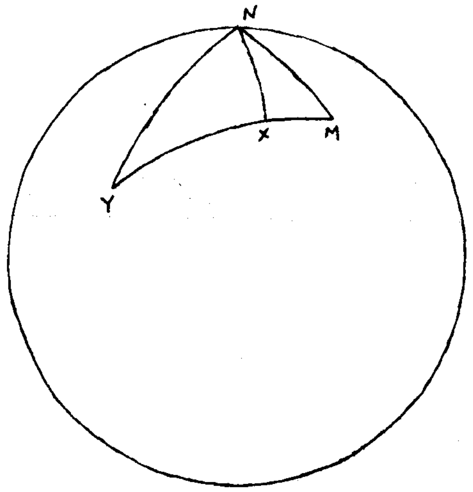
9.7.1 Navigational Example

A plane flies along a great circle route between Moscow ($55^\circ 45'$N, $37^\circ 43'$E) and New York ($40^\circ 43'$N, $73^\circ 59'$W). Calculate

(a) the starting bearings at each end of the route

(b) the most northerly latitude reached

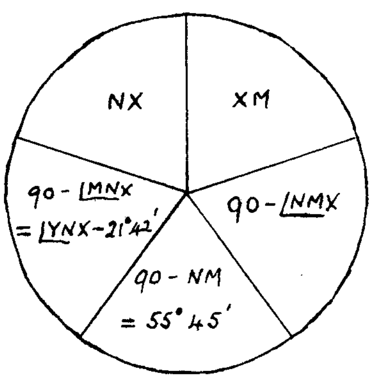
and (c) the length of the route assuming the earth's radius is 6378 Km



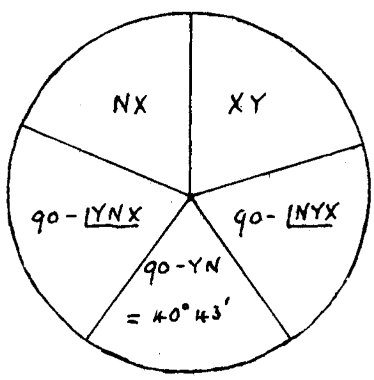
In the diagram N is the North Pole, M Moscow and Y New York. X is the most northerly point on the route i.e. the nearest point to N and hence the great circle NX will be at right angles to YXM at X.

$NM = 90^\circ - 55^\circ 45'$ $NY = 90^\circ - 40^\circ 43'$ $\angle MNY = 37^\circ 43' + 73^\circ 59' = 111^\circ 42'$

Consider the two right angled triangles NMX and NYX



Δ NMX



Δ NYX

ΔNMX $\qquad\qquad$ ΔNYX

(i)

$\sin(\underline{/YNX} - 21^{\circ}42') = \tan NX \tan 55^{\circ}45'$ $\qquad$ $\cos \underline{/YNX} = \tan NX \tan 40^{\circ}43'$

Dividing these two expressions gives

$$\tan \underline{/YNX} = \left(\frac{\tan 55^{\circ}45'}{\tan 40^{\circ}43'} + \sin 21^{\circ}42'\right) \Big/ \cos 21^{\circ}42' = \left(\frac{1.468697}{.860642} + .369747\right) \Big/ .929133$$

$= 2.234622$

$\underline{/MNX} = 45^{\circ}48'31''$ $\qquad$ $\underline{/YNX} = 65^{\circ}53'29''$

$\tan NX = \dfrac{\cos 45^{\circ}48'31''}{1.468697}$ $\qquad$ $= \dfrac{\cos 65^{\circ}53'29''}{.860642}$

$= .474609$ $\qquad$ $= .474608$

$NX = 25^{\circ}23'22''$

The most northerly latitude reached on the route is N64°36'38"

(ii)

$\sin 55^{\circ}45' = \cot 45^{\circ}48'31'' \cot \underline{/NMX}$ $\qquad$ $\sin 40^{\circ}43' = \cot 65^{\circ}53'29'' \cot \underline{/YNX}$

$\tan \underline{/NMX} = \dfrac{.972165}{.826590} = 1.176116$ $\qquad$ $\tan \underline{/YNX} = \dfrac{.447502}{.652319} = .686017$

$\underline{/NMX} = 49^{\circ}37'37''$ $\qquad$ $\underline{/NYX} = 34^{\circ}27'03''$

Initial Bearing at Moscow 310°22'23" $\qquad$ at New York 34°27'03"

(iii)

$\sin MX = \tan NX \cot \underline{/NMX}$ $\qquad$ $\sin XY = \tan NX \cot \underline{/NYX}$

$= .474609 \times .850256$ $\qquad$ $= .474609 \times 1.457687$

$= .403539$ $\qquad$ $= .691832$

$MX = 23^{\circ}47'59''$ $\qquad$ $XY = 43^{\circ}46'31''$

Distance $MX = 6378 \times (23^{\circ}47'59'') \times \dfrac{\pi}{180}$ $\qquad$ Distance $XY = 6378 \times (43^{\circ}46'31'') \times \dfrac{\pi}{180}$

$= 2649.32$ Km $\qquad$ $= 4872.94$ Km

Total Distance = 7522.26 Km

Since these three calculations are interrelated the complete calculation can be checked by computing the total distance from the formula

$$\cos YM = \sin 55^\circ 45' \sin 40^\circ 43' + \cos 55^\circ 45' \cos 40^\circ 43' \cos 111^\circ 42'$$

$$= .539200 - .157725 = .381475$$

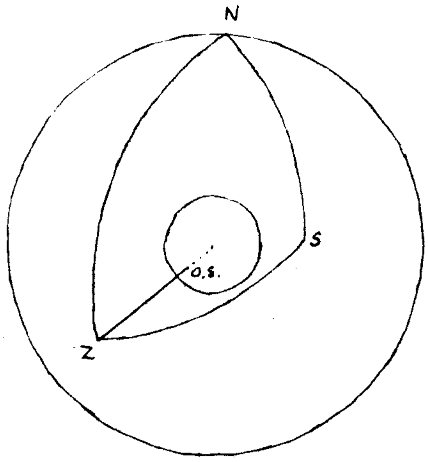
$$YM = 67^\circ 34' 30''$$

$$\text{Distance } YM = 6378 \times (67^\circ 34' 30'') \times \frac{\pi}{180} = 7522.25 \text{ km}$$

9.7.2 Astronomical Example - Sun Azimuth

It will be assumed that the student has no knowledge of astronomical surveying and the problem is dealt with purely as an exercise in the use of Spherical Trigonometry.

At a survey station, the latitude of which is approximately S32°33'20", observations are taken of the angle between a fixed reference station (R.O) and the sun and the times of the observations are also noted. The position of the sun at any time is determined by its Declination and Right Ascension which correspond to Latitude and Longitude in Geographical Co-ordinates. The zero points for Right Ascension and Longitude are however not the same and so for the purposes of the calculation, Right Ascension is replaced by the Hour Angle which is the angle, expressed in units of time, measured westward from the meridian through the observing station to the sun's position. The Declination is tabulated in the Star Almanac for Land Surveyors and the Hour Angle can be obtained from the quantity E which is also tabulated. In this problem the Declination is N 0°47'54", the Hour Angle 20 hrs.12 m 56.0s and the clockwise angle from the R.O to the sun is 50°14'26"



In the diagram OS is the Observing Station on the earth, Z the Zenith, i.e. the point vertically above it on the celestial sphere, N the North Pole and S the sun's position on the same sphere.

The latitude of Z will be the same as that of the observing station and hence NZ = 90 - $(-32^\circ 33'20")$ = $122^\circ 33'20"$

NS = 90 - $(0^\circ 47'\ 54")$ = $89^\circ 12'06"$

The hour angle converted into angular measure (multiply by 15 since 24 hours = 360°) is $303^\circ 14'00"$ but the angle of a spherical triangle cannot exceed 180° and hence we must deal with the angle ($360 - 303^\circ 14'00"$) i.e. $56^\circ 46'$. This is the angle at N and it should be noted that it places the sun to the east of the observing station and so the azimuth of the sun i.e. the angle at Z must lie between 0 and 180°.

The calculation is made using the Four Parts formula

$$\sin N \cot Z = \cot NS \sin NZ - \cos NZ \cos N$$

or

$$\tan Z = \frac{\sin 56^\circ 46'}{\cot 89^\circ 12'06" \sin 122^\circ 33'20" - \cos 122^\circ 33'20" \cos 56^\circ 46'}$$

$$= \frac{.836446}{.013934 \times .842870 + .538117 \times .548050}$$

$$= \frac{.836446}{.306660} = 2.727599$$

Azimuth of the sun = Z = $69^\circ 51'57"$

Azimuth of the R.O. = $69^\circ 51'57" - 50^\circ 14'26" = 19^\circ 37'31"$

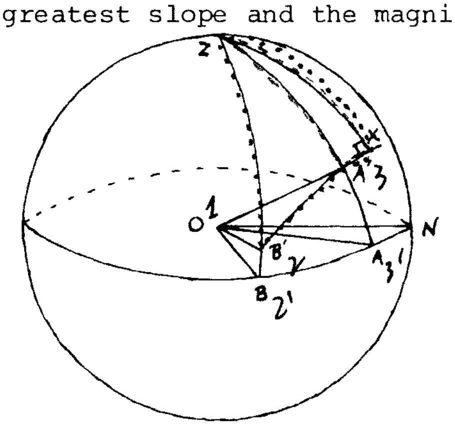
It should be noted that Z is the internal angle of the spherical triangle and that if the sun is to the west of the observing station, the external angle is required for the Azimuth. The following table enables the Azimuth to be placed in the correct quadrant.

	Hour Angle <12 hrs.	Hour Angle >12 hrs
Tan Z + ve	4th quadrant	1st quadrant
Tan Z - ve	3rd quadrant	2nd quadrant

9.7.3 Three Dimensional Problem

This type of problem is dealt with by taking the centre of the sphere as the point to which the relevant observations relate and the plane of the equator as the horizontal plane through this point.

Problem: To determine the dip and strike of an area of ground (assumed plane), two cross sections OA and OB were run out from O on bearings of 10° and 75° respectively. OA and OB were found to rise at 1 in 7.11 and 1 in 21.2 respectively. Find the bearing of the line of greatest slope and the magnitude of the slope.

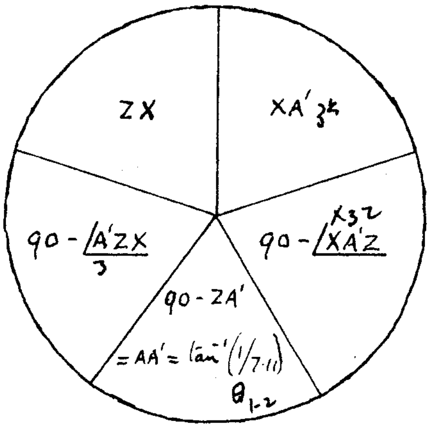


In the diagram ON is the North Direction at O, Z is the Zenith and OX is the line of greatest slope in the plane OA'B'

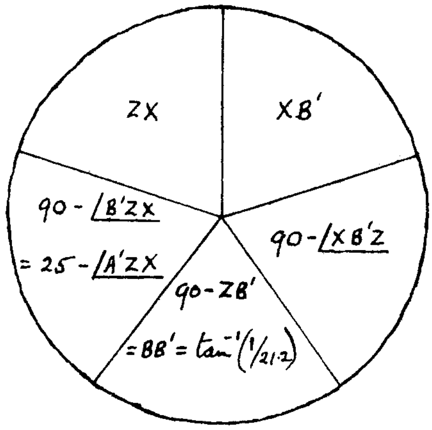
$AA' = \tan^{-1}(1/7.11)$ $BB' = \tan^{-1}(1/21.2)$ $\underline{/A'ZB'} = \underline{/AOB} = 75-10 = 65^{\circ}$

This problem is analagous to section (b) of the Navigational problem and can be treated in the same way.

Consider the two right angled triangles A'ZX and B'ZX



A'ZX



B'ZX

$\cos \underline{/A'ZX} = \tan ZX/7.11$ $\qquad$ $\sin (25-\underline{/A'ZX}) = \tan ZX/21.2$

Dividing the second equation by the first gives

$$\sin 25^{\circ} - \cos 25^{\circ} \tan \underline{/A'ZX} = 7.11/21.2 = .335377$$

$$\tan \underline{/A'ZX} = \frac{\sin 25^{\circ} - .335377}{\cos 25^{\circ}} = \frac{.422618 - .335377}{.906308} = .096260$$

$\underline{/A'ZX} = 5^{\circ}29'54"$ $\qquad$ Bearing OX = $4^{\circ}30'06"$

$ZX = 7.11 \cos 5^{\circ}29'54"$ $\qquad = 21.2 \sin 19^{\circ}30'06"$

$= 7.077$ $\qquad = 7.077$

∴ the magnitude of the greatest slope is 1 in 7.08 and it lies on a bearing of $4^{\circ}30'$

9.7.4 Geodetic Problem

From two known stations A & B, a third station C to the east of them has been fixed by observation of all the directions and the following information is available.

Spheroidal Distance AB = 28866.149 metres

Mean radius of curvature in the area = 6,369,750 metres

Observed angles A $59^\circ\ 34'\ 16.14''$

B $52^\circ\ 13'\ 22.06''$

C $68^\circ\ 12'\ 25.04''$

$180^\circ\ 00'\ 03.24''$

Calculate the Spherical Excess and the triangular misclosure.

Area of plane triangle $= \frac{1}{2}\ c^2\ \frac{\sin A \sin B}{\sin C}$

Spherical Excess $= \frac{1}{2} \left(\frac{c}{R}\right)^2 \frac{\sin A \sin B}{\sin C}$ radians

$= 103132.4'' \left(\frac{28866.149}{6369750}\right)^2 \frac{\sin 59^\circ 34' 16.14'' \sin 52^\circ 13' 22.06''}{\sin 68^\circ 12' 25.04''}$

$= 1.55''$

Triangular misclosure $= 180^\circ 00' 03.24'' - 180^\circ 00' 01.55''$

$= 1.69''$

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Appendix - Least Square Adjustment in other Survey Disciplines

Least square adjustment can be applied in any situation in which more observations have been made than are necessary to determine the required unknowns and in photogrammetry its main use is in the derivation of ground co-ordinates from machine co-ordinates by a similarity transformation, but it can also be used in minor computations such as parallax bar heighting.

Both these operations come under the general heading of curve fitting in which it is desired to represent a series of observations by a particular mathematical formula, the unknowns being the constants in the formula. As an example consider parallax bar heighting using the formula

$$(\text{Ground Height} - \text{Parallax Bar Height}) = a_o + a_1x + a_2y + a_3xy + a_4x^2$$

where x & y are the photo co-ordinates and information is available for 7 control points

Control Point	Photo Co-ordinates x_{mm}	y_{mm}	Parallax Bar Ht	Ground Ht
A	10	85	237.1	237.1
B	71	83	198.6	180.7
C	42	7	174.7	200.7
D	65	-80	250.0	271.8
E	12	-77	246.4	230.0
F	32	41	240.0	265.0
G	50	-17	270.0	292.3

It is obvious that, using these figures, the coefficients of a_3 and a_4 in the observation equations will be very much larger than those of a_o and this effect will be squared in the normal equations resulting in a very unbalanced normal equation matrix. For this reason it is desirable to use different units in order to make all coefficients of approximately the same order and in this case we will work in decimetre units. The observation equations will then be

a_o	a_1	a_2	a_3	a_4	Constant
1	.10	.85	.0850	.0100	0
1	.71	.83	.5893	.5041	+17.9
1	.42	.07	.0294	.1764	-26.0
1	.65	-.80	-.5200	.4225	-21.8
1	.12	-.77	-.0924	.0144	+16.4
1	.32	.41	.1312	.1024	-25.0
1	.50	-.17	-.0850	.2500	-22.3

giving the normal equations

a_o	a_1	a_2	a_3	a_4	Constant
7.000	2.8200	.4200	.1375	1.4798	-60.8000
2.8200	1.4798	.1375	.0896	.8671	-29.5630
.4200	.1375	2.8462	1.1188	.0896	+11.3900
.1375	.0896	1.1188	.6587	.0743	+18.2202
1.4798	.8671	.0896	.0743	.5370	-12.6723

The solution of these equations is

$a_o = -37.2821$ $a_1 = 297.0451$ $a_2 = 19.1957$ $a_3 = -53.5520$ $a_4 = -349.0953$

and these constants are applicable if the photo co-ordinates are measured in decimetres. For working in millimetres, the a_1 and a_2 values should be multiplied by 10^{-2} and the a_3 and a_4 values should be multiplied by 10^{-4}.

Least Square Adjustment is not often used in astronomical work but a simple example would be the determination of the clock correction at any moment in a timed series of astronomical observations. This is a further example of curve fitting, the curve in this case being a straight line and the formula to be used being $C_i = C_o + R(T_i - T_o)$

where C_i is the clock correction at time T_i, C_o is the clock correction at some datum time T_o and R is the amount by which the clock correction changes per unit of time. R and C_o are the unknown quantities which we have to find.

The observation equations will then be

$$v_i = C_o + R(T_i - T_o) - C_i \qquad\qquad i = 1 \text{ to } n$$

and these give the normal equations

$$nC_o + R\left[T_i - T_o\right]_1^n - \left[C_i\right]_1^n = 0$$

$$C_o\left[T_i - T_o\right]_1^n + R\left[(T_i - T_o)^2\right]_1^n - \left[C_i(T_i - T_o)\right]_1^n = 0$$

For hand calculations the arithmetic can be simplified by choosing T_o as the mean value of the observed T_i and if this is done then $\left[T_i - T_o\right]_1^n = 0$ and the normal equations reduce to

$$nC_o - \left[C_i\right]_1^n = 0$$

$$R\left[(T_i - T_o)^2\right]_1^n - \left[C_i(T_i - T_o)\right]_1^n = 0$$

giving C_o and R directly.

As a numerical example consider the following set of observations

Eastern Standard Time			Observed Clock Time			Clock Correction
h	m	s	h	m	s	secs
17	50	28.0	17	49	45.0	+43.0
18	59	31.0	18	58	59.7	+31.3
19	14	08.5	19	13	39.7	+28.8
19	16	08.5	19	15	40.1	+28.4
19	47	31.6	19	47	08.3	+23.3
19	56	30.4	19	56	08.6	+21.8

The calculation is carried out as follows

T_i	C_i	T_i-T_o	$C_i(T_i-T_o)$	$(T_i-T_o)^2$
17.829	+43.0	-1.339	-57.5770	1.7929
18.967	+31.3	- .201	- 6.2913	.0404
19.228	+28.8	+ .060	1.7280	.0036
19.261	+28.4	+ .093	2.6412	.0086
19.786	+23.3	+ .618	14.3994	.3819
19.936	+21.8	+ .768	16.7424	.5898
115.007	176.6		-28.3573	2.8172

$T_o = 115.007 \div 6 = 19.168$

$C_o = 176.6 \div 6 = 29.43$ seconds

$R = -28.3573 \div 2.8172 = -10.066$ second per clock hour

N.B. The simplification of the arithmetic by transferring the origin to the centres of gravity of the systems of known values can be used in all cases of curve fitting but its value is most apparent when the curve is linear as in the above example or in similarity and affine transformations.

B I O G R A P H Y

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