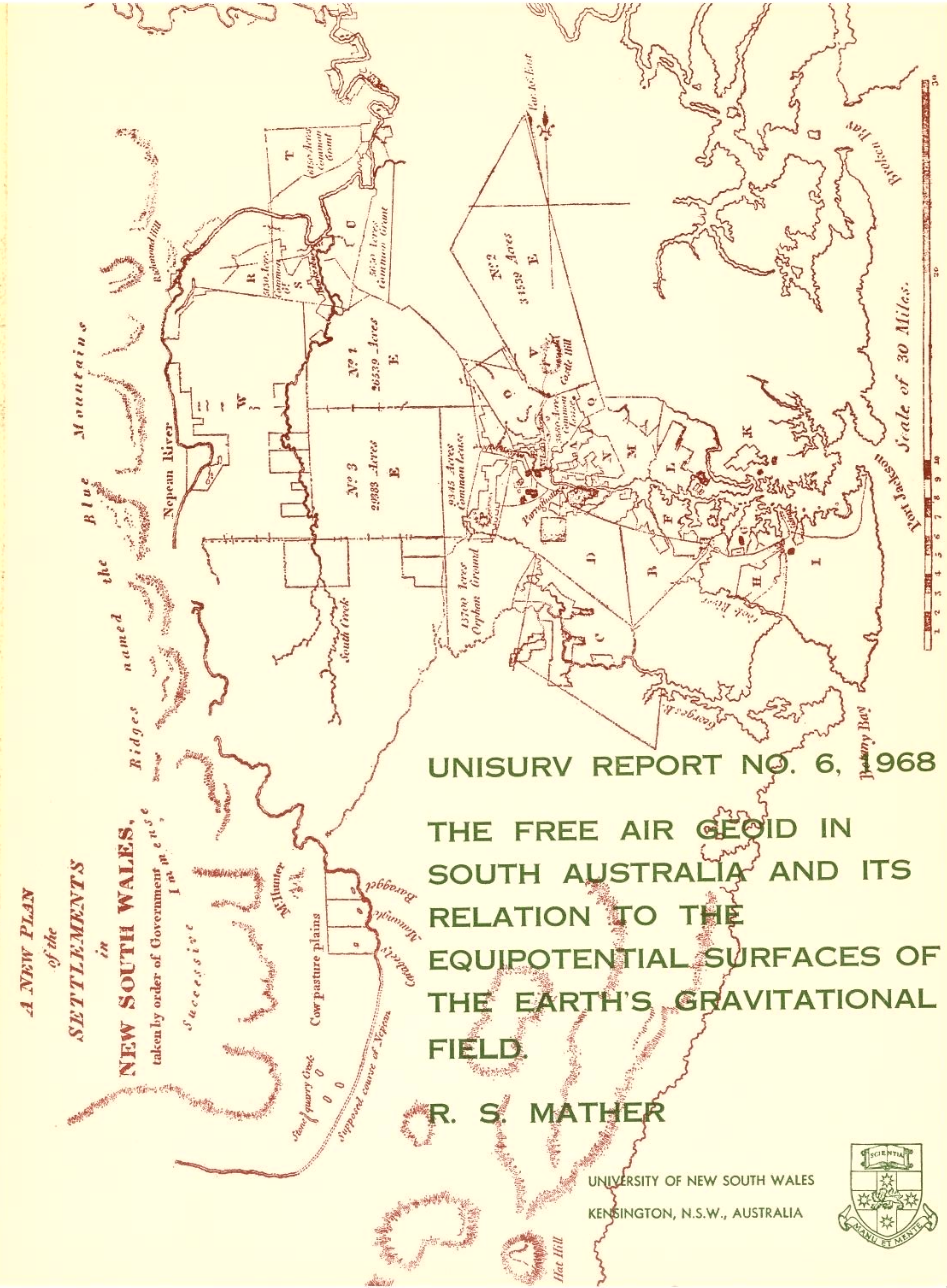


UNISURV REPORT NO. 6, 1968

THE FREE AIR GEOID IN SOUTH AUSTRALIA AND ITS RELATION TO THE EQUIPOTENTIAL SURFACES OF THE EARTH'S GRAVITATIONAL FIELD.

R. S. MATHER

UNIVERSITY OF NEW SOUTH WALES
KENSINGTON, N.S.W., AUSTRALIA

Reference to Districts.

A Northern Boundaries

B Liberty Plains

C Banks Town

D Parramatta

EEEE Ground reserved
for Govt. purposes

F Concord

G Petersham

H Bulanaming

I Sydney

K Hunters Hills

L Eastern Farms

M Field of Mars

N Ponds

O Toongabbey

P Prospect

Q

R Richmond Hill

S Green Hills

T Phillip

U Nelson

V Castle Hill

W Evan

The cover map is a reproduction in part of a map noted as follows:

London: Published by John Booth, Duke Street, Portland Place, July 20th, 1810

Reproduced here by courtesy of The Mitchell Library, Sydney

UNISURV REPORT No 6.

THE FREE AIR GEOID IN SOUTH AUSTRALIA AND ITS RELATION TO THE EQUIPOTENTIAL SURFACES OF THE EARTH'S GRAVITATIONAL FIELD

R.S. Mather

Department of Surveying - The University of New South Wales.
Received :- January 1968.

SUMMARY

The free air geoid is the surface defined by the use of free air anomalies in the Stokes and Vening Meinesz integrals. The relation between this surface and the equipotential surfaces of the earth's gravitational field, with special reference to the geoid, has been developed from basic vector theorems. The full set of conditions which apply to the solution are defined. Expressions are also derived for the complete indirect effect in the case of the free air geoid.

A complete set of relations is obtained for the deflections of the vertical at the geoid, for a postulated model of the topography exterior to the geoid. The relation between these deflections and the equivalent values at the surface of the earth is derived and related to astro-geodetic values.

The validity of the principle of evaluating area means for gravity anomalies from satellite data is discussed and the relevance of the solution to the determination in hand assessed. The assembly of gravity data over the Australian mainland to obtain a network of gravity stations for geodetic calculations is described and an acceptable method developed for the extension of the available gravity field to unsurveyed regions.

The interpretation of the gravimetric solution so obtained is investigated for strict geometrical significance of the solution. The inconsistencies of the international gravity formula are studied and a system, based on

SUMMARY (ctd.)

a set of parameters adopted by the International Astronomical Union, is proposed.

The significance of the zero order term in the geoid spheroid separation is studied and the results analysed for a value for the potential of the existent geoid.

The solution is compared for consistency with astro-geodetic results obtained from horizontal survey methods on the Australian National Spheroid and proposals are made for effecting the absolute orientation of the latter.

The method of astro-geodetic levelling is evaluated as a means for eliminating scale errors in control networks. The implications of the existence of the term $W_o - U_o$ on the solution of the boundary value problem at the physical surface of the earth are studied with a view to obtaining solutions which are geometrically exact.

PREFACE

The geoid which has been computed and shown on pages 339 to 341 is based on a theory which stems from the assumption that equation (5.21) on page 67 is valid for the non-regularised earth. The result was the derivation of an expression for the geoid spheroid separation given by equation (5.54) on page 76. The resulting indirect effect, shown on page 333, was quite large. As discussed in section (14.3), the total indirect effect for near zones tended to be zero but that arising from distant zones had a cumulative effect and contributed to the magnitude of N_{ip}.

The geoid-spheroid separation was then computed on the basis that the I.A.U. spheroid had the same volume as the geoid when the results shown on pages 339 - 341 were obtained. If this were so, the free air geoid is no longer a good first approximation to the geoid itself as is commonly held. Further, an anomaly did exist in that the solution for the physical surface of the earth, for which the free air geoid provided an adequate first approximation, did not coincide with the geoid spheroid separation obtained by the use of a modified form of equation (5.21) on carrying out comparisons at sea where the two surfaces should be identical. This was tentatively explained as being due to the fact that the matter on the reference system exterior to the surface being mapped had not been taken into account in the derivation of the equations.

Preface (ctd)

In this work, piecemeal attempts have been made at attempting to improve the solution by

(a) allowing for non-equality of W_o and U_o (sec. (13.5) ; 306 - 317) ;

(b) considering the existent of matter on the reference system exterior to the surface being mapped (App. 18).

Subsequent investiagtions show that neither of these effects have been fully considered in this report and, for a complete solution of the problem, interested readers are referred to UNISURV Rep. No 11 titled " THE NON-REGULARISED GEOID AND ITS RELATION TO THE TELLUROID AND REGULARISED GEOIDS." in which the errors in equation (5.21) are investigated and a complete solution set out for the non-regularised geoid as well as for the regularised geoid and the physical surface of the earth.

This latter investigation shows that the free air geoid is, indeed, a good approximation to the non-regularised geoid and hence figs (14.3) to (14.5) should be taken as being' fairly representative of the non-regularised geoid. Figs (14.6) to (14.11) should be ignored.

In attempting to use this report, the reader is advised to consider sections (4), (5), (13.6), (14.4) and App. (18) as superseded by and subsidiary to UNISURV Rep. 11 which sets out the complete solution free from error.

Preface (conclusion)

The value assigned to the potential of the geoid on page 338 should be ignored as should the value of N at the Johnston Origin (page 369). The latter, according to fig (14.4) is approximately 10 metres. To this value will have to be added a topographical correction which should not exceed 2 metres. The conclusions drawn in section (15.5) with the exception of those concerning astro-geodetic levelling are based on incbmplete derivations and should be considered to be superseded by the conclusions embodied in section (7) of UNISURV Rep 11. and this preface.

No attempt has been made to wrrite the now superseded portions of the report and the related conclusions out of the text as it illustrates with telling clarity the danger of accepting conventional assumpti ons without very careful invetsigations into the circumstances under which they hold.

My thanks are due to Dr. W.M.Kaula who persisted with his reservations regarding the nature of the indirect effect, prompting me to re-investigate the solution.

March 15th, 1968,
Sydney,
Australia

CORRIGENDA

This series of corrections sets out only those amendments which are not obvious in reading the text and does not include typing errors. I am grateful for the assistance given by Dr. P.V.Angus-Leppan and Mr. J.G.Fryer in compiling this list of errors.

Page No.	Correction
2	Equation (1.1) :- Precede right hand side by symbol for volume integration.
2	On line following equation (1), insert "ρ" between "constant" and "the".
5	On penultimate line replace "1966" by "1967".
8	Section should be numbered "1.4".
8	Insert "free" between "were" and "from" in the third line of section (1.4).
13	Insert a bracket around the term "$-\frac{x_i}{r^3}$" in penultimate line.
26	5th line the number should be "1000" and not "10,000".
28	6th last line replace "202" by "204".
35	Fig (3.1):- The distance P_{1_D} should read "h_D".
36	Fifth last line :- For "Q" read "$P_{R'}$".
38	3rd last line :- Replace "149" by "148".
46	2nd line :- Replace "46" by "52".
54	7th line :- Delete "a" in the reference.
61	Delete "γ" from the term multiplying the surface integral in equation (5.2).

Corrigenda(Page 2)

Page No	Correction
63	Last equation :- Denominator should be "R^2".
76	Equation (5.55) :- the L.H.S. should be "Δ g".
35	In fig (3.2) :- "g_i" should read "ζ_i".
54	Equation (4.20):- the quantity "$\partial \gamma/\partial h$" should be unsigned.
58	Equation (4.30):- The first sign in the integral should be "+".
72	Equation (5.41):- Insert "$\frac{2n+1}{n-1}$" between summation and integral signs.
79	Equation (6.1) :- Last term on R.H.S. should be multiplied by h.
87	Equation (6.29):- A set of brackets to be inserted after "$\frac{1}{2}\text{cosec}^2\frac{1}{2}\psi$" and terminating after "$A+c_1$".
106	In all three equations insert "ρ_{ij}" after Σ_j.
125.	The constant introduced in equation (7.15) and subsequent development should be "n(n+1)!".
126	Equation (7.19):- The last term is "$\{(n+1)n \sin^2\theta - m^2\}G(\theta)$"
127	Equation (7.23):- Insert "μ" between "2" and "$\partial G/\partial\mu$".
141	Line 7:- after "necessary to" add " set out".
142	Line3:- Reference should read "1963a".
144	Fig(7.5) :- Insert "γ" before " = equinox". Also arc length NP = ω.
157	In range of values given the magnitude of numbers overule signs in case of inconsistency.
159	In (a)(i) :- 5th line :- The relevant quantity in two places is "$\delta \Delta g$".

Corrigenda(Page 3)

Page No.	Correction
204	The last word on this page should be "differentiation".
212	Line 13 :- after "as" add " ψ " .
217	2nd para, 3rd line :- replace "$Z'_G P Z_G$" by " $Z_A P Z_G$ " .
228	Last line :- after "at" add the words "the geoid by the relation".
254	Numerical value of $C_{N\xi}$:- The index of 10 is "-1" and not "-2".
285	Insert "give variations" at the commencement of the page.
286	6th line:- Replace " ξ_i " and " η_i " by " $\Delta\xi_i$ " and " $\Delta\eta_i$ ".
300	12th line :- Replace "three" by "two".
338	Insert " ξ" after "N" and, on the second line after equation (14.5). Also insert "η " after " and" on this same line.
349	In penultimate line of second para insert "$\Delta\xi_o$ " after "magnitude of".

TABLE OF CONTENTS

Table of contents(2)

Table of contents(3)

Table of contents (5)

Table of Contents (6)

Table of contents (7)

NOTATION

Certain symbols, used repeatedly throughout this thesis, are, in addition to definition in context, described below for ease of reference.

A = azimuth, measured from north, through east.

A_{nm} = coefficient associated with the general surface harmonic for potential.

a = semi-major axis of meridian ellipse.

a_{nm} = coefficient associated with the general surface harmonic for disturbing potential.

C = a constant.

C_{nm} = coefficient of conventional Legendre functions.

c = a constant.

$E\{X\}$ = predicted value of X .

e = eccentricity of ellipse.

f = flattening of the reference spheroid.

g = the acceleration of gravity due to the existent earth.

g_{nm} = coefficient associated with the general surface harmonic for the gravity anomaly.

h = elevation.

h_D = height anomaly.

$\underline{i}$ = unit vector along the x_i axis.

i = inclination of the orbital plane to the plane of the earth's equator.

Notation(2)

J_{nm} = alternate form for the coefficients of conventional Legendre functions.

K_{nm} = as above.

k = the gravitational constant = 6.673×10^{-8} cm^3 $gm^{-1} sec^{-2}$.

M = mass of the earth.

$M\{X\}$ = the mean value of X.

N = separation of geoid and spheroid.

$\underline{N}$ = unit normal vector.

$N\{X\}$ = normalised value of X.

n = length of side of basic square in degrees.

$p_n(\mu)$ = Legendre function.

$p_{nm}(\mu)$ = associated Legendre function.

$P_{nm}(\mu)$ = $\frac{n!}{(n-m)!}\ p_{nm}(\mu)$.

R = distance from centre of mass.

R = reference surface.

R_m = mean radius of the earth.

r = distance of variable element from computation point.

r_o = radius of inner zone.

T = existent physical surface.

S_{nm} = coefficient of the conventional Legendre function.

$S_{nm}(\phi_c, \lambda_c)$ = surface harmonic.

U = gravitational potential due to a reference model of the earth.

V = volume.

W = gravitational potential due to the existent earth.

Notation(3)

z	=	variable height.
α	=	bearing, measured clockwise from north.
β	=	angle of elevation of ground slope.
γ	=	gravity on the reference system ; normal gravity.
δ_{ik}	=	Kronecker delta.
ζ	=	angle between the normal to the reference surface and that to the existent surface, + ve N, E.
η	=	component of the deflection of the vertical in in the Prime Vertical.
θ	=	co-latitude.
Θ	=	Greenwich Sidereal Time (sec. (7)).
λ	=	longitude, positive east.
μ	=	cos θ.
ν	=	radius of curvature of a spheroid in the Prime Vertical.
ξ	=	component of the deflection of the vertical in the meridian.
π	=	3.1415926...
ρ	=	density of matter in the earth's crust.
ρ	=	radius of curvature on a spheroid in the meridian.
σ	=	standard deviation of a sample.
ϕ	=	latitude.
ϕ	=	a scalar, usually potential (sec (2)).
ψ	=	angular displacement between the vectors from the origin of coordinates to the computational element and the computation point.

Notation(4)

Ω	=	right ascension of the ascending node.
ω	=	angular velocity of rotation of the earth.
ω	=	argument of perigee (sec (7)).
Δg	=	gravity anomaly.
ΔN	=	correction to the geoid spheroid separation on changing orientation of reference spheroid. at the origin of computation.
$\Delta \xi$	=	correction to the meridian component of the deflection for the above reason.
$\Delta \eta$	=	correction to the prime vertical component of the deflection of the above reason.
dA	=	element of surface area.
dg_P	=	correction to the Potsdam datum.
dm	=	element of mass.
dR	=	element of area on the reference surface.
dS	=	element of surface area.
dT	=	element of surface area on the existent surface.
$d\gamma$	=	correction to normal gravity for changes in the parameters of the reference spheroid.
$d\sigma$	=	element of surface area on a unit sphere.
$\prod_i$	=	product over the index i
$\sum_i$	=	summation over the index i.
$\underline{X}$	=	the vector X
$\underline{X} . \underline{Y}$	=	the scalar product of the vectors X and Y.
$\underline{X} \times \underline{Y}$	=	the vector product of the vectors X and Y.
$\overline{X}$	=	the normalised value of X .
$\overline{X}$	=	the mean value of X.

Notation(5)

$$\underline{\nabla} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \underline{i}$$

$$\underline{\nabla}^2 = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2}$$

Suffixes.

s = refers to spheroid or associated spherop. E.g., h_s is the elevation with respect to the spheroid.

P = values at the computation point.

Q = values at the variable general point.

A = astro-geodetic values.

G = gravimetric values.

o = values at the existent geoid; values at origin of triangulation spheroid.

F = refer to free air anomalies, e.g., the free air geoid.

m = mean value.

I = refers to the indirect effect.

i = values obtained from a set of masses internal to a given surface.

e = values obtained from a set of masses external to a given surface.

D = the " disturbing " value, due to the difference between values on the reference and existent systems.

c = correction to free air geoid to obtain values on the spherop.

co = differential topographical correction.

1. THE EQUIPOTENTIAL SURFACES OF THE EARTH'S GRAVITATIONAL FIELD AND THE REFERENCE SYSTEM.

1.1 Introduction.

Newtonian gravitation explained precession as being due to a flattened earth. This was initially verified by the French Academy of Sciences (Heiskanen and Vening Meinesz, 1958, 227) which sent out expeditions under Bouguer and la Condamine to Peru in 1735 and under Maupertius to Lapland in 1736 to measure the length of meridional arcs. These investigations established the shape of the earth as being spheroidal in character with a flattening of 1 part in 310.

The physical surface of the earth has deviations from the spheroidal shape which are of the order of the flattening f, i.e. i.e., 3 parts in 10^3. This is due to the topography of the earth, which, over the continental areas, departs considerably from a simple mathematical model. For the purpose of the conventional representation of position, geodesists have introduced the concept of a regular reference surface which conforms closely to the earth's surface over the ocean areas. The surface adopted (Bomford, 1962, 83) is the oblate spheroid, commonly called the reference spheroid. Ideally, the latter is centred at the earth's centre of mass, with its minor axis coincident with the earth's rotational axis. So far, the definition of a mathematical model with reference to physical realities presents no great difficulties. This poses the question "Is there a physical reality which is spheroidal in shape ?". The extent to which such agreement occurs is also of relevance.

The only physical reality which approaches a spheroidal shape is an equipotential surface of the earth's gravitational field. Such a surface, as will be shown later, has variations from an oblate spheroidal shape of the order of f^2 (1 in 10^5). The equipotential surface commonly adopted as a reference surface is that corresponding to mean sea level (M.S.L.), as this forms the datum for elevations (at least in theory, if not in practice).

The gravitational potential (V_P) at a point P in space is given by

$$V_P = \frac{k \rho \, dv}{r} \qquad \text{...........(1.1)}$$

where k is the gravitational constant, the density of matter situated in the element of volume dv and r is the distance from dv to P. Thus the shape of the equipotential surface

$$V = C \qquad \text{..........(1.2)}$$

where C is a constant, is extremely susceptible to changes in shape due to local density anomalies. These anomalies could cause either localised effects, with regional characteristics extending up to 60 miles from a localised centre, or give rise to regional effects which could extend over continental areas.

Conventionally, the parameters of the reference spheroid are chosen (Heiskanen and Vening Meinesz, 1958, 42) so that the latter has the same mass as the existent earth and the same volume and flattening as the geoid. Hence it appears predictable that the geoid rises above the spheroid over continental areas. This, however, is not necessarily so (Veis, 1965) as seen from a study of provisional "free air" geoids, determined from the analyses of the orbital perturbations of artificial earth satellites.

These studies seem to indicate that the chief factors influencing the shape of the geoid arise from deep seated density anomalies and not from the effect of topography alone. Thus, it can be concluded that the fluctuations of the geoid with respect to a reference spheroid are generally unpredictable and bear no relation to topographical features alone. The latter, however, do influence the shape of the geoid through the higher harmonic terms.

1.2 Mapping the geoid

Geoidal mapping can be carried out using two different techniques. The first method which has been quite commonly used , is known as astro-geodetic levelling (e.g., Fischer, 1966a). This method uses expressions for the deflections of the vertical (Bomford, 1962, 89) in comparing the positional parameters on the reference spheroid, established by horizontal methods, with the equivalent quantities determined astronomically. The quantities evaluated using this technique are "surface" deflections of the vertical (ζ_A in fig.(1.1)), being the angle between the normal to the surface equipotential (or geop) and the spheroidal normal. The true deflection of the vertical(ζ_0) is that between the normal to the geoid and the spheroidal normal. The difference ($\zeta_A - \zeta_0$) varies from very small values in relatively flat areas to larger ones in more rugged topography. The accuracy of astro-geodetic levelling is also dependent on the orientation, in space,

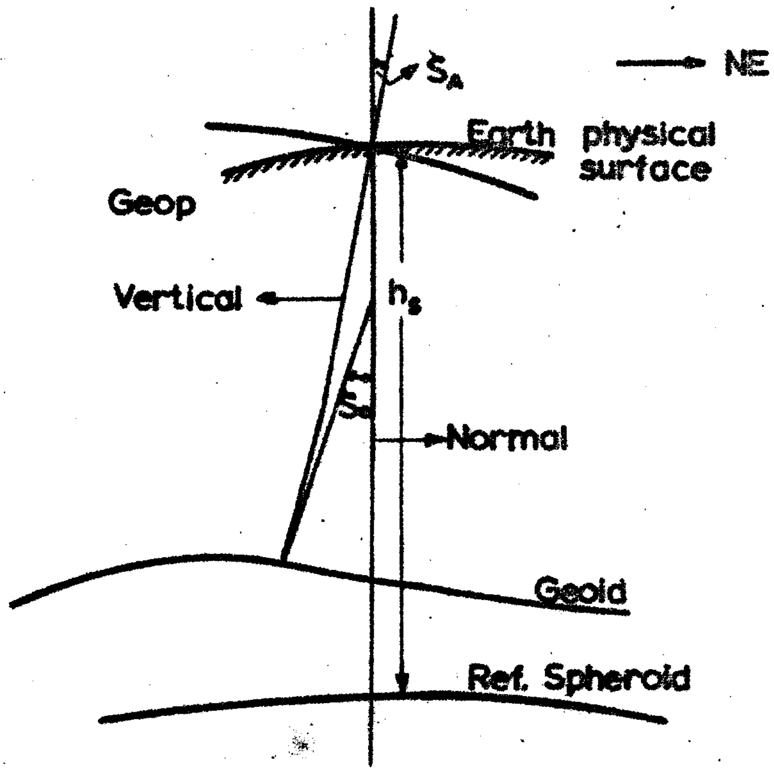


FIG. 1.1

THE REFERENCE SPHEROID AND THE EQUIPOTENTIAL SURFACES OF THE EARTH'S GRAVITATIONAL FIELD.

of the reference spheroid on which the horizontal geodetic network is calculated. Hence, any results obtained by the use of this method, cannot be considered to be absolute.

The second method is the gravimetric one in which the problem reduces to one of determining the vector of separation between equivalent points on the geoid and the spheroid of reference. The basic approach is one of vector algebra and the results are expressed as a combination of changes in the direction cosines of normals and linear normal displacements. These results, together with the relevant astronomical observations, will, in theory, give an absolute solution to the problem. This method was restricted in scope up to the present decade due to insufficient coverage of the gravity field over ocean areas.

Recent developments in the study of the perturbations of the orbits of artificial earth satellites have enables values to be established for the low order harmonics of the earth's gravitational field and these, in turn, have been used to obtain a representation of the gravity anomalies over ocean areas. Gravimetric investigations, in the past, have always hinged on the use of Stokes' theorem (Stokes, 1849, 693-4), which was derived on the assumption that there was no matter "exterior to the geoid". This has resulted in most subsequent researches into the problem being preoccupied with methods for "smoothing" or "removing" the topography of the earth exterior to the geoid. In fact, the first book, fully devoted to the subject of physical geodesy (Heiskanen and Moritz, 1966), develops the subject from the point of view of a regularised earth.

Such treatment of the matter exterior to the geoid causes changes in the position and the shape of the equipotentisl surfaces of the earth's gravitational field and resulted in a new bounding equipotential, called a co-geoid (Hunter, 1935, 378) , which was exterior to all matter. The solutions so obtained were complex and, due to numerous side effects being neglected, somewhat lacking in accuracy. In general, the use of Stokes' integral gave the separation between the co-geoid and the reference spheroid. The contribution of the correction terms, arising from the "mass transfers", to this co-geoid spheroid separation was called the indirect effect.

Molodenskii (1962) was the first geodesist to attempt the development of a solution unencumbered by such preconcieved ideas and Russian geodesy in the nineteen thirties and the nineteen forties did a lot to spearhead the revision of methods for solving this problem. This led to other European geodesists (e.g., Hunter, 1960; Hirvonen, 1960) having fresh ideas about the method of solution, which, however, in the writer's opinion, are restricted in scope due to the introduction of certain limiting concepts. These included "doing away with the geoid". This solution, however, is not widely acceptable to the geodetic community as the geoid still is the most clearly understood datum surface which approximates very closely to the accepted geodetic reference figure - the spheroid.

In recent years, more careful studies have been made into the density of the earth's crust (Hunter, 1966) and the clear analysis of Moritz (1965) has produced the following complete theoretical development which provides a solution of the gravimetric problem, correct to the order of the flattening. The extension of the solution to the order of the square of the flattening (1 part in 10^5) can be effected by the extension of the spherical considerations to spheroidal ones. The lack of precision in the gravity data available at present makes this tedious development a theoretical exercise of no practical significance.

1.3 The "Free air geoid".

The term "free air" geoid (e.g., Kaula, 1959, 108 et seq.) is applied by geodesists analysing satellite orbital perturbations, to the co-geoid they map when using free air anomalies, calculated from solutions for the low order harmonics of the earth's gravitational field, in Stokes' integral. The free air co-geoidal height (N_f) is given (Heiskanen and Vening Meinesz, 1958, 65) by

$$N_f = \frac{R_m}{4\pi\gamma_m} \int_0^{\sigma=4\pi} f(\psi)\ \Delta g_f\ d\sigma \quad \ldots.(1.3)$$

where Δg_f is the free air anomaly (Heiskanen and Vening Meinesz, 1958, 148), $f(\psi)$ is Stokes' function (see equation (5.53), ψ is the angular displacement of the computational element of surface area $d\sigma$ on a unit sphere, to the computation point, R_m is the mean radius of the earth and γ_m is the mean value of gravity over a spheroidal model of the earth.

The separation (N) between the geoid itself and the reference spheroid is given by

$$N = N_f + N_I \quad \text{.................... (1.4)}$$

where N_I is the indirect effect.

1.5 The definition of the geoid in continental areas.

The geoid admits readily to definition over ocean areas, being the free surface of the ocean itself, if the latter were from tidal, current and wave motion effects. This coincidence with a physical reality is not available over continental areas as the geoid is located below the existent surface of the earth. Further, as potential is a scalar quantity given by equation (1.1), it is extremely susceptible to local mass anomalies. Hence the shape of the geoid is dependent on the stratification of matter in the earth's crust. Consequently, an uncertainty will always exist in any solution for the geoid unless the internal density gradient is known.

A unique solution is, however, possible for the geoid if a definite model is adopted for the topography above a certain depth (to be specified later). Conversely, no geoid can be defined unless a reasonable model is adopted for the earth's topography. The search for such a model has, to date, been rather a fruitless one and, from a geodetic point of view, is a field in which sufficient research has not been done. The analysis of seismograph records, if carried out systematically, should provide valuable information. The Mohole project is certainly likely to give good first approximations, which will, however, prove useful only if the existence of a definite relation between surface topography and density gradient is established. Variations in the density of surface topography can be established by the Nettleton technique

(Nettleton, 1939), which is commonly adopted in geophysical prospecting to obtain the best possible density for use in computing the Bouguer reduction. The sampling techniques used in practice, however, cover only limited extents and can only be considered representative of surface sedimentary rocks. In Australia, this method gives variations in the value of the density (ρ) determined in the range

$$1.9 \leq \rho \leq 2.7 \text{ gm. cm}^{-3}$$

Hunter (1966) suggests certain relationships which may be of relevance and may very well replace the current practice of adopting the uniform value of 2.67. The two formulae suggested by him, based on 5000 samples from 47 regions of the world, are

$$\rho = 2.75 - h/18 \quad \ldots\ldots\ldots\ldots\ldots (1.5)$$

for $-4500 < h \text{ (metres)} < 1500$, and

$$\rho = 2.77 - h/21 \quad \ldots\ldots\ldots\ldots\ldots (1.6)$$

for $100 < h \text{ (metres)} < 2100$.

The value of h used in equations (1.5) and (1.6) is the elevation of the area in km. These formulae may very well overestimate the density of the topography in coastal regions where the density of sedimentary rocks is likely to be smaller. It is estimated that the error in the Bouguer reduction due to the density uncertainties in these regions is unlikely to exceed 4 per cent giving errors in the anomaly which should not exceed 1/4 mgal. This effect is much smaller than the uncertainty in the elevations of many gravity stations. On the other hand, the effect on rounding off errors tends to be systematic and may well give rise to a small systematic error in the final result.

1.5 The South Australian investigation

South Australia lies between the meridians 129°E and 141°E and the parallels 26°S and 40°S. Its main feature of interest is the satellite tracking station complex located at Woomera. Tentative solutions have been made for the absolute position of the former (Veis, 1965) based on satellite orbital analysis. The relative undulations of surface equipotentials have also been mapped (Bomford, 1963) in the tracking station area. It is of considerable significance to determine the relation between the free air geoid and the equipotential surfaces of the earth's gravitational field in this area as it will enable a more precise determination of the absolute location of the tracking stations to be effected. In this manner, the positions of the tracking stations could be fixed absolutely on a three dimensional cartesian system of coordinates and thus not be dependent on the orientation of the triangulation spheroid. This would, in turn, enable greater precision in satellite tracking.

2. THE FUNDAMENTAL THEOREM IN PHYSICAL GEODESY.

2.1 Introductory vector theorems.

All derivations in the theory of gravitational fields stem from the divergence theorem also known as Green's Lemma, which states (Jeffreys, 1962b, 193) that

$$\iiint_V \underline{\nabla} \cdot \underline{F} = \iint_S \underline{F} \cdot \underline{N} \; dS \qquad \ldots\ldots\ldots\ldots\ldots (2.1)$$

where $\underline{F}$ is any vector whose divergence exists everywhere within the volume V, and $\underline{N}$ is the unit normal vector, reckoned positive outward. The other quantities in equation (2.1) are defined by the following equations.

$$\underline{\nabla} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \underline{i} \qquad \ldots\ldots\ldots\ldots\ldots (2.2)$$

where $\underline{i}$ is the unit vector along the x_i axis.

$$\underline{F} = \sum_{i=1}^{3} F_i \underline{i} \qquad \ldots\ldots\ldots\ldots\ldots (2.3)$$

$$\underline{\nabla} \cdot \underline{F} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} F_i \qquad \ldots\ldots\ldots\ldots\ldots (2.4)$$

It should be noted that there is no objection to the volume V being bounded by two surfaces. Thus , a certain volume which has both an inner and outer surface, both of which are designated as the surface S, can be considered as one which the divergence theorem applies. When evaluating $\underline{N}$ in such a case, the outward normal is taken as positive in a direction away from the enclosed volume.

The divergence theorem is capable of direct extension to Green's theorem which considers 2 scalars (U_i i=1, 2) whose Laplace operators ($\underline{\nabla}^2 U_i$ i=1, 2) exist everywhere within a volume V, when

$$\iiint_V \left[U_1 \underline{\nabla}^2 U_2 - U_2 \underline{\nabla}^2 U_1 \right] dV = \iint_S \left[U_1 \underline{\nabla}\cdot\underline{N}\, U_2 - U_2 \underline{\nabla}\cdot\underline{N}\, U_1 \right] dS \quad \ldots\ldots\ldots\ldots(2.5)$$

where

$$\underline{\nabla}^2 = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \underline{i} \cdot \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \underline{i} = \sum_{i=1}^{3} \frac{\partial^2}{\partial x_i^2} \quad \ldots\ldots(2.6).$$

Equation (2.5) holds when $U_1 \underline{\nabla}^2 U_2$ and $U_2 \underline{\nabla}^2 U_1$ exist everywhere within the region V bounded by the surface S. The quantities $U_1 \underline{\nabla}\cdot\underline{N}\, U_2$ and $U_2 \underline{\nabla}\cdot\underline{N}\, U_1$ must also exist at every point on the surface S. As in the case of the divergence theorem, the bounding surface can be a continuous one or it can be in two parts, e.g., the volume contained between two nearly concentric spheres.

2.2 Laplace's theorem and its application to gravitational potential.

Consider the scalar 1/r, where r is the distance of the variable element Q from a fixed point P. Three possibilities exist.

(i) If P is exterior to the surface S which encloses the volume V in which all the points Q_i (i=1, n) lie, then 1/r exists for all points Q_i (i=1, n) in V and on the surface S.

(ii) If P lies on S and coincides with one of the Q_i, there exists one point on the surface at which 1/r is indeterminate and an infinitesimal region in V adjoining this point at which 1/r is unstable.

(iii) If P lies within the region V, the surface integral of 1/r is evaluable, but there may exist one point in V at which 1/r is indeterminate.

In case (i),

$$\underline{\nabla} \frac{1}{r} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \frac{1}{r} \underline{i}$$

where

$$r = \left[\sum_{i=1}^{3} x_i^2 \right]^{\frac{1}{2}}$$

Thus

$$\underline{\nabla} \frac{1}{r} = \sum_{i=1}^{3} - \frac{x_i}{r^3} \underline{i} \quad \ldots\ldots\ldots(2.7)$$

and $$\underline{\nabla}^2 \frac{1}{r} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} - \frac{x_i}{r^3} = \sum_{i=1}^{3} \frac{3x_i^2}{r^5} - \sum_{i=1}^{3} \frac{1}{r^3}$$

$$= \frac{3r^2}{r^5} - \frac{3}{r^3} = 0$$

1/r is said to satisy Laplace's theorem (Jeffreys, 1962b, 199)

$$\underline{\nabla}^2 \phi = 0 \quad \ldots\ldots\ldots\ldots(2.8)$$

where ϕ is a scalar.

If Q_i (i=1,n) are points with coordinates ξ_{ij} (j=1,3 i=1,n) on a three dimensional system and $r_i = PQ_i$ (i=1,n), where P is (x_j j=1,3), then

$$\underline{\nabla}^2 \frac{1}{r} = 0$$

at all points except $\xi_{ij} = x_j$

$$\text{where} \quad r_i = \sum_{j=1}^{3} (\xi_{ij} - x_j)^2 = 0$$

If a_i (i=1,n) are constants associated with the points Q_i (i=1,n), consideration of each of the Q_i s, in turn, gives

$$\underline{\nabla}^2 \frac{a_i}{r_i} = 0 \quad i=1,n \quad \ldots\ldots\ldots(2.9)$$

As each of the Q_i s are independent points,

$$\sum_{i=1}^{n} \underline{\nabla}^2 \frac{a_i}{r_i} = \underline{\nabla}^2 \sum_{i=1}^{n} \frac{a_i}{r_i} = 0 \quad \ldots\ldots(2.9)$$

This relation holds at any of the points Q_i which do not coincide with the origin of length r (i.e., P). Thus, if equation (2.9) is to represent gravitational potential, it can only do so at a point not occupied by matter. However, if the composite term a/r tends to zero as r tends to zero, or to any finite limit, it is possible to assign a value to the function defined in equation (2.9).

Conclusion:- Gravitational potential at any point not occupied by matter satisfies Laplace's equation.

2.3 Gauss' theorem.

Given a surface S which encloses a volume V (fig (2.2)). Consider the vector $\underline{F}$, given by

$$\underline{F} = \frac{\underline{R}}{r^3} \qquad \text{where}$$

$$\underline{R} = \sum_{i=1}^{3} x_i \underline{i} \quad .$$ Applying divergence

theorem (equation (2.1)) to $\underline{F}$,

$$\iiint_V \left[\frac{1}{r^3} \underline{\nabla} \cdot \underline{R} + \underline{R} \cdot \underline{\nabla} \frac{1}{r^3} \right] dV = \iint_S \underline{F} \cdot \underline{N} \, dS$$

If $r \neq 0$,

$$\iiint_V \left[\frac{3}{r^3} + \sum_{i=1}^{3} \left(-\frac{3x_i^2}{r^5} \right) \right] dV = 0 = \iint_S \underline{F} \cdot \underline{N} \, dS$$

Thus $$\iint_S \frac{\underline{R}}{r^3} \cdot \underline{N} \, dS = 0 \quad \ldots\ldots\ldots\ldots (2.10)$$

If $r = 0$ at a point within the surface S, exclude this point by

(i) a sphere of radius ε which tends to 0, if P is within S ;

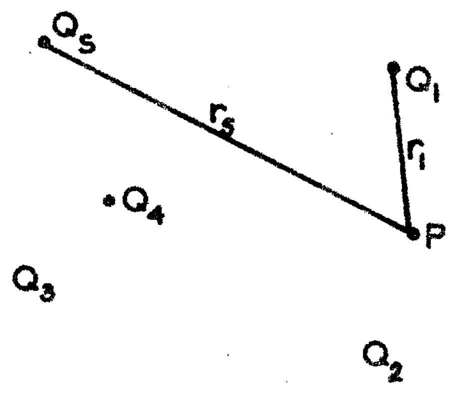


FIG. 2.1

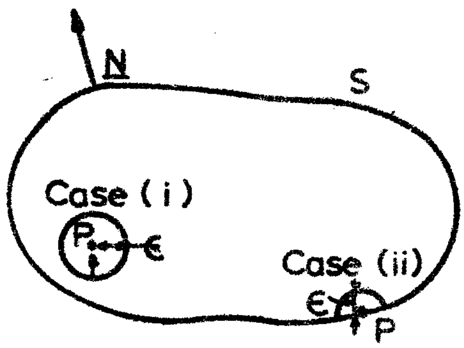


FIG. 2.2

(ii) a hemisphere of radius ε , which tends to 0, if Pis on the surface S.

In such cases, treat V as being enclosed by two surfaces S and S', where the latter is the surface of a sphere/ hemisphere of radius ε , centred on P, which is situated within / on S.

The application of equation (2.1) gives

$$\iiint_V \underline{\nabla} . \underline{F} \quad dV = \iint_S \underline{F} . \underline{N} \quad dS + \iint_{S'} \underline{F} . \underline{N}' \, dS'$$

where $\underline{N}'$ is the unit outward normal to the surface S'. From equation (2.10),

$$0 = \iint_S \underline{F} . \underline{N} \quad dS + \iint_{S'} \underline{F} . \underline{N}' \, dS' .$$

In the case of S' being a sphere, the direction cosines of the surface normal are the direction cosines of $\underline{R}$. Thus,

$$\iint_{S'} \underline{F} . \underline{N}' \, dS' = - \iint_{S'} \frac{\underline{R}}{\varepsilon^3} . \frac{\underline{R}}{\varepsilon} \quad dS' = - \iint_{S'} \frac{\varepsilon^2}{\varepsilon^4} \, dS'$$

$$= - \frac{1}{\varepsilon^2} \iint_{S'} dS'$$

Thus $$\iint_{S'} \underline{F} . \underline{N}' \quad dS' = - \begin{cases} 4\pi & \text{when S' is a sphere} \\ 2\pi & \text{when S' is a hemisphere.} \end{cases}$$

Substitution of the above result in the earlier equation gives

$$\iint_S \frac{\underline{R}}{r^3} \cdot \underline{N}\, dS = \begin{cases} 0 & \text{if P is outside the surface S} \\ 2\pi & \text{if P is on the surface S} \\ 4\pi & \text{if P lies within the surface S} \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.11).$$

It should be noted that the following results also hold.

$$\iint_S \sum_{i=1}^{n} \frac{a_i}{r_i^3}\, \underline{R}_i \cdot \underline{N}\, dS = \begin{cases} 0 & \text{if P is outside S} \\ 2\pi \sum_{i=1}^{n} a_i & \text{if the } Q_i\text{'s are on S} \\ 4\pi \sum_{i=1}^{n} a_i & \text{if the } Q_i\text{'s lie within S} \end{cases} \qquad \ldots\ldots\ldots\ldots(2.12),$$

where a is a constant associated with r at a point Q within S, the magnitude of the constant being a function of Q.

2.4 The function $\underline{\nabla}^2 \phi$ in regions occupied by matter.

Let the gravitational potential (ϕ) at a point P be given by

$$\phi = \sum_{i=1}^{n} \frac{a_i}{r_i} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.13),$$

where a_i is the relevant constant associated with the gravitating point $Q_i (i=1, n)$ at a distance r_i from the point P. The study of equation (2.7) gives

$$\underline{\nabla}\phi = \underline{\nabla}\sum_{i=1}^{n}\frac{a_i}{r_i} = \sum_{i=1}^{n}\sum_{j=1}^{3} - \frac{x_{ij}a_i}{r_i}\,\underline{i}$$

$$= - \sum_{i=1}^{n}\frac{a_i\underline{R}_i}{r_i^3} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.14).$$

This is the general expression for the gravitational force due to a general distribution of matter, without any restriction, except that no matter exists at P. If matter exists at P, the expression will still hold if the ratio a/r associated with P tends to zero as r tends to zero.

Consider the integral

$$I = \iint_S \underline{N}.\underline{\nabla}\phi\, dS = -\iint_S \sum_{i=1}^{n}\frac{a_i\underline{N}\cdot\underline{R}_i}{r_i^3}\, dS\ldots\ldots$$

The evaluation of this integral using equation (2.11) gives

$$\iint_S \underline{N}\cdot\underline{\nabla}\phi\, dS = -\begin{cases} 0 & \text{if P lies outside S} \\ 2\pi\sum_{i=1}^{n} a_i & \text{if the } Q_i\text{'s lie on S} \\ 4\pi\sum_{i=1}^{n} a_i & \text{if the } Q_i\text{'s lie within S} \end{cases}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots(2.15).$$

The second and third conditions apply when P lies within or on the bounding surface S and coincides with one of the gravitating points Q_i (i=1, n). The bounding surface S is taken to be exterior to all matter. If the point P does not coincide with a Q_i, the first condition holds irrespective of the location of P.

If, on the other hand, certain of the Q_i were situated external to S, but within another surface S' which was exterior to all matter. Consider those Q_i $(i=1, m)$ situated between S and S'. If P did not coincide with any of the above Q_i, where $m < n$, and all the Q_i $(i=1, m)$ are included in the set Q_i $(i=1, n)$, the condition

$$\iiint_{V(S,S')} \sum_{i=1}^{m} a_i \underline{\nabla} \cdot \frac{\underline{R}_i}{r_i^3} dV = 0$$

is satisfied. See fig (2.3). When P coincides with one of the points Q_i, exclude P from the volume considered between the two surfaces S and S', as before, and if $\underline{N}'$ is the unit outward normal to the surface S, the use of equation (2.1) gives

$$\iiint_{V(S,S')} \sum_{i=1}^{m} a_i \underline{\nabla} \cdot \frac{\underline{R}_i}{r_i^3} dV = \iint_S \sum_{i=1}^{m} a_i \underline{N}' \cdot \frac{\underline{R}_i}{r_i^3} dS + \ldots$$

$$+ \iint_{S'} \sum_{i=1}^{m} a_i \underline{N} \cdot \frac{\underline{R}_i}{r_i^3} dS' \quad \ldots\ldots.(2.16) .$$

$$= - \iint_S \sum_{i=1}^{m} a_i \underline{N} \cdot \frac{\underline{R}_i}{r_i^3} dS + \iint_{S'} \sum_{i=1}^{m} a_i \underline{N} \cdot \frac{\underline{R}_i}{r_i^3} dS'$$

In the specific case where P lies on S, equation (2.14) gives

$$\iint_S \underline{N}' \cdot \underline{\nabla} \phi_e \, dS = - \iint_S \sum_{i=1}^{m} a_i \underline{N} \cdot \frac{\underline{R}_i}{r_i^3} \, dS = -2\pi \sum_{i=1}^{m} a_i ,$$

where ϕ_e is the potential of matter external to the surface S but within the surface S'. If ϕ_i is the gravitational potential due to matter within S, the total vector integration over the surface Sof both the internal and external gravitational potential gives

$$\iint_S \underline{N} \cdot \underline{\nabla} \phi \, dS = \iint_S \underline{N} \cdot \underline{\ }(\phi_i + \phi_e) \, dS$$

$$= -2\pi \left[\sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i \right]$$

$$= -2\pi \sum_{i=1}^{n} a_i \quad \text{.........(2.17).}$$

The use of equation (2.1) on the vector $\underline{F} = \underline{\nabla} \phi$ gives

$$\iiint_V \underline{\nabla} \cdot \underline{\nabla} \phi \, dV = \iint_S \underline{N} \cdot \underline{\nabla} \phi \, dS .$$

The evaluation of the right hand side of the above equation using equation (2.15) gives

$$\iiint_V \underline{\nabla}^2 \phi \, dV = \iint_S \sum_{i=1}^{n} a_i N_i \cdot \frac{\underline{R}_i}{r_i^3} \, dS$$

$$= - \begin{cases} 0 & \text{if P is at a point not occupied by matter} \\ 2\pi \sum_{i=1}^{n} & \text{if the } Q_i\text{'s lie on the bounding surface} \\ 4\pi \sum_{i=1}^{n} & \text{if the } Q_i\text{'s lie within the bounding surface} \end{cases}$$

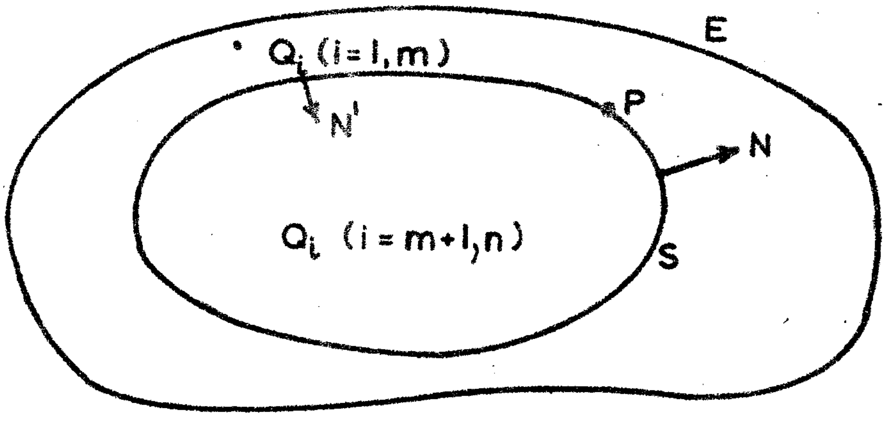
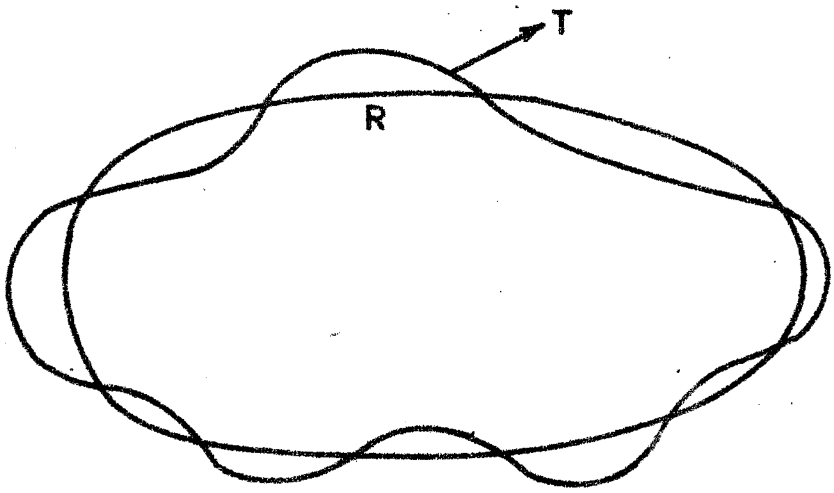


FIG. 2.3



T = True Surface
R = Reference Surface

FIG. 2.4

If ρ is the density of matter in the element of volume dV, it follows, from the definition of the a_i's, that

$$\lim_{n \to \infty} \sum_{i=1}^{n} a_i = \iiint_V k \rho \, dV,$$

where k is the gravitational constant. Thus

$$\nabla^2 \phi = -\begin{cases} 0 & \text{if there is no matter at P} \\ 2\pi k \rho & \text{if matter exists at P and } Q_i\text{'s lie on S} \\ 4\pi k \rho & \text{if } Q_i\text{'s lie within S with matter at P} \end{cases}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.18).$$

There is no necessity for the gravitating masses to lie entirely within the usrface S. The last of the relations in equation (2.18) is also known as Poisson's equation.

2.5 Green's third identity.

The application of Green's theorem (equation (2.5)) with $U_2 = \phi$ and $U_1 = \frac{1}{r}$ gives

$$\iiint_V \left[\frac{1}{r} \nabla^2 \phi - \phi \nabla^2 \frac{1}{r} \right] dV$$

$$= \iint_S \left[\frac{1}{r} \nabla . \underline{N} \phi - \phi \nabla . \underline{N} \frac{1}{r} \right] dS \ldots\ldots (2.19),$$

where the volume $V = V_i$ is interior to the surface S. As $\nabla^2 \frac{1}{r}$ is zero at all points except P, the origin of the length vector r, from the discussion prior to equation (2.11), it can be seen that the second term on the left hand side of equation (2.19) is given by

$$\phi_p \iiint_{V_i} \underline{\nabla}^2 \frac{1}{r} \, dV_i = \phi_p \iint_S \underline{N}.\underline{\nabla} \frac{1}{r} \, dS \quad .$$

If **P**, the origin of the length vector, is enclosed by an infinitesimal sphere (S') of radius ε , then

$$\phi_p \iiint_{V_i} \underline{\nabla}^2 \frac{1}{r} \, dV_i = 0 - \iint_{S'} \phi_p \sum_{i=1}^{3} \frac{x_i}{\varepsilon} \underline{i} \, . \sum_{i=1}^{3} \frac{x_i}{\varepsilon^3} \underline{i} \, \varepsilon^2 d\sigma$$

where $d\sigma$ is the element of surface area on a unit sphere. Simplification of the right hand side of the above equation gives

$$\phi_p \iiint_{V_i} \underline{\nabla}^2 \frac{1}{r} \, dV_i = - \phi_p \iint_{S'} d\sigma$$

$$= \begin{cases} 0 & \text{if P lies outside S} \\ -2\pi \, \phi_p & \text{if P lies on S} \\ -4\pi \, \phi_p & \text{if P lies within S.} \end{cases}$$

Thus

$$\iiint_{V_i} \frac{1}{r} \underline{\nabla}^2 \phi \, dV_i = -\phi_p \begin{cases} 0 & \text{if P lies outside S} \\ 2\pi & \text{if P lies on S} \\ 4\pi & \text{if P lies within S} \end{cases} + \ldots\ldots$$

$$+ \iint_S \left[\frac{1}{r} \underline{\nabla} \cdot \underline{N} \phi - \phi \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right] dS \quad \ldots\ldots(2.20).$$

Note :- r is the distance from the variable point Q to the fixed point P, which may be located anywhere The Q's, however, lie only within and on S. ϕ, on the left hand side of equation (2.20) , refers to potential at the internal point

If, on the other hand, the volume V_e were exterior to S, two factors have to be taken into consideration.

(i) The direction of the outward normal reverses its sign . The new unit normal $\underline{N}' = -\underline{N}$.

(ii) If P is external to S, it now lies within a region of matter, and not without, as before.

Thus, on lines similar to the derivation of equation (2.20),

$$\iiint_{V_e} \frac{1}{r} \underline{\nabla}^2 \phi \, dV = \phi_P \begin{cases} 0 & \text{if P is internal to S} \\ -2\pi & \text{if P is on S} \\ -4\pi & \text{if P is external to S} \end{cases} -$$

$$- \iint_S \left[\frac{1}{r} \underline{\nabla} \cdot \underline{N} \phi - \phi \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right] dS \; \ldots\ldots (2.21)$$

Equations (2.20) and (2.21) are known as Green's third identity.

2.6 The fundamental theorem in physical geodesy.

The basic problem in physical geodesy is that of locating a physical reality of unknown location and locating it to a known reference system which has no reality, so far as direct observations go. The reference surface best suited to solve the geodetic problem is the oblate spheroid.

The problem, in the case of the physical surface of the earth, is further complicated by the presence of the topography exterior to the geoid, which could cause deviations from the spheroidal shape. These could be as great as 10,000 metres or 2 parts in 10,000. In the geodetic problem as postulated, consideration of the geoid as the primary physical reality, could reduce the order of the deviation between the true surface and the reference surface to 1 part in 10^5. In this latter case, the residual discrepancies will be due to the irregular variations of the equipotential surfaces of the earth's gravitational field arising from local and regional density anomalies. These variations in shape are of vital moment in geodesy as every single geodetic observation is made with reference to the local vertical. These local verticals cannot be expected to have simple geometrical relationships to one another, analagous to spheroidal normals and their relative orientation cannot be determined without a definition of the earth's gravitational field.

Consider the projection of the sheroidal normal and the local vertical on the celestial sphere. Fig (2.5) shows that if N is the celestial pole and Z_A and Z_G are the zeniths for the local vertical and spheroidal normal respectively, the resulting discrepancy (ξ) in astronomical latitude, called the deflection of the vertical in the meridian and treated as positive if the outward vertical is north of the spheroidal normal, is the meridian component of the angle (ζ) between the spheroidal normal and the local vertical.

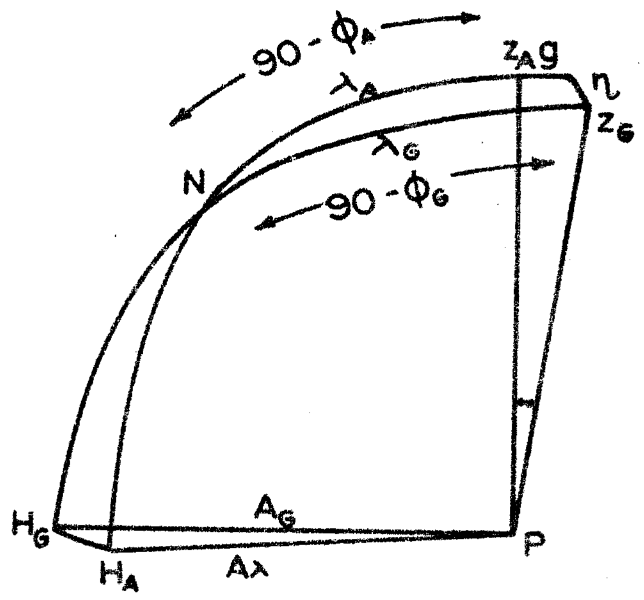

90 - ϕ_A
z_Ag
η
λ_A
z_G
λ_G
N
90 - ϕ_G
H_G
A_G
P
H_A
$A\lambda$

FIG. 2.5

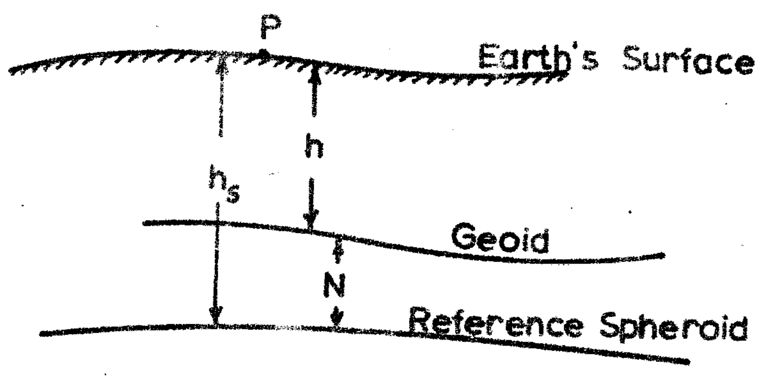

P
Earth's Surface
h
h_s
Geoid
N
Reference Spheroid

FIG. 2.6

The discrepancy (η) in the prime vertical is a function of the difference in geodetic and astronomical longitudes, being the prime vertical component of ζ . The difference in longitude has to be scaled by a factor of cos ϕ . The discrepancy is considered positive if the outward vertical lies to the east of the normal.

The effect of these deviations on the measurement of horizontal angles between normal sections is seldom of consequence. In general, triangulation and other methods of horizontal survey give relative positions unaffected by the non-coincidence of the local vertical and spheroidal normal. The position of any point on the earth's surface is fixed uniquely in three dimensions by the use of the spheroidal coordinates and the elevation of the point above the point on the spheroid whose normal passes through the former. The elevation of the point will thus be the length of the spheroidal normal intercepted between the spheroid and the point in question (Mather, 1966a). This quantity cannot be measured directly in the field. The elevations determined by field measurements are with respect to "mean sea level". Orthometric elevations (Bomford, 1962, 202) are, at present, purely arbitrary, being dependent on the model adopted for the earth's crust exterior to the geoid. Differences in potential can be determined (Baeschlin, 1960) and these constitute the only quantity related to displacements along the normal, which can be determined without ambiguity in the field.

The spheroidal elevation (h_s) is given by (fig (2.6))

$$h_s = h + N \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2.22),$$

where h is the orthometric elevation
and N the geoid spheroid separation.

An adequate spheroid of reference is therefore a pre-requisite for the complete definition of a point on the earth's surface. This is chosen using the arguments set out in section (1.1). Given this datum surface, the absolute location of a point is completely specified by its astronomical latitude (ϕ_A), astronomical longitude (λ_A), geopotential, N and the components ξ and η of the deflections of the vertical in the meridian and prime vertical. The last three quantities cannot be determined without a complete study of the earth's gravitational field.

See fig (2.4). Consider two surfaces R, capable of mathematical representation and hereafter called the reference surface, and T which is an existent physical surface of irregular form. R and T have the same volume and hence T lies partly within and partly outside R. The form of the surface T is not known, but its departures from R are slight. Both surfaces are situated with respect to a system of masses which, in general, lie partly within and partly outside the surface T. This would apply, by definition to R, which, however, has no physical reality.

As the departure of T from the regular form of R in the case of the geoid-spheroid system is expected to be of the order of 1 part in 10^5 (e.g., Veis, 1965), the gravitational potential of matter exterior to T (ϕ_e) and that of matter within it (ϕ_i) are harmonic both internally and externally with respect to T and the total gravitational potential (ϕ) is given by scalar addition

$$\phi = \phi_i + \phi_e \quad \ldots\ldots\ldots(2.23)$$

where $\phi_e << \phi_i$.

The application of Green's third identity to this system in the case where the volume V_e is exterior to the surface T at a point P on T gives, by the use of equations (2.14), (2.12) and (2.21),

$$\iiint_{V_e} \frac{1}{r} \underline{\nabla}^2 \phi \; dV = \iiint_{(V_e - \Delta V)} \frac{1}{r} \underline{\nabla}^2 \phi \; dV + \ldots$$

$$\ldots + \iiint_{\Delta V} \frac{1}{r} \underline{\nabla}^2 \phi \; d\Delta V$$

where the volume Δ V is a hemisphere centred at P on S. The first term on the right hand side is zero. The second term has a value at P only, when $\underline{\nabla}^2 \phi = -2\pi k \rho$.

Further, at P, $\frac{1}{r} d(\Delta V) = f(r) = 0$.

Thus, equation (2.21), in the case of ϕ_i, reduces to

$$0 = -2\pi \phi_{i_P} - \iint_T \left[\frac{1}{r} \underline{\nabla} \cdot \underline{N} \phi_i - \phi_i \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right] dT \qquad \ldots\ldots.(2.24)$$

where P is on the surface T,

r is the distance of the surface element dS from P and

ϕ_{i_P} is the potential at P due to masses within T.

Similar consideration of the masses exterior to T and their effect as set out in equation (2.20), over the volume V_i, interior to T gives

$$0 = -2\pi \phi_{e_P} + \iint_T \left[\frac{1}{r} \underline{\nabla} \cdot \underline{N} \phi_e - \phi_e \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right] dT \qquad \ldots\ldots(2.25)$$

where ϕ_{e_P} is the potential at P, on the surface S due to masses outside T.

The expressions for potential within the integral on the right hand side of equations (2.24) and (2.25) refer to values at the surface element dS. If ω is the angular velocity of rotation of the earth, the potential (ϕ_r) due to the rotation of the earth (Heiskanen and Vening Meinesz, 1958, 34) is given by

$$\phi_r = \sum_{i=1}^{2} \frac{1}{2} \omega^2 x_i^2$$

the x_3 axis coinciding with the rotational axis.

$$\underline{\nabla}^2 \phi_r = \underline{\nabla} \cdot \underline{\nabla} \phi_r = \underline{\nabla} \cdot \sum_{i=1}^{2} \frac{1}{2} \cdot 2\omega^2 x_i \underline{i}$$

$$= 2\omega^2 \qquad \ldots\ldots\ldots\ldots\ldots (2.26)$$

The application of equation (2.20) to the rotational potential gives

$$\iint_T \left[\frac{1}{r} \underline{\nabla} \cdot \underline{N} \phi_r - \phi_r \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right] dT - 2\pi \phi_r - \iiint_V \frac{1}{r} \underline{\nabla}^2 \cdot \phi_r dV$$

$$= 0 \qquad \ldots\ldots\ldots (2.27)$$

The total gravitational potential (W_P) at P is given by

$$W_P = \phi_e + \phi_i + \phi_r \qquad \ldots\ldots\ldots\ldots (2.28)$$

The subtraction of equations (2.25) and (2.27) from equation (2.24) gives

$$-2\pi(\phi_{i_P} - \phi_{e_P} - \phi_{r_P}) - \iint_T \left[\frac{1}{r} \underline{\nabla} \cdot \underline{N}\, W - W \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right] dT$$

$$+ \iiint_V \frac{1}{r} \underline{\nabla}^2 \phi_r \, dV = 0$$

Further simplification, using equation (2.28) gives

$$- 2\pi (W_P - 2\phi_{e_P}) + 4\pi \phi_{r_P} - \iint_T \left[\frac{1}{r}\underline{\nabla} \cdot \underline{N}\, W - W \underline{\nabla} \cdot \underline{N} \frac{1}{r}\right] dT$$

$$+ \iiint_V \frac{1}{r} \underline{\nabla}^2 \phi_r \, dV = 0 \quad \ldots\ldots\ldots\ldots (2.28)$$

Equation (2.28) is the basic equation in physical geodesy. It should be noted that

(i) the equation applies to a point P on the actual surface T ;

(ii) masses are located both internal and external to T;

(iii) W is the total gravitational potential and ϕ_{e_P} is the potential at P due to masses exterior to T ;

(iv) ϕ_{r_P} is the rotational potential at P

(v) V is the volume enclosed by the surface T, r referring to distances from P.

Equation (2.28) is basic and involves no approximations. W_P is a function related to geopotential (Baeschlin, 1960). If T were the geoid, $W_P = W_0$ is the potential of the equipotential surface corresponding to mean sea level. If, on the other hand, T were the physical surface of the earth, W_P would be variable and given (Jeffreys, 1962, 130) by

$$W_P = W_0 - \int_0^P g \, dh \quad \ldots\ldots (2.29)$$

where g is the value of gravity over the difference in height dh.

3. THE SOLUTION OF THE FUNDAMENTAL BOUNDARY CONDITION.

3.1 Basic definitions

Two problems arise in attempting to solve equation (2.28). Firstly, the integrals are non-linear and direct solution is not possible. Secondly, the surface T is not a known surface. A solution can be obtained by linearising the system. This is effected by introducing the concept of a mathematical model surface R which is spheroidal in nature and has dimensions approximately equal to those of the existent system. Astronomical coordinates (ϕ_A, λ_A) and geopotential (W), obtained by a combination of gravimetry and levelling afford a system of coordinates for every point on the earth's surface. In the case of such a point P, whose coordinates will be denoted by the use of the suffix P with the appropriate symbol, it will be possible to define an equivalent point Q on the reference system such that

$$\phi_Q = \phi_{A_P}$$

$$\lambda_Q = \lambda_{A_P} \qquad \text{...........(3.1)}$$

$$U_Q = W_P$$

where U_Q is the potential of the reference system at Q (See fig (3.1)). Let the equipotential surfaces

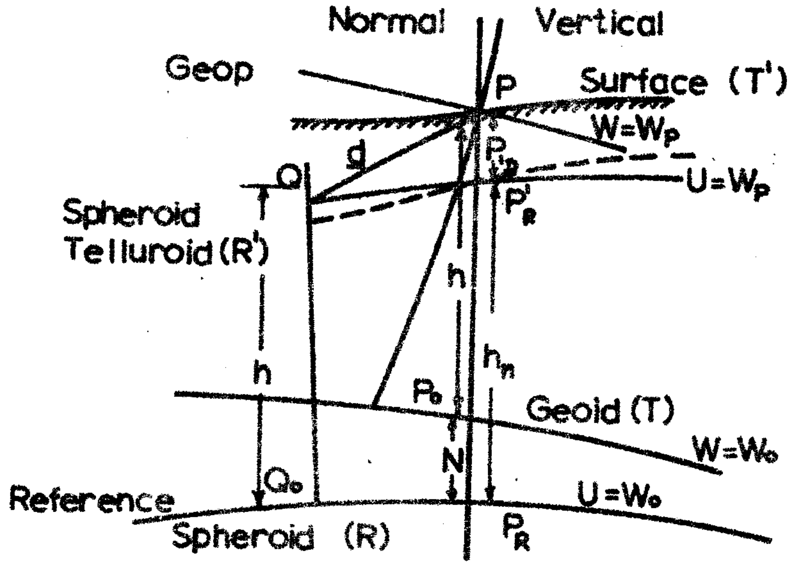

Normal
Vertical
Geop
P
Surface (T')
W=Wp
d
Q
U=Wp
Spheroid
Telluroid (R')
h
h
h_n
P_o
Geoid (T)
W=Wo
N
Q_o
Reference
U=Wo
Spheroid (R)
P_R

FIG. 3.1

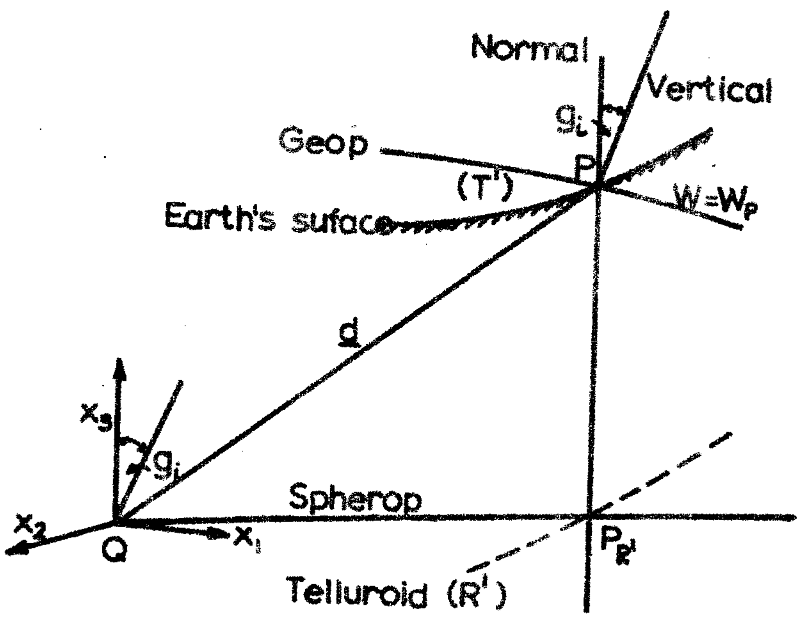

Normal
Vertical
g_i
Geop
P
(T')
W=Wp
Earth's suface
d
x_3
g_i
Spherop
x_2
Q
x_1
$P_{R'}$
Telluroid (R')

FIG. 3.2

of the reference system, which has the same rotational characteristics as the existent earth, have equations of the form

$$U = \text{constant} \quad \ldots\ldots\ldots (3.2).$$

Such surfaces were called spherops by Hunter (1960, 4). The term geop was used to describe the equipotential surfaces of the existent earth which are represented by equations of the form

$$W = \text{constant} \quad \ldots\ldots\ldots (3.3).$$

If the surface R is chosen to represent the surface T, the discrepancy in point to point representation can be fully defined by the vector $\underline{d}$, being the line PQ truly orientated in space. As shown in fig (3.1), equipotential surfaces of both the true system ($W = W_P$) and the reference system ($U = U_P$) pass through the point P.

Two possible systems of existent-reference surfaces are possible. Firstly there is the geoid - spheroid system where the true and reference surfaces will be referred to by the symbols T and R respectively. A second possibility is the system composed of the physical surface of the earth (T') traced out by the locus of the point P and the reference surface formed by the locus of the associated point Q, called (Hirvonen, 1960, 40) the telluroid (R'). Also see section (4.3). If $P_{R'}$ is the intersection of the spherop through Q and the normal through P, which intersects the spheroid at P_R, then $P_R P_{R'}$ ($= h_n$) is called the normal height.

If the normal PP_R through P intersects the geoid in P_0, then PP_0 (= h) is the orthometric height. h is approximately equal to h_n, and it is common practice to assume that the discrepancies do not exceed 1 to 2 metres. The disturbing potential (V_D) is defined by

$$V_D = W_P - U_P \quad ......(3.4).$$

If PP_R, (= h_D) is defined as the height anomaly,

$$h_D + h_n = N + h \quad ....(3.5).$$

Adopting the convention that longitude is positive east of Greenwich, and azimuth (A) is measured from north through east, a direct consideration of fig (2.5) and the discussion in section (2.6) gives

$$\xi = \phi_A - \phi_G \quad ....(3.6)$$

$$\eta = (\lambda_A - \lambda_G) \cos \phi \quad .....(3.7)$$

$$A_A - A_G = (\lambda_A - \lambda_G) \sin \phi = \eta \tan \phi$$

Alternatively,

$$\eta = (A_A - A_G) \cot \phi \quad .....(3.8).$$ In equations (3.6) to (3.8), the suffix G refers to geodetic values on the reference spheroid.

The vector $\underline{d}$ is given by

$$\underline{d} = R\xi\ \underline{1} + R\eta\ \underline{2} + h_D\ \underline{3} \quad ....(3.9)$$

where R is the radius of curvature of R' in azimuth A at P, i (i=1, 3) are the direction cosines of a local 3-dimensional cartesian system with the 3-axis orientated vertically, the 1 and 2-axes being in the horizon plane, being orientated north and east respectively.

3.2 Brun's theorem

The potential U of the reference system is given by Taylor's theorem as

$$U = U(h) \quad ....(3.10)$$

and

$$\begin{aligned} U_P &= U(h_n + h_D) \\ &= U(h_n) + h_D U'(h_n) + \frac{1}{2!} h_D^2 U''(h_n) + \ldots \\ &= U(h_n) + h_D \frac{\partial U}{\partial h_n} + \frac{1}{2} h_D^2 \frac{\partial^2 U}{\partial h_n^2} + \ldots\ldots. \end{aligned}$$

$$.........(3.10).$$

$\frac{\partial^2 U}{\partial h_n^2} \doteq \frac{\partial \gamma}{\partial h} \doteq 3.086 \times 10^{-6}\ \sec^{-2}$ (Heiskanen and Vening Meinesz, 1958, 148). As $h_D \simeq 10^4$ cm (Kaula, 1959, 114),

$$h_D^2 \frac{\partial^2 U}{\partial h_n^2} \simeq 3 \times 10^4\ \text{cm}^2\sec^{-2} = 3 \times 10^{-1}\ \text{g.p.u.}$$

The third term on the right hand side of equation (3.10) is equivalent to approximately 15 cm in distance. If this term is ignored, the magnitude of the change in potential is given by the second term. $\partial U/\partial h_n$ is of the order of γ and is of magnitude 10^3 cm sec^{-2}. Therefore the difference $U_P - U_Q$ is of order 10^7 cm^2 sec^{-2} which is equal to 100 g.p.u. Hence

$$U_P = U_Q - h_D \gamma_P + o\{10^{-1} \text{g.p.u}\} \quad ..(3.11),$$

where γ_P is the value of normal gravity at P. The use of equation (3.4) in equation (3.11) gives

$$V_{D_P} = W_P - U_P = h_{D_P} \gamma_P \quad .....(3.12)$$

This equation holds if the potential at Q on the reference system is numerically equal to that at P due to the existent earth. As a corollary,

$$h_{D_P} = \frac{V_{D_P}}{\gamma_P} + 0\{10^{-3}\} \quad ....(3.13).$$

Equations (3.12) and (3.13) represent Brun's theorem which relates the disturbing potential to the height anomaly and is correct to 15 cm if $h_D \simeq 100$ metres.

3.3 The free air anomaly in terms of the disturbing potential.

Consider equation (3.4). Differentiation of the potential function with respect to normal height gives

$$\frac{\partial V_D}{\partial h} = \frac{\partial W}{\partial h} - \frac{\partial U}{\partial h} = -g + \gamma \quad ....(3.14)$$

as potential decreases with increase of height. The application of equation (3.14) at P gives

$$\frac{\partial V_{D_P}}{\partial h} = -g_P + \gamma_Q + \frac{\partial \gamma}{\partial h} h_D \quad \ldots\ldots\ldots(3.15)$$

where $\partial\gamma/\partial h$ increases with increase of h_D. In reality, however, gravity decreases with increase of height. Thus, $\partial\gamma/\partial h$ is less than 0. Equation (3.15) can thus be rewritten as

$$\frac{\partial V_{D_P}}{\partial h} = -g_P + \gamma_Q - \left|\frac{\partial \gamma}{\partial h}\right| h_D \quad \ldots\ldots\ldots(3.16)$$

The free air anomaly (Δg_F) is defined as

$$\Delta g_F = g_P - \gamma_Q \quad \ldots\ldots\ldots\ldots(3.17)$$

The combination of equations (3.16) and (3.17) gives the relation between the disturbing potential and the free air anomaly as

$$\Delta g_F = -\left[\frac{\partial V_D}{\partial h} + \left|\frac{\partial \gamma}{\partial h}\right| h_D\right] + o\{10^{-3}\} \quad \ldots\ldots\ldots\ldots(3.18)$$

As the magnitude of the free air anomaly seldom exceeds 150 mgal, equation (3.18) defines the free air anomaly to the nearest mgal if the vertical gradient of the disturbing potential, the vertical gradient of gravity and the height anomaly are known.

If R is the length of the radius vector from the centre of mass to P, then Newtonian gravity gives

$$\gamma_P = \frac{k\,M}{R^2} \quad , \text{ where M is the mass of the}$$

earth, and

$$\frac{\partial\gamma}{\partial h} = -\frac{2kM}{R^3} = -\frac{2\gamma}{R}$$

Equation (3.18) can be re-written as

$$\Delta g_{F_P} = -\left[\frac{\partial V_D}{\partial h} + \frac{2\,\gamma_P h_{D_P}}{R}\right] \quad ....(3.19)$$

Equation (3.19) can be expressed as

$$\Delta g_{F_P} = -\left[\frac{\partial V_D}{\partial h} + \frac{2V_{D_P}}{R}\right] \quad ....(3.20)$$

It should be noted that equation (3.20) holds only if the potential on the reference surface equals the potential on the surface to be mapped. Both equations are correct to 1 part in 10^3. More generally, equation (3.20) can be written as

$$\Delta g_{F_P} = -\left[\frac{\partial V_D}{\partial h} + \frac{2(V_{D_P} - \overline{W_P - U_Q})}{R}\right] \quad ..(3.20a)$$

3.4 The telluroid

The telluroid was postulated by Hirvonen (1960) as the locus of the points $P_{R'}$ on the normal through the points P on the earth's surface at which the potential of the reference system $U_{P_{R'}}$ was given by

$$U_{P_{R'}} = W_P \qquad \ldots\ldots\ldots\ldots(3.21).$$

More precisely, the telluroid is the locus of the points Q defined in section (3.1). This type of reference system was suggested by Hunter in a modified form (1960) for his model earth in which he gave it the name Terroid. The general situation of the telluroid with respect to the physical surface of the earth is shown in fig (3.1). An enlargement of the upper portion of this figure is shown in fig. (3.2). Consider a localised three dimensional cartesian system of coordinates (x_i, i=1,3) centred on Q, which is defined in section (3.1). Assume the x_3 axis coincident with the local normal, the x_1 and x_2 axes being orientated north and east respectively. The non-coincidence of P and $P_{R'}$ is a function of h_D, while the non-coincidence of P and Q is dependent on the relative positions of the local vertical and the spheroidal normal as well as on h_D. In addition, the rate of change of Q with respect to P is dependent both on local mass anomalies and on the slope of the topography in the direction in which the change is considered. Under limiting conditions suitable for the application of differential calculus, the deflections of the vertical ξ_i (i=1,2 ; $\xi_2 = \eta$) are given by

$$\xi_i = - \frac{\partial h_D}{\partial x_i} \quad , \quad i=1,2 \quad \ldots\ldots(3.21a)$$

Equation (3.21a) is based on the same sign convention as defined in section (2.6). The equation of the spherop $U = W_P$ in the limit at Q is

$$x_3 = 0 \quad \ldots\ldots\ldots\ldots\ldots (3.22).$$

The equation of the telluroid in the neighbourhood of Q can be represented by

$$x_3 = \Delta h_s(x_i, i=1,2) \quad \ldots\ldots\ldots (3.23),$$

where Δh_s is the increment of normal height from Q. The height anomaly h_D can be expressed as

$$h_D = h_D(x_i, i=1,3) = h_D(x_1, x_2, \Delta h_s(x_1, x_2)) \quad \ldots\ldots\ldots\ldots (3.24)$$

The expansion of equation (3.24) by MacLaurin's theorem gives

$$h_D = \left[h_D(x_i, i=1,3) \right]_{x_3=0} + \Delta h_s \frac{\partial h_D}{\partial h_s} + o\{ \frac{\partial^2 h_D}{\partial h_s^2} \} .$$

Differentiaition with respect to x_i, i=1, 2, gives

$$\frac{\partial h_D}{\partial x_i} = \left[\frac{\partial h_D(x_i, i=1,3)}{\partial x_i} \right]_{x_3=0} + \frac{\partial h_s}{\partial x_i} \frac{\partial h_D}{\partial h_s} + o\{ \frac{\partial^2 h_D}{\partial x_i \partial h_s} \}$$

$$\ldots, i=1,2 \quad \ldots\ldots\ldots (3.25).$$

More specifically,

$$\left[\frac{\partial h_D}{\partial x_i} \right]_{\text{Telluroid}} = \left[\frac{\partial h_D}{\partial x_i} \right]_{U=W_P} + \frac{\partial h_D}{\partial h} \frac{\partial h}{\partial x_i} \quad \ldots . (3.26)$$

The first term of the pair which form the second expression on the right , is a small quantity, being the change in the height anomaly with elevation. The second term of the pair is the topographical gradient which is comparatively large. From equations (3.13) and (3.20),

$$\frac{\partial h_D}{\partial h} = \frac{\partial}{\partial h}\left[\frac{V_D}{\gamma}\right] = -\frac{1}{\gamma^2} V_D \left|\frac{\partial \gamma}{\partial h}\right| + \frac{1}{\gamma}\frac{\partial V_D}{\partial h}$$

$$= \frac{1}{\gamma}\left[\frac{V_D}{\gamma}\left|\frac{\partial \gamma}{\partial h}\right| - \Delta g_F - \left|\frac{\partial \gamma}{\partial h}\right| h_D\right]$$

$$= -\frac{\Delta g_F}{\gamma} \qquad \ldots\ldots\ldots\ldots\ldots(3.27).$$

Δg_F and γ refer to the values of these quantities at the point considered. Again, this equation holds where $U_Q = W_P$. Equation (3.21a) reduces to

$$\xi_i = -\left[\frac{\partial h_D}{\partial x_i}\right]_{U=W_P}$$

$$= -\left[\frac{\partial h_D}{\partial x_i}\right]_{\text{Telluroid}} - \frac{\Delta g_F}{\gamma}\frac{\partial h}{\partial x_i}, \quad i=1,2;\ \xi_2 = \eta$$

$$\ldots\ldots\ldots\ldots(3.28)$$

Equation (3.28) gives the deflections of the vertical in terms of the variations of the height anomaly with horizontal position and the topographical gradient. If the reference surface were a spheroid which is a bounding equipotential, only the first equality in equation (3.28) would be relevant.

3.5 Possible reference surfaces.

The basic equation can thus be considered to be of applicability in two different sets of circumstances :-

(i) The surface T can be the equipotential surface with closest correspondence to the earth's true surface, which, over 70 per cent of the latter , is mean sea level.

(ii) The surface T can be the actual surface of the earth, which coincides with the earlier definition over 70 per cent of the earth's surface, but may depart from the equipotential of reference by amounts as great as 1 part in 10^4 in certain instances on land. It should also be borne in mind that land areas are of prime importance in geodesy.

The gravitational potential of a rotating earth, which is spheroidal in shape, gives rise to a family of equipotential surfaces exterior to it (Heiskanen and Vening Meinesz, 1958, 46), which are spheroids of varying equatorial radii and flattenings. Thus, in both cases described above, it is convenient to consider the difference between the actual system and a postulated reference system of best fit which satisfies external astrophysical ans surface geodetic conditions. This has the effect of reducing the quantities in the basic integral equation (2.28) to magnitudes which are of the order of the square of the flattening and has the resulting effect of linearising the integrals.

The commonly adopted reference datum is the international spheroid (Heiskanen and Vening Meinesz, 1958, 46) and its associated family of spherops. The validity of this datum is more fully discussed in section 13. The choice of such a datum system defines either the reference spheroid required in case (i) or the telluroid required in case (ii). Thereafter, the use of equations (3.13) and (3.28) gives the displacement of points on the actual system in relation to equivalent points on the curvilinear reference system, as detailed in section (3.1). These displacements are of the second order of magnitude in geodesy i.e., of order f^2.

The international spheroid is postulated as satisfying the above conditions. Its equatorial radius (a) and flattening (f) were adopted by the International Association of Geodesy at the I.U.G.G. Madrid assembly in 1930. The parameters of the international spheroid are

$$a = 6,378,388 \text{ metres} \;; \qquad f = 1/297.0 = 3.367 \times 10^{-3} \quad \ldots\ldots\ldots\ldots\ldots (3.29)$$

The value of gravity on the surface of this spheroid on the then accepted value for kM is given by the international gravity formula

$$= 978.0490(1 + 5.2884 \times 10^{-3}\sin^2\phi - 5.9 \times 10^{-6}\sin^2 2\phi) \ldots\ldots (3.30)$$

The first term was obtained (Heiskanen and Vening Meinesz, 1958, 1958, 52) from an analysis of gravity over the surface of the earth. The second was obtained from the value for f using Clairaut's theorem.

The international spheroid is defined as

(a) having the same mass as the existent earth ;

(b) having a common centre of mass with the latter ;

(c) being bounded with an equipotential surface with the same potential as the geoid ;

(d) having no masses exterior to this bounding surface ;

(e) having the same rotational potential as the existent earth.

The full implications of the adoption of the international spheroid as the surface of reference and the correctness of the parameters chosen for it are discussed in section (13). In the following development, it should be borne in mind that the concept of the reference system is that of a hypothetical model which is merely impressed on the existent system to remove the zero order and first order effects of the existent earth's potential.

4. MODIFICATION OF THE BASIC EQUATION FOR SOLUTION

4.1 The reference system in the basic equation.

Consider the reference system where the surface R is exterior to all matter. In equation (2.28), $\phi_{e_P} = 0$. If R encloses matter which is rotating at the same angular velocity as the true surface T and if the total gravitational potential at a point on the surface R is U, then equation (2.28), in the case of this reference system becomes

$$- 2\pi U_P + 4\pi \phi_{r_P} - \iint_R \left[\frac{1}{r} \underline{\nabla} \cdot \underline{N}\, U - U \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right] dR + \iiint_V \frac{1}{r} \underline{\nabla}^2 \phi_r \, dV = 0 \quad \ldots\ldots.(4.1)$$

Equation (4.1) can apply with equal validity to either the reference spheroid or the telluroid. Equation (2.28) itself, represents the actual system, where the surface area T differs from surface area R by amounts of the order of 1 part in 10^5. Subtracting equation (4.1) from equation (2.28) on the basis that integration over the surface R is identical with integration over the surface T (thus, the physical surface of the earth cannot be referred directly to the reference spheroid)

$$- 2\pi (W_P - U_P - 2\phi_{e_P}) - \iint_R \left[\frac{1}{r} \underline{\nabla} \cdot \underline{N} (W-U) - (W-U) \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right] dR = 0 \quad \ldots\ldots\ldots.(4.2)$$

integration being taken over the reference surface, whose mathematical form is known. The use of equation (3.4) in equation (4.2) gives

$$- 2\pi \, (V_{D_P} - 2\phi_{e_P}) - \iint_R \left[\frac{1}{r} \, \underline{\nabla} \cdot \underline{N} \, V_D - V_D \, \underline{\nabla} \cdot \underline{N} \, \frac{1}{r} \right] dR = 0$$

...............(4.3)

or

$$V_{D_P} = 2\phi_{e_P} - \frac{1}{2\pi} \iint_R \left[\frac{1}{r} \, \underline{\nabla} \cdot \underline{N} \, V_D - V_D \, \underline{\nabla} \cdot \underline{N} \, \frac{1}{r} \right] dR = 0$$

...............(4.4).

The complete solution of equation (4.4) requires a knowledge of the disturbing potential over the surface R. Further, the latter must be capable of complete mathematical definition as the direction cosines of the normal are required to evaluate $\underline{N}$.

4.2 The evaluation of $\underline{\nabla} \cdot \underline{N} \, V_D$.

See fig (4.1). The point Q is that on the reference system, and hence on the telluroid, which represents the point P on the existent system. Let $W = W_P$ be the geop which passes through P. Then, as described in section (3.1) and figs (3.1) and (3.2), the spherop passing through Q is $U = W_P$. The equation of any equipotential surface in the family of spherops associated with the reference spheroid is of the form

$$U(x_i, i=1,3) = 0 \quad .$$

More specifically, restricting the investigation

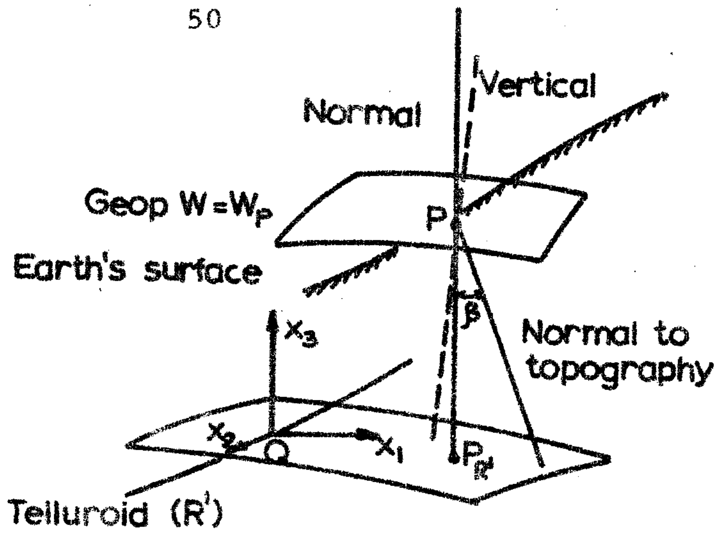

Normal
Vertical
Geop $W=W_P$
P
Earth's surface
X_3
β
Normal to topography
X_2
O
X_1
$P_{R'}$
Telluroid (R')

FIG. 4.1

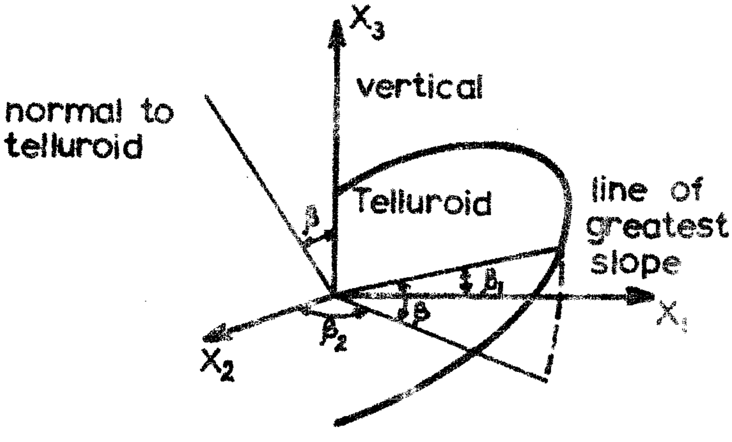

X_3
vertical
normal to telluroid
β
Telluroid
line of greatest slope
β_1
β
β_2
X_1
X_2

FIG. 4.2

to the neighbourhood of P and Q, noting that these points are generally held to be of the order of 100 metres apart, if the x_3 axis is orientated along the spherop normal at Q, then $PP_{R'}$, for all practical purposes, will have the property of being parallel to the x_3 axis. Any point on the telluroid, in this region can be represented by the equation

$$x_3 = \delta h_s(x_i, i=1,2) \quad \ldots\ldots\ldots\ldots\ldots (4.5).$$

The equation of the telluroid, in this restricted vicinity will be of the form

$$U(x_i, i=1,2 ; \delta h_s(x_i, i=1,2)) = 0 \ldots (4.6).$$

In equations (4.5) and (4.6), h_s will be the normal height and δh_s, to the relevant order of accuracy, the change in orthometric elevation. The rate of change of potential with position on the telluroid is obtained by the differentiation of equation (4.6), when

$$\left[\frac{\partial U}{\partial x_i}\right]_{\text{Telluroid}} = -\frac{\partial U}{\partial x_3}\frac{\partial h}{\partial x_i}, \quad i=1,2 \ldots\ldots (4.7)$$

In the immediate vicinity of any point on the telluroid, the relationship between the geop $W = W_P$ and its associated spherop $U = W_P$ is, for all practical purposes, the same as that between the physical surface of the earth and the telluroid, so far as the relative orientation and displacement of surface normals are concerned. In fig (4.1), if β is the angle between the telluroid normal and the normal to the spherop $U = W_P$, and if the direction cosines $(l_i, i=1,3)$ of the spherop normal are known, those of the telluroid are deflected through a further angle β. Along any equipotential surface,

$$\frac{\partial U}{\partial s} = 0 \quad \ldots\ldots\ldots\ldots\ldots (4.8),$$

where s is the element of length. If $\underline{X}$ is the vector from the origin to the general point on the spherop,

$$\underline{X} = \sum_{i=1}^{3} x_i(s)\,\underline{i} \quad \ldots\ldots\ldots (4.9).$$

The combination of equations (4.8) and (4.9) gives

$$\sum_{i=1}^{3} \frac{\partial U}{\partial x_i}\,\frac{\partial x_i}{\partial s} = \underline{\nabla} U \cdot \frac{d}{ds}\underline{X} = 0 \quad \ldots\ldots (4.10),$$

where $\frac{d}{ds}\underline{X}$ represent the direction cosines of the tangent to the line of greatest slope on the telluroid at the point $x_i(s)$, $i=1,3$, s being a parameter of length. From elementary differential geometry, the sum of the products of like direction cosines of mutually perpendicular lines must be zero. By analogy, $\underline{\nabla} U$ must be proportional to the direction cosines of the surface normal. As

$$\underline{N} = \sum_{i=1}^{3} l_i\,\underline{i} \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.11),$$

the following proportionality relationship holds :-

$$l_1 : l_2 : l_3 = \frac{\partial U}{\partial x_1} : \frac{\partial U}{\partial x_2} : \frac{\partial U}{\partial x_3} = -\frac{\partial h}{\partial x_1} : -\frac{\partial h}{\partial x_2} : 1 \quad \ldots (4.12).$$

The last equality in equation (4.12) is obtained from equation (4.7). Therefore, the direction cosines of the normal to the telluroid are equal to

$$-\frac{\partial h}{\partial x_1}\cos\beta, \quad -\frac{\partial h}{\partial x_2}\cos\beta \text{ and } \cos\beta \quad \text{respectively,}$$

as the last equality in equation (4.12) gives the direction cosines of the normal to the spherop $U = W_P$. Therefore,

$$\underline{N} \cdot \underline{\nabla} V_D = \sum_{i=1}^{3} \frac{\partial V_D}{\partial x_i}\, \underline{i} \cdot \left[-\frac{\partial h}{\partial x_1}\cos\beta\, \underline{1} - \frac{\partial h}{\partial x_2}\cos\beta\, \underline{2} + \cos\beta\, \underline{3} \right]$$

$$= \cos\beta \left[\frac{\partial V_D}{\partial x_3} - \sum_{i=1}^{2} \frac{\partial V_D}{\partial x_i}\frac{\partial h}{\partial x_i} \right] \quad \ldots\ldots (4.13),$$

where β is the angle beweeen the normal to the telluroid and the spherop normal. For all computational purposes, this is the slope of the topography. If $\beta_i (i=1,2)$ are the angles between the tangent to the telluroid and the $x_i (i=1,2)$ axes, and if the latter were orientated in the north and east directions, respectively, this would represent the slope of the topography in the north and east directions. Then,

$$\frac{\partial h}{\partial x_i} = \tan\beta_i \quad , \; i=1,2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.15).$$

Simple consideration of the direction cosines of the telluroid normal in fig (4.2) give

$$\cos^2\beta = \cos^2\beta_1 + \cos^2\beta_2 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4.16)$$

The substitution of values for $\frac{\partial V_D}{\partial x_3}$ from equation (3.16), together with the use of equation (4.15) transforms equation (4.13) into

$$\underline{N} \cdot \underline{\nabla} V_D = \cos\beta \left[- \Delta g_F - \frac{1}{\gamma} \left|\frac{\partial\gamma}{\partial h}\right| V_D - \sum_{i=1}^{2} \frac{\partial V_D}{\partial x_i} \tan\beta_i \right] \quad ..(4.17).$$

From equation (3.21),

$$\frac{\partial h_D}{\partial x_i} = - \xi_i = \frac{\partial}{\partial x_i}\left[\frac{V_D}{\gamma}\right]$$

$$= \frac{1}{\gamma} \frac{\partial V_D}{\partial x_i} - \frac{V_D}{\gamma^2} \frac{\partial\gamma}{\partial x_i} \qquad i=1,2; \ \xi_2 = \eta \qquad .....(4.18).$$

The second term on the right hand side of the above equation is very small, as the horizontal gravity gradient, which is less than 20 mgal per 100 km (Mather, 1967a, 13), has a magnitude of 2×10^{-7} gal met^{-1}. The term, therefore, has a magnitude of order 2×10^{-9}, while the magnitude of the first term is three orders of magnitude larger, the horizontal height anomaly gradient being about 1 metre per 15 km. (Mather, 1966a, 4).

Thus

$$\frac{\partial V_D}{\partial x_i} = - \gamma\,\xi_i + O\{10^{-9}\} \qquad ....(4.19)$$

$$i=1,2\ ; \ \xi_2 = \eta \ .$$

Substituting from equation (4.19) in (4.17),

$$\underline{N} \cdot \underline{\nabla} V_D = \cos\beta\left[\gamma\,(\xi \tan\beta_1 + \eta \tan\beta_2) - \Delta g_F - \frac{1}{\gamma}\frac{\partial\gamma}{\partial h} V_D\right]$$

$$...............(4.20).$$

Equation (4.20) modifies equation (4.4) to give the final form of the basic equation as

$$V_{D_P} = 2\phi_{e_P} + \frac{1}{2\pi}\iint_R \left[\left\{ \frac{1}{\gamma}\left|\frac{\partial\gamma}{\partial h}\right| \frac{\cos\beta}{r} + \underline{\nabla}\cdot\underline{N}\,\frac{1}{r} \right\} V_D + \right.$$

$$\left. + \left\{ \Delta g_F - \gamma(\xi\tan\beta_1 + \eta\tan\beta_2) \right\} \frac{\cos\beta}{r} \right] dR \quad \ldots (4.21)$$

Equation (4.21) gives the fundamental equation in terms of the disturbing potential, the free air anomaly, the deflections of the vertical, the vertical gradient of gravity and the slope of the reference surface R with respect to a spherop, as well as its components in the north and east directions.

A study of the derivation shows that no restrictions have been introduced regarding the nature of R. The latter could be either the telluroid itself or , for that matter, a spherop, more specifically, the reference spheroid itself. The only restriction is that the surface being mapped should not depart from the reference surface by amounts exceeding the order of the square of the flattening. Thus, the physical surface of the earth can only be mapped from the telluroid, while the geoid can only be mapped from the reference spheroid.

4.3 The outline of the solution for the physical surface of the earth.

In the case where the reference surface is the telluroid, there is no matter exterior to the surface being mapped as the latter is the physical surface of the earth. Thus

$$\phi_{e_P} = 0 .$$

The replacement of the disturbing potential in equation

(4.21) by the height anomaly using Brun's theorem(equation (3.13)) gives

$$h_{D_P} = \frac{1}{2\pi\gamma} \iint_R \left[\left\{ \frac{1}{\gamma} \left|\frac{\partial\gamma}{\partial h}\right| \frac{\cos}{r} + \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right\} \gamma \, h_D + \right.$$

$$\left. + \left\{ \Delta g_F - \gamma \, (\xi \tan\beta_1 + \eta \tan\beta_2) \right\} \frac{\cos\beta}{r} \right] dR \quad \ldots.(4.22).$$

Many attempts have been made by geodesists to solve equation (4.22) with the aid of certain approximations. The starting point itself, in most such solutions, has never been the fundamental equation but some approximate model. The only rigorous attempt to date is that of Molodenskii (1962). This very elegant method of solution has been set out in full on pages 81 to 99 of the monograph quoted.

Equation (4.22) can only be solved by the method of succesive approximations as any solution requires not only a knowledge of the disturbing potential; which, being related to the gravity anomaly, as will be shown in section (5), is capable of evaluation, but also a knowledge of the values of the surface deflections of the vertical and the slope of the topography. In the second term on the right hand side of equation (4.22), except in regions of very steep topography, the expression with the deflections of the vertical is of order $10^{-3}/r$, while that due to the term containing the gravity anomaly is likely to be of order $2 \times 10^{-2}/r$. Hence, a solution, correct to a reasonable order of accuracy can be obtained by ignoring the last term within the surface integral. From this solution, values can be assigned for the deflections of the vertical and succesive iterated solutions should converge rapidly.

Molodenskii effects the solution by introducing certain functions of a system of three-dimensional curvilinear coordinates $q_i (i=1, 3)$ where the q_1 coordinate is the orthometric height h. If

$$q_1 = h(q_i, i=2, 3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.23),$$

and the element of length (ds) is given by

$$ds^2 = \sum_{i=1}^{3} c_i^2 \, dq_i^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4.24),$$

where

$$c_i^2 = \sum_{j=1}^{3} \left[\frac{\partial x_j}{\partial q_i} \right]^2, \quad i=1,3 \quad \ldots\ldots\ldots\ldots (4.25),$$

$x_j (j=1, 3)$ being the reference three dimensional cartesian coordinate system. Molodenskii defined the operators

$$\frac{\partial_2}{\partial q_i} = \frac{\partial}{\partial q_i} + \frac{\partial}{\partial q_1} \frac{\partial h}{\partial q_i}, \quad i=1,3 \ldots\ldots (4.26)$$

and

$$\Delta_2 = \frac{1}{c_2 c_3} \left[\sum_{i=2}^{3} \frac{\partial_2}{\partial q_i} \left\{ \frac{c_1 c_{i+1}}{c_i} \frac{\partial_2}{\partial q_i} \right\} \right] \ldots (4.27).$$

In equation (4.27), if index exceeds 3, subtract 2. He also defined the operator $\bar{D}(F, h)$ as

$$\bar{D}(F, h) = \sum_{i=2}^{3} \frac{c_1}{c_i^2} \frac{\partial_2 F}{\partial q_i} \frac{\partial_2 h}{\partial q_i} \quad \ldots\ldots\ldots\ldots (4.28),$$

where F is a function given by

$$F = F(q_i, i=1,3) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(4.29).$$

The fundamental equation is then derived as (Molodenskii, 1962, 94)

$$h_D = \frac{1}{2\pi\gamma} \iint_R \left[\left\{ \underline{\nabla} \cdot \underline{N} \frac{1}{r} - \frac{1}{\gamma} \left| \frac{\partial\gamma}{\partial h} \right| \frac{\sec \beta}{r} - \bar{D}\left(\frac{1}{r}, h\right) \cos \beta - \Delta_2 h \frac{\cos \beta}{r} \right\} \gamma \, h_D + \Delta g_F \frac{\sec \beta}{r} \right] dR \quad \ldots(4.30).$$

Molodenskii defines the solution obtained by the use of equation (4.30) as the quasi-geoid, as the height anomalies obtained do not refer to any regular surface. The quasi-geoid is the surface traced out by the ordinates of height h_D above the reference spheroid. This quasi-geoid is identical with the telluroid as defined in section (3.1). However, the term telluroid was initially used by **Hirvonen who** defined it as the surface obtained by the use of free air anomalies in Stokes' integral. The calculations involved in this method of solution are no easier than those completely developed in the next section. Hirvonen's telluroid is in reality the free air geoid. As will be shown later, this surface is quite useful in that it is closely related, though only approximately so, to the quantities determined in astro-geodesy. The free air geoid does not have the same physical meaning as the geoid itself, which however, is dependent on the definition of the topography exterior to it for its complete specification.

In attempting to solve the problem defined in the title of this investigation, no advantage was apparent in adopting either the approach of Molodenskii or the solution defined in equation (4.22) as neither give solutions of direct relevance. Instead, it is more to the point to revert to the basic equation (4.21) and consider its application to the geoid-spheroid system.

The geoid coincides with the physical surface of the earth over oceans and thus admits simple definition. In continental areas, the geoid, being a surface of equal potential, apparently presents problems of definition. The potential ϕ at any point is given by

$$\phi = \iiint_V \frac{k \rho \, dV}{r}$$

and when $r \to 0$, $1/r \to \infty$. There appears to be an inconsistency at the point $r = 0$, which could be occupied by matter. On the other hand, the existence of potential at the geoid in continental areas is admissible in the case of a mine shaft driven down to sea level. The geoid, as defined in the next section will be the locus of points in continental areas where the potential of the existent earth system would be equal to the potential of the oceans had there been no matter "at the point." As the value of $k \simeq 10^{-7}$, the radius of the locatopn defined by the term " at the point " can be as small as desired as

$$\lim_{r \to 0} \frac{dV}{r} \simeq r^2 .$$

As such, it is possible to define the

potential at points which are occupied by matter by the conventional mathematical model, there being no error of practical significance in the value computed using equation (1.1).

5. THE GEOID SPHEROID SYSTEM

5.1 Modifications to the basic system.

Unlike the telluroid - physical surface of the earth system, both the spheroid and the geoid are equipotential surfaces. As such, the angle between the reference surface and the equipotential surface of reference is zero. Thus,

$$\beta_1 = \beta_2 = \beta = 0 \quad \ldots\ldots\ldots\ldots\ldots(5.1),$$

and equation (4.21) reduces to

$$V_{D_P} = 2\phi_{e_P} + \frac{1}{2\pi\gamma}\iint_{R}\left[\left\{\frac{1}{\gamma}\left|\frac{\partial\gamma}{\partial h}\right|\frac{1}{r} + \underline{\nabla}\,.\,\underline{N}\,\frac{1}{r}\right\}V_D + \frac{\Delta g_o}{r}\right]dR \quad \ldots\ldots\ldots\ldots\ldots(5.2),$$

where ϕ_{e_P} is the potential of all matter exterior to the geoid as calculated at a point $\bar{P}$ on the geoid.

All other terms refer to the geoid itself and the major change of interpretation is that of the gravity anomaly. The definition of the quantity Δg_F used in equation (4.21) is given in equation (3.17) as the difference between gravity at a given point on the actual surface due to the existent earth and that due to the reference system at the associated point. In equation (5.2), the former value is not the value of gravity as observed at the surface of the earth, but is the value of gravity had it been observed at the geoid.

Bruns' theorem, when applied to the geoid spheroid separation gives the actual separation of the geoid and the spheroid, provided the potential of the spheroid is equal to that of the geoid. If N is this separation, equation (5.2) reduces to

$$N_P = 2\frac{\phi_{e_P}}{\gamma} + \frac{1}{2\pi\gamma}\iint_R \left[\left\{\frac{1}{\gamma}\left|\frac{\partial\gamma}{\partial h}\right|\frac{1}{r} + \underline{\nabla}\cdot\underline{N}\,\frac{1}{r}\right\}V_D + \frac{\Delta g_o}{r}\right] dR \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.3)$$

In practice, the integral set out in equation (5.3) is solved by resorting to the adoption of a spherical model for the surface R. As f is of order 3×10^{-3}, the resulting **error** introduced in the value of N_P is about **30** cm if N_P is not greater than 100 metres. On this basis, considering the earth to be a sphere of radius R_m, where R_m is the radius of a sphere with with the same volume as the spheroid of reference, given by (Bomford, 1962, 497)

$$R_m = a(1 - \frac{1}{3}f) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.4) \; ,$$

where a is the equatorial radius of the reference spheroid and f is its flattening. From fig (5.1), it can be seen that the arc distance on such a sphere between a surface element dR situated at Q (ϕ , λ) and the computation point P (ϕ_P, λ_P) is given by

$$\psi = \cos^{-1}\left\{\sin\phi\sin\phi_P + \cos\phi\cos\phi_P\cos d\lambda\right\} . (5.5),$$

where

$$d\lambda = \lambda_P - \lambda \quad \ldots\ldots\ldots\ldots\ldots(5.6).$$

In computations at any one point, the sphere is orientated at the point P and the evaluation of $\underline{\nabla} \cdot \underline{N} \frac{1}{r}$ is only dependent on the variation of r with the displacement of the radius vector with respect to the moving point. The true length r in the general case is given by

$$r = \left[R_P^2 + R_Q^2 - 2 R_P R_Q \cos\psi \right]^{\frac{1}{2}} \quad ....(5.7),$$

where R with the appropriate suffix is the length from the centre of the reference system.

$$\underline{\nabla} \cdot \underline{N} \frac{1}{r} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} \underline{i} \cdot \sum_{i=1}^{3} \frac{\partial x_i}{\partial n_i} \underline{i} \quad \frac{1}{r} \quad ,$$

where differentials with respect to n denote normal derivatives. In the spherical case, putting $R_P = R_Q = R$

$$\underline{\nabla} \cdot \underline{N} \frac{1}{r} = \frac{\partial}{\partial n} \frac{1}{r} = \frac{\partial}{\partial R} \frac{1}{r}$$

$$= - \frac{1}{r^3}(2 R \sin^2 \psi /2) \quad ....(5.8),$$

and

$$r = 2 R \sin\psi /2 \quad ....(5.9)$$

Equation (5.8) simplifies to

$$\underline{\nabla} \cdot \underline{N} \frac{1}{r} = - \frac{1}{2 R r} \quad ....(5.10).$$

The mean value of gravity over a sphere of radius R (γ_m) is given by

$$\gamma_m = \frac{k M}{r^2}$$

, where M is the mass of the earth, assumed to be a sphere and k is the gravitational constant. As shown in section (3.3),

$$\left|\frac{\partial\gamma}{\partial h}\right| \doteq \frac{2\,\gamma_m}{R} \quad .$$

Substitution of these values in equation (5.3) gives

$$N_P = \frac{2\,\phi_{e_P}}{\gamma_P} + \frac{1}{2\pi\,\gamma_P} \iint_R \left[\frac{1}{\gamma_m} \frac{2\,\gamma_m}{R} \frac{1}{r} - \frac{1}{2\,R\,r} V_D + \frac{\Delta g_o}{r} \right] dR$$

$$= \frac{2\,\phi_{e_P}}{\gamma_m} + \frac{1}{2\,\pi\,\gamma_m} \iint_R \left[\frac{3\,V_D}{2\,Rr} + \frac{\Delta g_o}{r} \right] dR \quad ..(5.11),$$

as $\gamma_P = \gamma_m + o\{\gamma \times 10^{-3}\}$. The surface element of the mean sphere is given by

$$dR = R^2\, d\sigma \quad ....(5.12),$$

where $d\sigma$ is the element of surface area on a unit sphere. A slight re-arrangement of terms gives

$$N_P = \frac{2\,\phi_{e_P}}{\gamma_m} + \frac{R}{4\pi\gamma_m} \int_0^{\sigma=4\pi} \left[\frac{3V_D}{\Delta g_o} + 2R \right] \frac{\Delta g_o}{r} \cdot d\sigma$$

$$............(5.14).$$

All quantities in equation (5.14) refer to the geoid, Δg_o being the difference between actual gravity as reduced to the geoid, without removal of topography and theoretical gravity obtained from the concept of the reference surface.

5.2 A generalised relationship between the disturbing potential and the gravity anomaly

The disturbing potential (V_D) is given by equation (3.4) and is due to deviations in the variation of density as specified by the reference model of the earth. These are due to structural differences between the existent earth and the reference model. These deviations would be due to

(i) the effect of the topography, as there are no masses in the reference system exterior to the reference spheroid ;

(ii) mass anomalies below the spheroidal surface which cause the existent earth to deviate from the perfectly stratified model. Let V_D be expressed as

$$V_D = \phi_e + V_{D_i} \qquad \text{.................(5.15)} ,$$

where ϕ_e is the potential at the geoid due to matter exterior to it and the term V_{D_i} is due to internal mass anomalies.

The application of Green's third identity (equation (2.20)) to the potential arising from the internal density anomalies at points on the geoid gives

$$\iiint_{V_i} \frac{1}{r} \underline{\nabla}^2 V_{D_i} \, dV = - 2\pi V_{D_i} + \iint_S \left[\frac{1}{r} \underline{\nabla} \cdot \underline{N} V_{D_i} - V_{D_i} \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right] dS \qquad \text{.............(5.16)}$$

At any point on the bounding surface,

$$\underline{\nabla} \cdot \underline{N} \frac{1}{r} = - \frac{\underline{N} \cdot \underline{R}}{r^3}$$

The use of this result in equation (2.11) gives

$$\iint_S \underline{\nabla} \cdot \underline{N} \frac{1}{r} \, dS = -2\pi \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5.16a).$$

The term on the left hand side of equation (5.16) can be simplified as follows by the use of equation (2.18).

$$\iiint_{V_i} \frac{1}{r} \underline{\nabla}^2 V_{D_i} \, dV_i = -\iiint_{V_i} \frac{4\pi k \, d\rho}{r} \, dV_i$$

$$= -4\pi V_{D_i} \qquad \ldots\ldots\ldots\ldots\ldots\ldots (5.17).$$

The quantity $d\rho$ used in the above development is the density anomaly which gives rise to the disturbing potential. The us e of these results transform equation (5.16) into

$$-4\pi \, V_{D_i} = \iint_S \frac{1}{r} \frac{\partial}{\partial n} V_{D_i} \, dS \quad \ldots\ldots\ldots\ldots (5.18).$$

Alternately,

$$V_{D_i} = -\frac{1}{4\pi} \iint_S \frac{1}{r} \frac{\partial V_{D_i}}{\partial n} \, dS \qquad \ldots\ldots (5.19).$$

V_{D_i} can thus be expressed as

$V_{D_i} = \frac{1}{4\pi k}$ Potential at P on a surface S of a coating of density $\partial V_{D_i} / \partial n$ at all points on S .... (5.20).

The entire disturbing mass internal to the geoid can be considered as condensed on the geoid itself and hence its distribution and stratification internally need not be known. Similarly, ϕ_e is a function of

$$\iint_S \frac{k\rho h}{r}\, dS$$ as shown in section (6), where

h is the elevation of the tcpography exterior to the geoid and at a distance r from the computation point. Thus ϕ_e is also capable of the same sort of definition and the total disturbing potential can be considered as a function of position on the surface S and hence, a function of the form

$$V_D = \iint_S \frac{a}{r}\, dS \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.21).$$

Equation (5.21) is structured in a manner which is similar to equation (2.13). 1/r, from equation (5.7) can be expressed as

$$\frac{1}{r} = \frac{1}{R}(1 - 2t\cos\psi + t^2)^{-\frac{1}{2}} \qquad \ldots\ldots\ldots\ldots(5.22),$$

where

$$t = \frac{R_P}{R} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.23).$$

Equation (5.22) can be expressed (Whittaker and Watson, 1963, 302) as

$$\frac{1}{r} = \frac{1}{R}\sum_{n=0}^{\infty} t^n P_n(\cos\psi) \qquad \ldots\ldots\ldots\ldots\ldots(5.24),$$

where $$P_n(\cos\psi) = \sum_{r=0}^{m} (-1)^r \frac{(2n-2r)!}{2^n\, r!\,(n-r)!\,(n-2r)!}\cos^{n-2r}\psi$$

$$m = \begin{cases} n/2 & \text{if n is even} \\ (n-1)/2 & \text{if n is odd} \end{cases} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(5.25).$$

For any given point P, the expression for $1/r$, given in in equation (5.24) can be expanded in the form

$$\frac{1}{r} = \frac{1}{R} \sum_{i=0}^{\infty} \frac{A_i'}{R^i} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.26).$$

Thus, equation (5.21) can be re-written as

$$V_D = \frac{1}{R} \sum_{i=0}^{\infty} \frac{A_i}{R^i} \quad \ldots\ldots\ldots\ldots\ldots\ldots (5.27)$$

as $t \to 1$ in the case of a spherical approximation of the earth. If R is measured from the centre of mass, the use of equations (5.7) and (5.21) gives

$$V_D = \iint_S \frac{k\,dm}{r}\,dS = \iint_S \frac{k\,\Delta\rho}{R}\left(1 + t\cos\psi - \frac{1}{2}t^2 + \right.$$

$$\left. + \frac{3}{2}t^2\cos^2\psi + \ldots\right) dS$$

$$= \iint_S \frac{k\Delta\rho}{R}\,dS + \iint_S k\Delta\rho\,\frac{t\cos\psi}{R}\,dS + k\iint_S \Delta\rho\,\frac{3}{2}t^2(\cos^2\psi -$$

$$- 1)\,dS + \ldots\ldots \quad \ldots\ldots\ldots (5.28),$$

where dm is the mass anomaly of density $\Delta\rho$ condensed on the surface element dS. The first term is equal to zero if the mass of the reference spheroid is equal to that of the existent earth. However, the implication of this term in the geoid - spheroid system where the exterior topography has not been smoothed is discussed more fully in section (13). The swecond term

$$= \iint_S \frac{R_P\cos\psi \quad k\,\Delta\rho}{R^2}\,dS \qquad \text{and will be zero}$$

if the earth's centre of mass coincided with the origin of length (i.e., the centre of the reference spheroid). If these two conditions are defined in the postulation of the parameters of the reference figure whose gravitational potential is U, the disturbing potential V_D can be represented by

$$V_D = \sum_{i=2}^{\infty} \frac{A_i}{R^{i+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5.29),$$

$$\text{where} \quad A_i = a_i S_i \quad \ldots\ldots\ldots\ldots\ldots(5.30),$$

the quantities a_i and S_i are defined in section (7.4). From equation (3.20), if the potential on the reference spheroid is equal to that on the geoid,

$$\Delta g_o = - \left[\frac{\partial V_D}{\partial h} + \frac{2V_D}{R} \right] .$$

Differentiation of equation (5.29) with respect to R gives

$$\frac{\partial V_D}{\partial h} = - \sum_{n=2}^{\infty} (n+1) \frac{A_n}{R^{n+2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots(5.31).$$

Thus Δg_o is given by

$$\Delta g_o = \sum_{n=2}^{\infty} \frac{(n-1) A_n}{R^{n+2}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(5.32)$$

From equation (5.27), it can be seen that the disturbing potential (V_D) has the same form as equation (2.13) and the quantity a in the latter will vary about a mean value of zero, both positive and negative values being admissible.

The disturbing potential and the related set of gravity anomalies could also be represented by a spherical harmonic series (Jeffreys, 1962b, 635) of the form

$$\Delta g_o = g_{nm} S_{nm}$$

$$= \sum_{n=2}^{\infty} \sum_{m=0}^{n} p_{nm}(\sin \phi) \left\{ g_{c_{nm}} \cos m\lambda + g_{s_{nm}} \sin m\lambda \right\} \quad \ldots\ldots\ldots (5.33),$$

where the associated Legendre function $p_{nm}(\mu)$ is given by

$$p_{nm}(\mu) = \frac{(n-m)!}{2^n (n!)^2} \cos^m \phi \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n \quad \ldots\ldots\ldots (5.34)$$

If the disturbing potential were then represented by equation (5.29),

$$\frac{(n-1) A_n}{R^{n+2}} = g_{nm} S_{nm} \quad \ldots\ldots\ldots\ldots\ldots (5.35)$$

or

$$A_n = \frac{R^{n+2}}{(n-1)} g_{nm} S_{nm} \quad \ldots\ldots\ldots\ldots (5.36).$$

As A_n represents the mass deviations of the existent earth from the adopted model which totals zero over the whole surface, it can also be expressed as a set of surface harmonics as shown in equation (5.30), as

$$V_D = \sum_{n=2}^{\infty} \frac{1}{R^{n+1}} \sum_{m=0}^{n} p_{nm}(\sin \phi) \left\{ a_{nm} \cos m\lambda + b_{nm} \sin m\lambda \right\} \quad \ldots\ldots\ldots (5.37).$$

Combining equations (5.36) and (5.37),

$$V_D = \sum_{n=2}^{\infty} \frac{R}{(n-1)} \sum_{m=0}^{n} p_{nm}(\sin \phi) \left\{ g_{c_{nm}} \cos m\lambda + \right.$$

$$\left. + g_{s_{nm}} \sin m\lambda \right\} \quad \ldots\ldots\ldots(5.38).$$

In equations(5.33) through (5.38), the quantities $g_{c_{nm}}$, $g_{s_{nm}}$, a_{nm} and b_{nm} are harmonic coefficients.

5.3 The evaluation of the basic integral for the geoid spheroid system.

The quantity to be evaluated in equation (5.14) is

$$F = 3V_D + 2R \, \Delta g_o$$

$$= 3 \sum_{n=2}^{\infty} \frac{A_n}{R^{n+1}} + 2R \sum_{n=2}^{\infty} \frac{(n-1) A_n}{R^{n+2}},$$

where V_D and Δg_o have been evaluated using equations (5.29) and (5.32). The use of equation (5.36) gives

$$F = \sum_{n=2}^{\infty} \frac{(2n+1) A_n}{R^{n+1}} = \sum_{n=2}^{\infty} \frac{(2n+1)}{(n-1)} R \, g_{nm} S_{nm} \quad \ldots\ldots\ldots\ldots\ldots(5.39)$$

The combination of the results obtained in equations (5.7), (5.24), (5.39) with equation (5.14) gives, on evaluation with R_m as the mean radius of the earth and R as the geocentric distance in the appropriate expressions,

$$N_P = \frac{2\phi_{e_P}}{\gamma_m} + \frac{R_m}{4\pi \gamma_m} \int_0^{\sigma=4\pi} \frac{R_m}{R} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} g_{nm} S_{nm} \, .$$

$$\sum_{i=0}^{\infty} \left[\frac{R_m}{R} \right]^i p_i(\cos\psi) \, d\sigma \, . \, .$$

Thus

$$N_P = \frac{2\phi_{e_P}}{\gamma_m} + \frac{R_m}{4\pi\gamma_m} \int_0^{\sigma=4\pi} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left[\frac{R_m}{R}\right]^{n+1} g_{nm} S_{nm} \cdot p_n(\cos\psi)\, d\sigma \quad \ldots\ldots..(5.40),$$

as (Jeffreys, 1962a, 636) the product of the surface harmonics of different orders integrated over the surface of a sphere is zero. I.e.,

$$\iint_S S_m S_n \, dS = 0 \quad \text{if } m \neq n .$$

Extending this deduction further using equation (5.33), it can be seen that Δg_o can **replace** $g_{nm} S_{nm}$ in equation (5.40) without affecting the result. Thus,

$$N_P = \frac{2\phi_{e_P}}{\gamma_m} + \frac{R_m}{4\pi\gamma_m} \sum_{n=2}^{\infty} \int_0^{\sigma=4\pi} \left[\frac{R_m}{R}\right]^{n+1} \Delta g_o \, p_n(\cos_\psi)\, d\sigma \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots..(5.41).$$

Equation (5.41) holds on the geoid and the gravity anomaly used is the free air anomaly <u>on the geoid</u>, i.e., the difference between observed gravity, had it been observed on the geoid and normal gravity at the spheroid on the reference system. Thus observed gravity must be correctly reduced to the geoid prior to its use in the above integral. $p_n(\cos_\psi)$ is a zonal harmonic and $d\sigma$ is unit element of surface area on a sphere of unit radius.

Consider the second expression ($Q(\psi)$) on the right hand side of equation (5.41), given by

$$Q(\psi) = \frac{R_m}{4\pi\gamma_m} \lim_{R \to R_m} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \int_0^{\sigma=4\pi} \left[\frac{R_m}{R}\right]^{n+1} .$$

$$\Delta g_o \; p_n(\cos\psi) \; d\sigma$$

$$= \frac{R_m}{4\pi\gamma_m} \int_0^{\sigma=4\pi} f(\psi)\Delta g_o \, d\sigma \quad \text{.........(5.42)} ,$$

where

$$f(\psi) = \lim_{R \to R_m} \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \, p_n(\cos\psi) \left[\frac{R_m}{R}\right]^{n+1} \quad \text{..(5.43)}$$

$$= \lim_{R \to R_m} R_m \left[f_1(\psi) + f_2(\psi) \right] \quad \text{.........(5.44).}$$

The quantities $f_1(\psi)$ and $f_2(\psi)$ in equation (5.44) are given by

$$f_1(\psi) = 2 \sum_{n=2}^{\infty} \frac{R_m^n}{R^{n+1}} \, p_n(\cos\psi) \quad \text{.......(5.45)}$$

and

$$f_2(\psi) = 3 \sum_{n=2}^{\infty} \frac{R_m^n}{(n-1)R^{n+1}} \, p_n(\cos\psi) \quad \text{....(5.45a)}$$

Using equation (5.24) in equation (5.45),

$$f_1(\psi) = 2\left[\frac{1}{r} - \frac{1}{R} - \frac{R_m}{R^2}\cos\psi\right] \quad \text{......(5.46)}$$

The right hand side of equation (5.45a) can be re-written as

$$f_2(\psi) = \frac{3}{R^2} \int_R^{\infty} R \sum_{n=2}^{\infty} \frac{R_m^n}{R^{n+1}} \, p_n(\cos\psi) \; dR$$

$$f_2(\psi) = \frac{3}{R^2} \int_R^\infty R \left[\frac{1}{r} - \frac{1}{R} - \frac{R_m}{R^2} \cos\psi \right] dR \quad ..(5.47).$$

The right hand side of equation (5.47) is integrated (Stokes, 1849) by rationalisation. The first term is given by

$$\int_R^\infty \frac{R}{r}\, dR = \int_R^\infty \frac{R\, dR}{(R^2 + R_m^2 - 2 R R_m \cos\psi)^{1/2}}$$

The differentiation of equation (5.7) gives

$$dr = \frac{R - R_m \cos\psi}{r}\, dR \quad ....(5.48)$$

Thus,

$$\int_R^\infty \frac{R}{r}\, dR = \int_R^\infty \frac{R - R_m \cos\psi}{r} dR + \int_R^\infty \frac{R_m \cos\psi}{r}\, dR$$

$$= \left[r\right]_R^\infty + R_m \cos\psi \int_R^\infty \frac{r+R-R_m \cos\psi}{r(r+R-R_m \cos\psi)}\, dR$$

$$= \left[r\right]_R^\infty + R_m \cos\psi \int_R^\infty \frac{1}{r+R-R_m \cos\psi} \left[1 + \frac{R - R_m \cos\psi}{r}\right] dR .$$

From equation (5.48), it can be seen that

$$\int_R^\infty \frac{R}{r} dR = \left[r\right]_R^\infty + R_m \cos\psi \int \frac{d(r+R-R_m \cos\psi)}{r+R-R_m \cos\psi} \quad .$$ Thus

$$f_2(\psi) = \frac{3}{R^2} \left[r + R_m \cos\psi \ln(r+R-R_m \cos\psi) - R_m \cos\psi \ln R - - R \right]_R^\infty \quad ....(5.49).$$

As $R \to \infty$, $r - R \to R(1+\frac{R_m^2}{R} - 2R_m \cos\psi)^{1/2} - R$

Thus, $r - R \to -R_m \cos\psi$ ....(5.50).

Further,

$$\lim_{R \to \infty} \ln \frac{r + R - R_m \cos\psi}{R} = \ln 2.$$ Thus, the upper

limit of $f_2(\psi)$, on evaluation gives $R_m \cos\psi$ (ln 2 - 1). Hence,

$$f_2(\psi) = \frac{3}{R^2}\left[R - R_m \cos\psi - r - R_m \cos\psi \ln \frac{r+R-R_m \cos\psi}{2R}\right]$$

....................(5.51).

The use of equations (5.46) and (5.51) in equation (5.44) gives

$$f(\psi) = \lim_{R \to R_m} R_m \left[2\left\{\frac{1}{r} - \frac{1}{R} - \frac{R_m \cos\psi}{R^2}\right\} - \frac{3}{R^2}\left\{r + R_m \cos\psi \ln \frac{r+R-R_m \cos\psi}{2R} + R_m \cos\psi - R\right\}\right] \ldots (5.52).$$

When $R \to R_m$, $r = 2R_m \sin\frac{\psi}{2}$. Thus,

$$f(\psi) = \operatorname{cosec}\frac{\psi}{2} - 2 - 2\cos\psi - 6\sin\frac{\psi}{2} + 3 - 3\cos\psi - 3\cos\psi \ln\left\{\sin\frac{\psi}{2}\left(1 + \sin\frac{\psi}{2}\right)\right\}.$$

Regrouping of terms gives the final form of $f(\psi)$ as

$$f(\psi) = \operatorname{cosec}\frac{\psi}{2} + 1 - 5\cos\psi - 6\sin\frac{\psi}{2} - 3\cos\psi \ln\left\{\sin\frac{\psi}{2}\left(1 + \sin\frac{\psi}{2}\right)\right\}$$ ....(5.53).

$f(\psi)$, as determined in equation (5.53) is commonly known as Stokes' function, as it was first derived by Stokes in 1849, though under certain specific conditions. Equation (5.53) provides the required solution for a spherical approximation which is introduced in setting the conditions for the limit. The substitution of this result in equations (5.41) and (5.42) gives the complete expression for N_P as

$$N_P = \frac{2\,\phi_{e_P}}{\gamma_m} + \frac{R_m}{4\,\pi\gamma_m} \int_0^{\sigma=4\pi} f(\psi)\ \Delta g_o\ d\sigma \quad \ldots\ldots\ldots(5.54),$$

where $f(\psi)$, to the order of the flattening, is given by equation (5.53) and

Δg_o is the difference between observed gravity on the geoid and normal gravity on the reference spheroid. It is important to note that, for a complete solution, it is necessary to consider two terms :-

(i) The Stokesian effect, due to the term on the right, which, in the specific case of a bounding equipotential is Stokes' integral. If, on the other hand, the existent earth is considered, the anomaly to be used is the free air anomaly at the geoid. This is not the same quantity as the anomaly commonly referred to as the free air anomaly and used in compiling the free air geoid. The conventional free air anomaly is defined as

$$g_F = g_P - \gamma_Q \doteq g_P - \gamma_o + \left|\frac{\partial\gamma}{\partial h}\right| h \quad \ldots(5.55),$$

where g_P is the value of observed gravity at a point P on the earth's surface and γ_Q is the value of normal gravity on the reference system at Q as defined in section (3.1). For the purpose of determining the magnitude of the free air reduction, it would suffice to consider the reference system as being spherical in shape, when Newtonian gravitation is given by

$$\gamma = \frac{k\,M}{R^2} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5.56),$$

where R is the distance of the point from the centre of mass.

$$\left|\frac{\partial \gamma}{\partial h}\right| = \frac{2\gamma}{R} \doteqdot 0.3086 \times 10^{-5}\ \text{sec}^{-2} \ldots\ldots\ldots (5.57)$$

if mean values for γ and R. The term $\left|\frac{\partial \gamma}{\partial h}\right| h$ is called the free air reduction and the terms excluded in formulating equation (5.55) are over two orders smaller than the term considered.

(ii) The effect of the topography exterior to the geoid. This contributes in two ways. Firstly, there is the first term on the right hand side of equation (5.54). Secondly, Δg_o in equation (5.54) is not the same as Δg_F expressed in equation (5.55). The observed value of gravity has to be reduced to the geoid. The value g_P has to be reduced to the value g_o for gravity at the geoid before comparson with the value of normal gravity γ_o on the reference spheroid. The true anomaly to be used in the case of the geoid spheroid system is Δg_o, given by

$$\Delta g_o = g_o - \gamma_o \quad \ldots\ldots\ldots\ldots\ldots\ldots (5.58).$$

Let the quantity Δg_{co} be defined by the equation

$$\Delta g_o = \Delta g_F + \Delta g_{co} \quad \ldots\ldots\ldots\ldots (5.59).$$

From equation (1.3), it can be seen that the actual separation of the geoid and the spheroid (N_P) and the separation of the spheroid and the free air geoid (N_{F_P}) is given by

$$N_P = \frac{2\,\phi_{e_P}}{\gamma_m} + N_{F_P} + \frac{R_m}{4\,\pi\gamma_m} \int_{0}^{\sigma=4\,\pi} f(\psi)\,\Delta g_{co}\,d\sigma \quad \ldots(5.60).$$

The indirect effect (N_i) defined in equation (1.4), in the case of the free air geoid is given by

$$N_{i_P} = \frac{2\,\phi_{e_P}}{\gamma_m} + \frac{R_m}{4\,\pi\gamma_m} \int_{0}^{\sigma=4\pi} f(\psi)\,\Delta g_{co}\,d\sigma \quad \ldots(5.61),$$

where Δg_{co} would be the differential effect in the attraction of the topography between the point P on the earth's surface and its equivalent point P_o on the geoid.

6. THE INDIRECT EFFECT FOR THE FREE AIR GEOID

6.1 The nature of the correction Δg_{co}.

Consider the system of masses that constitute the the existent earth to be divided into two sections corresponding to matter within and external to the geoid (S). If the internal matter alone gave rise to values of gravity g_{io} at P_o, situated on the geoid at an elevation h below an equivalent point P on the earth's surface. Let the value of gravity at P due to the internal masses only be g_i. Then

$$g_i = g_{io} - c'_F \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.1),$$

where

$$c'_F = \left|\frac{\partial g_i}{\partial h}\right| = \frac{2\, g_i}{R_m} + o\{\, c' \times 10^{-3}\} \ldots\ldots\ldots(6.2)$$

If the topography exterior to the geoid were now superimposed, an increased (algebraic) attractive effect is introduced at both P (Δg_e) and P_o(Δg_{eo}). The true value of gravity at the geoid (g_o) is given by

$$g_o = g_{io} + \Delta g_{eo} \quad \ldots\ldots\ldots\ldots(6.3)$$

The value of gravity (g) at P due to the existent earth is given by

$$g = g_i + \Delta g_e \ldots\ldots\ldots\ldots\ldots\ldots(6.4).$$

The use of equations (6.1), (6.2) and (6.4)

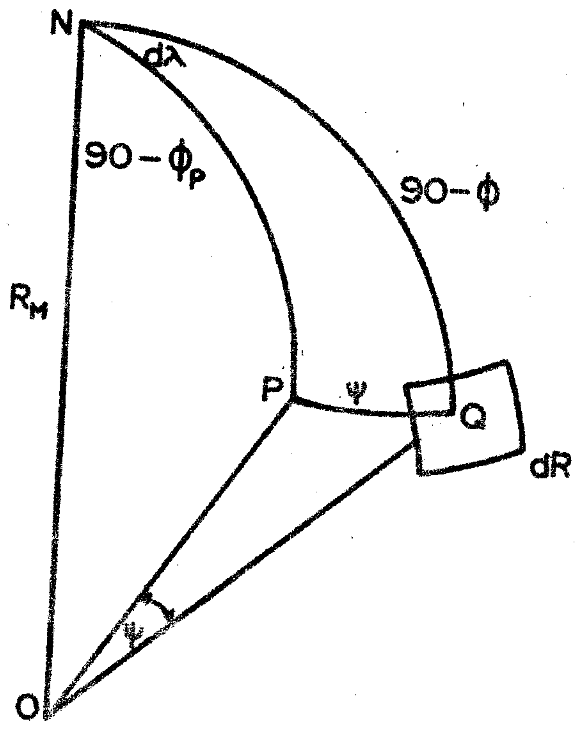


FIG. 5.1

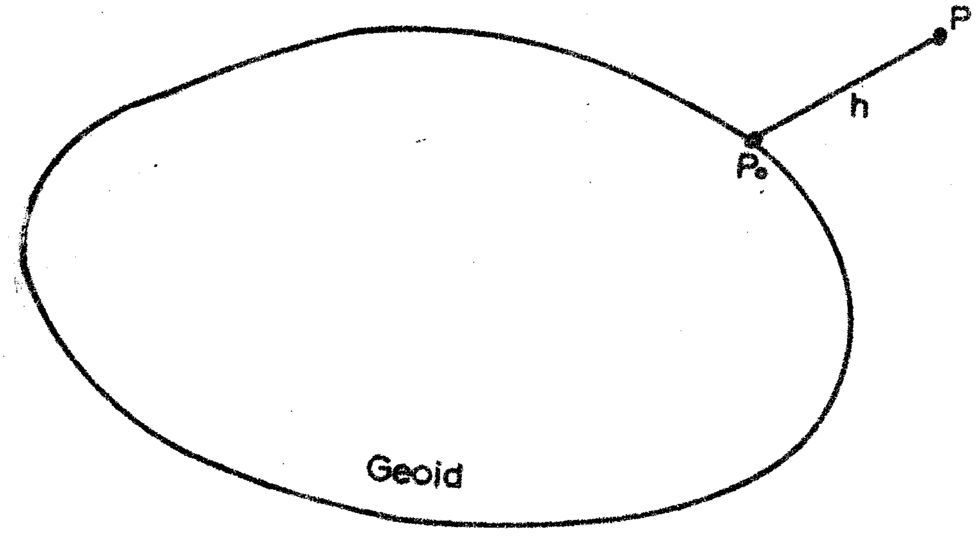


FIG. 6.1

gives

$$g \quad = \quad g_{io} \quad - \frac{2\,g_i}{R_m}\,h \quad + \quad \Delta\,g_e \qquad \ldots\ldots\ldots\ldots(6.5).$$

The combination of equations (6.3) and (6.5) gives

$$g_o \quad = \quad g \quad + \frac{2\,g_i}{R_m}\,h + \Delta\,g_{eo} - \Delta\,g_e \quad \ldots.(6.6).$$

Equation (6.6) is fundamental and the attractions are considered vectorially, directed toward the centre of mass. The term $2g_ih/R_m$ is, to three significant figures, the free air reduction, outlined in equations (5.55) to (5.57). The gravity anomaly on the geoid is obtained by substituting equation (6.6) in equation (5.58).

$$g_o \quad = \quad g_o \quad - \quad \gamma_o$$

$$= \quad g \; + \frac{2\,\gamma_m}{R_m}\,h \; - \; \gamma_o \quad + \quad \Delta\,g_{eo} \; - \Delta\,g_e \ldots(6.7).$$

Comparison of equation (6.7) with equation (5.59) shows that the first three terms constitute the free air anomaly ($\Delta\,g_F$) and hence the quantity $\Delta\,g_{co}$ defined therein is given by

$$\Delta\,g_{co} \quad = \quad \Delta\,g_{eo} - \quad \Delta\,g_e \qquad \ldots\ldots\ldots\ldots\ldots\ldots(6.8).$$

$\Delta\,g_{co}$ is the difference between the attraction of the topography exterior to the geoid as exerted at a point on the geoid and that at the point on the earth's surface which lies on the same normal (see fig (6.1)). This correction is called the differential topographical effect and differs from all the conventional reduction previously used in the solution of the boundary value problem in physical geodesy.

6.2 The evaluation of the differential topographical effect Δg_{co}

The matter exterior to the geoid can be considered to consist of a series of columns of matter on an elemental base, which is the element of surface area dS on the geoid. The base areas at the variable point Q on the geoid, as shown in fig (6.2) are given by

$$dS = R_Q^2 \, d\psi \, d\alpha \, \sin\psi$$
$$= R_Q^2 \, d\phi \, d\lambda \, \cos\phi \quad \ldots\ldots\ldots\ldots\ldots (6.9),$$

R_Q being the distance from the geocentre O to Q. In equation (6.9), ψ is the angular displacement at O between the lines OQ and OP, P being the computation point, α is the azimuth of Q from P, ϕ and λ being the latitude and longitude of Q. From fig (6.3) it can be seen that the geocentric distance R is related to the equatorial radius a and the flattening f of the meridian ellipse by the relation (Bomford, 1962, 496)

$$R_{P(Q)} = a\,(1 - f \sin^2 \phi_{P(Q)} + o\{f^2\})\ldots\ldots (6.10),$$

where R_P is the distance from the geocentre to the point P_o on the geoid which represents the point P on the surface of the earth. The equation (6.10) holds at both P and Q if the appropriate suffix is used. Using the expression for the radius of a sphere of volume equal to that of the geoid (R_m) and hereafter called the mean radius, the combination of equations (5.4) and (6.10) gives

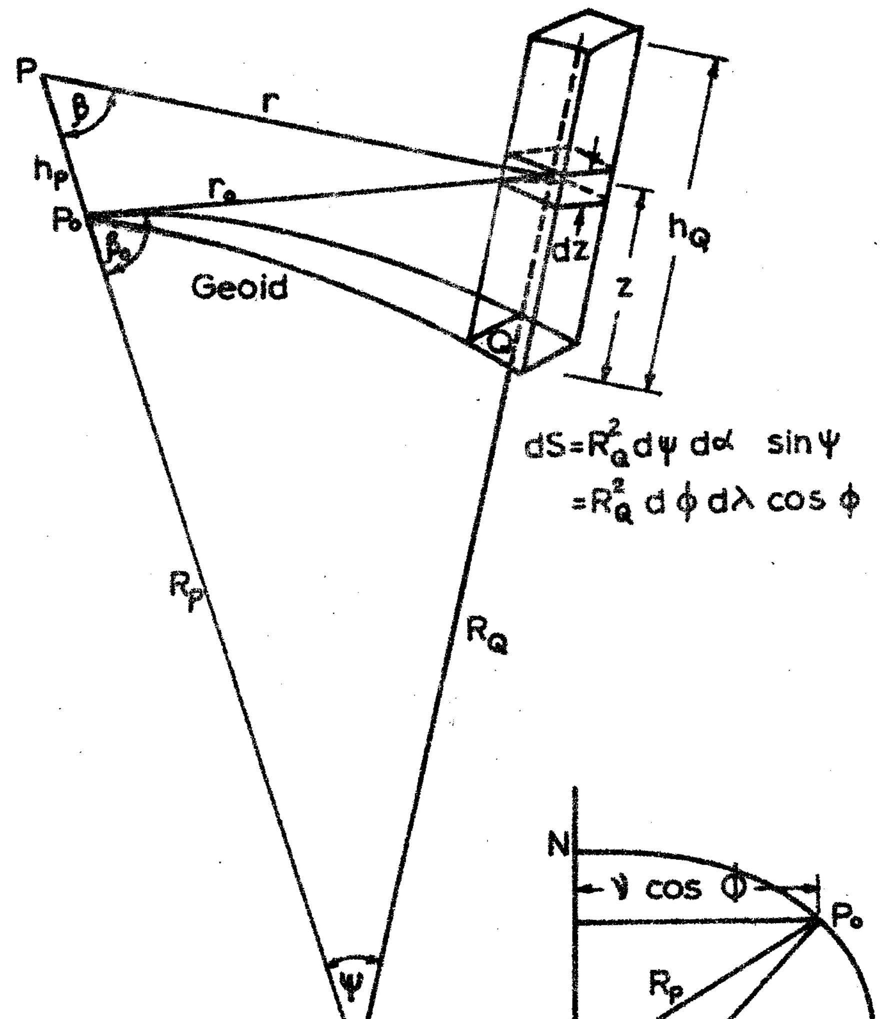

P
β
r
h_p
r_0
P_0
β_0
Geoid
dz
h_Q
z
Q
$dS = R_Q^2 \, d\psi \, d\alpha \, \sin\psi$
$= R_Q^2 \, d\phi \, d\lambda \, \cos\phi$
R_P
R_Q
ψ
O

FIG. 6.2

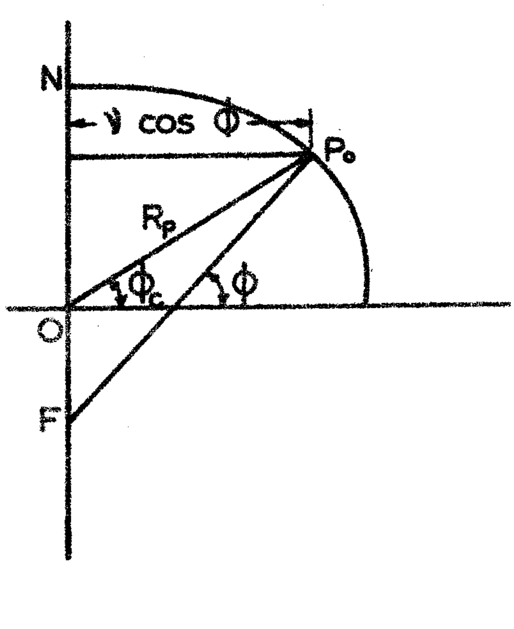

N
ν cos φ
P_0
R_P
ϕ_c
φ
O
F

FIG. 6.3

$$R_{P(Q)} = R_m(1 + c_{P(Q)}) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.11),$$

where

$$c_{P(Q)} = f\left(\frac{1}{3} - \sin^2 \phi_{P(Q)}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.12)$$

Let the column of matter on a spheroidal base area dS extending to a height h_Q above the geoid at Q be composed of a series of unit areas of thickness dz at a height z above the geoid and at distances r from P and r_o from P_o.

As $0 \leqq \psi \leqq \pi$,

$$r_o = \left\{(R_Q + z)^2 + R_P^2 - 2\,(R_Q + z)\,R_P \cos\psi\right\}^{1/2} \quad \ldots(6.13).$$

Defining

$$c = \frac{z}{R_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.14),$$

the substitution of values for R_P, R_Q and z from equations (6.11), (6.12) and (6.14) in equation (6.13) gives

$$r_o = R_m\left[1 + 2(c_Q + c) + 1 + 2c_P - 2(1 + c_Q + c_P + c)\cos\psi + o\{f^2\}\right]^{1/2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(6.15)$$

$$= 2R_m \sin\psi/2\left[1 + \frac{1}{2}(c_P + c_Q + c) + o\{f^2\}\right] \ldots.(6.16).$$

Defining

$$A = c_P - c_Q - c \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.17),$$

the substitution of equation (6.17) in (6.16) gives

$$r_o = 2R_m \sin\frac{\psi}{2}\left[1 - \frac{1}{2}(A - 2c_P) + o\{f^2\}\right] \ldots(6.18).$$

r is determined on similar lines. Defining c_1 as

$$c_1 = \frac{h_P}{R_m} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.19),$$

$$r = \left[(R_Q+z)^2 + (R_P+h_P)^2 - 2(R_Q+z)(R_P+h_P)\cos\psi\right]^{1/2} \quad \ldots (6.20).$$

The use of equations (6.11), (6.12), (6.14) and (6.19), as before, in equation (6.20) gives

$$r = R_m \left\{ 2(1 - \cos\psi)\left[(1 + c_P + c_Q + c + c_1) + o\{f^2\}\right]\right\}^{1/2}$$

$$= 2 R_m \sin\tfrac{1}{2}\psi \left[1 + \tfrac{1}{2}(c_P + c_Q + c + c_1) + o\{f^2\}\right] \ldots (6.21)$$

$$= 2 R_m \sin\tfrac{1}{2}\psi \left[1 - \tfrac{1}{2}(A - 2c_P - c_1) + o\{f^2\}\right] \ldots (6.22).$$

A consideration of fig (6.2) shows that the angle β_o is given by

$$\cos\beta_o = \frac{r_o^2 + R_P^2 - (R_Q + z)^2}{2 r_o R_P} \qquad \ldots\ldots\ldots\ldots (6.23).$$

The substitution of expressions for R_P, R_Q, z and r_o from equations (6.11), (6.12), (6.14) and (6.18) in equation (6.23) gives

$$\cos\beta_o = \frac{4\sin^2\tfrac{1}{2}\psi\left[1 - (A-2c_P)\right] + 1 + 2c_P - 1 + 2(c_Q+c) + 0\{f^2\}}{4\sin\tfrac{1}{2}\psi\left[\{1 - \tfrac{1}{2}(A-2c_P)\}(1+c_P) + 0\{f^2\}\right]}$$

$$= \sin\tfrac{1}{2}\psi\left[1 - (A-2c_P) + \tfrac{1}{2}A\,\mathrm{cosec}^2\tfrac{1}{2}\psi + o\{f^2\}\right]\left[1 + \right.$$

$$\left. + \tfrac{1}{2}(A - 4c_P) + o\{f^2\}\right] \ldots\ldots\ldots (6.24).$$

It can be seen that, in equation (6.24), the order of the third term in the numerator, for values of $\psi < 5^{o}$, is one greater than that of the second (10^{-3}). In the case of very small angles, the order of the neglected term of order f^2 when multiplied by $\operatorname{cosec}^2 \frac{1}{2}\psi$, for values of $\psi < 30'$ is greater than 10^{-3}. As such, it is necessary to define equation (6.11) more completely. Consideration of terms of order f^2 gives

$$R_{P(Q)} = R_m(1 + c_{P(Q)} + c'_{P(Q)} + o\{ f^3 \} \quad \ldots\ldots\ldots (6.25)$$

where

$$c'_{P(Q)} = f^2\left(\frac{2}{9} + \frac{5}{8}\sin^2 2\phi \right) \quad \ldots\ldots\ldots\ldots\ldots (6.26).$$

The inclusion of t erms of order f^2 in the second factor of equation (6.24) in those instances where multiplication by $\operatorname{cosec}^2 \frac{1}{2}\psi$ occurs gives

$$\cos\beta_0 = \sin\frac{1}{2}\psi \left[1 - (A-2c_P) + \frac{1}{4}\operatorname{cosec}^2\frac{1}{2}\psi \left[1 + 2c_P + c_P^2 + \right.\right.$$
$$\left.\left. + 2c'_P - \left\{ 1 + 2(c_Q+c) + (c_Q+c)^2 + 2c'_Q \right\} \right] + o\{ f^2, f^3\operatorname{cosec}^2\frac{1}{2}\psi\} \right]$$
$$\left[1 + \frac{1}{2}(A - 4c_P) \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.27)$$

The use of equation (6.17) simplifies equation (6.27) to

$$\cos\beta_o = \sin\frac{1}{2}\psi \left[1 - \frac{1}{2}A(1 - \operatorname{cosec}^2\frac{1}{2}\psi) + \frac{1}{4}\operatorname{cosec}^2\frac{1}{2}\psi \left\{ A(2c_P - \right.\right.$$
$$\left.\left. - A) + A(A-4c_P) + 2(c'_P - c'_Q) \right\} + 0\{ f^2, f^3\operatorname{cosec}^2\frac{1}{2}\psi\} \right]$$
$$\ldots\ldots\ldots\ldots\ldots\ldots (6.28)$$

Further simplification of equation (6.28) gives

$$\cos \beta_o = \sin \frac{1}{2}\psi \left[1 - \frac{1}{2} A(1 - \operatorname{cosec}^2 \frac{1}{2}\psi) + \frac{1}{2}\operatorname{cosec}^2 \frac{1}{2}\psi(c'_P - c'_Q - c_P A) + o\{f^2, f^3 \operatorname{cosec}^2 \frac{1}{2}\psi\} \right] \quad \ldots\ldots.(6.28).$$

On similar lines, β in fig (6.2) is given by

$$\cos \beta = \frac{r^2 + (R_P + h_P)^2 - (R_Q + z)^2}{2\, r\, (R_P + h_P)}$$

$$= \sin \frac{1}{2}\psi \Big\{ 1 - (A - 2c_P - c_1) + \frac{1}{4}\operatorname{cosec}^2 \frac{1}{2}\psi \Big[1 + 2(c_P + c_1) + (c_P + c_1)^2 + 2c'_P - \{1 + 2(c_Q + c) + (c_Q + c)^2 + 2c'_Q\} \Big] \Big\} \Big[1 - \frac{1}{2}(A - 2c_P - c_1)(1 + c_P + c_1) \Big]^{-1}$$

$$= \sin \frac{1}{2}\psi \Big[1 - \frac{1}{2}(A + c_1) + \frac{1}{2}(A + c_1)\operatorname{cosec}^2 \frac{1}{2}\psi + \frac{1}{4}\operatorname{cosec}^2 \frac{1}{2}\psi \big[(A + c_1)(A - 4c_P - 3c_1) + (A + c_1)(2c_P - A + c_1) + 2(c'_P - c'_Q) \big] + o\{f^2, f^3 \operatorname{cosec}^2 \frac{1}{2}\psi\} \Big]$$

Thus

$$\cos \beta = \sin \frac{1}{2}\psi \Big[1 - \frac{1}{2}(A + c_1)(1 - \operatorname{cosec}^2 \frac{1}{2}\psi) + \frac{1}{2}\operatorname{cosec}^2 \frac{1}{2}\psi \big(c'_P - c'_Q - (c_P + c_1)(A + c_1) \big) + o\{f^2, f^3 \operatorname{cosec}^2 \frac{1}{2}\psi\} \Big] \quad \ldots\ldots\ldots\ldots\ldots\ldots.(6.29)$$

Equations (6.28) and (6.29), while having more terms than necessary for any single evaluation of cos β and cos β_o adequately do so in every possible case to order f^2.

The differential topographical correction is given by equation (6.8)

$$\Delta g_{co} = \Delta g_{e_o} - \Delta g_e \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.8)$$

$$= \iiint \frac{k\, dm \cos \beta_o}{r_o^2} - \iiint \frac{k\, dm \cos \beta}{r^2}$$

$$= k \iiint dm \left[\frac{\cos \beta_o}{r_o^2} - \frac{\cos \beta}{r^2} \right],$$ and is

directed along the downward vertical at P. dm refers to the mass of the element considered in fig (6.2) and situated in the column of matter at Q. Substituting values for r_o, r, β_o and β from equations (6.18), (6.22), (6.28) and (6.29) in the above equation,

$$\Delta g_{co} = k \iiint \frac{dm \sin \frac{1}{2}\psi}{4R_m^2 \sin^2 \frac{1}{2}\psi} \left[\left\{ 1 - \frac{1}{2} A(1 - \operatorname{cosec}^2 \frac{1}{2}\psi) + \right.\right.$$

$$\left. + \frac{1}{2} \operatorname{cosec}^2 \frac{1}{2}\psi \,(c'_P - c'_Q - A\, c_P) \right\} \left\{ 1 + A - 2c_P \right\} - \left\{ 1 - \right.$$

$$\left. - \operatorname{cosec}^2 \frac{1}{2}\psi + \frac{1}{2} \operatorname{cosec}^2 \frac{1}{2}\psi \left[c'_P - c'_Q - (A + c_1)(c_P + c_1) \right] \right\}$$

$$\left. \left\{ 1 + (A - 2c_P - c_1) \right\} \right]$$

$$= k \iiint \frac{dm \operatorname{cosec} \frac{1}{2}\psi}{4 R_m^2} \left[\frac{3}{2} c_1 - \frac{1}{2} c_1 \operatorname{cosec}^2 \frac{1}{2}\psi - \frac{1}{2} \operatorname{cosec}^2 \frac{1}{2}\psi \right.$$

$$\left. \left\{ (A + c_1)(A - 3c_P - 2c_1) - A(A - 3c_P) \right\} + o\{ f^2, \operatorname{cosec}^2 \frac{1}{2}\psi\, f^3 \} \right]$$

Thus

$$\Delta g_{co} = k \iiint \frac{dm\, h_P \operatorname{cosec} \frac{1}{2}\psi}{8 R_m^3} \left[3 + \operatorname{cosec}^2 \frac{1}{2}\psi \;(2A + 3c_P + 2c_1) - \operatorname{cosec}^2 \frac{1}{2}\psi \;+ o\{ f^2, f^3 \operatorname{cosec}^2 \frac{1}{2}\psi \} \right] \quad ..(6.30).$$

The integration is carried out by substituting for A from equation (6.17) and evaluating dm by the relation

$$dm = \rho dA \left[1 + \frac{z}{R_m} \right]^2 dz \quad ,$$

where ρ is the density of matter. dm could be evaluated in terms of α, ψ and z using equation (6.9). It is more convenient, for purposes of computation on an electronic computer, to express dm as

$$dm = \rho\, dA \left[1 + 2 \frac{z}{R_m} \right] dz + o\{ f^2 \} ...(6.31).$$

The combination of equations (6.30) and (6.31) gives

$$\Delta g_{co} = \frac{k h_P}{8R_m^3} \iiint dA \operatorname{cosec} \frac{1}{2}\psi \left[3 - \operatorname{cosec}^2 \frac{1}{2}\psi + \operatorname{cosec}^2 \frac{1}{2}\psi \left\{ 5c_P - 2c_Q - \frac{4z}{R_m} + 2c_1 \right\} + \frac{6z}{R_m} + o\{ f^2, f^3 \operatorname{cosec}^2 \frac{1}{2}\psi \} \right] dz$$

.........(6.32).

Integration with respect to dz gives

$$\Delta g_{co} = \frac{k h_P}{8R_m^3} \iiint h_Q \operatorname{cosec} \frac{1}{2}\psi \; dA \left[3(1 - \frac{h_Q}{R_m}) - \operatorname{cosec}^2 \frac{1}{2}\psi + \operatorname{cosec}^2 \frac{1}{2}\psi \left\{ 5c_P - 2c_Q - \frac{2h_Q}{R_m} + 2c_1 \right\} + o\{ f^2, f^3 \operatorname{cosec}^2 \frac{1}{2}\psi \} \right]$$

.................(6.33)

Replacing dA by a square bounded by meridians and parallels of latitude of side $n^o \times n^o$, the use of equation (6.11) gives

$$dA = R_m^2 \left(1 + 2c_Q \right) \frac{\pi^2}{180^2} n^2 \cos \phi_Q \quad \text{........(6.34)}.$$

The combination of equations (6.33) and (6.34) gives

$$\Delta g_{co} = \frac{k \pi^2 h_P}{8 R_m 180^2} \Sigma n^2 \Sigma \rho h_Q \operatorname{cosec} \frac{1}{2}\psi \cos \phi_Q \left[3(1 - \frac{h_Q}{R_m}) + 6c_Q - \operatorname{cosec}^2 \frac{1}{2}\psi + \operatorname{cosec}^2 \frac{1}{2} \psi (5c_P - 4c_Q - \frac{2h_Q}{R_m} + 2c_1) + o\{ f^2, f^3 \operatorname{cosec}^2 \frac{1}{2} \psi\} \right] \quad \text{....(6.35)},$$

where $k = 6.673 \times 10^{-8} \ cm^3 g^{-1} sec^{-2}$

and $R_m = 6{,}371.2$ km. For normal computations, equation (6.35) can be reduced to

$$\Delta g_{co}^{(mgal)} = C h_P^{(met)} \Sigma n^2 \Sigma \rho h_Q^{(met)} \operatorname{cosec} \frac{1}{2} \psi \cos \phi_Q \left[3 - \operatorname{cosec}^2 \frac{1}{2}\psi + o\{f, f\operatorname{cosec}^2 \frac{1}{2} \psi\} \right] \quad \text{.....(6.36)},$$

where $C = 3.99 \times 10^{-12} \ cm^{-1} sec^{-2} \times 10^3$.

In attempting to estimate the magnitude of this correction, it must be borne in mind that the topography exterior to the geoid is the only part of the earth's crust that need be considered. Thus, ocean areas do not contribute to its magnitude.

For values of $\psi > \frac{1}{2}\pi$, put $n = 5^o$. The magnitude of

$$\left[\Delta g_{co}\right]_{\psi > \frac{1}{2}\pi} < 10^{-2} \text{ mgal}$$

and can be ignored. A similar consideration of the area $10^{\circ} < \psi < 90^{\circ}$ shows that topography in this region is likely to contribute less than 0.1 mgal to the final result. In relatively low lying regions like Australia, the effect of the region $1.5^{\circ} < \psi < 10^{\circ}$ is also unlikely to contribute more than 0.1 mgal to the differential topographical effect. In more mountainous regions, where both h_P and h_Q are large, the magnitude of the contribution of this zone may be significant and should be considered where necessary, as the error will be of the same sign and give rise to a systematic error in the final result.

Both equations (6.35) and (6.36) are indeterminate at $\psi = 0$ and the expressions are relatively unstable as $\psi \to 0$. This requires calculations of the effect due to the closer zones to be evaluated using smaller values of n. The innermost zones are usually of size $n = 0.1^{\circ}$, the four innermost squares being excluded from the calculation and considered separately. The inner squares can be replaced by a cylinder and the topographical effect considered to be due to the differential attraction of a cylinder whose height equals the mean elevation of the region.

6.3 Evaluation of the differential topographical effect for the inner zone.

Let the attractive matter in the inner zone comprise a cylinder of height h_m, where h_m is the mean elevation of the region about the computation point P, the elevation of P being h_P. Consider an element of mass dm at an azimuth α and height z above the point P_o on the geoid, situated at the variable point Q, a distance s from the axis PP_o of the cylinder, whose radius is r, the former being orientated along the vertical as shown in fig (6.4). If the lines P_oQ and PQ make angles β_o and β with PP_o, as the attractions considered are always directed along the downward vertical directed toward the centre of mass, the application of equation (6.8) gives the differential topographical attraction of the inner zone (Δg^i_{co}) as

$$\Delta g^i_{co} = - k \iiint \left[\frac{\cos \beta_o}{P_oQ^2} + \frac{\cos \beta}{PQ^2} \right] dm \quad ..(6.37).$$

$$\text{As } \cos \beta_o = \frac{z}{\left[z^2 + s^2 \right]^{\frac{1}{2}}} \quad ....(6.38)$$

$$\text{and} \quad \cos \beta = \frac{h_P - z}{\left[(h_P - z)^2 + s^2 \right]^{1/2}} \quad ....(6.39),$$

where the denominators of equations (6.38) and (6.39) are the lengths P_oQ and PQ respectively. The mass element dm is given by

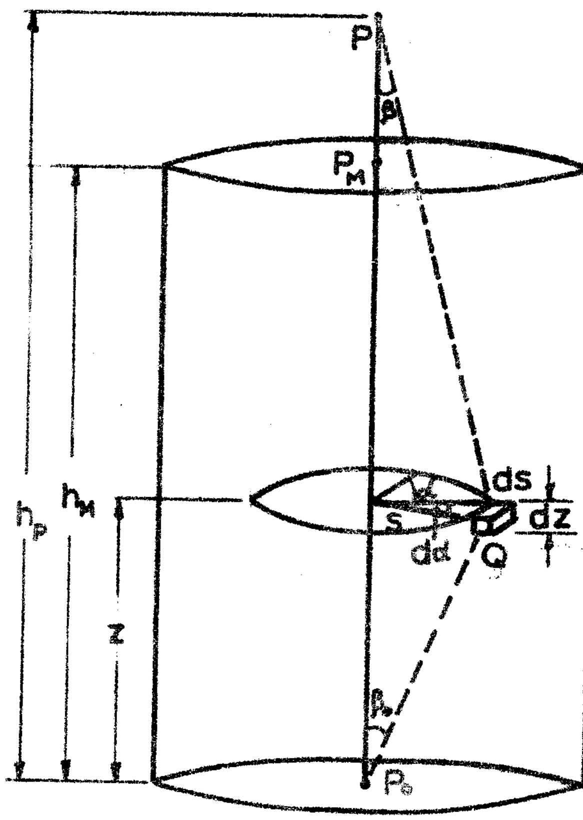


FIG. 6.4

$$dm = \rho \; dz \; s \; ds \; d\alpha \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (6.40),$$

where $d\alpha$ is the increment in azimuth and ρ the density. The insertion of equations (6.38), (6.39) and (6.40) in equation (6.37) together with the appropriate limits of integration gives

$$\Delta g^{i}_{co} = - k \int_0^{h_m} \int_0^{r} \int_0^{2\pi} \rho \left[\frac{z \, dz \, s \, ds \, d\alpha}{(z^2 + s^2)^{3/2}} + \frac{(h_P - z) dz \, s \, ds \, d\alpha}{\left[(h_P - z)^2 + s^2\right]^{3/2}} \right] .$$

If the density is assumed to be constant over the volume considered,

$$\Delta g^{i}_{co} = - 2 \pi \, k \, \rho \int_0^{h_m} \int_0^{r} \left[\frac{z \, dz \, s \, ds}{(z^2 + r^2)^{3/2}} + \frac{(h_P - z) dz \, s \, ds}{\left[(h_P - z)^2 + s^2\right]^{3/2}} \right]$$

$$= - 2 \pi \, k \, \rho \int_0^{h} \left[dz - \frac{z \, dz}{z^2 + r^2 \; 1/2} + dz - \frac{(h_P - z) dz}{(h_P - z)^2 + r^2 \; 1/2} \right]$$

$$= - 2 \pi \, k \, \rho \left[2 h_m + r - (h_m + r^2)^{\frac{1}{2}} + \left\{ (h_P - h_m)^2 + r^2 \right\}^{1/2} - (h_P + r^2)^{1/2} \right] \ldots\ldots\ldots\ldots (6.41).$$

In Australia, $h_P - h_m$ is seldom expected to exceed $\pm$ 100 metres. The ratio $h/r < 0.2$ as $r \simeq 10$ km.

Using these considerations, the expansion of equation (6.41) including those terms whose magnitudes are of order 10^{-4} or larger gives

$$\Delta g_{co}^{i} = -4\pi k \rho r \left[\frac{h_m}{r} - \frac{h_m^2}{4r^2} + \frac{h_m^4}{16r^4} + \frac{(h_P - h_m)^2}{4r^2} - \frac{h_P^2}{4r^2} + \frac{h_P^4}{16r^4} \right] \qquad \ldots\ldots\ldots\ldots\ldots(6.42).$$

In the case of the small order terms, h_P can be assumed to be equal to h_m and hence the correction reduces to

$$\Delta g_{co}^{i} = -4\pi k \rho h_m \left[1 - \frac{h_P}{2r} + \frac{h_m^3}{8r^3} + o\{\frac{h^5}{r^5}\} \right] \qquad \ldots\ldots\ldots(6.43)$$

As each term in equation (6.43) is at least an order smaller than that immediately preceding it, the major term is $-4\pi k \rho h_m$ which (Heiskanen and Vening Meinesz, 1958, 152) is twice the Bouguer reduction and is hence of magnitude 0.2236 mgal per metre. Thus the refinement in distinguishing between h_P and h_m is necessary only in areas of rugged topography where the gravity station elevation may be totally unrepresentative of the region. In the case of calculations in Australia, where the gradient of the topography is reasonably smooth except in certain limited regions and for normal work, the equation (6.43) can be replaced by

$$\Delta g_{co}^{i} = -4\pi k \rho h_P \left[1 - \frac{h_P}{2r} + \frac{h_P^3}{8r^3} \right] \qquad \ldots\ldots(6.44)$$

Equation (6.44) is an adequate working formula for most purposes, provided the gravity station is reasonably representative of the area.

6.4 The accuracy of the cylindrical assumption for the inner zone.

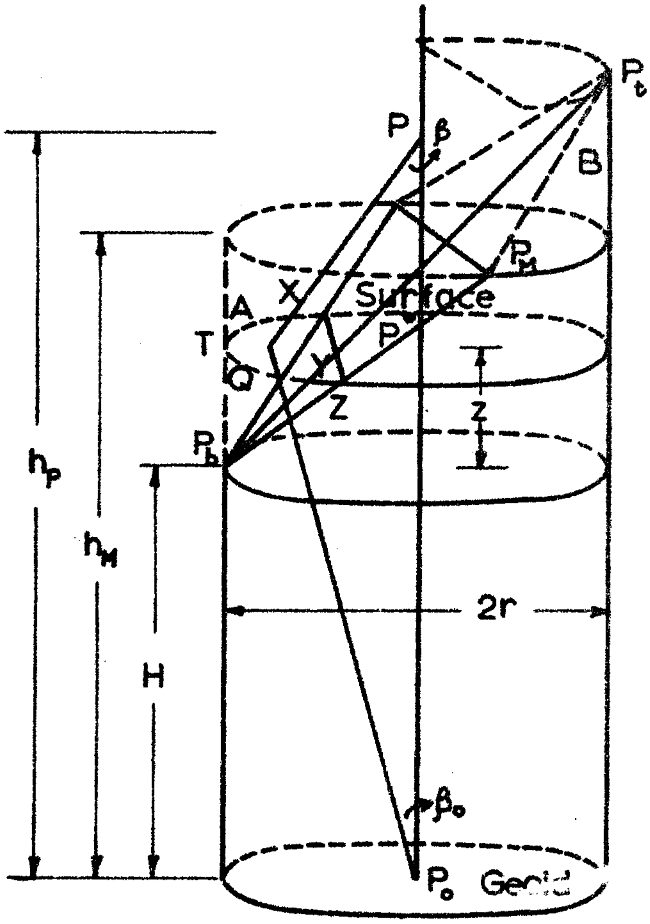
In practice, the inner zone can deviate considerably from a cylinder due to the slope of the terrain at the upper end. The configuration producing the maximum disturbance (Δg_d) from the cylindrical model is that where the ground has a steep and constant grade over the surface of the model. Consider the inner zone to be a cylinder of radius r and height h_m as used in the previous section, but modified as shown in fig (6.5), with maximum and minimum surface elevations of $(2h_m - H)$ and (H) respectively, where h_m is the mean elevation. The true differential attraction of the inner zone is

$$\Delta g^{i}_{co_{true}} = \Delta g^{i}_{co} + \Delta g_d \quad \ldots\ldots\ldots\ldots.(6.45) ,$$

where Δg_d is given by

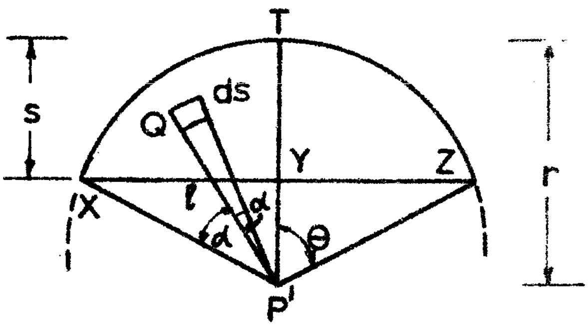
$$\Delta g_d = \Delta g_{d_B} - \Delta g_{d_A} \quad \ldots\ldots\ldots\ldots\ldots.(6.46).$$

In equation (6.46), Δg_{d_A} is the differential attraction of the area A by which the true terrain departs from the cylindrical model below h_m and the correction Δg_{d_B} is the differential attraction of the area B which represents the actual topography which does not lie within the cylindrical model.



P
β
Pt
B
PM
X
Surface
A
P'
T
Y
Q
Z
z
Pb
hp
hM
2r
H
βo
Po
Geoid

FIG. 6.5



T
s
ds
Q
Y
Z
r
X
ℓ
α
α
θ
P'

FIG. 6.6

The volumes A and B are, by definition, identical , but the differential gravitational attractions on P and P_o are slightly different depending on the departure of P from the point P_m, which is at the mean elevation h_m on the vertical PP_o.

The solid A can be considered to consist of a series of segments of circular discs of radius r and thickness dz. Consider the segment XYZT at a height z above the minimum elevation H. Let the bounding chord XYZ subtend an angle 2 θ at the centre P' of the disc. Consider (fig (6.6)) a mass element dm at Q such that P'Q = l. Then

$$dm = \rho \, d\alpha \, l \, dl \, dz \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.47),$$

where ρ is the density of matter and α is the azimuth of the element recknoned clockwise from the line P'X. The attraction at P_o due to the mass element at Q is

$- k \, dm \cos \beta_o / P_oQ^2$ directed along the downward vertical and that at P is

$k \, dm \cos \beta / PQ^2$ in the same direction. The differential attraction due to the element is

$$- k \, dm \left[\frac{P_oP'}{P_oQ^3} + \frac{PP'}{PQ^3} \right] \qquad \text{and } \Delta g_{d_A}$$

is given by

$$\Delta g_{d_A} = - k \rho \sum_{i=1}^{n} dz \sum_{j=1}^{2} \int_0^{2\theta_i} \int_w^r \frac{f_j(z_i) \, d\alpha \, l \, dl}{\left[f_j(z_i)^2 + l^2 \right]^{\frac{3}{2}}}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (6.48),$$

where

$$w = (r - s_i) \sec(\theta_i - \alpha) \quad \ldots\ldots\ldots\ldots\ldots\ldots(6.48a),$$

$$s_i = \frac{z_i}{h_m - H} r \quad \ldots\ldots\ldots\ldots\ldots\ldots(6.49),$$

$$\theta_i = \cos^{-1}\left[1 - \frac{s_i}{r}\right] \quad \ldots\ldots\ldots\ldots\ldots\ldots(6.50),$$

$$f_1(z_i) = H + z_i \quad \ldots\ldots\ldots\ldots\ldots\ldots(6.51),$$

and $f_2(z_i) = h_P - H - z_i$ ....(6.52).

In equation (6.48), n represents the number of contiguous circular discs. Integration with respect to l gives

$$\Delta g_{d_A} = - k \rho\, dz \sum_{i=1}^{n} \sum_{j=1}^{2} \int_0^{2\theta_i} \left[\frac{f_j(z_i)\, d\alpha}{\left\{ f_j(z_i)^2 + (r-s_i)^2 \sec^2(\theta_i - \alpha)\right\}^{\frac{1}{2}}} - \frac{f_j(z_i)\, d\alpha}{\left\{[f_j(z_i)]^2 + r^2\right\}^{\frac{1}{2}}} \right]$$

$$= - k \rho\, dz \sum_{i=1}^{n} \sum_{j=1}^{2} \left[\int_0^{2\theta_i} f_j(\theta_i)\, d\alpha - \frac{2\,\theta_i\, f_j(z_i)}{\left\{[f_j(z_i)]^2 + r^2\right\}^{\frac{1}{2}}} \right] \quad \ldots\ldots\ldots\ldots(6.53),$$

where

$$f_j(\theta_i) = - \frac{d \sin(\theta_i - \alpha)}{\left[\frac{\{f_j(z_i)\}^2 + (r - s_i)^2}{\{f_j(z_i)\}^2} - \sin^2(\theta_i - \alpha) \right]^{\frac{1}{2}}} ,$$

$j=1,2$ ....(6.54)

The integral in equation (6.53), on evaluation (Lamb, 1940, 164) gives

$$\int_0^{2\theta_i} f_j(\theta_i)\, d\alpha = \left[- \sin^{-1} \frac{\sin(\theta_i - \alpha)\, f_j(z_i)}{\left[\{f_j(z_i)\}^2 + (r - s_i)^2\right]^{\frac{1}{2}}} \right]_{\alpha=0}^{\alpha=2\theta_i} ,$$

$j=1, 2$

$$= 2 \sin^{-1} \frac{f_j(z_i) \sin \theta_i}{\left[\{f_j(z_i)\}^2 + (r - s_i)^2\right]^{\frac{1}{2}}} , \; j=1, 2 \quad ..(6.55),$$

as $\sin^{-1}(-\theta) = - \sin^{-1}\theta$.

Substitution of the results of equation (6.55) in (6.53) gives

$$\Delta g_{dA} = - 2 \rho k \quad dz \sum_{i=1}^{n} \sum_{j=1}^{2} \left[\sin^{-1} \frac{\sin \theta_i\, f_j(z_i)}{\left[\{f_j(z_i)\}^2 + (r - s_i)^2\right]^{\frac{1}{2}}} - \frac{\theta_i\, f_j(z_i)}{\left[\{f_j(z_i)\}^2 + r^2\right]^{1/2}} \right] \quad \ldots\ldots\ldots\ldots\ldots (6.56).$$

It should be noted that in equation (6.56), the $\sin^{-1}$ function takes values between $-\pi/2$ and $+\pi/2$ (Lamb, 1940, 59). Further,

(i) $0 \leqq \theta_i \leqq \frac{1}{2}\pi$;

(ii) $f_2(z_i)$ can both be greater than 0 as well as less than 0. As such, there is no necessity to consider any further the effect of the location of P with respect to P_m as the attraction computed using equation (6.56) is of general application.

In calculating the attraction due to the matter in area B (Δg_{d_B}), z is measured from the plane of P_m. In this case,

$$f_1(z_i) = h_m + z_i \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.57)$$

$$f_2(z_i) = h_P - h_m - z_i \quad \ldots\ldots\ldots\ldots\ldots (6.58).$$

Proceeding on lines similar to the derivation of equation (6.48),

$$\Delta g_{d_B} = - k \rho \, dz \sum_{i=1}^{n} \sum_{j=1}^{2} \int_{0}^{2\theta_i} \int_{w}^{r} \frac{f_j(z_i) \, d\alpha \, l \, dl}{\left[\{f_j(z_i)\}^2 + l^2\right]^{\frac{3}{2}}} \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.59),$$

where w is defined by equation (5.48a) and

$$r - s_i = \frac{z_i}{h_m - H} r \quad \ldots\ldots\ldots\ldots\ldots\ldots (6.60).$$

The integration of equation (6.59) will give the same result as that of equation (6.48). Thus,

$$\Delta g_{d_B} = - 2 k \rho \, dz \sum_{i=1}^{n} \sum_{j=1}^{2} \left[\sin^{-1} \frac{f_j(z_i) \sin \theta_i}{\left[\{f_j(z_i)\}^2 + (r - s_i)^2\right]^{\frac{1}{2}}} - \frac{\theta_i f_j(z_i)}{\left[\{f_j(z_i)\}^2 + r^2\right]^{\frac{1}{2}}} \right] \quad \ldots\ldots\ldots\ldots (6.61),$$

where $f_j(z_i)$, $j=1, 2$, are given by equations (6.57) and (6.58) and s_i is given by equation (6.60) for any given z_i. The total attractive differential effect is then given by equations (6.45), (6.46), (6.56) and (6.61). Computations made using different ground models are shown in table (6.1).

Topographical gradient 1 in	Station height h_P(met)	Mean height h_m(met)	dz (met)	Differential correction Δg_d(mgal)
31	400	300	50	0.13
62	250	200	50	0.00
125	175	150	50	0.00
31	300	300	50	0.10
62	200	200	50	0.00
125	150	150	50	0.00
31	200	300	50	0.06
62	150	200	50	0.00
125	125	150	50	0.00
31	300	400	50	0.10
62	250	300	50	0.00
125	225	250	50	0.00
12	1000	1100	50	0.84
31	720	800	50	0.23
62	630	700	50	0.00
125	590	650	50	0.00
6	1900	2100	50	0.16
12	1450	1600	50	0.95
31	1150	1300	50	0.34
62	1100	1200	50	0.00
6	2400	2600	50	0.19
12	1900	2100	50	0.94
31	1600	1800	50	0.44
62	1500	1700	50	0.00
125	1500	1650	50	0.00

TABLE (6.1)

Effect of ground topography on the differential topographical correction

A general study shows that in the worst possible case where the gravity station P is situated on the side of a mountain range, the error introduced into the differential topographical correction on assuming a cylindrical shape is less than 0.5 mgal if the ground slope, assumed uniform, is less than 1 in 50. This assumption is quite acceptable for Australian conditions and all further consideration of the topographical correction Δg_{co} will assume it to be structured according to the equation

$$\Delta g_{co} = \Delta g_{co}^{e} + \Delta g_{co}^{i} \quad \ldots\ldots\ldots(6.62),$$

where Δg_{co}^{e} is computed using equation (6.36) with the four innermost $0.1^{o} \times 0.1^{0}$ squares excluded and Δg_{co}^{i} is obtained from equation (6.43).

The radius of the cylinder r is chosen so that the volume of the cylinder equals that of the four innermost $0.1^{o} \times 0.1^{o}$ squares which it is to replace. This condition is achieved if the surface area of both are identical. In equation (6.34), $n^2 = 10^{-2}$ and

$$dA = R_m^2 \frac{\pi^2}{180^2} 10^{-2} \cos \phi$$

$$= \pi r^2 \quad \text{, from which}$$

$$r = R_m \frac{(\pi \cos \phi)^{\frac{1}{2}}}{1800} \quad \ldots\ldots\ldots\ldots(6.63).$$

The use of $R_m = 6,371.2$ km. gives

$r = 12.55 \cos^{\frac{1}{2}} \phi$ km , which is a value adequate for normal computational purposes.

6.5 The contribution to the indirect effect of the term $\frac{2\phi_e}{\gamma m}$.

The principal quantity computed in the evaluation of this term is ϕ_{e_P} which is the potential at the point P, situated on the geoid, of matter which lies exterior to it. Consider fig (6.7) which is similar to fig (6.2) except in that the distance and angle from the variable element Q to the surface point P are not considered as the exterior potential has only to be computed at the point P_o which is the point on the geoid equivalent to P. The exterior potential ϕ_{e_P} will thus refer to the potential of matter exterior to the geoid as computed at P_o.

$$\phi_{e_P} = \iiint \frac{k\, dm}{r_o} \qquad \ldots\ldots\ldots\ldots(6.64).$$

Evaluating r_o using equations (6.11), (6.12), (6.14), (6.17) and (6.18) and dm using equations (6.31) and (6.34),

$$\phi_{e_P} = \frac{k R_m \pi^2}{2 \times 180^2} \sum_i n_i^2 \sum_j \rho \operatorname{cosec} \psi_{ij} \cos \phi_{ij} \int_0^{h_Q} \frac{1 + 2(c_Q + c)\, dz}{1 - \frac{1}{2}(A - 2c_P + o\{f^2\}}$$

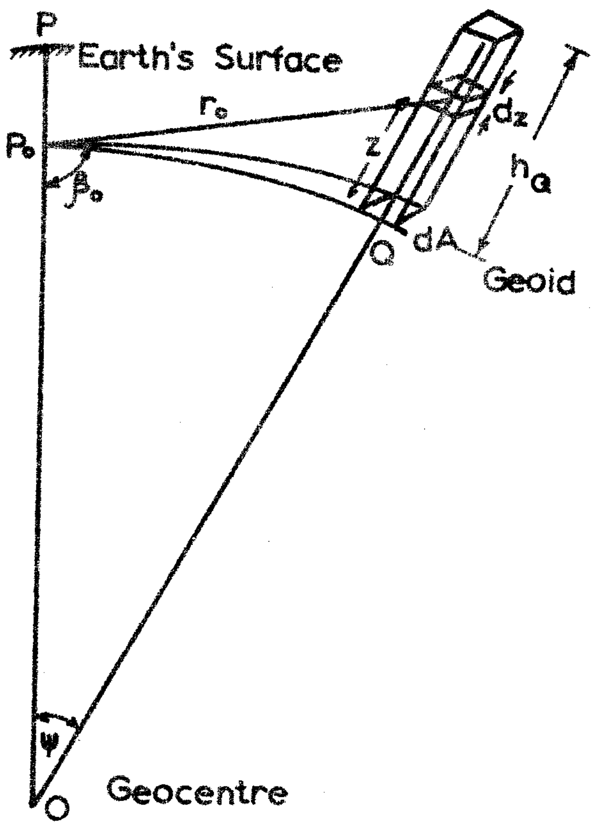


FIG. 6.7

$$\phi_{e_P} = \frac{k R_m \pi^2}{2 \times 180^2} \sum_i n_i^2 \sum_j \operatorname{cosec} \psi_{ij} \cos \phi_{ij} \int_0^{h_Q} \left[1 + \frac{1}{2}\left(3c_Q - c_P + \frac{3z}{R_m} + o\{ f^2 \} \right) \right] dz \quad ,$$

which, on integration and generalisation, gives the contribution to the indirect effect as

$$\frac{2\phi_{e_P}}{\gamma_m} = \frac{k R_m \pi^2}{\gamma_m 180^2} \sum_i n_i^2 \sum_j \operatorname{cosec} \psi_{ij} \cos \phi_{ij} h_{ij} \left[1 + \frac{1}{2}\left(c_{ij} - c_P + \frac{3h_{ij}}{2R_m} + 0\{ f^2 \} \right) \right] \quad \ldots(6.65),$$

the suffix Q being replaced by the indexes i and j for summation. The evaluation of the constant term in the above expression gives

$$\frac{2\phi_{e_P}}{\gamma_m} = C \sum_i n_i^2 \sum_j \operatorname{cosec} \psi_{ij} \cos \phi_{ij} h_{ij}^{(met)} \left[1 + \frac{1}{2}(3c_{ij} - c_P + \frac{3h_{ij}}{2R_m} + o\{ f^2 \} \right] \text{ cm} \quad \ldots\ldots\ldots(6.66),$$

where $C = 3.529 \times 10^{-3}$ for values of the density in $g.cm^{-3}$.

Equation (6.66) does not converge as rapidly as equation (6.36) as the constant coefficient is nine orders larger. Consequently it is necessary, in any computation, to take into account the effect of all distant zones up to $\psi = \pi$.

Equation (6.66) is also indeterminate at $\psi = 0$, being rapidly variable as $\psi \to 0$. This instability in the expression as derived can be eliminated by adopting the same procedure as followed in the case of the differential topographical effect. A cylindrical assumption is adopted for the four innermost $0.1^{\circ} \times 0.1^{\circ}$ squares. As in sections (6.3) and (6.4), using the same symbols as before, the potential at P_o ($\phi^{i}_{e_P}$) due to the cylinder is given by

$$\phi^{i}_{e_P} = \iiint \frac{k\,dm}{P_oQ}$$

$$= k\,\rho \int_0^{h_m} \int_0^{r} \int_0^{2\pi} \frac{dz\ s\ ds\ d\alpha}{\left[z^2 + s^2\right]^{\frac{1}{2}}}$$

$$= 2\,\pi k\,\rho \int_0^{h_m} \left\{\left[z^2 + r^2\right]^{\frac{1}{2}} - z\right\} dz \quad \ldots (6.67).$$

The first term to be integrated is operated on by parts.

Put
$$u = (z^2 + r^2)^{\frac{1}{2}}.$$

As
$$d(uz) = u\,dz + z\,du, \text{ integration will give}$$

$$\int u\,dz = uz - \int z \frac{du}{dz}\,du\ .\quad \text{Hence,}$$

$$\int (z^2 + r^2)^{\frac{1}{2}}\,dz = (z^2 + r^2)^{\frac{1}{2}}\,z - \int \frac{z^2\,dz}{\left[z^2 + r^2\right]^{\frac{1}{2}}}$$

$$\ldots\ldots\ldots (6.68)$$

In addition,

$$\int (z^2 + r^2)^{\frac{1}{2}} dz = \int \frac{(z^2 + r^2) \quad dz}{(z^2 + r^2)^{1/2}}$$

$$= \int \frac{z^2 \ dz}{(z^2 + r^2)^{1/2}} + \int \frac{r^2 \ dz}{(z^2 + r^2)^{1/2}}$$

$$\ldots\ldots\ldots\ldots(6.69).$$

The addition of equations (6.68) and (6.69) gives

$$\int (z^2 + r^2)^{\frac{1}{2}} dz = \frac{1}{2} (z^2 + r^2)^{\frac{1}{2}} z + \frac{1}{2} \frac{r^2 dz}{\left[r^2 + z^2\right]^{1/2}}$$

$$= \frac{1}{2} (z^2 + r^2)^{1/2} z + \frac{1}{2} r^2 \sinh^{-1} \frac{z}{r} \ldots(6.70).$$

Expansion of the second term in equation (6.70) gives (Lamb, 1940, 164)

$$\frac{1}{2} r^2 \sinh^{-1} \frac{z}{r} = \frac{1}{2} r^2 \ln \frac{z + (z^2 + r^2)^{1/2}}{r} \quad \ldots.(6.71).$$

The use of the above results in equation (6.67) gives

$$\phi^i_{e_P} = 2\pi k \rho \left[\frac{h_P}{2} (h_P^2 + r^2)^{1/2} + \frac{r^2}{2} \ln \frac{h_P + (h_P^2 + r^2)^{1/2}}{r} - \right.$$

$$\left. - \frac{1}{2} h_P^2 \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6.72).$$

As the ratio h_P/r converges rapidly in all but very mountainous regions, it is convenient, from the point of view of calculations, to expand the relevant series which comprise (6.72). In Australia, $h_P/r \approx 0.1$ or less and hence it is valid to replace x in the expansion of

$$\frac{h_P}{r} = \sinh x = x + \frac{x^3}{6} + \frac{x^5}{120} + o\,\{10^{-7}\}$$

as h_P/r in the smaller terms of the expansion.

As a first approximation,

$$\sinh^{-1}\frac{h_P}{r} = x = \frac{h_P}{r} - \frac{h_P^3}{6r^3} - \frac{h_P^5}{120r^5} .$$

More accurately,

$$x = \frac{h_P}{r}\left(1 - \frac{h_P^2}{6r^2}\right) \quad \text{which gives}$$

$$\sinh^{-1}\frac{h_P}{r} = \frac{h_P}{r} - \frac{h_P^3}{6r^3} + \frac{3h_P^5}{40r^5} \qquad \ldots\ldots\ldots\ldots(6.73).$$

The substitution of equation (6.73) in (6.72) gives

$$\phi^i_{e_P} = \pi k\, \rho h_P r \left[1 + \frac{h_P^2}{2r^2} - \frac{h_P^4}{8r^4} + 1 - \frac{h_P^2}{6r^2} + \frac{3h_P^4}{40r^4} - \frac{h_P}{r}\right]$$

The contribution of the inner zone to the indirect effect is

$$N_i^i = \frac{4\pi\, k \rho\, h_P r}{\gamma_{\Omega}} \left[1 - \frac{h_P}{2r} + \frac{h_P^2}{6r^2} - \frac{h_P^4}{40r^4}\right] \ldots\ldots(6.74) ,$$

where r is given by equation (6.63) for the same reasons as before.

The magnitude of this effect depends both on the latitude and elevation of P. While the relation is not a linear one, for moderate elevations it varies from approximately 28 cm per 100 metres of elevation at a latitude of 5^{o} to 8.4 cm per 100 metres at a latitude of 85^{o}(see table (6.2).

Lat. o	ρ = 2.67				ρ = 2.77 - h/21 h in km			
	Height (met)	N_i^i (cm)	Height (met)	N_i^i (cm)	Height (met)	N_i^i (cm)	Height (met)	N_i^i (cm)
5	100	28.5	2000	526.6	100	29.5	2000	527.5
15	100	28.1	2000	517.8	100	29.0	2000	518.8
25	100	27.2	2000	500.2	100	28.1	2000	501.1
35	100	25.8	2000	473.3	100	26.8	2000	474.1
45	100	24.0	2000	436.5	100	24.9	2000	437.2
55	100	21.6	2000	388.5	100	22.4	2000	389.3
65	100	18.5	2000	327.0	100	19.2	2000	327.6
75	100	14.5	2000	246.0	100	15.0	2000	246.4
85	100	8.4	2000	123.6	100	8.7	2000	123.8

TABLE (6.2)

Variation of the indirect effect of the inner zone with latitude, elevation and density

This, however, is not of intrinsic interest. Of greater value is the study of the variation in the magnitude of N_i^i due to a change in the density from the internationally accepted uniform crustal density of 2.67 to the Hunter formula given in equation (1.6). Table (6.2) shows that there is a constant difference between the values of N_i^i

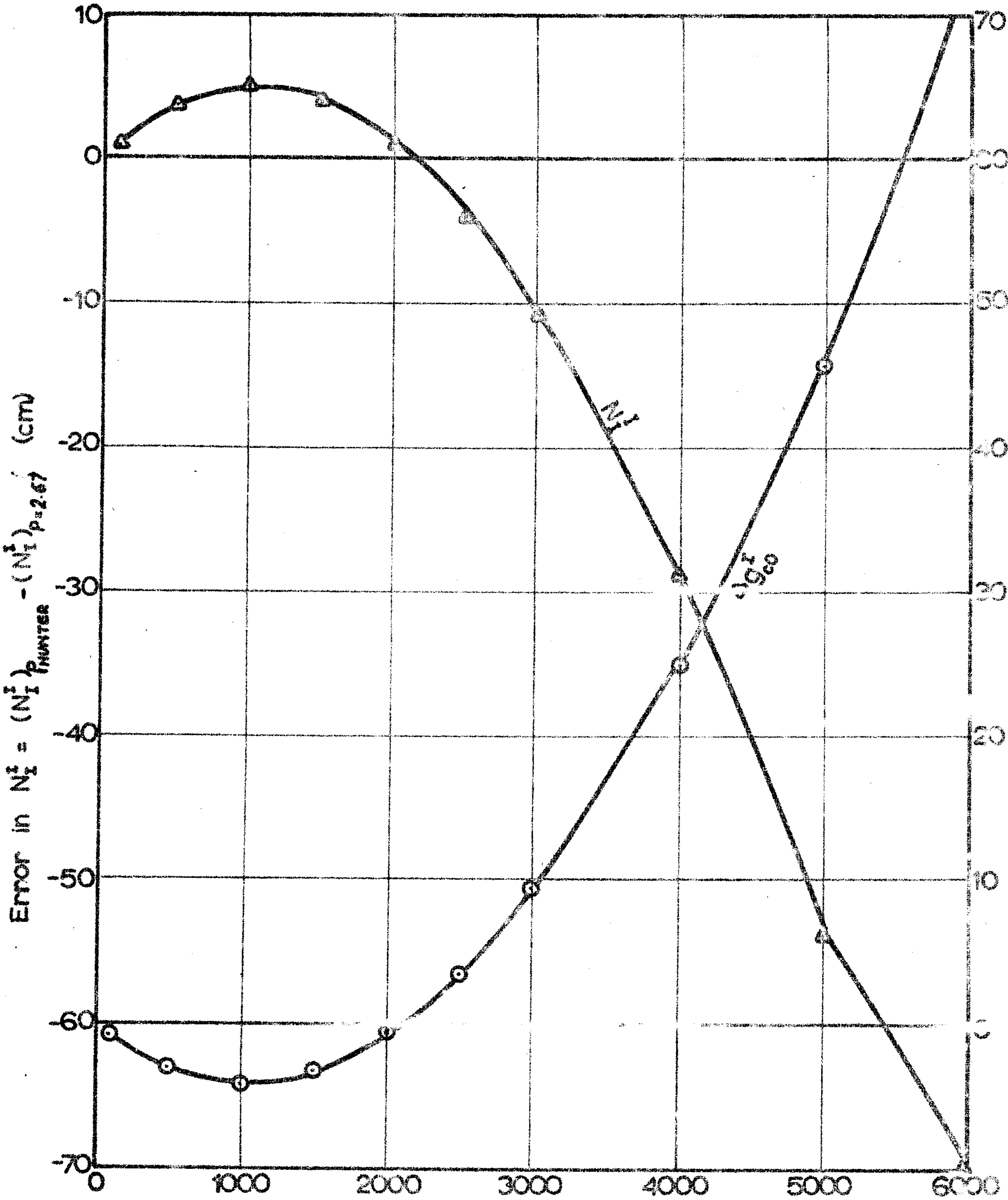
computed using different elevations at the same latitude for the value of r as defined in equation (6.63). Consider the variation of N_i^i with elevation at latitude 35^o for the two different density relations and also consider the differences between the values of Δg_{co}^i under the same circumstances as set out in table (6.3).

	$\rho = 2.67$		$\rho = 2.77 - h/21$, h in km.	
Elevation met.	N_i^i cm.	Δg_{co}^i mgal.	N_i^i cm.	Δg_{co}^i mgal.
100	25.8	-22.3	26.8	-23.0
500	126.9	-109.5	130.5	-112.6
1000	248.0	-214.0	252.9	-218.2
1500	363.5	-313.7	367.4	-317.0
2000	473.3	-408.4	474.1	-409.1
2500	577.3	-498.1	573.1	-494.6
3000	675.6	-583.0	664.7	-573.6
4000	855.1	-737.8	826.1	-712.8
5000	1052.5	-884.9	998.0	-839.2
6000	1215.0	-1013.2	1130.2	-942.7

TABLE (6.3)

Variation of the values of N_i^i and Δg_{co}^i at latitude 35^o with changes in elevation and density

The graph of differences is shown in fig (6.8). A study of this graph shows that the Hunter formula is quite unstable for elevations greater than 2500 metres and its use



GRAPH SHOWING DIFFERENCES IN THE VALUES OF N_I^I & Δg_{co} FOR CHANGES IN DENSITY.

in world wide investigations must be carefully controlled. For the Australian region the Hunter formula is quite adequate but in the consideration of world wide investigations, it is prudent to use the following set of relations to define topographical density.

If $h \leq 2500$ metres , use $\rho = 2.77 - \frac{h}{21}$

If $h > 2500$ metres, use $\rho = 2.67$

Test calculations made using world wide $5^{\circ} \times 5^{\circ}$ height means show that the resulting variations in the value of N_i range from $5\frac{1}{2}$ metres to $6\frac{1}{2}$ metres. The density relations set out above are used in effecting the calculations carried out during the current investigation.

7. THE AVAILABLE GRAVITY DATA.

7.1 Introduction.

Until ten years ago, there was only one method available for determining the gravitational field over the surface of the earth. This consisted of measuring the acceleration due to gravity (g) or differences in this same quantity by a variety of methods. The first category of measurements, where gravity is determined at a point on the earth's surface without reference to the value of gravity at any other point, is called absolute. Absolute determinations fall into two bread classifications :-

(a) Systems which enable g to be determined by measuring the acceleration of falling bodies, sometimes coupled with the deceleration of falling ones (e.g., Faller, 1965 ; Cook, 1965 ; Agaletskii et al., 1959 ; Rose et al., 1959).

(b) Pendulous systems which have been developed to afford portable forms which permit the measurement of g at field stations (e.g., Jackson, 1959 ; Heiskanen and Vening Meinesz, 1958, 84 et seq)

While the accuracy of the pendulous systems is generally accepted to be of the order of 1 - 3 mgal, greater precision has been claimed in the case of determinations which use interferometric systems (Faller, 1965a). However, it has not been clearly established whether the precision claimed is merely a reflection of internal consistency or a measure of the absolute accuracy of the method.

The second category of determinations establishes the differences in the value of g between two points on the earth's surface which, if the value of g is known at one of the former, enables a value to be established for g at the other point. These relative determinations have an accuracy which is at least an order greater than that obtainable from the use of a pendulous system as the basic reading unit on most commercial gravimeters is approximately 0.1 mgal with interpolation possible to 10^{-2} mgal. This is an order of accuracy of at least 1 part in 10^7, which is approximately four times greater than that afforded by any other geodetic method. In this manner, it is possible to establish a framework of first order gravity stations analagous to a first order level network. In general, gravity control networks are of three types :-

(a) intercontinental connections made with either pendulous systems or batteries of gravimeters ;

(b) continental control networks ;

(c) local survey networks , the accuracy of establishment of each type of network decreasing from (a) to (c).

7.2 The Australian control network.

The gravity reference network in Australia has been established entirely by the Commonwealth of Australia's authority, the Bureau of Mineral Resources, Geology and Geophysics (B.M.R.). The original reference network was a network of pendulum stations (Dooley et al, 1961). This has now been superceded by a more extensive survey called the Isogal Regional Gravity Survey. The name "Isogal" was used because of the predominantly east - west gravity traverses which comprise the survey. The gravity traverses were surveyed using a battery of gravimeters (Dooley, 1965). In this method of traversing, with one exception, all gravity differences on a particular traverse were within the small dial range of the gravimeters used (i.e., 50 - 100 mgal.). All traverses were connected to the National Gravity Base Station at Melbourne which was assigned the value 979,979.0 mgal. Thirty Isogal stations were established in South Australia (Mather, 1966b, Fig (1)).

7.3 Existing gravity data.

In addition to the B.M.R.'s control network of first order accuracy,there exist, on the Australian mainland, numerous regional gravity surveys carried out for prospection purposes. Some of these surveys establish gravity values using difference methods while others are merely relative surveys which have both arbitrary datums for both elevation and gravity. In general, however, there is adequate gravity data available over the entire continent to make the gravimetric solution for the geoid a practicable proposition.

A study of the values of the function f(ψ) in equation (5.54) (Lambert and Darling, 1936) shows that f(ψ) is not negligible for large values of ψ . It is therefore necessary to consider world wide effects for the evaluation of N by equation (5.54). This necessitates a complete knowledge of the gravity field over the surface of the earth to effect a solution. It is generally agreed that the breakdown of the earth's surface area into elements whose sides are five degree arc length of meridians and parallels is adequate (Uotila, 1960) for the more distant regions for which $\psi > 20^{\circ}$ from the computation point.

Up to 1960, there was very little hope of effecting a realistic calculation of N_P due to the very poor gravity coverage available at sea. In addition, the available gravity data on land often tended to be concentrated in small regions where active mineral prospection has occured. Sea gravimeters (e.g., Caputo et al (1962) , Harrison and Spiers (1963)) are making gravity determinations at sea a more practicable proposition, though with reduced accuracy, claimed to be of the order of 3 - 5 mgal. It is unlikely that much headway can be expected in this generation in the establishment of adequate gravity coverage at sea using earth based techniques as the oceans cover approximately 70 per cent of the earth's surface area. It is also estimated that about 40 per cent of the approximately 2600 $5^{o} \times 5^{o}$ units of surface area over the earth have no gravity values determined in them. Of the balance 60 per cent, it is expected that a high proportion have only 2 or 3 readings in them. Thus the sample of surface gravity determinations available are, at best, of marginal sufficiency.

Many attempts have been made to extend the available field to unrepresented areas (e.g., Jeffreys (1941), Kaula (1959), Uotila (1962)) but all have their limitations due to the inadequacy of the sample available. The launching of the first artificial earth satellite in 1957 gave a new lease of life to attempts at solving the boundary value problem in physical geodesy.

7.4 The gravitational field from the observation of the orbital perturbations of artificial earth satellites.

The gravitational potential at a point P exterior to the earth satisfies Laplace's theorem, set out in equation (2.18). Thus

$$\underline{\nabla}^2 W_P = 0 \quad ,\ldots\ldots\ldots\ldots\ldots\ldots (7.1).$$

It will be shown in section (7.5) that the solution of Laplace's equation is given completely (Jeffreys, 1962a, 133) by

$$W_P = \sum_{n=0}^{\infty} \frac{1}{r_P^{n+1}} \sum_{m=0}^{n} A_{nm} S_{nm}(\phi_c, \lambda_c) \quad ..(7.2),$$

where P is expressed in geocentric coordinates (r_P, ϕ_c, λ_c) and the surface harmonic is given by

$$A_{nm} S_{nm}(\phi_c, \lambda_c) = \sum_{m=0}^{n} p_{nm}(\sin \phi_c) \left[a_{nm} \cos m \lambda_c + \right.$$

$$\left. + b_{nm} \sin m \lambda_c \right], n = 0, \quad ...(7.3).$$

In equation (7.3), $p_{nm}(\sin \phi_c)$ is given by equation (5.34). λ_c is measured from the principal inertial axis in the equatorial plane. The entire solution has (2n+1) terms such that none could be expressed as a linear combination of the rest. The movement of a particle (satellite) in the earth's gravitational field is subject to central force motion (McCuskey, 1963, 19 et seq). This problem has been dealt with in detail by others (e.g., Kaula (1966b), Mueller (1964)) and can be summarised as follows. If the earth were a regularised sphere, the orbit of any artificial earth satellite would be an ellipse, fixed in space in relation to the parent body. The exact orientation of the orbit with respect to any system of earth bound coordinates would be determined by the launching details.

The existent earth is spheroidal to order 3 in 10^{-3}, with further irregular deviations from the spheroidal model due to topography and mass anomalies. These cause the shape of the exterior equipotential surfaces of the earth's gravitational field to depart from the exact spheroidal reference figure by amounts of magnitude 1 part in 10^5. The orbital parameters of a given satellite undergo change due to (Kaula, 1962)

(a) the earth's oblateness ;

(b) atmospheric drag ;

(c) electromagnetic effects ;

(d) localised large scale mass anomalies and

(e) other short period gravitational effects.

If the non-gravitational influences could be minimised, the residual perturbation of a satellite's orbit would be due entirely to gravitational causes. The rates of change of the orbital parameters could then be related to the gravitational field enabling a determination of the latter to be effected. For this purpose, it is relevant to formulate the exterior gravitational potential using a set of mathematical functions which are then evaluated in terms of the measured rates of change of the orbital parameters. This mathematical representation of the exterior gravitational potential is conventionally carried out using the functions set out in equations (7.1) to (7.3). Section (7.5) shows how equations (7.2) and (7.3) are solutions of equation (7.1).

7.5 The solution of Laplace's equation in terms of spherical harmonics.

The solution of Laplace's equation as applied to gravitational potential is very completely dealt with by Jeffreys (1962b, 529 et seq). Consider a set of curvilinear coordinates (ξ_i, i=1,3) , where (see fig (7.1))

$$\xi_1 = r$$
$$\xi_2 = \theta \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.4),$$
$$\xi_3 = \lambda$$

where r is the geocentric distance, θ the geocentric co-latitude and λ the geocentric longitude. The element of volume (dV) in space is defined by the equation

$$dV = \prod_{i=1}^{3} h_i d\,\xi_i \qquad \ldots\ldots\ldots\ldots\ldots\ldots (7.5),$$

where

$$h_1 = 1$$
$$h_2 = r \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.[illegible]).$$
$$h_3 = r \sin\theta$$

The left hand side of equation (7.1) is expressed in terms of the curvilinear coordinates by the use of the divergence theorem , stated in equation (2.1), on the elemental volume set out in equation (7.5). If $\underline{\nabla}\; W_P$ is constant over the element of volume dV in space, the application of the divergence theorem to $\underline{\nabla} W_P$ gives

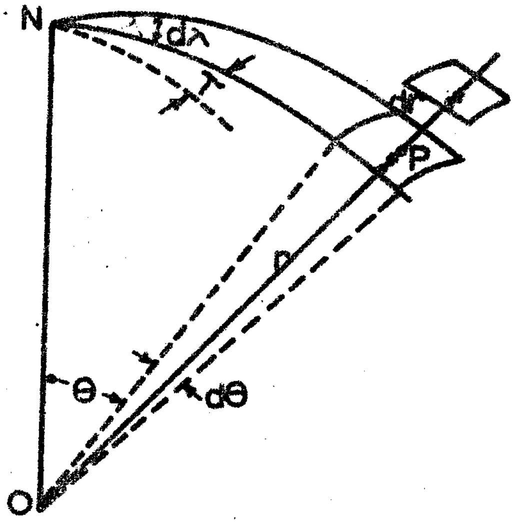


FIG. 7.1

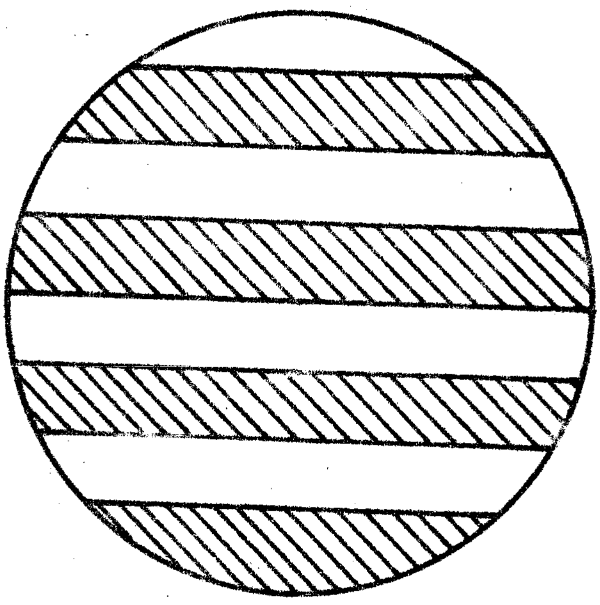


FIG 7.2

$$\underline{\nabla}^2 W_P \prod_{i=1}^{3} h_i \, d\xi_i = \iint \underline{\nabla} \cdot \underline{N} \, W_P \, dS \quad \ldots\ldots.(7.7).$$

The definitive values of ξ_i on the left hand side are made up of pairs of the form ($\xi_{i_o} \pm d\,\xi_i$, i=1,3). On considering the elemental volume only, the right hand side of equation (7.7) will consist of six terms, equivalent to the six sides of the solid. Thus,

$$\underline{\nabla} \cdot \underline{N} \, W_P = \sum_{i=1}^{3} \frac{\partial}{h_i \partial \xi_i} W_P \quad \ldots\ldots\ldots.(7.8).$$

The contribution to the integral by pairs of opposite surfaces is obtained by the application of Taylor's theorem as

$$d\xi_{i+1} d\xi_{i+2} h_{i+1} h_{i+2} \left[\left[\underline{\nabla} \cdot \underline{N} \, W_P \right]_{\xi_i = \xi_{i_o}} + \frac{1}{2} d\xi_i \frac{d}{d\xi_i} \left[\underline{\nabla} \underline{N} \, W_P \right] \right.$$

$$\left. + \left[\underline{\nabla} \cdot \underline{N}' \, W_P \right]_{\xi_i = \xi_{i_o}} - \frac{1}{2} d\xi_i \frac{d}{d\xi_i} \left[\underline{\nabla} \cdot \underline{N}' \, W_P \right] \right] ,$$

which reduces to

$$\prod_{j=1}^{3} d\xi_j \frac{d}{d\xi_i} \left[\frac{h_{i+1} h_{i+2}}{h_i} \frac{\partial}{\partial \xi_i} W_P \right] , i=1,3 \quad \ldots.(7.9).$$

In expression (7.9), if the index exceeds 3, subtract 3. This expression holds as $\underline{N} = -\underline{N}'$. Summing up these results and inserting them in equation (7.7), the expression in the case of gravitational potential at an external point becomes

$$\underline{\nabla}^2 W_P \prod_{i=1}^{3} h_i \, d\xi_i = \sum_{i=1}^{3} \prod_{j=1}^{3} d\xi_j \frac{\partial}{\partial \xi_i}\left[\frac{h_{i+1} \, h_{i+2}}{h_i} \frac{\partial W_P}{\partial \xi_i}\right]$$

$$= 0 \quad , \quad \text{if index} > 3, -3.$$

Substitution for $\xi_i (i=1,3)$ from equation (7.4) and $h_i (i=1,3)$ from equation (7.6) in the above equation gives

$$\underline{\nabla}^2 W_P = \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial r}\left[r^2 \sin\theta \frac{\partial W_P}{\partial r}\right] + \frac{\partial}{\partial\theta}\left[\sin\theta \frac{\partial W_P}{\partial\theta}\right] + \right.$$

$$\left. + \frac{\partial}{\partial\lambda}\left[\frac{1}{\sin\theta} \frac{\partial W_P}{\partial\lambda}\right]\right]$$

$$= \frac{1}{r^2} \frac{\partial}{\partial r}\left[r^2 \frac{\partial W_P}{\partial r}\right] + \frac{1}{r^2 \sin\theta}\left[\frac{\partial}{\partial\theta}\left\{\sin\theta \frac{\partial W_P}{\partial\theta}\right\} + \right.$$

$$\left. + \frac{\partial}{\partial\lambda}\left\{\frac{1}{\sin\theta} \frac{\partial W_P}{\partial\lambda}\right\}\right] = 0 \; \ldots\ldots.. (7.10).$$

Let

$W_P = W_P(r, \theta, \lambda)$ be a solution of the above equation and let the form of this solution be

$$W_P(r, \theta, \lambda) = R(r)\, S(\theta, \lambda) \; \ldots\ldots\ldots\ldots. (7.11),$$

where $S(\theta, \lambda)$ is a function of the surface coordinates only. Further simplification of equation (7.10) gives

$$r^2 \sin\theta \frac{\partial^2 W_P}{\partial r^2} + 2r\sin\theta \frac{\partial W_P}{\partial r} + \cos\theta \frac{\partial W_P}{\partial\theta} +$$

$$+ \sin\theta \frac{\partial^2 W_P}{\partial\theta^2} + \frac{1}{\sin\theta} \frac{\partial^2 W_P}{\partial\lambda^2} = 0 \; \ldots\ldots (7.12)$$

Adopting the functions defined in equation (7.11) in equation (7.12),

$$r^2 \sin\theta\, S(\theta,\lambda) \frac{\partial^2 R}{\partial r^2} + 2 r \sin\theta\, S(\theta,\lambda) \frac{\partial R}{\partial r} + \cos\theta\, R(r) \frac{\partial S}{\partial \theta} +$$

$$+ \sin\theta\, R(r) \frac{\partial^2 S}{\partial \theta^2} + \frac{1}{\sin\theta} R(r) \frac{\partial^2 S}{\partial \lambda^2} = 0 \quad ..(7.13)$$

Dividing equation (7.13) by $\sin\theta\, R(r)\, S(\theta, \lambda)$ and re-arranging the terms,

$$\frac{r^2 \frac{\partial^2 R}{\partial r^2}}{R(r)} + \frac{2 r \frac{\partial R}{\partial r}}{R(r)} = - \frac{1}{S(\theta,\lambda)} \left[\frac{\partial^2 S}{\partial \theta^2} + \right.$$

$$\left. + \cot\theta \frac{\partial S}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 S}{\partial \lambda^2} \right] \quad .....(7.14).$$

The solutions of both sides of the above equation can hold simultaneously if the constituent differential equations are both equal to a common value which is a constant. The commonest form used for this constant is $n(n-1)$. Thus, in the general case, equation (7.14) generates two differential equations, one of which is in the variable r only and the other is in the variables θ and λ. They are

$$r^2 \frac{\partial^2 R}{\partial r^2} + 2 r \frac{\partial R}{\partial r} - n(n-1) R(r) = 0 \quad .....(7.15)$$

and

$$\frac{\partial^2 S}{\partial \theta^2} + \cot\theta \frac{\partial S}{\partial \theta} + \frac{1}{\sin^2\theta} \frac{\partial^2 S}{\partial \lambda^2} + n(n-1) S(\theta, \lambda) = 0$$

$$......................(7.16)$$

In a similar manner, equation (7.16) can be further subdivided into two differential equations in each of the surface coordinates θ and λ . If

$$S(\theta, \lambda) = G(\theta)\, F(\lambda) \quad \ldots\ldots\ldots\ldots\ldots\ldots(7.17),$$

appropriate substitution in equation (7.16) gives

$$\frac{d^2G}{d\theta^2} F(\lambda) + \cot\theta\, F(\lambda) \frac{dG}{d\theta} + \operatorname{cosec}^2\theta\, G(\theta) \frac{d^2F}{d\lambda^2} + $$

$$+ n(n-1)\, G(\theta)\, F(\lambda) = 0 \;.$$

Re-arrangement and separation of variables gives

$$\frac{\sin\theta}{G(\theta)} \frac{d^2G}{d\theta^2} + \frac{\sin 2\theta}{2G(\theta)} \frac{dG}{d\theta} + n(n-1)\sin^2\theta = -$$

$$- \frac{1}{F(\lambda)} \frac{d^2F}{d\lambda^2} \;.$$

Both sides of the above equation can hold independently if they equal some common constant value, say m^2. The resulting differential equations are

$$\frac{d^2F}{d\lambda^2} + m^2 F(\lambda) = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots(7.18)$$

and

$$\sin^2\theta \frac{d^2G}{d\theta^2} + \frac{\sin 2\theta}{\;} \frac{dG}{d\theta} + (n-1)n\sin^2\theta - m^2 G(\theta) = 0$$

$$\ldots\ldots\ldots\ldots\ldots(7.19)$$

The complete solution of Laplace's equation (7.1) can be expressed in terms of the product of the solutions of the three differential equations (7.15), (7.18) and (7.19). Putting

$$\mu = \cos\theta \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7.20),$$

then

$$\frac{dG}{d\theta} = - \sin\theta \frac{\partial G}{\partial \mu} \quad \ldots\ldots\ldots\ldots\ldots\ldots (7.21),$$

and

$$\frac{d^2G}{d\theta^2} = - \cos\theta \frac{\partial G}{\partial \mu} + \sin^2\theta \frac{\partial^2 G}{\partial \mu^2} \quad \ldots\ldots\ldots (7.22).$$

The substitution of equations (7.21) and (7.22) in equation (7.19) gives

$$(1 - \mu^2) \left[- \mu \frac{\partial G}{\partial \mu} + (1 - \mu^2) \frac{\partial^2 G}{\partial \mu^2} \right] - \mu(1 - \mu^2) \frac{\partial G}{\partial \mu} +$$

$$+ \left[n(n - 1)(1 - \mu^2) - m^2 \right] G(\theta) = 0 .$$

Division of the above equation by $(1 - \mu^2)$ and the grouping of terms gives

$$(1 - \mu^2) \frac{\partial^2 G}{\partial \mu^2} - 2 \frac{\partial G}{\partial \mu} + \left[n(n - 1) - \frac{m^2}{1 - \mu^2} \right] G(\theta) = 0$$

$$\ldots\ldots\ldots\ldots\ldots (7.23).$$

When m = 0, equation (7.23) reduces to Legendre's differential equation, whose solution (Whittaker and Watson, 1963, 304) is

$$G(\theta) = p_n(\mu) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.24),$$

where

$$p_n(\mu) = \frac{1}{2^n n!} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n \quad \ldots\ldots\ldots\ldots\ldots (7.25).$$

If $m \neq 0$, the solution of equation (7.23) is (Whittaker and Watson, 1963, 324)

$$G(\theta) = (1 - \mu^2)^{m/2} \frac{d^m}{d\mu^m} p_n(\mu) \quad \ldots\ldots\ldots\ldots(7.2\_).$$

Conventionally, the solution is represented as

$$G(\theta) = P_{nm}(\mu) \quad , \text{ where}$$

$$P_{nm}(\mu) = \frac{1}{2^n\, n!} \sin^m\theta \frac{d^{m+n}}{d\mu^{m+n}} (\mu^2 - 1)^n \ldots\ldots(7.27).$$

In most literature, this solution, known as the associated Legendre function (Jeffreys, 1962a, 134), is represented by the function $p_{nm}(\mu)$ which is related to $P_{nm}(\mu)$ by the relation

$$p_{nm}(\mu) = \frac{(n - m)!}{n!} P_{nm}(\mu) \quad \ldots\ldots\ldots\ldots(7.28),$$

as in this form, the variability of the mean square values of the associated Legendre functions increase with m.

Equation (7.15) is known as Euler's equation, the solution of which is

$$R(r) = c_1 r^n + \frac{c_2}{r^{n+1}} \quad .$$

Setting the boundary conditions, $W_P \to 0$ as $r \to \infty$. Hence, $c_1 = 0$ and

$$R(r) = \frac{c_2}{r^{n+1}} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(7.29).$$

The solution of equation (7.18) is easily seen to be

$$F(\lambda) = c_3 \cos m\lambda + c_4 \sin m\lambda \quad \ldots\ldots\ldots(7.30).$$

Thus the following equations are solutions of Laplace's equation in the case of gravitational potential

$$W_{P_n} = \frac{c_2}{r^{n+1}} \left[c_3 \cos m\lambda + c_4 \sin m\lambda \right] p_{nm}(\mu) , \quad n=0, \infty \quad .$$

As the infinite number of solutions in the previous equation are of different orders of magnitude, a complete solution is given by

$$W_P = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} p_{nm}(\mu) \left[a_{nm} \cos m\lambda + b_{nm} \sin m\lambda \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.31),$$

where a_{nm} and b_{nm} are constant coefficients. Equation (7.31) can also be expressed as

$$W_P = \sum_{n=0}^{\infty} \frac{A_{nm}}{r^{n+1}} S_{nm}(\theta, \lambda) \quad \ldots\ldots\ldots\ldots\ldots (7.32),$$

where

$$A_{nm} S_{nm}(\theta, \lambda) = \sum_{m=0}^{n} p_{nm}(\mu) \left[a_{nm} \cos m\lambda + b_{nm} \sin m\lambda \right] \quad \ldots\ldots\ldots\ldots\ldots\ldots (7.33).$$

$A_{nm} S_{nm}$ is called a surface harmonic, the quantity W_{P_n} is called a solid harmonic of order n.

The solution obtained from equation (7.33) when m = 0 is called a zonal harmonic, $p_{n0}(\mu)$ being called a Legendre function. For $m \neq 0$, the solution obtained from equation (7.33) is called a tesseral harmonic, the quantity $p_{nm}(\mu)$ being called an associated Legendre function. The surface harmonic $A_{nn} S_{nn}$ is called a sectorial harmonic.

7.6 The physical interpretation of surface harmonics.

Surface harmonics are ca‚ able of physical interpretation on a regular closed surface, e.g., a sphere.

(i) Zonal harmonics

The zonal harmonic is obtained as the special case of the surface harmonic when m = 0.

$$p_{n0}(\mu) = \frac{1}{2^n\, n!} \frac{d^n}{d\mu^n} (\mu^2 - 1)^n \quad \text{........(7.34).}$$

There exist n values of θ at which the function $(\mu^2-1)^n$ will be zero at each of $\mu = 1$ and $\mu = -1$. Its first derivative has (n-1) real zeros each at $\mu = \pm 1$ and one other zero in between. In general, the n-th derivative which occurs in $p_{n0}(\mu) = p_n(\mu)$ will have n values of θ between 0 and π .t which $p_n(\mu) = 0$.

In the case where θ is the co-latitude, this affords the interpretation that zonal harmonics of order n keep the same sign over (n+1) belts of latitude, the boundary of each such belt being a parallel of latitude. Fig (7.2) illustrates the case when n = 7. The polar caps will also count as belts. The following corollaries hold.

(i) If n = 29, these belts occur at 6° intervals of latitude .

(ii) Even order zonal harmonics are symmetrical about the equator.

(iii) The existence of odd-order zonal harmonics indicates asymmetry about the equator.

(iv) If a function which varies over a sphere is represented by a set of harmonics, then low order zonal harmonics represent slow changes of the function with changes in μ. High order harmonics represent more rapid changes in the function for the same change in μ.

(v) If a function F has small variations from some value F_o over a sphere, and if these variations were a function of θ only, F could be adequately represented by a set of zonal harmonics

$$F = \sum_{n=0}^{\infty} A_n p_n(\mu) \quad ,$$ where A_n is the associated harmonic coefficient, A_o being equal to F_o as $p_o(\mu) = 1$.

The even zonal harmonics $p_2(\mu)$, $p_4(\mu)$ etc., will represent those variations which are symmetrical about the equator, while the odd order zonal harmonics will represent asymmetries about the equator. The zonal harmonics $p_1(\mu)$ and $p_3(\mu)$ will represent those asymmetries which vary very slowly with θ.

(ii) <u>The associated Legendre function.</u>

From equations (7.27) and (7.28), the associated Legendre function $p_{nm}(\mu)$ is given by

$$p_{nm}(\mu) = \frac{(n-m)!}{2^n (n!)^2} \sin^m\theta \frac{d^{n+m}}{d\mu^{n+m}} (\mu^2 - 1)^n \quad \ldots .(7.35)$$

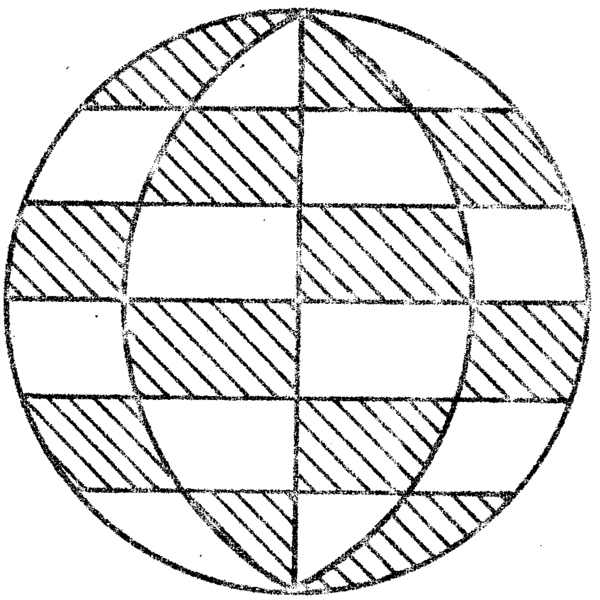
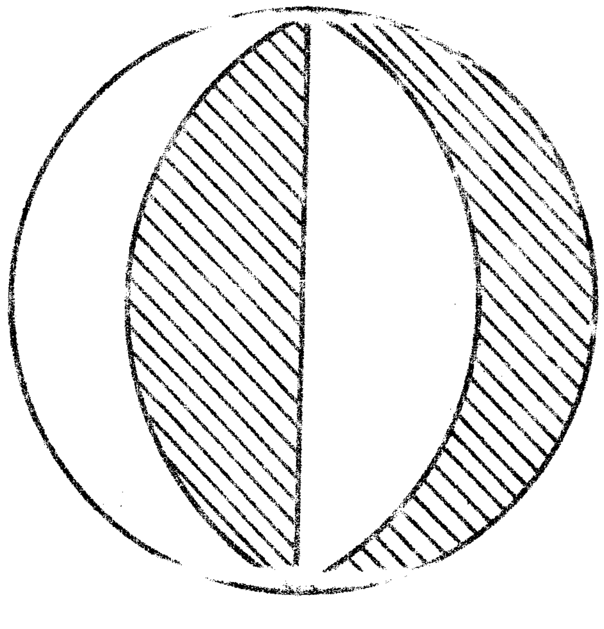


FIG. 7.3

THE TESSERAL HARMONIC $\sin 4\lambda P_{94}(\mu)$



In equation (7.35), $\sin\theta \neq 0$ unless $\mu = \pm 1$ in the range $0 \leq \theta \leq \pi$. Thus, if $m > 0$, $p_{nm}(\mu)$ has m zeros less than $p_{n0}(\mu)$, i.e., (n - m) zeros in the range $-1 \leq \mu \leq 1$, all of which are real and hence there are (n - m + 1) belts in all, inclusive of polar caps.

(iii) Tesseral harmonics

The tesseral harmonic T_{nm} is given by

$$T_{nm} = p_{nm}(\mu) \left\{ \cos m\lambda + \sin m\lambda \right\} \quad ....(7.36),$$

where $m \neq 0$. The combination of the associated Legendre function and the longitude dependent function gives rise to a surface harmonic which changes sign along (n - m + 1) parallels of latitude and also increases the number of meridians over which the longitude function changes sign from 2 to 2m. As such, the tesseral harmonic will change sign at m meridians in a hemisphere .

Fig (7.3) illustrates the tesseral harmonic $\sin 4\lambda \; p_{94}(\mu)$ over a hemisphere. Thus tesseral harmonics represent variations in both latitude and longitude of any function they are chosen to represent over a sphere.

(iv) Sectorial harmonics.

Sectorial harmonics are special cases of tesseral harmonics when $n = m$. From the above discussion it can be seen that sectorial harmonics represent those variations of a function which are longitude dependent. Fig (7.4) represents the sectorial harmonic $\sin 4\lambda \, p_{44}(\mu)$ over a hemisphere.

Tesseral harmonics in the special case of $n = m$ can also represent variations of the function with latitude, provided such variations are of the same sign.

Thus, any function which takes values on the surface of a sphere can be represented by a family of spherical harmonics, exact representation being ob ined by assigning appropriate values for the coefficients of the harmonic functions. The higher the order of harmonic functions used, the more accurately will the set of functions represent the local fluctuations of the quantity being represented. The order (n) to which the function being represented must be analysed depends entirely on the size of the adjacent blocks over which the function can be expected to have variations from the zero order value but of opposite sign.

In the case of the earth, if the size of the smallest such unit of surface area is a $u^{o} \times u^{o}$ square, whose bounding curves are meridians and parallels, the nexus between n and u is obtained by considering the variation (dF) of the function F from a mean value over the earth in relation to the variation of dF over a $u^{o} \times u^{o}$ square. If the mean values of dF for the population of $u^{o} \times u^{o}$ squares are part of a random distribution about a zero mean, then it would be sufficient to carry out the harmonic analysis to order n, where n is given by the relation

$$u\ n = 180^{o} \quad \ldots\ldots\ldots\ldots\ldots\ldots(7.37),$$

where u is expressed in degrees of arc. The analysis will exclude the zonal harmonic $p_{n0}(\mu)$.

If, for example, it is required to represent a function over a sphere using values for each $5^{o} \times 5^{o}$ square, it is not necessary to carry out an analysis to order n = 36 to obtain adequate values. The critical factor in such an analysis is not the size of the square but the value of u as defined in equation (7.37). The only requirement to be met is that the mean values of dF over the squares whose sides are of angular length u, are normally distributed about a mean value zero over the sphere.

7.7 Normalisation of Legendre functions.

The coefficient of a given harmonic is dependent on the structure of the harmonic and the nature of the function it seeks to represent. The conventional form of the general Legendre function is given in equation (7.27). The concept of normalising Legendre functions is introduced with the idea of making the values of certain integrals containing Legendre functions equal to a chosen quantity between set limits of integration. The value usually adopted is unity and the introduction of the correcting factor to satisfy this condition is called the normalisation of the Legendre function . This results in the amendment of the values of the coefficients a_{nm} and b_{nm} of the surface harmonic defined in equation (7.31).

Jeffreys (1962b, 632) uses the expression

$$p_{nm}(\mu) = \frac{(n - m)!}{n!} \sin^m\theta \frac{d^m}{d\mu^m} p_n(\mu) \quad ....(7.38).$$

In this form, the associated Legendre function has the property

$$p_{n(-m)}(\mu) = (-1)^m p_{nm}(\mu) \quad ....(7.39).$$

Other researchers (e.g., Kaula, 1959, 39) prefer to to use the Legendre functions in their normalised form ($\overline{P}_{nm}(\mu)$) where the latter is constructed to satisfy the relation

$$\frac{1}{4\pi} \int_0^{\pi} \int_0^{2\pi} \left[\overline{P}_{nm}(\mu) \left\{ \begin{matrix} \cos m\lambda \\ \sin m\lambda \end{matrix} \right\} \right]^2 \sin\theta \, d\theta \, d\lambda = 1$$

$$..............(7.40).$$

If $m \neq 0$, this can be achieved (Jeffreys, 1962b, 638) if

$$\overline{P}_{nm}(\mu) = P_{nm}(\mu) (4\pi)^{\frac{1}{2}} \left[\frac{(2n + 1)}{2\pi} \frac{(n - m)!}{(n + m)!} \right]^{\frac{1}{2}}$$

$$= \left[\frac{2(2n + 1)(n - m)!}{(n + m)!} \right]^{\frac{1}{2}} P_{nm}(\mu) \quad ....(7.41).$$

If m = 0, and the function reduces to a simple Legendre function,

$$\overline{P}_{n0}(\mu) = (2n + 1)^{\frac{1}{2}} P_{n0}(\mu) \quad ....(7.42).$$

In general, the fully normalised Legendre function which satisfies equation (7.40) is given by

$$\overline{P}_{nm}(\mu) = \left[\frac{(2 - \delta_{0m})(2n + 1)(n - m)!}{(n + m)!} \right]^{1/2} P_{nm}(\mu)$$

$$..............(7.43),$$

where $P_{nm}(\mu)$ is the conventional Legendre function and δ_{0m} is the Kronecker delta (Jeffreys, 1962b, 59) given by

$$\delta_{ik} = \begin{cases} 1 & \text{when } i = k \\ 0 & \text{when } i \neq k \end{cases} \quad \ldots\ldots\ldots(7.44)$$

7.8 Gravity anomalies expressed in terms of the harmonics of the earth's gravitational field.

From equation (7.31), the external potential of the earth at a point P (W_P) can be expressed as

$$W_P = \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \sum_{m=0}^{n} P_{nm}(\mu) \left[a_{nm} \cos m\lambda + b_{nm} \sin m\lambda \right] \quad \ldots\ldots\ldots(7.31).$$

This particular expression uses the conventional Legendre functions given by equation (7.27) (Kaula, 1962, 35)

$$P_{nm}(\mu) = \sin^m\theta \frac{d^m}{d\mu^m} P_n(\mu)$$

$$= \frac{(1-\mu^2)^{m/2}}{2^n\, n!} \sum_{t=0}^{k} \frac{(2n-2t)!}{(n-m-2t)!} \binom{n}{t} (-1)^t \ldots \ldots\ \mu^{(n-m-2t)} \quad \ldots\ldots\ldots\ldots\ldots(7.45),$$

where $k = \frac{n-m}{2}$ the value of k being the truncated integer. In literature on satellite geodesy, the above expression for potential is expressed in the form (e.g., Kaula, 1962, 35)

$$W_P = \frac{kM}{r_P} \sum_{n=0}^{\infty} \left(\frac{a_e}{r_P} \right)^n \sum_{m=0}^{n} P_{nm}(\mu) \left[C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \right] \quad \ldots\ldots\ldots\ldots(7.46),$$

where a_e is the equatorial radius of the earth and M its mass. An alternative form is

$$W_P = \frac{kM}{r_P} \left[1 - \sum_{n=0}^{\infty} \left[\frac{a_e}{r_P} \right]^n \sum_{m=0}^{n} P_{nm}(\mu) \left[J_{nm} \cos m\lambda + \right. \right.$$

$$\left. \left. + K_{nm} \sin m\lambda \right] \right] \qquad \ldots\ldots\ldots\ldots(7.47),$$

where

$$a_{nm} = kMa_e^n \, C_{nm} = - kMa_e^n \, J_{nm} \qquad \ldots\ldots\ldots(7.48)$$

and

$$b_{nm} = kMa_e^n \, S_{nm} = - kMa_e^n \, K_{nm} \qquad \ldots\ldots\ldots(7.49).$$

As shown in section (3), the free air gravity anomaly at a point P exterior to the earth is the difference between observed gravity at P and the value of gravity at the equivalent point on the associated spherop of the international spheroid or adopted reference model. If this reference model is chosen so that its origin is at the centre of mass of the existent earth, McCullagh's formula (Bomford, 1962, 393) shows that the following relation is satisfied

$$\frac{kM}{r} \frac{a_e}{r} \sum_{m=0}^{1} P_{1m}(\mu) \left[C_{1m} \cos m\lambda + S_{1m} \sin m\lambda \right]$$

$$= 0 \qquad \ldots\ldots\ldots\ldots\ldots\ldots(7.50).$$

Jeffreys (1962a, 141) interprets this physically as follows. A change in the position of the reference axes does not alter the values of surface gravity.

The international spheroid, being a regular solid of revolution, takes into account only those variations of gravity with latitude. If the rotational potential is excluded, the potential at an exterior point is given by

$$V_P = \frac{kM}{r_P} P_{00}(\mu)\, C_{00} + \left[\frac{a_e}{r_P}\right]^2 P_{20}(\mu)\, C'_{20} + \left[\frac{a_e}{r_P}\right]^4 P_{40}(\mu)\, C'_{40} \quad \ldots\ldots\ldots (7.50).$$

The odd order zonal harmonics are excluded from this theoretical representation as the gravitational field of the regular reference spheroid is symmetrical about the equator. The zero order term is chosen to be the same as that for the existent earth as the term in the case of the international spheroid is obtained by a least squares fit of the mathematical model to observed gravity. This is more fully discussed in sections (13) and (15). It should be noted that C_{20} and C'_{40} are not necessarily the same as C_{20} and C_{40} respectively. Tesseral harmonic terms are not considered as the reference spheroid is a solid of revolution. The disturbing potential at the surface of comparison (V_{D_P}) is given by

$$V_{D_P} = \frac{kM}{r_P} \sum_{n=2}^{\infty} \left[\frac{a_e}{r_P}\right]^n \sum_{m=0}^{n} P_{nm}(\mu) \left[C_{nm} \cos m\lambda + S_{nm} \sin m\lambda\right] \quad \ldots\ldots\ldots (7.51),$$

as $V_{D_P} = W_P - V_P$.

C_{20} and C_{40} given in equation (7.51) are not the same as the quantities with identical descriptions in equation (7.45).

They are, however, related by the following equations

$$C_{20_{V_D}} = C_{20_W} - C'_{20}$$
$$C_{40_{V_D}} = C_{40_W} - C'_{40} \qquad \ldots\ldots\ldots\ldots(7.52).$$

The combination of equation (7.51) and equation (3.20) gives

$$\Delta g_F = -\frac{kM}{r_P} \sum_{n=2}^{\infty} \frac{a_e^n}{r_P^{n+1}} \{-(n+1)+2\} \sum_{m=0}^{n} P_{nm}(\mu) \left[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right]$$

$$= \frac{kM}{r_P^2} \sum_{n=2}^{\infty} (n-1) \left[\frac{a_e}{r_P}\right]^n \sum_{m=0}^{n} P_{nm}(\mu) \left[C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right] \qquad \ldots(7.53).$$

The quantities C_{nm} and S_{nm} can be replaced by the equivalent fully normalised coefficients given by equations (7.43) and (7.44). Denoting these coefficients as $\bar{C}_{nm}$ an $\bar{S}_{nm}$, equation (7.53) becomes

$$\Delta g_F = \frac{kM}{r_P^2} \sum_{n=2}^{\infty} (n-1) \left[\frac{a_e}{r_P}\right]^n \sum_{m=0}^{n} \bar{P}_{nm}(\mu) \left[\bar{C}_{nm}\cos m\lambda + \bar{S}_{nm}\sin m\lambda\right] \qquad \ldots\ldots\ldots(7.54)$$

where

$$\bar{C}_{nm} = \left[\frac{(n+m)!}{(2-\delta_{0m})(2n+1)(n-m)!}\right]^{1/2} C_{nm} \qquad \ldots\ldots(7.53)$$

From equation (7.48),

$$\bar{C}_{nm} = -\left[\frac{(n+m)!}{(2-\delta_{0m})(2n+1)(n-m)!}\right]^{1/2} J_{nm} \ldots (7.53a).$$

Similar expressions relate $\bar{S}_{nm}$, S_{nm} and K_{nm}.

Conclusion :- The free air anomalies on the earth's surface could be calculated if the coefficients of the Legendre functions, used in the calculation of the potential due to the earth's gravitational field at a point P exterior to the surface of the earth, were known. It would also be necessary to the international gravity formula expressed in spherical harmonics. The latter (Jeffreys, 1961, 5) can be expressed as

$$\gamma^{(mgal)} = C'_{00} + C'_{20}P_{20}(\mu) + C'_{40} \quad P_{40}(\mu) \ldots (7.56),$$

where

$$C'_{00} = 979770 ,$$

$$C'_{20} = 3446.0 ,$$

and

$$C'_{40} = 5.3.$$

In writing out equation (7.56), C'_{00} has not been assumed equal to C_{00}, though this is the conventional procedure. It should also be noted that the gravity anomaly deduced from the above relations can hardly represent the gravity anomaly on ground, in the sense of individual gravity readings in view of the methods used to determine the coefficients of the harmonic functions. However, such determinations should provide adequate estimates of area means.

7.9 Determination of the zonal and tesseral harmonic coefficients of the basic equation.

In the following discussion, all the coefficients $\bar{C}_{nm}$, $\bar{S}_{nm}$ to be determined are of order 10^{-6}, with the exception of $\bar{C}_{20}$, which (Kaula, 1963, 509) is given by

$$\bar{C}_{20} = - \frac{J_{20}}{5^{\frac{1}{2}}} = 484.2 \times 10^{-6} \quad \ldots\ldots(7.57),$$

or C_{20} (King-Hele and Cook, 1965, 17) by

$$C_{20} = - J_2 = 1082.64 \pm 0.02 \quad \ldots.(7.58) .$$

All determinations of the other harmonic coefficients (e.g., King Hele et al, 1963, 119 ; Guier and Newton, 1965, 4619) indicate that the orders of both tesseral and zonal terms are of order which approaches 10^{-6}. Such determinations are effected by the analysis of the rates of variation of the parameters of the instantaneous orbital ellipse. As explained in section (7.3), certain variations are primarily due to the nature of the earth's gravitational potential. The disturbing potential V_{D_P}, given by equation (7.51), can be expressed as

$$V_{D_P} = \frac{kM}{r_P} \sum_{n=2}^{\infty} \left[\frac{a_e}{r_P} \right]^n \sum_{m=0}^{n} P_{nm}(\mu) \left[C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \right]$$

$$= \sum_{n=2}^{\infty} \sum_{m=0}^{n} V_{D_{nm}} \quad \ldots\ldots\ldots(7.59).$$

The zonal harmonics are determined by a study of the perturbations of normally inclined orbits, while the tesseral harmonic coefficients are best determined from resonant ones having periods which approach 24 hours.

The even zonal harmonics are determined (King-Hele et al, 1963, 120) from secular variations while the odd order zonal coefficients are obtained from a study of the long period variations of the orbital elements. The Keplerian elements of an orbital ellipse are

a = semi-major axis of the orbital ellipse,

e = the eccentricity of the orbital ellipse ,

ω = the argument of perigee ,

i = the inclination of the orbit to the plane of the equator,

Ω = the right ascension of the node, and

$$M = E - e \sin E \quad \ldots\ldots\ldots\ldots\ldots\ldots (7.60),$$

which is Kepler's equation. These elements are completely defined in fig (7.5) and fig (7.6).

If ϕ and λ are the instantaneous latitude and longitude of the satellite with reference to the earth's equatorial plane - Greenwich meridian plane reference system,

$$\sin \phi = \sin i \sin(\omega + f) \ldots\ldots\ldots (7.61),$$

where f is called the true anomaly. E is called the eccentric anomaly and M, defined in equation (7.60) the mean anomaly. λ is given by

$$\lambda = \lambda(\Omega, \omega, M, \theta) \ldots\ldots\ldots\ldots (7.62),$$

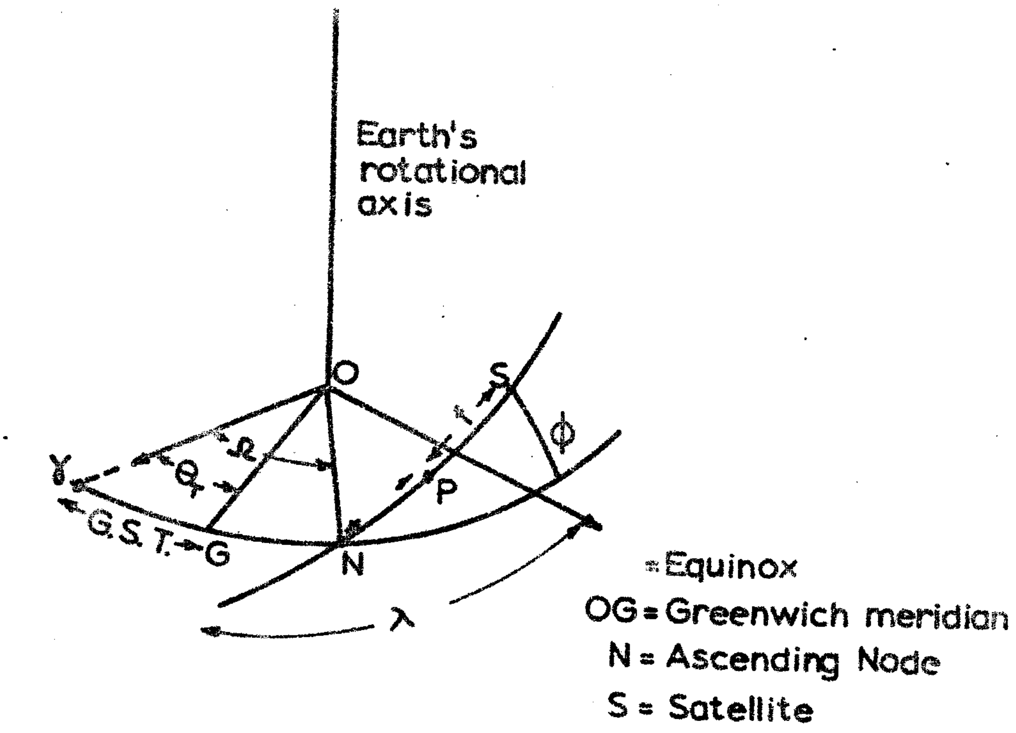

Earth's rotational axis
O
S
P
N
G
G.S.T.
λ
φ
= Equinox
OG = Greenwich meridian
N = Ascending Node
S = Satellite

FIG. 7.5

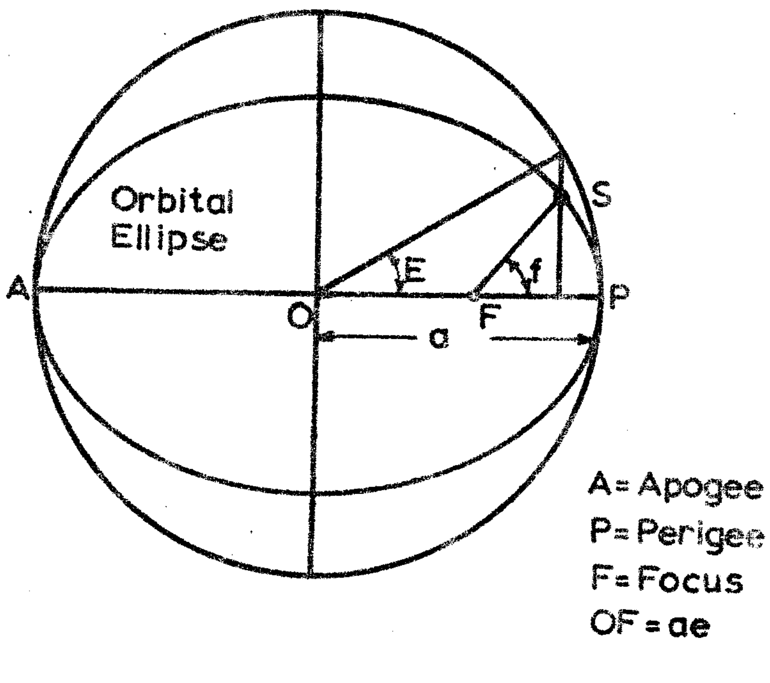

Orbital Ellipse
A
O
F
P
S
E
f
a
A= Apogee
P= Perigee
F= Focus
OF = ae

FIG. 7.6

where θ is Greenwich sidereal time. Evaluation and substitution in equation (7.59) gives the n-th order coefficient of the disturbing potential (Kaula, 1962, 35) as

$$V_{D_{nm}} = \frac{kMa_e^n}{a^{n+1}} \sum_{p=0}^{n} F_{nmp}(i) \sum_{q=-\infty}^{\infty} G_{npq}(e)$$

$$S_{nmpq}(\omega, M, \Omega, \theta) \quad \ldots\ldots\ldots\ldots (7.63),$$

where

$$F_{nmp}(i) = \sum_{t=0}^{p(\text{or } k \text{ if } k<p)} \frac{(2n-2t)!}{t!\,(n-t)!\,2^{(2n-2t)}} \sin^{n-m-2t} i$$

$$\sum_{s=0}^{m} \begin{bmatrix} m \\ s \end{bmatrix} \cos^s i \sum_{c} \begin{bmatrix} n-m-2t+s \\ c \end{bmatrix} \begin{bmatrix} m-s \\ p-t-c \end{bmatrix} \frac{(-1)^{c-k}}{(n-m-2t)!}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.64),$$

where $k = \frac{n-m}{2}$, the value being truncated if non-integral, and c is summed over all values making the two binomial coefficients non-zero;

for $q = 2p - n$,

$$G_{npq}(e) = (1-e^2)^{\frac{1}{2}-n} \sum_{d=0}^{p'-1} \begin{bmatrix} n-1 \\ 2d+n-2p' \end{bmatrix} \begin{bmatrix} 2d+n-2p' \\ d \end{bmatrix} \left[\frac{e}{2}\right]^{2d+n-2p'} \quad \ldots\ldots\ldots\ldots (7.65),$$

$$\text{where} \quad p' = \begin{cases} n-p & \text{if } p \geq 0.5n \\ p & \text{if } p \leq 0.5n \end{cases}$$

Equation (7.65) becomes more complex for $q \neq 2p - n$. In equation (7.63),

$$S_{nmpq}(\omega, M, \Omega, \theta) = \begin{bmatrix} C_{nm} \\ \\ -S_{nm} \end{bmatrix}_{(n-m)\ \text{odd}}^{(n-m)\ \text{even}} \cos\{(n-2p)\omega + (n-2p+$$

$$+q)M + m(\Omega-\theta)\} + \begin{bmatrix} S_{nm} \\ \\ C_{nm} \end{bmatrix}_{(n-m)\ \text{odd}}^{(n-m)\ \text{even}} \sin\{(n-2p)\omega + (n-2p+$$

$$+q)M + m(\Omega-\theta)\} \quad \ldots\ldots (7.66).$$

In equations (7.63) to (7.66), it should be noted that

$\begin{bmatrix} x \\ y \end{bmatrix}$ indicates summation over the relevant terms while

$\begin{bmatrix} x \\ y \end{bmatrix}$ $\begin{matrix} \text{condition} \\ \\ \text{condition} \end{matrix}$ indicate alternatives

as set by the conditions outside the grouping sign.

The effect of the disturbing potential on the variations of the Keplerian elements with time are given (Kaula, 1962, 9) by

$$\dot{a} = \frac{2}{n_g a} \frac{\partial V_D}{\partial M} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (7.67),$$

$$\dot{e} = \frac{1-e^2}{n_g a^2 e} \frac{\partial V_D}{\partial M} - \frac{(1-e^2)^{\frac{1}{2}}}{n_g a^2 e} \frac{\partial V_D}{\partial \omega} \quad \ldots\ldots (7.68),$$

$$\dot{\Omega} = \frac{1}{n_g a^2 (1-e^2)^{1/2} \sin i} \frac{\partial V_D}{\partial i} \quad \ldots\ldots\ldots\ldots (7.69),$$

$$\dot{\omega} = -\frac{\cos i}{n_g a^2 (1-e^2)^{\frac{1}{2}} \sin i} \frac{\partial V_D}{\partial \omega} + \frac{(1-e^2)^{\frac{1}{2}}}{n_g a^2 e} \frac{\partial V_D}{\partial e} \quad ..(7.70)$$

$$\dot{M} = n_g - \frac{1-e^2}{n_g a^2 e} \frac{\partial V_D}{\partial \varepsilon} - \frac{2}{n_g a} \frac{\partial V_D}{\partial a} \quad \cdot ..(7.71),$$

and

$$\frac{di}{dt} = \dot{i} = \frac{\cos i}{n_g a^2 (1-e^2)^{\frac{1}{2}} \sin i} \frac{\partial V_D}{\partial \omega} - \frac{1}{n_g a^2 (1-e^2)^{\frac{1}{2}} \sin i} \frac{\partial V_D}{\partial \Omega}$$

$$..................(7.72),$$

where

$$n_g \; (\doteqdot \dot{M}) = \left[\frac{kM}{a^3}\right]^{\frac{1}{2}} \quad ....)\,(7.73),$$

from Kepler's third law.

(ii) Discussion.

$\frac{\partial V_D}{\partial M}$, $\frac{\partial V_D}{\partial \omega}$ and $\frac{\partial V_D}{\partial \Omega}$ must be sinusoidal in character from the form of S_{nmpq}, which, if small, could be interpreted as allowing a, e and i on the right hand sides of equations (7.67) to (7.73) to be treated as constants. But this is merely conjecture as the Keplerian elements are instantaneous as can be seen from a study of the equations quoted. The entire problem becomes exceedingly complex due to the non-linear characteristics inroduced by these constantly varying Keplerian elements. The most convenient development is the use of the concept of an intermediate orbit which has an instantaneous geometrical definition. Once this intermediate model is assumed, different methods are available (Kaula, 1963, 532) for the solution of the problem.

(iii) The determination of zonal harmonics.

Zonal harmonics are obtained when $m = 0$, and equation (7.63) reduces to (Kaula, 1962, 37)

$$V_{D_{n0}} = \frac{a_e^n \, kM}{a^{n+1}} F_{n0k}(i) \, G_{nk(2k-n)}(e) \, C_{n0} \begin{cases} 1 & n \text{ even} \\ 2\sin\omega & n \text{ odd} \end{cases} \qquad \ldots\ldots\ldots\ldots(7.74),$$

where $k = \frac{n}{2}$ and $p = k$. The factor 2 appears in the alternative value when n is odd as equation (7.74) is the sum of two equal terms obtained when $p = k$ and $p = k + 1$.

The determination of the even C_{n0}'s is effected by analysing the secular changes in ω and Ω as the rates of change of these quantities are reasonable constant. $\dot{\Omega}$ is best to work with as the ascending node can be unambiguously defined as the point where the satellite orbit, on the celestial sphere, cuts the equator going north (King-Hele et al, 1963, 123). The observational data from a variety of satellites with different inclinations of orbits to the equator is used in effecting the solution. For the determination of the first n even zonal harmonics, it is necessary to have adequate observational data from the tracking of at least n satellites which is then analysed for the orbital parameters.

In practice, the values of a and e for any satellite lie within a restricted range and only the values of i have any variations of considerable magnitude. For example, King-Hele and Cook (1965, 17) used seven satellites with values of i in the range $28^{\circ} \leq i \leq 96^{\circ}$ for the determination of J_2, J_4, J_6 and J_8. The full formulae for the determination of the first n values of the even zonal harmonics are given by King-Hele (King-Hele et al., 1963, 123 et seq). Kaula (1963, 534) emphasises the necessity for

(a) accurate evaluation of the constants of integration and averaging the values of a and e for the elimination of drag effects ;

(b) accurate determinations of i in the cases where $i > 45^{\circ}$, as errors in i for near polar orbits, give rise to large effects on the value of $\dot{\Omega}$. This is seen from a direct consideration of equation (7.70) as an error e_i in i causes an error $e_{\dot{\Omega}}$ in $\dot{\Omega}$ given by

$$e_{\dot{\Omega}} = - \dot{\Omega} \tan i \, e_i \quad \ldots\ldots\ldots\ldots\ldots (7.75) ;$$

(c) avoiding satellites with low perigees, non-spherical shapes and large area-to-mass ratios as the calculation of drag and radiation pressure effects in such cases is complicated.

The assignment of weight coefficients to observation equations in orbital analysis has also been the subject of query. In fact, some researchers have tended to give such small weight coefficients to data from polar orbital satellites that such orbits have almost been weighted out of the solution.

The determination of the odd C_{n0}'s is effected by substituting the partial derivatives from equation (7.74) in equations (7.67) to (7.72). For odd values of n, variations occur in e, i, M, ω and Ω with frequency ω. The use of the concept of an intermediate orbit introduces a complication as the perturbations of e and i cause an interaction with C_{20} which, as the latter is three orders larger than the other harmonic coefficients, have a significant magnitude. Thus all terms of the form $C_{20}C_{n0}$ must also be included in the analysis.

It is also of consequence to have the drag corrections mentioned in (a) above to be correctly applied. If the values of a and e used are not the average values, it is necessary to correct the node and perigee for an average value on the assumption that the perigee height is constant.

(iv) The determination of the tesseral harmonics.

The C_{22} tesseral harmonic was first predicted by Jeffreys (1962a, 187) from surface gravity values. Surface gravity anomalies were expressed by Kaula upto the eighth order (1959, 51 et seq). The rotation of the earth has the effect of causing variations in θ ($\dot{\theta}$) which are many orders of magnitude larger than the variations ($\dot{\omega}$, $\dot{\Omega}$) in ω and Ω. Thus, the tesseral harmonics, which are longitude dependent, cannot be determined by a study similar to that described in section (ii) above, as, inevitably, the frequency of observation is not much greater than the frequency of orbital perturbations caused by the tesseral harmonics. Further, even if numerous observation stations were available, the result would be heavily dependent on the absolute accuracy of the positions of the tracking stations.

The problem is made more complex by drag effects affecting observations. In fact, most of the earlier determinations were bedevilled by the inability to carry out an adequate statistical analysis of the observational data and eliminate the covariance between observations. The separation of the orbital perturbations due to tesseral harmonics alone from those due to drag effects, mutual effects and tracking station positional errors has been effected by the Transit system where four satellites of ideal altitude and a good range of i are tracked electronically. The substitution of equations (7.63) to (7.66) in equations (7.67) to (7.72) through the equation (7.59) gives (Kaula, 1965b, 6)

$$\Delta\Omega_{nm} = \frac{1}{n_g a^2 (1-e^2)^{1/2} \sin i} \frac{kM a_e^n}{a^{n+1}} \sum_{p=0}^{n} \frac{\partial F_{nmp}(i)}{\partial i}$$

$$\sum_{q=-\infty}^{\infty} G_{npq}(e) \frac{1}{(n-2p)\dot{\omega} + (n-2p+q)\dot{M} + m(\dot{\Omega} - \dot{\theta})}$$

$$\left\{ \begin{bmatrix} -C_{nm} \\ -S_{nm} \end{bmatrix}_{(n-m)\ \text{odd}}^{(n-m)\ \text{even}} \sin\left\{(n-2p)\omega + (n-2p+q)M + m(\Omega - \theta)\right\} - \right.$$

$$\left. - \begin{bmatrix} S_{nm} \\ C_{nm} \end{bmatrix}_{(n-m)\ \text{odd}}^{(n-m)\ \text{even}} \cos\left\{(n-2p)\omega + (n-2p+q)M + m(\Omega - \theta)\right\} \right\}$$

..................(7.76)

For $\Delta\Omega$ to permit evaluation from observation,

(a) a must be small ,

(b) the index q must be small and the second nested series must converge rapidly,

(c) the rate $(n - 2p)\dot{\omega} + (n - 2p + q)\dot{M} + m(\dot{\Omega} - \dot{\theta})$ should be as small as possible. In general,

$$\dot{\omega} \;,\; \dot{\Omega} \;\simeq\; 10^{-2} \text{ cycles per day,}$$

$$\dot{M} \;\simeq\; 13 \text{ cycles per day}$$

$$\dot{\theta} \;\simeq\; 1 \text{ cycle per day.}$$

Hence, for the best possible results,

$$(n - 2p + q) \;\rightarrow\; 0.$$

The major term $m(\dot{\Omega} - \dot{\theta})$ cancels itself out m times per day. Assuming C_{nm} and S_{nm} to be of order 10^{-6} in the case of a typical close orbit, in the case of n = 8, the perturbations range in magnitude from 10 metres to a few hundred metres. For 24-hour satellites,

$$\lambda = \omega + M + \Omega - \theta \quad \ldots\ldots\ldots\ldots\ldots(7.77).$$

In the case of synchronous orbits, the use of equation (7.67) together with equations (7.63) to (7.66) and (7.67) gives (Kaula, 1965b, 8)

$$\lambda_c = \ddot{M} = 3n^2 \sum_{(n-m)\text{even}} \left[\frac{a_e}{a}\right]^n F_{nmp}(i)\; G_{np0}(e)$$

$$\left[C_{nm} \sin m(\omega + M + \Omega - \theta) - S_{nm} \cos m(\omega + M + \Omega - \theta) \right]$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7.78),$$

where $p = \dfrac{n - m}{2}$, and λ_c is the computed acceleration of longitude. The observed acceleration ($\ddot{\lambda}_o$) is determined from observational data obtained by the tr[illegible]ng [illegible]ite for many weeks using the second difference of week-to-week longitudes.

To separate the individual tesseral harmonics, it is desirable to use orbits of varying i values, such that the relative magnitude of the inclination function $F_{nmp}(i)$ is varied. Variations in a and e are not feasible because of the need to keep perigee sufficiently high enough to avoid drag effects (i.e., above 800 km.). It is simultaneously necessary to achieve a balance as the value of a must be kept small enough to avoid the $1/a^{n+1}$ having a dampening effect by reducing the magnitude of $\dot{\lambda}_o$.

(v) Limitations to the determination of harmonics from the perturbations of artificial earth satellites.

(a) Zonal harmonics.

The methods used by earlier researchers have been critcised (e.g., Cook, 1965, 181) on the grounds that the assumption of the hypothesis that higher order harmonics have negligible effects on the determination of those of lower order is invalid. King-Hele and Cook (1965, 17) remark on the inevitable changes which occur in the values of the zonal harmonics obtained in the solution when more harmonic coefficients are used. These fluctuations in the values determined are of the order of $\pm\ 0.1 \times 10^{-6}$ and increase with the magnitude of the order of the harmonic being determined. Thus, the higher the order of the harmonic coefficients being considered, the greater is likely to be the difference in the value obtained for a specific harmonic in comparison to one determined from an analysis to a lower order.

The use of spherical harmonics in the analysis of the disturbing potential has given rise to the following problems

(i) the seriesof harmonic coefficients is not rapidly convergent;

(ii) the solutions obtained by arbitrarily limiting the number of harmonics considered may be systematically affected by higher order harmonics which have been neglected, giving individual solutions of internally consistent terms but whose values, in an absolute sense, are inaccurate.

(b) Tesseral harmonics.

From a pragmatic point of view, the best determinations of tesseral harmonics are from resonant orbits (Kaula, 1963, 532) which have semi-major axes approximately equal to 42,000 km. Under such conditions, it is unlikely that any but the lowest order tesseral harmonics are capable of accurate determination. Drag effects are being reduced to a minimum on a new drag free satellite (Kaula, 1965b, 12) which uses a proof mass as a nulling device to control gas jets to cancel out drag accelerations.

The pre-requisites for an adequate determination are

(i) a network of observing stations whose positions are accurately known on the same datum ;

(ii) a number of artificial earth satellites with orbits ranging from polar to equatorial, some of which should be resonant.

Thus would give all zonal harmonics up to order 13. Those of order higher than 13 are capable of determination from near-synchronous orbits, as are the low order tesseral harmonics. The determination of spherical harmonics to date (Kaula, 1965a, 2) has yielded estimates of the values of zonal harmonic coefficients through to 7 and tesseral harmonics through to 6. The ignoring of higher order harmonics can be interpreted as smoothing out the gravity field and is equivalent to the representation of large areas by their mean values (Jeffreys, 1962a, 135).

Using the properties of spherical harmonics developed in section (7.5), it can be seen that the combination of zonal harmonics t the fifth order with tesseral harmonics through to 6, 6 gives a representation of the gravity field which can be considered adequate if the mean values of the free air anomalies of $30^{o} \times 30^{o}$ areas are normally distributed. This would imply that no covariance exists between the mean values of adjacent $30^{o} \times 3$ squares. This premise seems to be acceptable (Hirvonen, 1956). Kaula (1966a, 26-7) has combined such means with surface gravimetry to give values for $5^{o} \times 5^{o}$ square mean free air anomalies. See section (12.3)

8. ADAPTATION OF GRAVITY DATA FOR COMPUTATION

8.1 Introduction.

Gravity data is used in the calculation of the geoid spheroid separation vector only in the evaluation of the " Free Air" geoid , given by the term N_{F_P}, which by equation (1.3) is

$$N_{F_P} = \frac{R_m}{4\pi\gamma_m} \int_0^{\sigma=4\pi} f(\psi)\,\Delta g_F\,d\sigma \qquad \ldots\ldots\ldots(1.3),$$

where $f(\psi)$ is given by equation (5.33). N_{F_P} is related to the true geoid-spheroid separation by equation (5.60). Implicit in the evaluation is the knowledge of the free air anomaly at every element of surface area $d\sigma$ over the surface of the earth. The integral is usually evaluated by quadratures. Equation (1.3) can be re-written, dropping the suffix P, as

$$N_F = \frac{R_m}{4\pi\gamma_m} \sum_i f(\psi_i)\,\Delta g_{F_i}\,d\sigma_i \qquad \ldots\ldots\ldots\ldots(8.1),$$

where

$$d\sigma_i = \frac{\pi^2 n_i^2 \cos\phi_i}{180^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(8.2).$$

In equation (8.2), the element of surface area is considered to be bounded by meridians and parallels of latitude n^o apart in each case. Using

$$R_m = 6{,}371.2 \text{ km} \qquad \text{and} \qquad \gamma_m = 979.77 \text{ gal},$$

$$N_F^{(cm)} = 1.576 \times 10^{-2} \sum_i n_i^2 \sum_j f(\psi_{ij})\,\Delta g_{F_{ij}}^{(mgal)} \cos\phi_{ij} \qquad \ldots\ldots\ldots(8.3),$$

where the size of square represented by the mean anomaly $\Delta g_{F_{ij}}$ depends on the magnitude of $f(\psi_{ij})$ which takes the values

$$f(\psi) > 100 \quad \text{for} \quad 1.2^{\circ} > \psi \; ;$$
$$10 > f(\psi) > 100 \quad \text{for} \quad 1.2^{\circ} > \psi > 13^{\circ} \; ;$$
$$2 > f(\psi) > 10 \quad \text{for} \quad 13^{\circ} > \psi > 30^{\circ} \; ;$$
$$|2| > f(\psi) \quad \text{for} \quad 30^{\circ} < \psi \; .$$

It is necessary to investigate the error introduced into computations by the use of the mean value located at the centre of the basic area instead of individual values located at evenly distributed points over this same area. If this effect is small, it is in order to represent large areas by their means located at the centre of the area instead of using smaller subdivisions. This is quite vital as the saving on computer time is significant.

Consider a basic area within which is **available** a regularly spaced network of gravity stations $P_i (i = 1, n)$ at which the gravity anomalies are $\Delta g_i (i = 1, n)$, each of which is at an angular distance $\psi_i (i = 1, n)$ from the computation point. If the mean anomaly is given by

$$\overline{\Delta g} = M\{\Delta g_i\} = \frac{\sum_{i=1}^{n} \Delta g_i}{n} \quad \ldots\ldots\ldots\ldots (8.4),$$

the true contribution to N (N_t) due to this region is given by

$$N_t = C \sum_{i=1}^{n} f(\psi_i) \, \Delta g_i \quad \ldots\ldots\ldots\ldots\ldots (8.5),$$

where C is a constant. If δf_i and $\delta \Delta g_i$ are the departures of $f(\psi_i)$ and Δg_i from $f(\overline{\psi})$ and $\overline{\Delta g}$, equation (8.5) can also be written as

$$N_t = C \sum_{i=1}^{n} \left[f(\overline{\psi}) - \delta f_i \right] \left[\overline{\Delta g} - \delta \Delta g_i \right]$$

$$= C \left[\sum_{i=1}^{n} f(\overline{\psi}) \, \overline{\Delta g} - \overline{\Delta g} \sum_{i=1}^{n} \delta f_i - f(\overline{\psi}) \sum_{i=1}^{n} \delta \Delta g_i + \right.$$

$$\left. + \sum_{i=1}^{n} \delta f_i \, \delta \Delta g_i \right] \qquad \text{...........(8.6)} .$$

The first term on the right hand side gives the effect N_m obtained by the use of area means centred at the area centre. The second and third terms are, by definition, zero. It should be noted that, in the case of $f(\psi)$, this supposes that the function varies linearly over the area considered. Thus, the error that arises in the calculation due to the use of the area mean instead of individual values (e_n) is given by

$$e_n = \sum_{i=1}^{n} \delta f_i \; \delta \Delta g_i \qquad \text{..................(8.7).}$$

Both $f(\psi)$ and Δg vary systematically over the area considered, except in the case where the unit of area exceeds 30°, when the latter can be considered to belong to a population which exhibits zero covariance. For smaller basic areas, a systematic correlation between the gravity anomaly gradient and ψ could give rise to serious errors in adopting the adequacy of representation of the mean. While the former is dependent on the existent gravity field, the latter is strictly a function of position from the computation point.

If the square size was assumed to be equal to 5°, the variation of $f(\psi)$ with increase of ψ is negligible, affecting the fourth significant figure, for values of $\psi > 65^{o}$. Thus, in the case of any calculation, it is in order to use the assumption of adequate representation by the area mean for three-quarters of the earth, irrespective of the nature of the gravity anomaly gradient. Thus the covariance term in equation (8.6) can be taken to be zero for $\psi > 65^{o}$ if elemental area is the $5^{o} \times 5^{o}$ square.

This same assumption will hold to

(i) the third significant figure for $59^{o} < \psi < 65^{o}$;

(ii) the second significant figure for $16^{o} < \psi < 59$.

It will be necessary to work with smaller area means if the error due to the covariance term is to remain negligible and independent of any assumptions regarding the gravity anomaly gradient. These arguments are based on the assumption that a fully represented gravity field is available, which may not be the case. Summing up,

(a) <u>if the gravity field is fully represented</u>,

(i) For $\psi > 65^{o}$, adequate representation would be obtained by the use of $5^{o} \times 5^{o}$ square means as $\delta f_i \simeq 10^{-3}$, and even allowing for the definite improbability of a constant gravity anomaly gradient of the same sign and of magnitude δg, the resulting error $e_n \simeq 0.7 \ \delta g^{\text{mgal}}$cm. The resulting value can justifiably be expected not to exceed 0.1 cm.

(ii) In the region $16^{o} < \psi < 65^{o}$, the number of $5^{o} \times 5^{o}$ squares is approximately 650. The number of $1^{o} \times 1^{o}$ squares in this same area is approximately 1.6×10^{4}. As no covariance exists between $30^{o} \times 30^{o}$ square means, it can be expected that the gravity anomaly gradient changes sign at least twice along any particular azimuth in the region and hence the magnitude of e_n arising from covariance effects in this region is unlikely to exceed 2 cm.

(iii) In $10^{\circ} < \psi < 16^{\circ}$, it is quite reasonable to assume $\delta\ \Delta g$ to have the same sign over significant extents over a smaller unit of area over which means are to be taken. As the number of $1^{\circ} \times 1^{\circ}$ squares in this region is approximately 600 and the value of δf in this region for $1^{\circ} \times 1^{\circ}$ squares is the contribution to e_n is unlikely to exceed cm.

(iv) In $1.2^{\circ} < \psi < 10^{\circ}$, the quantity δf ranges from 4 as ψ varies between the limits. The use of a combination of $0.1^{\circ} \times 0.1^{\circ}$ squares for the inner area and $0.5^{\circ} \times 0.5^{\circ}$ for the rest will enable computations to be carried out such that the resulting errors due to the covariance term are of the same order as in the previous cases.

In general, it can be seen that, in the case of a fully represented field, the following representation of areas should give a value for N_F which will have errors which are an order smaller than those inherent in the spherical assumption.

Range of ψ	Size of basic area
$\psi > 20^{\circ}$	$5^{\circ} \times 5^{\circ}$
$10^{\circ} \leq \psi < 20^{\circ}$	$1^{\circ} \times 1^{\circ}$
$1.5^{\circ} \leq \psi < 10^{\circ}$	$0.5^{\circ} \times 0.5^{\circ}$
$0.1^{\circ} \leq \psi < 1.5^{\circ}$	$0.1^{\circ} \times 0.1^{\circ}$

(b) <u>If the gravity field is inadequately represented</u>

In such cases, $n \to 0$ and some assumptions have to be made about the distribution of gravity within individual units of area. In such a case, the mean gravity anomaly ($E\{\Delta g\}$) is only an estimate of $\overline{\Delta g}$ defined in equation (8.4) and $E\{\Delta g\}$ only represents a point central to the observed Δg_i ($i=1,n$). Let $E\{\Delta g\}$ be related to the true mean $\overline{\Delta g}$ by the relation

$$\overline{\Delta g} = E\{\Delta g\} + c_{\Delta g} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.8).$$

Substitution in equation (8.6) gives

$$N_t = C \sum_{i=1}^{n} \left[E\{\Delta g\} + c_{\Delta g} - \delta\Delta g_i \right]\left[f(\overline{\psi}) - \delta f_i \right]$$

$$= C \left[n\,E\{\Delta g\}\, f(\overline{\psi}) - E\{\Delta g\} \sum_{i=1}^{n} \delta f_i + n\, c_{\Delta g} f(\overline{\psi}) - \right.$$

$$\left. - c_{\Delta g} \sum_{i=1}^{n} \delta f_i + \sum_{i=1}^{n} \delta\Delta g_i\, \delta f_i \right] \ldots\ldots (8.9),$$

as $\sum_{i=1}^{n} \delta\Delta g_i = 0$ by definition.

In this case, it can be seen that any marked asymmetry about the centre of the square in the distribution of the readings introduces two additional terms in the expression for N_t. If the readings are symmetrically distributed about the centre of the square, equation (8.9) reduces to

$$N_t = C \left[\sum_{i=1}^{n} \delta\Delta g_i\, \delta f_i + n\, f(\overline{\psi}) \left[E\{\Delta g\} + c_{\Delta g} \right] \right] \ldots (8.10).$$

Equation (8.10) shows that if the available sample is evenly distributed over the area to be represented and if $E\{\Delta g\}$ is a reasonable estimate of the sample mean $\overline{\Delta g}$, no additional errors will be introduced into the computation over and above those for a fully represented sample. For distant zones where $\psi > 65^{\circ}$, even a bias in the distribution of individual samples is unlikely to cause systematic effects in view of the fact that the final result is the summation taken over a very large number of areal elements. In the critical zone $16^{\circ} < \psi < 65^{\circ}$ which is composed of approximately twenty five $30^{\circ} \times 30^{\circ}$ squares whose means are normally distributed without covariance, the effect of the summation of the terms which include $c_{\Delta g}$ is unlikely to be cumulative.

Conclusion :-

The computation of the free air geoid can be adequately carried out at any particular point in 4 stages :-

(a) Computations for an outer zone which is made up of approximately 2500 $5^{\circ} \times 5^{\circ}$ squares in the region $\psi > 20^{\circ}$.

(b) Computations for a mid zone situated in the region $10^{\circ} < \psi < 20^{\circ}$, composed of 1200 $1^{\circ} \times 1^{\circ}$ squares.

(c) Computations using means representing approximately 1600 $0.5^{\circ} \times 0.5^{\circ}$ squares comprising a near zone.

(d) Computation of an inner region for $\psi < 1.5^{\circ}$, in which a closer spacing of gravity data is necessary.

In the case where the field is not fully represented and the mean anomalies are computed from sparsely distributed populations, the computed mean is acceptable if

(a) the sample is evenly distributed about the centre of the square ;

(b) the computed mean anomaly is a reasonable estimate of the true square mean.

In view of the fact that the means of $30^{o} \times 30^{o}$ squares form a normally distributed population with no auto-correlation, the effect of slight biases in the individual means so computed, provided the sampling is carried out methodically, is unlikely to produce any cumulative effect on the final result.

8.2 The required form of the gravity data.

The following gravity data is required to compute N_F at any point :-

(i) Free air anomaly means for $5^{o} \times 5^{o}$ squares over the entirety of the earth's surface (approximately 2600 squares in all) ;

(ii) $1^{o} \times 1^{o}$ square means for all areas in the range $10^{o} < \psi < 20^{o}$ (1200 quantities in all) ;

(iii) 1600 $0.5^{o} \times 0.5^{o}$ square means in the range $1.5^{o} < \psi < 10^{o}$.

(iv) Iso-anomaly maps showing free air anomaly contours in the region within the circle of radius $\psi = 1.5^{o}$, centred on the computation point.

All these gravity values have to be tied into a common control network which must, therefore, have a datum of international significance. In general, the data used can be divided into two broad groups: -

(a) Gravity data on the Australian mainland and continental shelf regions.

(b) Overseas gravity data. This, in the case of calculations on the Australian mainland, will almost entirely fall into category (i) above. The nature of this data is discussed in section (12.3).

8.3 An Australian geodetic gravity network.

An Australian geodetic gravity network is defined as one which, when fully established, will provide adequate gravity data to enable geodetic calculations to be made at any point on the Australian mainland. Such a network will provide the fully defined field of n stations required in equation (8.9) to enable $c_{\Delta g}$, defined in equation (8.8) to be zero. For this purpose, it is desirable to choose the smallest possible interval in which the choice of a single representative gravity station will be adequate to represent, in toto, the variability of the gravity field. If this minimum interval is chosen as v^o, the criteria which fix the value of v are :-

(a) The extent of gravity data available. For example, if the gravity data is available in the form of iso-anomaly maps, v cannot be smaller than the interval between the gravity stations from which the iso-anomaly maps were drawn if no other sources of error are to be introduced into the chosen field.

(b) The accuracy with which the chosen anomaly can represent the chosen $v^o \times v^o$ area.

(c) The linearity of the function $f(\psi)$ over the interval chosen for the close field. For fields which are not fully represented, it would suffice if the linearity condition held to the third significant figure.

These factors are further controlled by the necessity for the realistic use of the computer to obtain results which aspire to an accuracy in keeping with that possible from the available field.

In Australia, the majority of the continental coverage has been established by the Bureau of Mineral Resources, Geology and Geophysics (Dooley, 1965) using a series of helicopter gravity traverses with a station spacing of 8 miles. This approximates well to the corners of a square where $v = 0.1^o$. This square size adequately satisfies the linearity condition in (c) even when $\psi < 1^o$ and does not overstrain the resources of a medium sized computer. The error in representing a $0.1^o \times 0.1^o$ square by a single reading (Hirvonen, 1956, 2) is about $\pm$ 3 mgal. The resultant error in the computation of N (equation (8.3)) is unlikely to exceed 10 cm, an amount which is considerably smaller than that introduced by the spherical assumption.

On this line of reasoning, the basic square size as required in equation (8.6) was chosen to be

$$v = 0.1^o \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.11)$$

in the case of the Australian gravity field.

8.4 The Australian Reference System.

The gravity network for geodetic calculations to be set out at corners of 0.1^{O} x $o.1^{O}$ squares , as discussed above, must be based on a common datum. This datum has to be established using a measuring accuracy which is substantially greater than that of the network being controlled. Such a network has already been established in Australia by the Bureau of Mineral Resources. Originally, the reference system was a network of pendulum stations (Dooley et al, 1961). This has now been superceded by a more extensive survey called the Isogal Regional Gravity Survey as explained in section (7.2). Provisional values known as the "May 1965 Isogal" values were assigned to all such control stations. These values, though accepted as final for the present, are subject to change due to

(a) any change in the value at the Potsdam datum ;

(b) any correction to the connection between Potsdam and Melbourne, estimated to be of magnitude 0.5 mgal.

(c) any international decision which could affect the value of the Mean Australian Milligal. This change (Mather, 1966b, 8) is expected to be of the order $\pm$ 1 part in 3 x 10^{-3}.

8.5 The Mean Australian Milligal.

All gravimeters in Australia are calibrated against standard gravimeter calibration ranges established in each of the major cities in Australia (Barlow, 1965). It is common experience that the instrument calibrations provided by the makers of American gravimeters are up to 0.4 per cent higher than the standard provided by the B.M.R. .

The Bureau's standard is based on a "standard gravity interval" established between two points in Victoria, the difference being originally determined by pendulum observations. This was further amended as a by-product of the Australian gravity network adjustment in 1961 (Dooley, 1965), using gravimeter ties between all pendulum stations in Australia. The revised value so obtained was used to establish the nationwide Australian calibration standards. Barlow reports (1965, sec. 7) that this new set of differences agrees "fairly well with both the 'Recent American' and the '1957 European' systems", described by Morelli (1957). This is also in agreement with values obtained by the United States Air Force, using a battery of four La Coste and Romberg gravimeters, when the results obtained were found to agree with the Mean Australian Milligal to approximately 2 parts in 5000 (Whalen, 1966) .

8.6 $1^{\circ} \times 1^{\circ}$ and $0.5^{\circ} \times 0.5^{\circ}$ square means

The discussion in section (8.1) shows that the even distribution of readings over a square is vital if the sample used in the calculation of the area mean is to be a representative one. Further, the reliability of the estimate $E\{\Delta g\}$ of the square mean obtained from the available sample will vary from square to square. Superficially, it appears to be in order to use all available gravity readings to compute sample means, without any consideration of the reliability of the sample (e.g., Heiskanen, 1962; Uotila, 1962). This is probably due to the paucity of available gravity data and the futility of trying assess the accuracy of woefully inadequate data samples. In the transition period till a complete network is obtained, some assessment of the accuracy of the results of gravimetric computations must be made. In the case of fields which are not fully represented, it is necessary to take into account an estimate of the accuracy of $E\{\Delta g\}$ if any estimate is to be made of the accuracy of the value of N computed. This matter is more closely investigated in section (9).

If a $u^o \times u^o$ area is to be fully represented by n gravity readings, each of which represent a $v^o \times v^o$ with a representation error E_s, in the case of full coverage,

$$n = \frac{u^2}{v} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (8.12),$$

and the error of representation E_u of the $u^o \times u^o$ square by the mean of the n readings is given by

$$E_u = \pm \frac{v E_s}{u} \quad \ldots\ldots\ldots\ldots\ldots\ldots (8.13).$$

As $v < u$, $E_u < E_s$. Hence the error of representation of a $u^o \times u^o$ square which is completely covered by a geodetic gravity network with specifications as laid down in section (8.3) is a negligible quantity. For example, for a fully represented $0.5^{o} \times 0.5^{o}$ square, using $v = 0.1^{o}$,

$$E_s = \pm 3 \text{ mgal and } E_u = \pm 0.6 \text{ mgal.}$$

On a similar basis, for a fully represented $1^{o} \times 1^{o}$ square,

$$E_u = \pm 0.1 \text{ mgal.}$$

8.7 The compilation of a geodetic gravity network for Australia.

The basic data is available in one of four forms :-

(a) Iso-anomaly maps prepared from close gravity surveys. These maps are available for surveys carried out by the B.M.R. as well as various geophysical prospecting groups. They are printed at scales varying from 1 : 50,000 to 1 : 500,000. The anomalies shown are, invariably, Bouguer anomalies, which are usually established with respect to the B.M.R's Isogal network, using M.S.L. as the datum for elevations. In instances where the station spacing is of the order of miles, the elevations are established (Mather, 1966b, 10 et seq) by altimeterwith frequent checks on bench marks established by third order levelling. The accuracy of station elevations in such cases is about ± 3 metres.

On certain surveys, both the gravity and elevations were on arbitrary datums. Such surveys were connected to the Isogal datum for gravity and M.S.L. for elevations prior to incorporation in the unified scheme. Anomaly values were then read at the corners of $0.1^{o} \times 0.1^{o}$ squares and these were loaded onto the University of N.S.W's I.B.M. 360/50 computer. The elevations of all $0.1^{\circ} \times 0.1^{\circ}$ squares were read off available topographical maps and also stored in the computer. As the relationship between Bouguer and free air anomalies can be assumed to be linear over a $0.1^{\circ} \times 0.1^{\circ}$ square, the mean free air anomaly for the $0.1^{\circ} \times 0.1^{\circ}$ square was obtained from the relation

$$\overline{\Delta g_F}^{(mgal)} = \overline{\Delta g_B} + 0.04187 \rho \, h_m \quad \ldots\ldots\ldots(8.14),$$

where $\overline{\Delta g}$ is the mean Bouguer anomaly in mgal and h_m the mean square elevation in metres. The quantity ρ is the density used in the preparation of the iso-anoamly map, expressed in g. cm^{-3}.

(b) Gravity traverses run along access routes. Station on such traverses do not necessarily fall on the corners of $0.1^{\circ} \times 0.1^{\circ}$ squares. In such cases, while an accurate value of station position was required to obtain a value of latitude for the computation of the free air anomaly, the gravity station was used to represent that $0.1^{\circ} \times 0.1^{\circ}$ square corner which was closest to it. No attempt was made to interpolate the value of gravity at the location of the square corner, as the spacing of the traverses made the gain in accuracy quite marginal. The error in the gravity value of square corners on the basis of representation instead of interpolation can be expected to be random and hence with little or no change in the estimated value of E_s. This conclusion is in agreement with the findings of Moritz (1964, 14).

(c) Gravity traverses run specifically to establish geodetic gravity networks (Mather, 1966b, 12). The gravity stations on such traverses are located at the corners of $0.1^{o} \times 0.1^{o}$ squares. The general accuracy of the established gravity values was greater than $\pm$ 0.25 mgal and the station elevations, established by altimeter had errors not exceeding $\pm$ 3 meters, provided the differences in elevation, established altimetrically, were controlled by more than one base altimeter, paying attention to the direction of movement of local pressure fronts and hence obtaining a better estimate of the slope of isobaric surfaces.

(d) Local gravity surveys with local datums for both gravity and elevation.

All gravity data so assembled from the records of the Bureau of Mineral Resources, the South Australian Department of Mines and certain geophysical prospecting companies, were processed on the computer along with the mean elevations of $0.1^{o} \times 0.1^{o}$ and the following quantities were computed

(a) free air anomalies representing $0.1^{o} \times 0.1^{o}$ squares ;

(b) free air anomaly means for $0.5^{o} \times 0.5^{o}$, $1^{o} \times 1^{o}$ and $5^{o} \times 5^{o}$ squares ;

(c) elevation means for the square units mentioned in (b) . This data, in the case of $1^{o} \times 1^{o}$ means , excluding any restricted information as on 1.3.67, has been used in compiling Appendix (1). Appendices (2) and (3) give the sample standard deviations and sample sizes respectively for the

Australian mainland.

The actual gravity data available is insufficient in the above form for calculations using equation (8.3) as the summation must be taken over every element of surface area. Further, a study of appendix (1) shows the inaccuracy of the assumption that the free air anomaly in unsurveyed areas is zero. Thus, it is vital to make some estimate of the value of the free air anomaly in un-surveyed regions on the basis that some correlation exists between the values of adjacent square means.

9. THE EXTENSION OF THE GRAVITY ANOMALY FIELD IN SOUTH AUSTRALIA

9.1 Introduction.

Gravity surveys have been in progress in Australia for over 20 years (Thyer, 1963). The boundaries of South Australia have been described in section (1.5). It is (see fig (9.1)) a reasonably flat area with a mean elevation of 120 metres, the maximum and minimum $1^{o} \times 1^{o}$ square mean elevations being 565 metres and - 275 metres respectively. The western half of the state is part of the pre-Cambrian granitic shield, which extends further westward, and is, by and large, semi-desert in character. The rock formations are essentially sedimentary, being generally cainozoic with some proterozoic formations in the hilly regions.

In computing the geoid-spheroid separation vector from incomplete gravity data, it is of importance to make some assessment of the accuracy of the data from which the calculations are made. Thus, it is necessary not only to determine the best possible value for the mean value to represent a given unit of surface area but it is also of significance to assess the accuracy of the value adopted. This is dependent on two factors in addition to the sample variance itself, which gives a measure of the degree of variation of the gravity field over the area studied. These arethe size of the sample and its distribution over the area considered. As a result of accepting the criteria set out in section (8.1), only about one tenth of the available gravity data could be included in the sample used for computation. The distribution of approximately 4000 gravity stations used in an analysis carried out in early 1966 is shown in fig (9.2), being concentrated along the northern and eastern borders of the state. The area is essentially a region of negative free air anomalies (fig (9.3)), the field being extremely variable in the north west (fig (9.5)).

9.2 The correlation between mean free air anomalies and mean square elevations.

Earlier research (Uotila, 1960) indicated that the following relation between the free air anomaly and the station elevation was satisfied over limited regions.

$$\Delta g_F = C + 0.1118\, h \quad \ldots\ldots\ldots\ldots(9.1),$$

where Δg_F is in mgal and h in metres. C is a constant over the region. The scope of this type of expression is limited in relatively flat country where the variation, with position, of free air anomalies is considerably greater than can be accounted for by the height term. Nevertheless, it can still be of some use in the establishment of estimates of values for regional free air anomalies to be used in the low degree harmonic analysis of gravity material. The results of a least squares fit of equation (9.1) over 9 areas in South Australia is shown in fig (9.6), the value of C being obtained

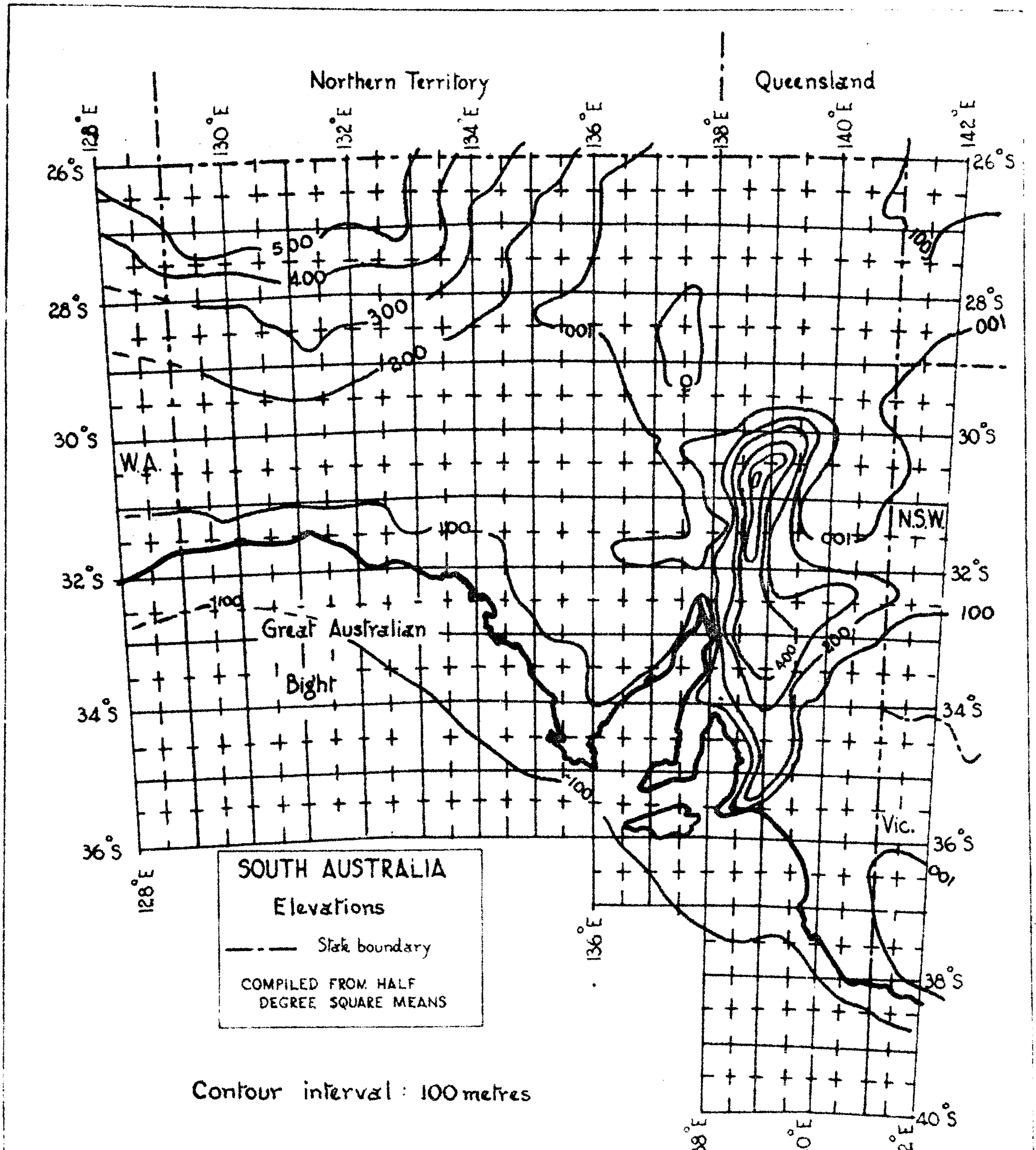

Northern Territory
Queensland
W.A.
N.S.W.
Vic.
Great Australian Bight
SOUTH AUSTRALIA
Elevations
State boundary
COMPILED FROM HALF DEGREE SQUARE MEANS
Contour interval : 100 metres

FIG.9.1.

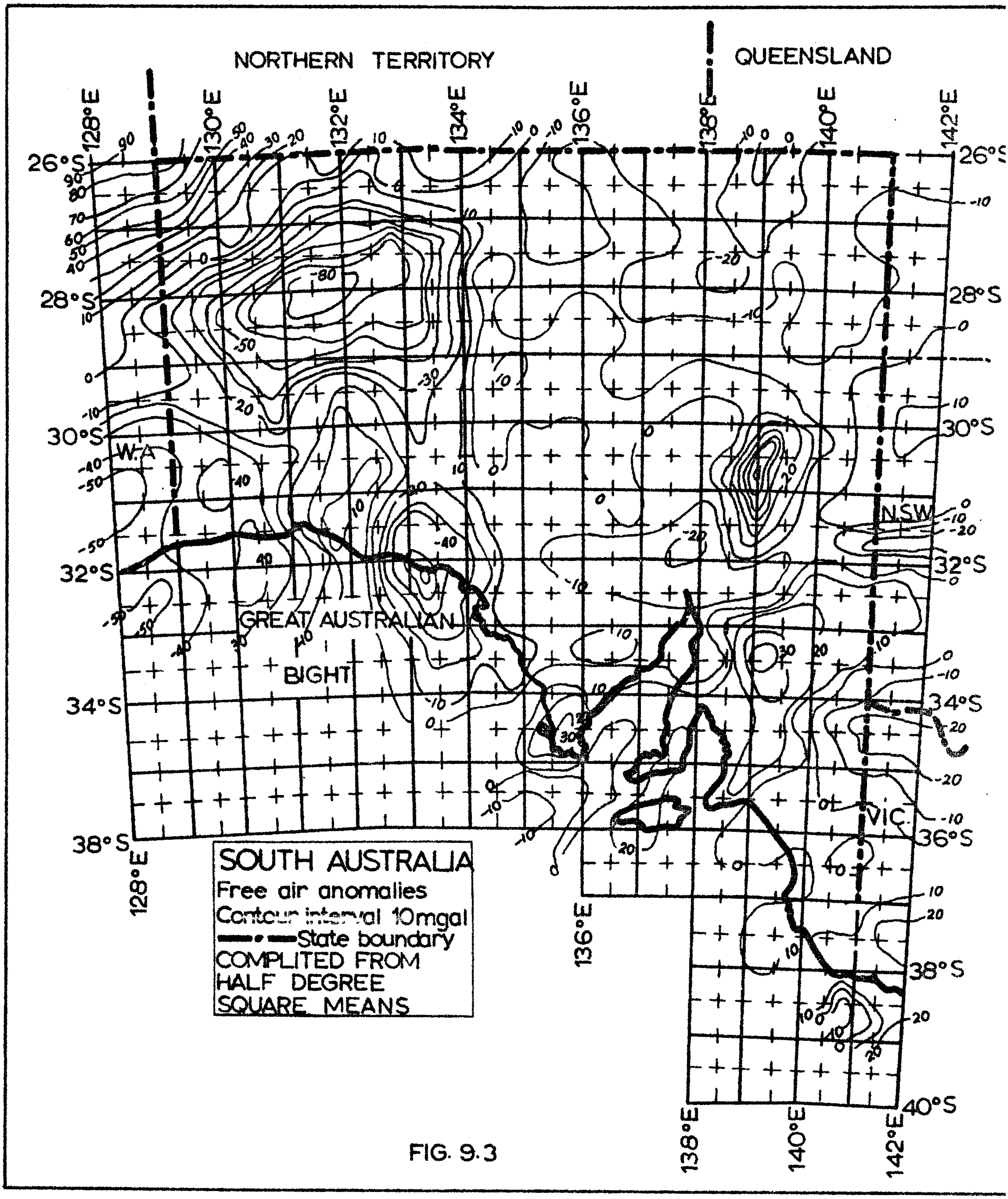

NORTHERN TERRITORY
QUEENSLAND
W.A
N.S.W
VIC.
GREAT AUSTRALIAN BIGHT
SOUTH AUSTRALIA
Free air anomalies
Contour interval 10mgal
State boundary
COMPLITED FROM
HALF DEGREE
SQUARE MEANS

FIG. 9.3

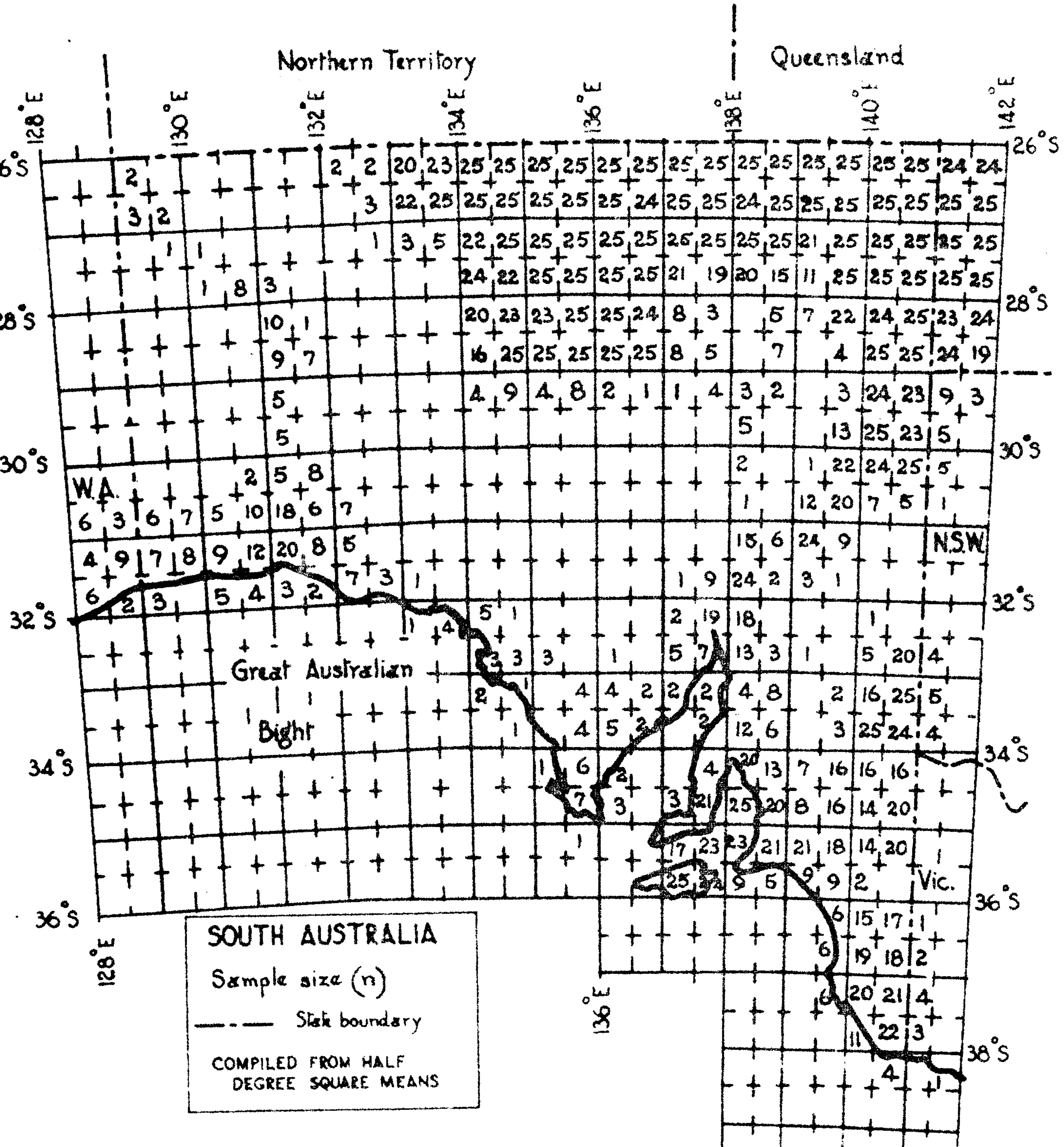

Northern Territory
Queensland
W.A.
N.S.W.
Vic.
Great Australian
Bight
SOUTH AUSTRALIA
Sample size (n)
State boundary
COMPILED FROM HALF DEGREE SQUARE MEANS
128°E
130°E
132°E
134°E
136°E
138°E
140°E
142°E
26°S
28°S
30°S
32°S
34°S
36°S
38°S
40°S

FIG. 9.4

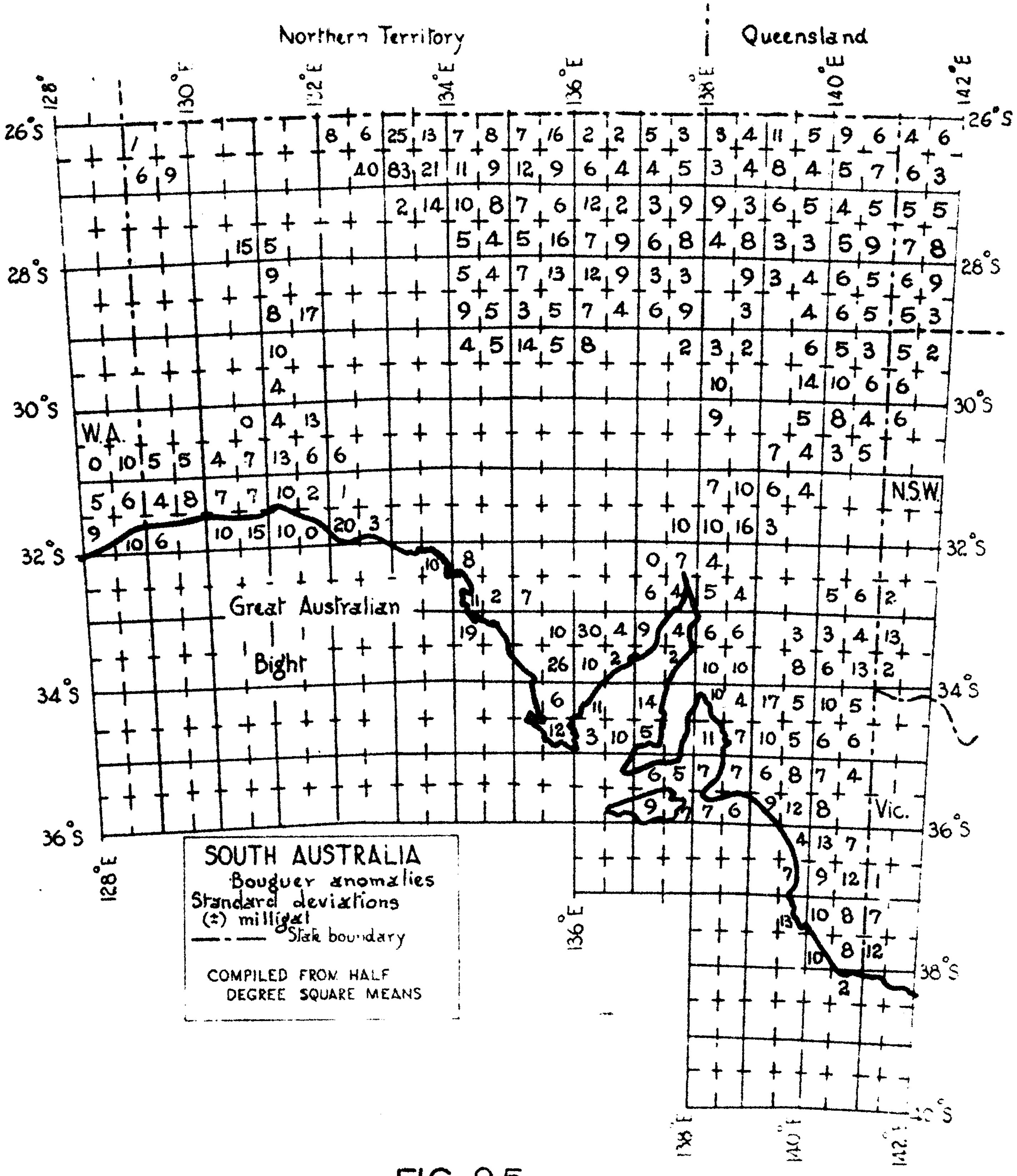

Northern Territory
Queensland
W.A.
N.S.W.
Vic.
Great Australian
Bight
26°S
28°S
30°S
32°S
34°S
36°S
38°S
128°
130°E
132°E
134°E
136°E
138°E
140°E
142°E
SOUTH AUSTRALIA
Bouguer anomalies
Standard deviations
(±) milligal
State boundary
COMPILED FROM HALF
DEGREE SQUARE MEANS

FIG. 9.5

Longitude (degrees east)

Latitude (degrees S)	128–132	132–136	136–142	142–
26–30	-64 (415)	-49 (262)	-14 (583)	
30–34	-26 (86)	-21 (40)	-19 (175)	
34–40	-	22 (-70)	2 (9)	
40–				

FIG (9.6)

Regional free air anomalies from height-anomaly correlation. Values of C in mgal. Elevations (in parentheses) in metres.

for $0.5^{\circ} \times .5^{\circ}$ means .

9.3 The spread of a sample.

A criterion already defined for the spread o a sample (Hirvonen, 1956, 1) is the "error of representation" E_s, given by

$$t E_s^2 = \sum_{i=1}^{t} \frac{\sum_{j=1}^{n} (\Delta g_{ij} - \overline{\Delta g_i})^2}{n_i} \qquad \text{........(9.2)},$$

where $\overline{\Delta g_i}$ is the mean anomaly in the i-th square,

Δg_{ij} are the observed anomalies,

n_i the number of readings in the i-th square

and t the total number of squares considered.

In the case of $0.5^{o} \times 0.5$ squares, as a fully represented square, on the basis of criteria set out in section (8.1), has only 25 readings in it, the denominator used in equation (9.2) was (n - 1) instead of n (Spiegel, 1961, 70). Analysis of a high proportion of the South Australian sample showed good agreement with the set of values obtained by Hirvonen, except possibly, in the case of $0.5^{o} \times 0.5^{o}$ squares. This is due to the highly variable field in the north west corner of the state, where the free air anomalies approach -100 mgal and the standard deviation of free air anomalies within a single square are as large as ± 82 mgal.

The E_s values in table (9.1) are compiled from free air anomalies and would apply, with relatively slight variations to both Bouguer and isostatic anomalies due to the relatively small elevation gradient in the region. Thus, while the variation of gravity is similar in magnitude to that of the European gravity field over large extents (Hirvonen, 1956, 3), it is slightly greater over limited ones.

9.4 Extension of the gravity field to unsurveyed areas.

The extension of the available gravity field to provide continuous gravity coverage has been investigated by Jeffreys (1941), Kaula (1959), Moritz (1964) and Uotila (1962). The methods of extension used are fully described in the first three researches. Much less data was available to Jeffreys than to Kaula who had the added advantage of an electronic

Square size	Sample Size (n)										Total Sample		Hirvonen's value
	A		B		C		D		E				
	E_s	t	E_s	t	E_s	t	E_s	t	E_s	t	E_s	t	E_s
$0.5^\circ \times 0.5^\circ$	$\pm$ 11	60	$\pm$ 11	61	$\pm$ 11	13	$\pm$ 9	15	$\pm$ 10	62	$\pm$ 10.1	211	$\pm$ 9.0
$1^\circ \times 1^\circ$	$\pm$ 12	21	$\pm$ 16	26	$\pm$ 13	10	$\pm$ 13	7	$\pm$ 9	12	$\pm$ 13.5	76	$\pm$ 12.7
$\times 2^\circ$	$\pm$ 21	13	$\pm$ 15	15	$\pm$ 14	6	$\pm$ 14	2	$\pm$ 11	2	$\pm$ 17.7	28	$\pm$ 17.6

TABLE (9.1)

The error of representation E_s in mgal.

Key to Classes

Square Size	A	B	C	D	E
$\frac{1}{2}^\circ \times \frac{1}{2}^\circ$	$0 < n \leq 5$	$5 < n \leq 10$	$10 < n \leq 15$	$15 < n \leq 20$	$n > 20$
$1 \times 1^\circ$	$0 < n \leq 10$	$10 < n \leq 30$	$30 < n \leq 50$	$50 < n \leq 75$	$n > 75$
$2^\circ \times 2^\circ$	$0 < n \leq 50$	$50 < n \leq 100$	$100 < n \leq 200$	$200 < n \leq 300$	$n > 300$

computer to handle many more unknowns in a truly simultaneous solution. Jeffreys sets out his solution explicitly. He worked on a $10^{o} \times 10^{o}$ square unit, assuming a singl value to represent each square after allowing for height correlation. A series of observation equations were then fitted to each parallel from which harmonic coefficients were evaluated up to the fourth order.

Kaula, on the other hand, used Markov theory to interpolate $1^{o} \times 1^{o}$ square means using all available values. Autocorrelation analysis was then used to assign values for the mean values for all $10^{o} \times 10^{o}$ squares. A set of low order spherical harmonics up to the eighth degree was fitted to these ten degree means by the simple orthogonal method using fully normalised coefficients (Kaula, 1959, 89). Uotila obtained a solution by fitting harmonic coefficients using a least squares analysis.

The application of spherical harmonics is unsuited to limited areas and Moritz (1964) used covariance analysis for carrying out the field extensions. Both Markovian predictions and covariance analysis tend to yield oversmoothened fields (Kaula, 1965a, 4) unless adequate samples are available over local extents to re-compute the operational functions so as to be relevant over local areas. While the inadequacy of the available samples is likely to make this possibility rather remote, the computer time expended in the process is seldom worth the effort.

In the analysis of the South Australian data, a two dimensional trigonometrical series was chosen to represent limited areas in two distinct stages:-

(i) The extension of the gravity anomaly field in a limited $u^{o} \times u^{o}$ area where single readings represent $v^{o} \times v^{o}$ areas. The total number of possible readings (N) is given by

$$N = \frac{u^2}{v^2} \quad \dots\dots\dots\dots\dots\dots (9.3).$$

(ii) The extension of $w^{o} \times w^{o}$ square means over a a larger area to unsurveyed portions.

Let the predicted value of the gravity anomaly $E\{\Delta g\}$ be given by an equation of the form

$$E\{\Delta g\,(\phi, \lambda)\} = \sum_{i=1}^{p} A_i f_i(\phi, \lambda) \quad \dots\dots\dots (9.4),$$

where A_i (i=1, p) are constant coefficients and $f_i(\phi, \lambda)$ (i=1, p) are functions of ϕ and λ whose form is known. The values of the p constant coefficients are determined by setting up the observation equations at each available gravity station in the area. If the number of available gravity stations is n (= N), let the residuals r_j(j=1, n) be given by

$$r_j = \Delta g_j(\phi, \lambda) - \sum_{i=1}^{p} A_i f_i(\phi, \lambda) \quad \dots\dots\dots (9.5).$$

The values for A_i (i=1, p) are chosen on the assumption that the r_j (j=1, n) are normally distributed. In this case, the most probable values of the residuals satisfy

$$F = \sum_{j=1}^{n} w_j r_j^2 = \text{minimum} \quad \ldots\ldots\ldots\ldots (9.6),$$

where w_j $(j=1, n)$ are the weight coefficients, which in the first case described above will be equal to unity. The required set of equations for solution are

$$\frac{\partial F}{\partial A_k} = 0 \quad k=1,p \quad \ldots\ldots\ldots\ldots\ldots\ldots (9.7).$$

The use of equation (9.5) in equation (9.6) to satisfy the conditions set out in equation (9.7) gives

$$\sum_{j=1}^{n} w_j \left[\Delta g_j(\phi, \lambda) - \sum_{i=1}^{p} A_i f_i(\phi, \lambda) \right] f_k(\phi, \lambda) = 0$$

$$k=1,p \quad \ldots\ldots\ldots\ldots (9.8).$$

The solution can also be expressed in matrix notation as

$$F A - G = R \quad \ldots\ldots\ldots\ldots\ldots\ldots (9.9),$$

where

$$F = \begin{vmatrix} f_{11} & f_{12} & \ldots\ldots\ldots & f_{1p} \\ f_{21} & f_{22} & \ldots\ldots\ldots & f_{2p} \\ \cdot & & & \\ \cdot & & & \\ \cdot & & & \\ f_{n1} & f_{n2} & \ldots\ldots\ldots & f_{np} \end{vmatrix}, \quad A = \begin{vmatrix} A_1 \\ A_2 \\ \cdot \\ \cdot \\ A_p \end{vmatrix}, \quad G = \begin{vmatrix} \Delta g_1 \\ \Delta g_2 \\ \cdot \\ \cdot \\ \Delta g_n \end{vmatrix}$$

and $R^T = \begin{vmatrix} r_1 & r_2 & . & . & . & r_n \end{vmatrix}$.

For a least squares solution,

$$\tfrac{1}{2}\ R^T\ W\ R \quad = \quad \text{minimum} \quad \ldots\ldots\ldots\ldots\ldots\ldots (9.10),$$

where

$$W = \begin{vmatrix} w_{11} & 0 & \ldots\ldots\ldots\ldots & 0 \\ 0 & w_{22} & \ldots\ldots\ldots\ldots & 0 \\ \cdot & & & \\ \cdot & & & \\ 0 & & \ldots\ldots\ldots\ldots & w_{nn} \end{vmatrix} \quad \ldots\ldots\ldots (9.11).$$

The substitution of equation (9.9) in (9.10) and differentiation with respect to A_k (k=1, p) gives

$$A = (F^T W F)^{-1} F^T W G \ldots\ldots\ldots\ldots (9.12).$$

9.5 The extension of fields in $u^o \times u^o$ areas where $v = 0.1^o$.

The choice of a suitable trigonometrical series depends not only on the available computer storage but, paradoxically, it is also dependent on the available gravity field. Field extensions obtained from limited amounts of data using functions with a high degree of resolution are generally unreliable. In the analysis of the South Australian data which was carried out on the C.S.I.R.O's C.D.C. 3200 computer, the available computer storage restricted the maximum value of u to 2^o. A series which gives adequate representation independent of the value of u is

$$\Delta g(\phi, \lambda) = \sum_{i=0}^{q} A_i \cos\{\pi\ (\phi - \phi_o) i\} + \sum_{i=q+1}^{2q} A_i \sin\{\pi (\phi - \phi_o)(i - q)\} + \sum_{i=2q+1}^{3q} A_i \cos\{\pi (\lambda - \lambda_o)(i-2q)\} + \sum_{i=3q+1}^{4q} A_i \sin\{\pi (\lambda - \lambda_o)(i-3q)\} \ldots\ldots\ldots (9.13),$$

where p in equation (9.4) is given by

$$p = 4q + 1 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (9.14),$$

ϕ_o, λ_o are the coordinates of the south west corner of the $u^o \times u^o$ area and (ϕ, λ) are the coordinates of the $0.1^o \times 0.1^o$ square which is represented by the gravity anomaly $\Delta g(\phi, \lambda)$.

Repeated application to varying sets of data showed that the minimum conditions for a non-trivial solution are :-

(i) A minimum of 5 readings should be available in every constituent $\frac{1}{2}^o \times \frac{1}{2}^o$ square.

(ii) At least one reading should be available in every row and column of the array.

While lesser amounts of data have provided seemingly acceptable solutions, the results are subject to fortuitous circumstances. Thus, in the case of a $2^o \times 2^o$ area, 80 well distributed values can give estimates of the balance 320 readings. This, in effect, provides a 1 to 5 extension.

The degree of resolution of the function defined in equation (9.4) is governed by the value of q in equation (9.13). While the maximum value of q is controlled by the available computer storage, the actual value used is determined by the amount of gravity data available. In a pilot investigation, it was found that increasing the ratio $u/v : q$ above 2 : 1, while requiring more computer time, did not materially improve the accuracy of the extensions. Extensions of reasonable adequacy were obtained by setting $u/v : q = 3$ in the case of well represented fields. In regions with inadequate representation, improved results were obtained by reducing q in equation (9.13) in the range of values $7 \leqq q \leqq 0$ for $u = 2^{\circ}$ as n in equation (9.8) reduces through the range $80 \leqq n \leqq 1$, the extreme case being one of direct representation.

These conclusions were used in the prediction of values of the gravity anomaly at unsurveyed locations. The anomaly to be used in these extensions must, ideally, be free from correlation with height as the series represented by equations (9.8) and (9.14) can only adequately represent variations in gravity anomaly with position. It is therefore preferable to used an anomaly such as the Bouguer anomaly in the extension if equation (9.13) is to be used in the stated form. If, on the other hand, free air anomalies are to be used directly in the extension, the series must be enlarged to represent variations in h. This was felt to be a wastage of effort as $0.1^{\circ} \times 0.1^{\circ}$ square mean heights were available for all squares in the region analysed.

The Bouguer anomalies so predicted were corrected for the height term (Heiskanen and Vening Meinesz, 1958, 153) using the estimated elevation of the tenth degree square. An attempt was made to check the accuracy of the field extension made by studying the comparisons made between predicted values satisfying conditions (i) and (ii) for non-trivial solutions and gravity data which became available subsequent to the computations. 154 comparisons, made from different solutions in a random manner were made in six different $2^{\circ} \times 2^{\circ}$ areas and the differences (predicted value - observed value) were found to be normally distributed with a standard deviation of ± 7.2 mgal.

In fitting the two-dimensional series defined in equation (9.13) to a field with a variable number (n) of gravity stations in it, the error of prediction (e_P) is given by

$$e_P = E\{\Delta g(\phi,\lambda)\} - \Delta g(\phi,\lambda) \quad ....(9.15).$$

e_P was found to increase with n for a fixed value of q. Table (9.2) sets out values of the mean error of prediction ($M\{e_P\}$) for q = 7. It can be seen that, while the available storage of the computer used restricts the maximum value of q possible, the magnitude of the error of prediction in well represented fields necessitates the "normalisation" of predicted values prior to use. This can be effected either manually using a graphical extension technique or by the use of Markov theory (Bartlett, 1960, 24 et seq) as the accuracy of the predicted value is dependent not only on the error of prediction at the adjacent gravity stations, but also on the average gravity anomaly gradient G, given by

$$G = \left|\frac{d\,\Delta g}{dl}\right| \quad ....(9.16),$$

where $d\,\Delta g$ is the change in gravity anomaly which occurs over the distance dl.

Following a procedure similar to that adopted by

Kaula (1959, 9), let the couplet c_{iu} be given by

$$c_{iu} = \begin{bmatrix} e_{P_i} \\ G_u \end{bmatrix} \qquad \text{..................(9.17).}$$

The expected error of prediction $E \; e_{P_i}$ is given by

$$E\{e_{P_i}\} = \frac{c_i \, p^{iu}_{jv}(\delta l)}{p^{u}_{jv}(\delta l)} \qquad \text{..................(9.1),}$$

where $p^{iu}_{jv}(\delta l)$ is the probability of the couplet c_{iu} occurring a distance δl away from the couplet c_{jv} ; suppression of an index denotes summation with respect to that index. The mean value of G over the area being investigated was 8.9 mgal / 50 km, with maximum, modal and minimum values of 48, 6 and 0 respectively. The mean comparison error $M\{e_c\}$, given by

$$\left[M\{e_c\} \right]^2 = \frac{1}{n} \sum_{i=1}^{n} \left[E\{\Delta g_i(\phi, \lambda)\} - E\{e_{P_i}\} - \Delta g_i(\phi, \lambda) \right]^2 \qquad \text{..............(9.19),}$$

was $\pm$ 3.2 mgal for the 154 comparisons.

Field extensions carried out under the above conditions can be expected to have estimated error of prediction of $\pm$ 3 mgal, which is of an accuracy comparable with the representation of a single tenth degree square by a single reading (Hirvonen, 1956, 2). Relaxation of the criteria at (i) to 3 stations within each of the constituent half degree squares, while maintaining those at (ii) gave estimated comparison errors of $\pm$ 8 mgal, which, on normalisation reduced to $\pm$ 6 mgal.

If these minimum conditions are not satisfied, the

$\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ squares		$2^{\circ} \times 2^{\circ}$ squares	
Sample size (n)	$M\{e_P\}$ mgal.	Sample size (n)	$M\{e_P\}$ mgal.
$0 < n \leq 5$	± 4.4	$0 < n \leq 20$	± 1.8
$5 < n \leq 10$	± 6.0	$20 < n \leq 50$	± 2.3
$10 < n \leq 15$	± 6.4	$50 < n \leq 100$	± 4.7
$15 < n \leq 20$	± 7.3	$100 < n \leq 200$	± 6.7
$20 < n \leq 25$	± 6.5	$200 < n \leq 300$	± 8.3
		$300 < n \leq 400$	± 8.3
Total sample	± 6.5	Total sample	± 8.3

TABLE (9.2)

Classification of errors of prediction

the error of field extension becomes much larger and, unless q in equation (9.13) is reduced proportionately, the functional representation becomes erratic.

9.6 The extension of the field in $w^{\circ} \times w^{\circ}$ areas where $v = 0.5^{\circ}$.

A field extension on lines similar to the above can be performed from $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ means, which have already been computed on the lines set out above to establish an estimate of $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ areas means in regions where no observed gravity is available. The field extensions, in this case, will be made from data which does not have the same reliability as

(i) the number (n) of readings used to evaluate the

the mean ;

(b) the standard deviation (σ) of each sample

vary from square to square. While n is dependent on the available gravity field, σ is a function of the variability of the latter and is not dependent on topography alone . The appropriate weight coefficient (w) is thus given by an expression of the form

$$w = w(n, \frac{1}{\sigma^2}) .$$

In attempting to formulate the nature of $w(n, 1/\sigma^2)$, it must be borne in mind that the final weight coefficient must reflect the distribution of the sample within the area to be represented as a high sample density in a restricted area may give too small a value of σ as it is not truly representative of the area. Further., the expression should reduce to

$$w = 1/ E_s^2 \qquad \text{..................(9.20)}$$

when n = 1 and $\sigma = 0$ and

$$w \rightarrow N/\sigma^2 \qquad \text{.............. (9.21),}$$

where N is the maximum number of readings possible, as $n \rightarrow N$.

In general, the weight coefficient should be inversely proportional to the variance of the sample mean. However, in squares where the sampl e only covers a small fraction of the total area, the use of n/σ^2 tends to overestimate the weight coefficient. An expression for the latter which not only satisfies the limiting conditions but also the general requirements is

$$w = \frac{n}{\sigma^2}\left[1 + \frac{(N-n)^2}{(N-1)^2\sigma^2} E_s^2\right]^{-1} \quad .....(9.22)$$

as

$$\frac{1}{w} = \frac{1}{n}\left[\sigma^2 + \frac{(N-n)^2}{(N-1)^2} E_s^2\right] \quad ....(9.23).$$

The individual weight coefficients for each $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ square were incorporated in equation (9.8), which was expressed in the form set out in equation (9.13) prior to solution. In this manner, the available field was extended to unrepresented areas.

The use of the weigh coefficients in the analysis of a given field with considerable variation locally was found to give a smoothened field when compared with a similar extension, but with weight coefficients set to unity for all available values. The values of $E\{\Delta g(\phi, \lambda)\}$ so obtained in the weighted solution were normalised as explained earlier using equations (9.15) to (9.18), and the final extended value accepted was

$$\Delta g(\phi, \lambda) = E\{\Delta g(\phi,\lambda)\} - E\{e_P(\phi,\lambda)\} \quad ...(9.24).$$

The assumption that the Bouguer anomalies used in the extension were free from height correlation is a justifiable one as the maximum mean elevation for $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ squares in the region considered was 803 metres.

The extended free air anomaly means were obtained from the extended Bouguer anomaly using the mean square elevation. These extended values were used to supplement the observed values in compiling fig (9.3) .

9.7 The accuracy of the field extension.

The precision of any prediction can be checked in one of two ways. Firstly, the extended values can be compared with actual ones. Alternatively, the extended values obtained by the use of any particular method could be compared with those obtained by any other acceptable method. Let the predictions be required in certain positions of a m x n array of Δg where certain values of the quantity are available. Three distinct cases of prediction of the anomaly are possible :-

(i) Interpolation :-

In this case, the readings $\Delta g(h, u)$, $\Delta g(j, u)$, $\Delta g(i, t)$ and $\Delta g(i, v)$ are available ; $h < i < j$; $t < u < v$.

(ii) Interpolation / extrapolation :-

In this circumstance, one only, of h, t, j and v is zero, on adopting the convention that

$\Delta g(0, u) = \Delta g(i, 0) =$ no reading available.

(iii) Extrapolation :-

At least one, each, of (h, j) and (t, v) is zero.

A preliminary study showed that the reliability of predictions was strongly affected by the variability of the gravity field and all predictions were normalised in terms of the average gravity gradient in the area, using the relation

$$N\{C\} = \frac{E\{C\}}{E\{G\}} M\{G\} \quad \ldots\ldots\ldots\ldots (9.25),$$

where $N\{C\}$ is the normalised prediction, $E\{G\}$ the predicted gravity anomaly gradient, $E\{C\}$ the predicted value and $M\{G\}$ the mean gravity anomaly gradient. The normalised predictions so obtained were classified according to

(i) the minimum interval (I_i), (j-h) or (v - t), as the case may be, for interpolations and interpolation/ extrapolations ;

(b) the minimum interval (I_e), i - h(or j) or u - v(or t), in the case of extrapolations.

Four different methods of prediction were used :-

(a) Graphical.
(b) Markov theory.
(c) Trigonometrical series with weighting.
(d) Trigonometrical series without weighting.

If the standard deviations of the comparisons between method (c) and any one of the others were σ_{c_i} (i=1, 3), reliability limits were set to the values of $I_{i(e)}$ accepted on the basis that

$$\sigma_{c_i} - \sigma_{c_{i+1}} \not> K \; M\{G\} \quad , \; i=1,3 \; (\text{if } i > 3, -3) \quad \ldots\ldots\ldots\ldots (9.26),$$

where K is a comparison factor. For interpolations, acceptable agreement. was obtained between the σ_{c_i} for

	Standard deviations of discrepancies in field extension ($\pm$ mgal) Numbers in parentheses represent sample size					
$I_{i(e)}$	1	2	3	4	5	6
Interpolation	2 (15)	5 (22)	5 (23)	5 (45)	5 (61)	6 (71)
Interpolation/ extrapolation	6 (2)	5 (7)	11 (20)	18 (37)	10 (41)	9 (44)
Extrapolation	10 (29)	12 (46)	14 (56)	15 (61)	15 (63)	-

TABLE (9.3)

Field extension discrepancies for $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ square means.

Case (a) / Case (c).

for each of the methods of extension when $I_i = 6$. For values of $I_i > 6$, the comparisons become more erratic and a greater K value had to be introduced, usually greater than 0.2, to enable the inequality given in equation (9.26) to be satisfied.

In case (ii), no comparison of $\sigma_{c_i} - \sigma_{c_{i+1}}$ could be considered acceptable unless the value of K was increased to 0.4. This value of K was also accepted as the value of for comparisons which came under class (iii).

The results in table (9.3) summarise the accuracy of comparisons made in one instance, after discarding those $I_{i(e)}$ values which were large enough to make the resulting extensions unreliable by the standards defined in inequality

(9.26) with K = 0.4.

While the samples are too small to draw definite conclusions, it would appear that, as a general rule, the weighting of extensions according to

Interpolations : Interpolation/ extrapolations : Extrapolations

$$= \frac{1}{1^2} : \frac{1}{2^2} : \frac{1}{3^2} = 1 : 0.25 : 0.11$$

is indicated. The factors to be considered in estimating the error of the predicted value are

(a) the nature of the extension , i.e., interpolation, interpolation/ extrapolation or extrapolation ;

(b) the interval from the most reliable value ;

(c) the accuracy of this value.

If the error of representation of the nearest basic square mean is e_{ref} and the estimated error of extension is e_{ext}, the estimated error of prediction ($E\{e_P\}$) is given by

$$\left[E\{e_P\}\right]^2 = e_{ref}^2 + e_{ext}^2 \quad \ldots\ldots\ldots\ldots(9.27).$$

The value of e_{ref} is also obtained from equation (9.23) .

In the use of such errors of prediction in the computation of estimates of error in values of N, ξ and η computed from the extended gravity anomaly field, it will also be necessary to allow for a certain degree of correlation between the values of adjacent predictions . This quantity is extremely difficult to assess as it is dependent on the location of the anomalies in the matrix from which the extension is made. As these are never easily predictable, and as the predictions are normalised, it is expected that the effect is not large, except in the case of extrapolation. This is elaborated on in section (12).

9.8 Conclusion.

The extension of gravity anomaly fields using mathematical functions, unless carried out under carefully controlled conditions, could produce results of questionable accuracy. The use of the two-dimensional trigonometrical series described in equation (9.13) is quite satisfactory for limited extents, provided that the degree of resolution of the function is adequately reduced when a paucity of data occurs. In the case where the field extension is eff cted using area means, which could be of differing reliability, the extension should be performed after adequate weighting. The accuracy of the values so predicted is dependent on whether the extension performed was an interpolation, interpolation/ extrapolation or an extrapolation. Field extensions performed over intervals more than 6 positions from the nearest available value (i.e., $I_{i(e)} > 6$) were found to be unreliable. For $I_{i(e)}$ 6, the ratios of the accuracies of interpolation : interpolation/ extrapolation : extrapolation = 1 : 2 : 3. Within this range, the error of the predicted value would not be materially larger than the error of representation of the nearest value used in the extension. Adjacent extended values can be expected to be correlated, especially in the case of extrapolation.

10. DEFLECTIONS OF THE VERTICAL FROM GRAVITY ANOMALIES.

10.1 Introduction.

In sections (2.6) and (3.1), it was shown that the discrepancy in position between equivalent points on the reference surface and the true surface could be represented in three dimensions by a vector. This vector is represented in fig (3.1) by $\underline{d}$ and is fully defined in equation (3.9) by three parameters, h_D, the separation between the existent surface and the true surface, the components ξ, in the meridian and η in the prime vertical, of the angle between the normals to the two surfaces in question. These departures of the existent surface from the mathematical model are of the order of f^2(1 part in 10^5).

The adoption of the telluroid as the reference surface specifies these quantities as

(a) the height anomaly ;

(b) the angle between the surface vertical and the normal to the associated spherop and its related resolutes into the meridian and prime vertical planes.

If the reference surface was the spheroid itself, these quantities become

(a) the separation N between the geoid and the reference spheroid ;

(b) the angle between the geoid and spheroid normals (ζ_o) and its related meridian (ξ_o) and prime vertical (η_o) components.

The vertical at an observation station on the earth's surface is the normal to the surface geop which passes through the former. The curvature of the family of geops of the existent earth can be expected to be irregular as it is subject to vagaries in the density distribution within the earth's crust and mantle.

10.2 Astro-geodetic deflections of the vertical.

The concept of deflections of the vertical in classical geodesy stem from origins far removed from th that occur in physical geodesy. The origin of the former is the interpretation required for the definition of a specific point on the spheroid which is to represent a given point on the earth's surface. In classical geodesy, a point P on the earth's surface is defined by that point P_G on the spheroid, such that the spheroidal normal at P_G passes through P.

In fig (10.1), P, Z_G and Z_A are as defined in fig (2.5). There is no equivalent definition in physical geodesy as the existent surface being mapped is always referred to a reference surface which approximates to it to order f^2. Thus, as shown in section (10.1), the surface vertical is referred to the normal of the associated spherop while the geoid normal is referred to the spheroid normal.

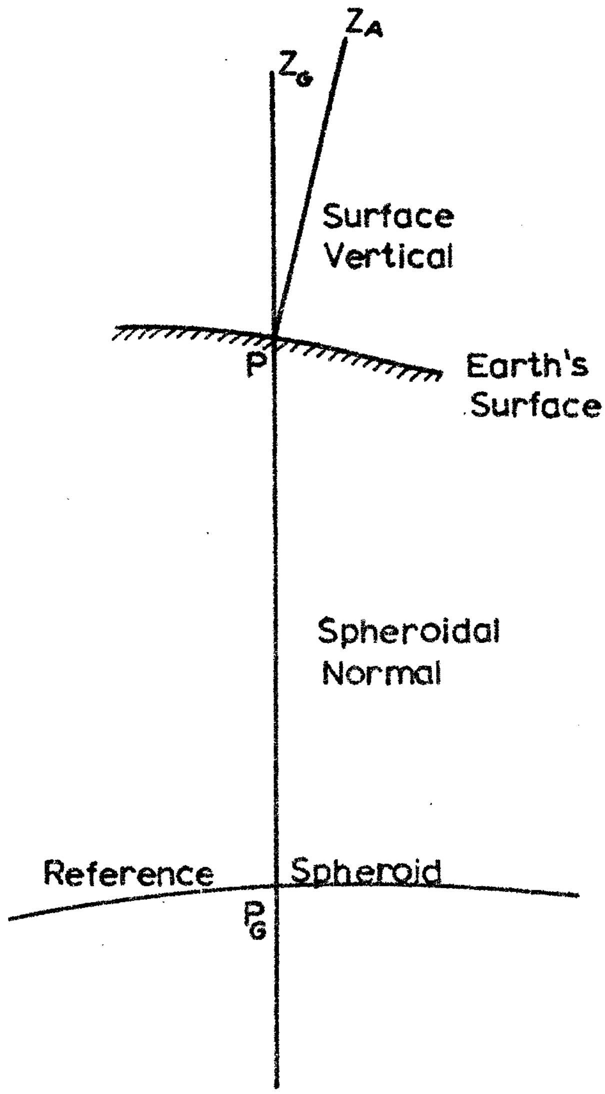

Z_A
Z_G
Surface
Vertical
P
Earth's
Surface
Spheroidal
Normal
Reference
Spheroid
P_G

FIG. 10.1

The geodetic zenith is defined by the geodetic co-latitude, as the latter is the angle between the celestial pole and the former on the celestial sphere. The geodetic co-latitude is a quantity computed from either geodetic triangulation or traversing. Further, the triangulation spheroid on which the triangulation or control geodetic survey is calculated is not necessarily the reference spheroid in the sense of the definition in section (3.5), as the orientation of such a reference spheroid is fixed by virtue of the definitions given in the section quoted. In the case of a triangulation spheroid, while its rotation axis is aligned parallel to the axis of rotation of the earth by virtue of the fact that all position determination in classical geodesy has to stem from astronomical determinations of latitude and longitude, its centre cannot, except by coincidence, be set to agree in location with the centre of mass of the earth, unless some additional information from an independent non-geometrical geodetic process is available.

Prior to 1950, the normal practice in classical geodesy was to carry out a comprehensive program of astronomical observations at a suitable triangulation station from which values of ϕ_A and λ_A were determined. To commence calculations, these values were put equal to ϕ_G and λ_G. This, in effect was equivalent to assuming that the deflections of the vertical and the separation of geoid and spheroid were zero at the origin of computations, as the motivation for the assumption was to orientate the reference spheroid correctly in space. If the resultant errors in N_o, ξ_o and η_o at the origin were e_N, e_ξ and e_η, it is possible to compute the errors in the astrogeodetic deflections of the vertical in the n-th station in a chain of triangulation, on the assumption that e_N, e_ξ and e_η were of the order of f^2, by the use of the principal term in any of the sets of geodetic formulae for converting measured lengths and azimuths to differences in latitude, longitude and convergence. For average length lines in a network, the length l is of order 50 km. The use of Puissant's formulae (Bomford, 1962, 107) to the appropriate order of accuracy gives

$$\Delta\phi_i = \frac{l_i}{\rho_{1i}} \cos A_i + o\{10^{-4}\} \quad i=1,n \quad \ldots\ldots (10.1),$$

$$\Delta\lambda_i = \frac{l_i}{\nu_{2_i}} \sin A_i \sec \phi_{2_i} + 0\{10^{-6}\} \quad i=1,n \quad \ldots\ldots (10.2),$$

and

$$\gamma_{s_i} = \Delta\lambda_i \sin \phi_{m_i} \sec \tfrac{1}{2}\Delta\phi_i + o\{10^{-6}\} \quad i=1,n \quad \ldots\ldots\ldots\ldots\ldots\ldots (10.3),$$

where A is the azimuth, the suffixes $_1$ and $_2$ refer to the two ends of the line, γ_s is the convergence on the spheroid and ϕ_m the mean latitude.

The difference in height anomaly between two triangulation stations at which astronomical determinations of latitude and longitude are available is given by a consideration of fig (10.2). If ξ_G and η_G are the astro-geodetic deflections of the vertical, with suffixes $_1$ and $_2$ referring to the two ends of the line, and if A and l are defined as in the previous

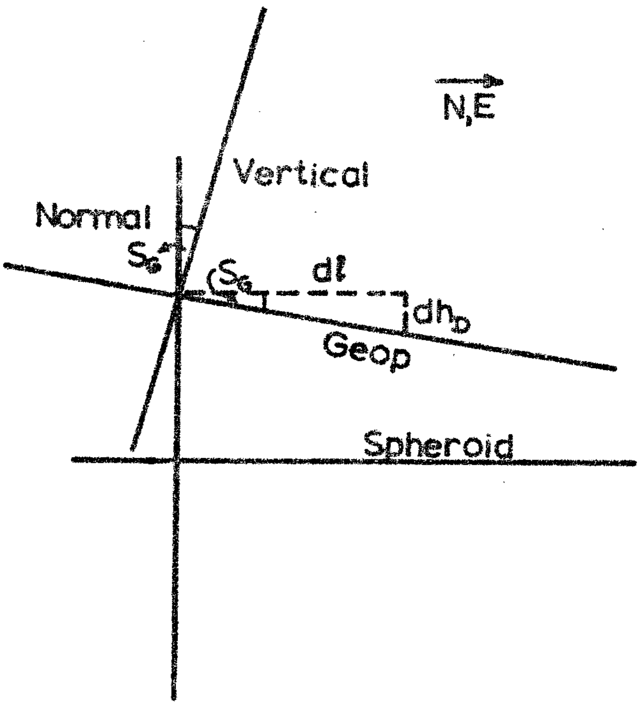


FIG. 10.2

ASTRO GEODETIC DEFLECTIONS OF THE VERTICAL.

paragraph, the component of the deflection of the vertical along the line and positive north and east is

$$\zeta_G = \xi_G \cos A + \eta_G \sin A \quad \ldots\ldots\ldots\ldots(10.4).$$

The difference in the height anomaly h_{D_G} is given by

$$h_{D_G} = - \int_{P_1}^{P_2} \zeta_G \, dl \quad \ldots\ldots\ldots\ldots\ldots(10.5),$$

where P_1 and P_2 are the two terminal triangulation stations. If these two stations are less than 50 km apart, it is valid to consider the difference in height anomaly on the i-th line of a chain of n triangulation lines as given by

$$h_{D_{G_i}} = - \zeta_{G_{m_i}} l_i \qquad i=1,n \quad \ldots\ldots\ldots\ldots(10.6),$$

where $\zeta_{G_{m_i}}$ is the mean astro-geodetic deflection of the vertical over the i-th line of a chain of n control stations. It should be noted that h_D and h_{D_G} are not the same because

(a) the differences in h_{D_G} obtained by the use of equation (10.5) are not the same as differences in h_D as there is a relative change between the orientation of adjacent spheroid normals which is regular while the change between consecutive associated spherop normals is also dependent on the change in elevation between the adjacent control points. This is developed in section (11.1).

The error $e_{\Delta\phi_i}$ in the i-th difference in latitude computed from equation (10.1) is obtained by partial of this equation.

$$e_{\Delta\phi_i} = -\Delta\phi_i\left[\tan A_i \sum_{j=2}^{i} e_{\gamma_{s_j}} + \frac{3e^2 \sin 2\phi_i}{2(1-e^2\sin^2\phi_i)}\left\{\sum_{j=1}^{i-1} e_{\Delta\phi_j} + \xi_o\right\}\right] \ldots\ldots \ldots\ldots\ldots (10.7),$$

where the i-th line is that between the i-th and (i+1) - th points in the chain of control points and $e_{\gamma_{s_i}}$ is the error in the computed convergence at the i-th point.

On a similar consideration of equation (10.2), the error $e_{\Delta\lambda i}$ in the i-th difference in longitude is given by

$$e_{\Delta\lambda_i} = \Delta\lambda_i\left[\cot A_i \sum_{j=2}^{i} e_{\gamma_{s_i}} + \left\{\sum_{j=1}^{i} e_{\Delta\phi_j} + \xi_o\right\}\left\{\tan\phi_{i+1} - \frac{e^2\sin 2\phi_{i+1}}{2(1-e^2\sin^2\phi_{i+1})}\right\}\right] \ldots (10.8).$$

The error in the i-th determined value of $_s$ is

$$e_{\gamma_{s_{i+1}}} = \gamma_{s_{i+1}}\left[\frac{e_{\Delta\lambda_i}}{\Delta\lambda_i} + \cot\phi_{m_i}\left\{\sum_{j=1}^{i-1} e_{\Delta\phi_i} + \xi_o + \tfrac{1}{2} e_{\Delta\phi_i}\right\} + \tfrac{1}{2}\tan\tfrac{1}{2}\phi_i\, e_{\Delta\phi_i}\right] \ldots (10.9).$$

These expressions assume that no other sources of error affect the calulcated quantities. For the first line, the only errors that affect the values of $e_{\Delta\phi_1}$, $e_{\Delta\lambda_1}$ and $e_{\gamma_{s_2}}$ arise from the adoption of the astronomical latitude as the equivalent geodetic quantities at the origin.

$$e_{\Delta\phi_1} = -\Delta\phi_1 \frac{3e^2 \sin 2\phi_1}{2(1 - e^2\sin^2\phi_1)} \xi_o \qquad \ldots\ldots(10.10).$$

$$e_{\Delta\lambda_1} = \Delta\lambda_1 \left[\tan\phi_2 - \frac{e^2 \sin 2\phi_2}{2(1 - e^2\sin^2\phi_2)}\right]\left[\xi_o + e_{\Delta\phi_1}\right]$$

$$\ldots\ldots\ldots\ldots(10.11)$$

and $$e_{\gamma s_2} = \gamma s_2 \left[\frac{e_{\Delta\lambda_1}}{\Delta\lambda_1} + \cot\phi_{m_1}\left(\xi_o + \tfrac{1}{2} e_{\Delta\phi_1}\right) + \right.$$

$$\left. + \tfrac{1}{2}\tan\tfrac{1}{2}\Delta\phi_1\, e_{\Delta\phi_1}\right] \qquad \ldots\ldots\ldots(10.12)$$

If ξ_o is assumed to be less than 10", then the order of $e_{\Delta\phi_1}$ for a standard first order geodetic line situated in a mid-latitude region where $\phi \simeq 10^{-2}$, is of order 10^{-9}. The error $e_{\Delta\lambda_1}$ in $\Delta\lambda_1$ is of order 5×10^{-7} as is the error in γ_s^1.

It can therefore be concluded that the errors introduced into the computed differences in latitude, longitude and convergence due to errors in the deflections of the vertical at the origin are negligible, never exceeding the order of observational accuracy, even though the effect is cumulative, as the error in azimuth is localised by the re-establishment of azimuth at every Laplace station. In geodetic traversing, Laplace stations are much closer together than in triangulation networks, where they could be upto 500 km. apart. In either case, however, the accumulated error in the computed value of the latitude will seldom exceed $0''.01$ due to this cause.

More serious errors are introduced into the value of the computed geodetic longitude due to errors in the value of the scaling factor $\cos \phi$ in equation (10.2). Once again, these will give rise to errors in the computed longitude, which will be given by

$$e_{\lambda} = \eta_{o} \sec \phi_{o} + \xi_{o} \sum_{i} \Delta\lambda_{i} \tan \phi_{i+1}$$

.................(10.13),

where ϕ_o is the latitude of the crigin. The nature of equations (10.7) to (10.9) ensure that any very large errors in the orientation of the reference spheroid at the origin will show in the latitude and longitude equations in the adjustment of the control network.

When comparing astro-geodetic deflections ofthe vertical with gravimetric ones, it will be necessary to effect the comparison by setting up a series of observation equations for each Laplace station at which gravimetric deflections of the vertical have been computed. If the residuals for these observation equations are not normally distributed and exhibit a definite correlation between position on the computation chain and magnitude, it could be inferred that the observed quantities are subject to a systematic effect.

In the event of the gravity anomaly field being fully represented and the triangulation spheroid arbitrarily orientated in space, the residuals in the observation equations could be made members of a normally distributed population by introducing the parameters ξ_o and η_o, which are the corrections to the values of the deflections of the vertical adopted at the origin for the arbitrarily orientated reference spheroid, which was used for the calculation of the control network. The general observation equations at the comparison points will be

$$v_{\phi_j} = \xi_{A_{c_j}} - \xi_{G_j} + \xi_o + \sum_{i=1}^{j} e_{\Delta\phi_i} + C_{1_j} + C_{2_j} ,$$

$$j=1,n \quad \ldots\ldots\ldots\ldots(10.14)$$

and

$$v_j = \eta_{A_{c_j}} - \eta_{G_j} + \eta_o \sec\phi_o \cos\phi_j + \cos\phi_j \sum_{i=1}^{j} e_{\Delta\lambda_i} +$$

$$+ C_{3_j} + C_{4_j} \quad \ldots\ldots\ldots\ldots\ldots(10.15),$$

where $e_{\Delta\phi_i}$ and $e_{\Delta\lambda_i}$ are given by equations (10.7) and (10.8). The suffix $_A$ refers to astro-geodetic values, computed on the reference spheroid used to compute the control network. The suffix $_G$ refers to quantities computed gravimetrically. The quantities C_1 and C_3 are introduced are introduced to allow for the fact that astro-geodetic deflections are computed with respect to the spheroid normal, while gravimetric values are with respect to the normal of the associated spherop. The constants C_2 and C_4 are corrections for possible differences in the dimensions of the reference spheroids used in computing the two sets of deflections of the vertical. In addition, it is necessary to consider equations (13.17) to (13.19). These 2n observation equations are then solved by the method of least squares, using equations (9.9) to (9.12) to obtain the best possible values for ξ_o and η_o.

In the absence of gravimetric deflections, the following method may be used to obtain the best possible values of ξ_o and η_o. It is assumed that in the event of the spheroid used in the computation of the control network (triangulation spheroid) being truly orientated in space, deflections of the vertical over a large enough area will be part of a normally distributed population. Equations (10.14) and (10.15) reduce to

$$v_{\phi_j} = \xi_{A_{c_j}} + \xi_o + \sum_{i=1}^{j} e_{\Delta\phi_i} \qquad j=1,n \ldots.(10.16)$$

$$v_{\lambda_j} = \eta_{A_{c_j}} + \eta_o \sec \phi_o \cos \phi_j + \sum_{i=1}^{j} e_{\Delta\lambda_i} \qquad j=1,n \ldots\ldots.(10.17).$$

These 2n observation equations are solved as before. Upto date, no reliable study has been carried out to determine whether the hypothesis of a normally distributed population of deflections of the vertical over a continental extent is a tenable one.

All solutions for free air geoids from the analysis of the perturbations of the orbits of artificial earth satellites show a marked slope of the co-geoid across Australia, indicating that the earlier proposition is a false one. The use of the equations (10.16) and (10.17) would fit the triangulation spheroid onto the average slope of the free air co-geoid over the region in which the equations are used.

Conclusions :-

Astrogeodetic deflections of the vertical are dependent on the nature of the triangulation spheroid on which the geodetic latitude and longitude have been computed. This spheroid may differ from the spheroid of best fit in one of two ways :-

(i) Its dimensions may be incorrect ;

(ii) Its centre may not coincide with the centre of the earth. This second effect arises in practice by the adoption of the commonly used assumption that the astronomically determined astronomical coordinates at the origin of computation represent the true geodetic coordinates at this point.

Astro-geodetic deflections of the vertical are angles between the surface vertical, which is normal to the surface geop, and the spheroidal normal. These deflections are not directly comparable with gravimetric deflections in either the case of the geoid - spheroid system or the physical surface - telluroid system.

10.3 Deflections of the vertical at the geoid.

The geoid, as explained in section (2.2), has no definition in continental areas unless a model is postulated for the topography exterior to it. Once this model has been defined, the deflections of the vertical for the geoid-spheroid system have a simple relation to the reference spheroid, being the rate of change of the geoid-spheroid separation (N) with distance along the reference system (x_i, $i=1, 2$). The conventional directions considered are x_1 orientated north and

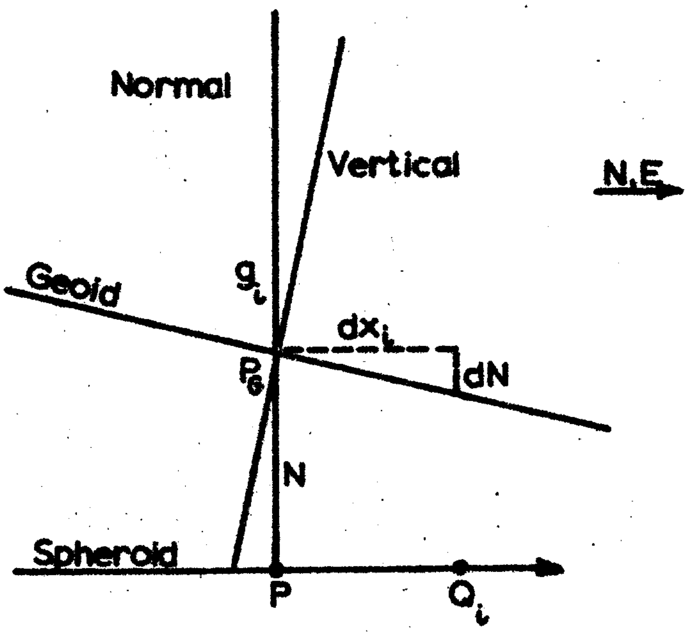


FIG. 10.3

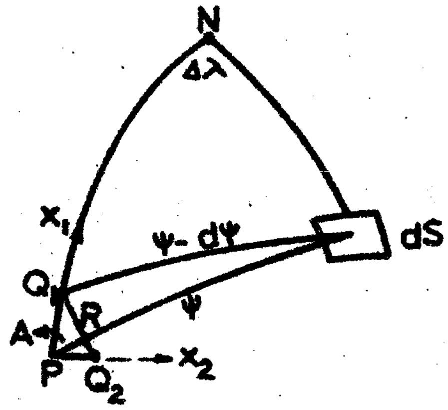


FIG. 10.4

x_2 orientated east. If the components of the deflections of the vertical in these directions are given by ξ_i(i=1, 2),

$$\xi_1 = \xi \qquad \text{.....................(10.18)}$$

and

$$\xi_2 = \eta \qquad \text{.....................(10.19).}$$

Adopting the same sign convention as in section (2.6), a consideration of fig (10.3) gives

$$\xi_i = - \frac{dN}{dx_i} \quad i=1,2 \qquad \text{...............(10.20),}$$

as N decreases with x_i for positive values of ξ_i. N is given by equation (5.54) and can be expressed as

$$N = N(\psi) \qquad \text{......................(10.21).}$$

It is therefore convenient to convert the differential on the right hand side of equation (10.20) as a differential with respect to ψ . This is a purely geometrical relationship as decreases with increase of x_i for all values of $\alpha_i < 90^\circ$. In this notation,

$$\alpha_1 = A \qquad \text{........................(10.22)}$$

and

$$\alpha_2 = \tfrac{1}{2}\pi - A \qquad \text{....................(10.22a),}$$

where , from fig (10.4), A iz the azimuth of the element of surface area dS from the point P at which the computation is being carried out. Thus

$$\cos \alpha_i = - \frac{R_m \, d\psi}{dx_i} \qquad \text{..............(10.23),}$$

where R_m is the mean radius of the earth. The use of equation (10.23) in equation (10.20) gives

$$\xi_i = \frac{dN}{R_m d\psi} \cos \alpha_i \quad i=1,2 \qquad \text{.............(10.24)}$$

The complete expression for N_P in terms of ψ is obtained from equations (5.53), (5.54) and equation (6.65) as

$$N_P = \frac{k R_m}{\gamma_m} \int_0^{\sigma=4\pi} \rho \, h_Q \operatorname{cosec} \tfrac{1}{2}\psi \left[1 + \tfrac{1}{2}(3c_Q - c_P + \frac{3h_Q}{2R_m}) \right] d\sigma$$

$$+ \frac{R_m}{4\pi\gamma_m} \int_0^{\sigma=4\pi} f(\psi) \, \Delta g_o \, d\sigma \quad \ldots (10.25),$$

where c_Q' and c_P are given by equation (6.12).

This equation can be expressed in the following form which is suitable for evaluation by quadratures when the use of equation (6.34) gives, on dropping the suffix $_Q$ and using instead the parametric indices i and j,

$$N_P = \frac{k R_m \pi^2}{180^2 \gamma_m} \sum_i n_i^2 \sum_j T_{ij} \operatorname{cosec} \tfrac{1}{2}\psi_{ij} \; +$$

$$+ \frac{R_m \pi}{4 \gamma_m 180^2} \sum_i n_i^2 \sum_j f(\psi_{ij}) \, \Delta g_{o_{ij}} \cos \phi_{ij} \quad \ldots (10.26),$$

where the basic unit of surface area is expressed as an $n^o \times n^o$ square and T_{ij} is given by

$$T_{ij} = \rho_{ij} \cos \phi_{ij} \, h_{ij} \left[1 + \tfrac{1}{2}(3c_{ij} - c_P + \frac{3h_{ij}}{2R_m}) \right] \quad \ldots (10.27).$$

The deflections of the vertical are given by

$$\xi_k^{(sec)} = \frac{\pi^2 k \, 206265}{180^2 \gamma_m} \sum_i n_i^2 \sum_j T_{ij} \left[\frac{\partial}{\partial\psi} \operatorname{cosec} \tfrac{1}{2}\psi \right]_{ij} \cos \alpha_{k_{ij}} \; +$$

$$+ \frac{206265\,\pi}{180^2 . 4\gamma_m} \sum_i n_i^2 \sum_j \Delta g_{o_{ij}} \left[\frac{\partial}{\partial\psi} f(\psi) \right]_{ij} \cos \phi_{ij} \ldots$$

$$\ldots \cos \alpha_{k_{ij}} \qquad k = 1, 2 \; \ldots\ldots\ldots\ldots (10.28),$$

where

$$\frac{\partial}{\partial\psi} \operatorname{cosec} \tfrac{1}{2}\psi = -\tfrac{1}{2}\cos\tfrac{1}{2}\psi \operatorname{cosec}^2\tfrac{1}{2}\psi \quad ....(10.29).$$

Differentiation of equation (5.53) with respect to ψ gives

$$\frac{\partial}{\partial\psi} f(\psi) = -\tfrac{1}{2}\cos\tfrac{1}{2}\psi \operatorname{cosec}^2\tfrac{1}{2}\psi - 3\cos\tfrac{1}{2}\psi + 5\sin\psi - $$

$$- 3\cos\psi \frac{\cot\frac{1}{2}\psi\,(1 + 2\sin\frac{1}{2}\psi)}{2\,(1 + \sin\frac{1}{2}\psi)} +$$

$$+ 3\sin\psi \ln\left\{\sin\tfrac{1}{2}\psi\,(1 + \sin\tfrac{1}{2}\psi)\right\} \quad .... (10.30)$$

For purposes of economy in the use of computer time it is convenient to adopt the following working formulae.

$$\xi_k^{(sec)} = \xi_{i_k}^{(sec)} + \xi_{s_k}^{(sec)} \quad k=1,2 \quad .....(10.31),$$

where the components ξ_{i_k} (k=1, 2) of the deflection of the vertical due to the indirect effect are given by

$$\xi_{i_k}^{(sec)} = K_i \sum_i n_i^2 \sum_j T_{ij} \cos\tfrac{1}{2}\psi_{ij} \operatorname{cosec}^2\tfrac{1}{2}\psi_{ij} \cos\alpha_{k_{ij}}$$

$$k=1,2 \quad ....(10.32),$$

where

$$K_i = -\frac{\pi^2\, k\, 206265 \times 10^2}{2 \times 180^2\, \gamma_m} \quad ....(10.33),$$

which holds when the value of γ_m is in gal and that of k is in $cm^3 gm^{-1} sec^{-2}$, T_{ij} being expressed as

$$T_{ij} = \rho_{ij}^{(gm.cm^{-3})} \cos\phi_{ij}\, h_{ij}^{(met)} \left[1 - \tfrac{1}{2}\left(c_P - 3c_{ij} - \frac{3h_{ij}}{2R_m}\right)\right]$$

The values of $\alpha_{k_{ij}}$ (k=1, 2) are given by equation (10.22).

The component due to the Stokesian separation is given by

$$\xi_{s_k}^{(sec)} = K_s \sum_i n_i^2 \sum_j \Delta g_{o_{ij}}^{(mgal)} \cos \phi_{ij} \left[\frac{\partial}{\partial \psi} f(\psi)\right]_{ij} \cos \alpha_{k_{ij}} \quad \ldots\ldots\ldots\ldots\ldots(10.33), \quad k=1, 2.$$

where

$$K_s = \frac{\pi \times 206265 \times 10^{-3}}{4 \times 180^2 \; \gamma_m} \quad \ldots\ldots\ldots\ldots(10.34),$$

γ_m being in gal. $\Delta g_{o_{ij}}$ is given by

$$\Delta g_{o_{ij}} = \Delta g_{F_{ij}} + \Delta g_{co_{ij}} \quad \ldots\ldots\ldots(10.35)$$

and $\frac{\partial}{\partial \psi} f(\psi)$ is given by equation (10.30). Equation (10.33), if the nature of the anomaly used is ignored, is commonly known as an integral set named after Vening Meinesz.

Equations (10.31) through (10.35) combined with equation (5.54) give a complete solution for the separation vector of the geoid-spheroid system. The above equations do not hold in their stated form for very small values of ψ as functions which include cosec $\frac{1}{2}\psi$ become indeterminate when $\psi = 0$. On the other hand, as the functions always appear as a product with the element of surface area, evaluation of the function is possible and this is dealt with under section (12).

11. THE COMPARISON OF ASTRO-GEODETIC DEFLECTIONS OF THE VERTICAL WITH THOSE AT THE GEOID FOR A POSTULATED MODEL OF THE TOPOGRAPHY.

11.1 Introduction.

The nature of astrogeodetic deflections of the vertical were fully discussed in section (10.2). At a first glance, it would appear that, provided the triangulation spheroid were correctly orientated in space with its centre at the earth's centre of mass, astro-geodetic deflections of the vertical would be the same as those obtained from a consideration of equation (4.21) in the case where the reference surface was the telluroid. The height anomaly h_{D_P} was obtained from

$$h_{D_P} = \frac{1}{2\pi\gamma} \iint_R \left[\left\{ \frac{1}{\gamma} \left|\frac{\partial\gamma}{\partial h}\right| \frac{\cos\beta}{r} + \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right\} \gamma h_D + \left\{ \Delta g_F - \gamma \left(\xi \tan\beta_1 + \eta \tan\beta_2 \right) \right\} \frac{\cos\beta}{r} \right] dR \qquad \ldots\ldots\ldots\ldots(4.22)\ .$$

At the earth's surface, the deflections of the vertical are defined as the angle between the local vertical and the normal to the associated spherop as resolved into meridian and prime vertical components. From equation (3.28) it is seen that this quantity is given by

$$\xi_i = -\left[\frac{\partial h_D}{\partial x_i}\right]_{\text{Telluroid}} - \frac{\Delta g_F}{\gamma_P} \tan\beta_i \quad , \; i=1,2 \quad , \; \xi_2 = \eta \; . \qquad \ldots\ldots \quad (3\ 28)$$

h_D could be used in equations (10.20) through (10.24) in the evaluation of $\xi_i (i=1,2)$ with the additional term given in equation (3.28) to give the meridian and prime vertical resolut es of the deflection of the vertical at the earth's surface . A study of fig (3.2) shows that the quantities $\xi_i (i=1,2)$ so defined are not the same as the astrogeodetic deflections of the vertical , shown in fig (10.2) as, in the former case, the angle ζ is that between the local vertical and the normal to the associated spherop while in the latter, it is the angle between the local vertical and the spheroid normal which passes through the ground point. Thus, the astro-geodetic deflections of the vertical (ξ_{A_i} , i=1, 2) are the components of angle represented by $Z_G P Z_A$ in fig (11.1) , where the three points mentioned have the same definition as in fig (2.5)

The surface deflections of the vertical obtained by a devlopment of equations (4.22) and (3.28) refer to the angle $Z'_G\ PZ_G$, where PZ_G is the normal to the associated spherop . The non-coincidence of the normals PZ_G and PZ'_G is due to the convergence of equipotential surfaces with increase of latitude. In fig (11.1), consider the point P' on the spherop $U = W_P$ whose normal passes through the surface point P. Let Q in fig (11.2) be a consecutive point on the meridian through P'. As all spherops will be solids of revolution for a spheroid, it is obvious that the effect described in fig (11. 1) has no magnitude in the prime vertical as a section in this plane has no differential curvature. Adoption of the conventions used in equations (10. 18) and (10. 19) gives

$$\xi_{2_S} - \xi_{2_A} = 0 \quad \ldots\ldots \ldots\ldots (11.1)$$

The effect in the meridian is considered in fig (11.2). If the angle between the spheroid normal and the spherop normal is ξ_c,

$$\xi_c = \xi_{1_S} - \xi_{1_A} \quad \ldots\ldots \ldots\ldots (11.2),$$

Provided PP' is small in comparison to $P'P_s$, it is justifiable to assume that

$$P_sP = P_sP' + P'P$$

with sufficient accuracy in f[illegible] (11.2) and ξ_c can be assumed to be the angle between the spheroid and spherop normals as shown thereon. If Q is a consecutive point on the spherop $U = W_P$ and its equivalent point on the reference spheroid ($U = W_o$) is Q_s, let the height $P'P_s = h$. Defining $QQ_s = h + dN$ and the difference in latitude between P and Q as $d\phi$, the linear distance P_sQ_s ($= dx_1$) is given by

$dx_1 = \rho d\phi$, where ρ is the radius of curvature in the meridian. As the reference spheroid is an equipotential surface, let the mean value of gravity along the normal P_sP' be γ and that along the normal QQ_s be $\gamma - d\gamma$. The use of equation (2.29) gives

$$W_P - W_o = -\gamma h = -(\gamma - d\gamma)(h + dN) \quad \ldots (11.3).$$

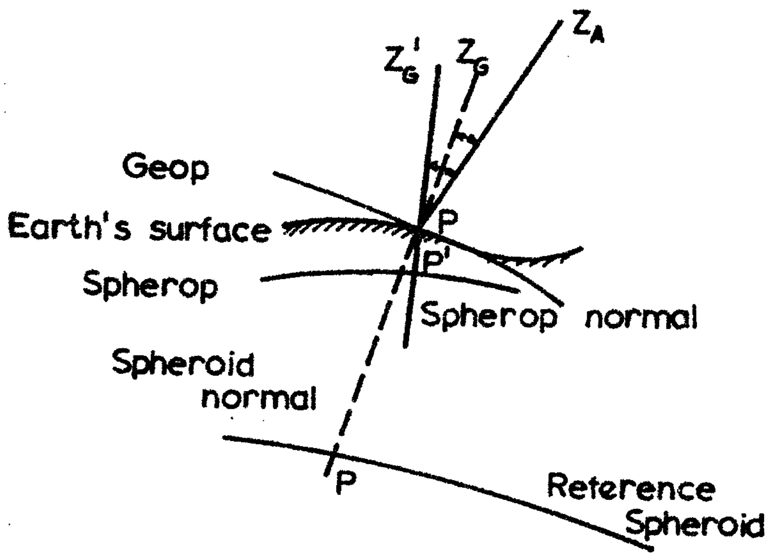


FIG. 11.1

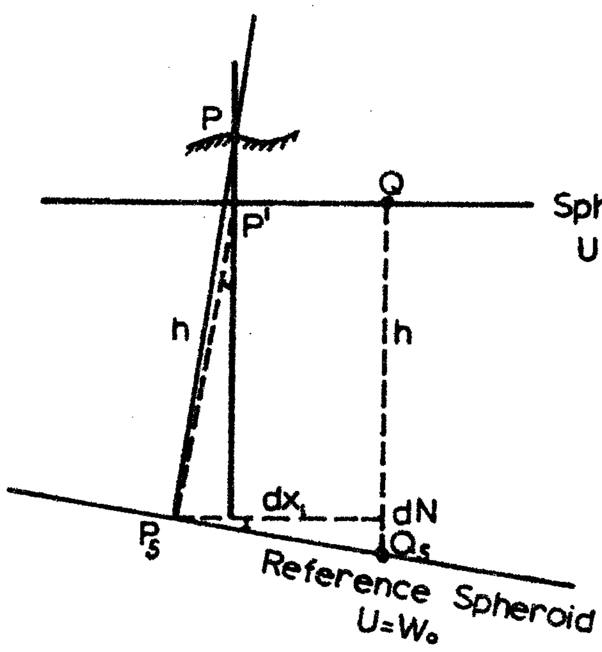


FIG. 11.2

$$\gamma\, dN - h\, d\gamma = 0 + o\{d\gamma\, dN\} \quad \ldots\ldots\ldots\ldots (11.4),$$

and

$$dN = \frac{h\, d\gamma}{\gamma} .$$

γ is given as a combination of equations (3.18) and (3.30) as

$$\gamma = \gamma_e \left(1 + \beta \sin^2 \phi + \frac{h}{R_m} + o\{f^2\} \right) \ldots . (11.5),$$

where $\beta = 5.2884 \times 10^{-3}$ for the international spheroid. $d\gamma$ is obtained by the differentiation of equation (11.5) which gives

$$d\gamma = \gamma_e \beta \sin 2\phi\, d\phi \quad \ldots\ldots\ldots\ldots (11.6).$$

The ratio $d\gamma / \gamma$ can be replaced with adequate accuracy by appropriate evaluation using equation (11.6) giving

$$dN = h\, \beta \sin 2\phi\, d\phi \quad \ldots\ldots\ldots\ldots (11.6a)$$

The use of equation (11.6a) gives, on considering fig (11.2),

$$-\xi_c = \frac{dN}{\rho\, d\phi} = \frac{h\, \beta \sin 2\phi}{\rho}$$

This expression is commonly accepted as defining the "curvature of the vertical" (Bomford, 1962, 410), having an approximate magnitude of 0".17 per km elevation and it is adequate to express ξ_c as

$$\xi_c^{(sec.)} = -0''.00017\, h^{(met)} \sin 2\phi \quad \ldots\ldots . (11.7).$$

Thus the astro-geodetic deflection of the vertical is related to the surface deflection of the vertical obtained gravimetrically by the relation

$$\xi_{1_s}^{(sec)} = \xi_{1_A}^{(sec)} - 0''.00017\, h^{(met)} \sin 2\phi \qquad \ldots\ldots\ldots(11.8)$$

provided the following conditions are satisfied :-

(i) The dimensions of the triangulation spheroid used in the calculation of the astro-geodetic values are the same as those of the reference spheroid used in calculating the gravity anomalies used in the evaluation of equation (4.22) .

(ii) The centre of the triangulation spheroid coincides with the centre of mass of the earth.

These two conditions have been dealt with in full by Vening Meinesz (1950a; 1950b) and Vincenty (1965) and will be summarised in section (13).

11.2 The relation between surface deflections of the vertical and those at geoid level for a postulated model.

Consider a point P on the earth's surface and the point P_o on the geoid which is equivalent to P. If the height anomaly at P is h_D and the geoid spheroid separation at P_o is N, the latter has been shown in section (5) to be given by the relation

$$N = \frac{2\,\phi_e}{\gamma_m} + \frac{R}{4\,\pi\gamma_m} \int_0^{\sigma=4\pi} \left[\frac{3V_D}{\Delta g_o} + 2R \right] \frac{\Delta g_o}{r} \, d\sigma \qquad \ldots\ldots\ \ldots\ldots.(5.14),$$

where ϕ_e is the potential of matter exterior to the geoid at P_o and Δg_o is the free air anomaly at the geoid.

This expression, on continued development using the spherical approximation, as shown in sections (5.2) and (5.3) gave N by the relation

$$N = \frac{2\phi_e}{\gamma_m} + \frac{R_m}{4\pi\gamma_m} \int_0^{\sigma=4\pi} f(\psi)\ \Delta g_o\ d\sigma \quad ..(5.54),$$

where f(ψ) is given by equation (5.53). The expression for h_D is obtained by a slight re-arrangement of terms from equation (4.22) as

$$h_D = \frac{1}{2\pi\gamma} \iint_R \left[\left\{ \frac{1}{\gamma} \left|\frac{\partial\gamma}{\partial h}\right| \frac{1}{r} + \underline{\nabla} \cdot \underline{N} \frac{1}{r} \right\} \gamma\ h_D + \frac{\Delta g_F}{r} \right] dR -$$

$$- \frac{1}{2\pi\gamma} \iint_R \left[\left[\frac{1}{r} \left|\frac{\partial\gamma}{\partial h}\right| h_D + \frac{\Delta g_F}{r} \right] \text{versine}\ \beta + \right.$$

$$\left. + \gamma(\xi \tan\beta_1 + \eta \tan\beta_2) \frac{\cos\beta}{r} \right] dR \quad ..(11.9).$$

The first integral on the right hand side of equation (11.9) is identical with that in equation (5.2) which, on development, gives equations (5.53) and (5.54). The only difference is that the anomaly used is the free air anomaly at the surface of the earth while the earlier integral required the free air anomaly at the geoid. Thus, the first integral, on evaluation, gives the free air geoid, the quantity determined being N_F as defined in equation (5.60). Let the second integral in equation (11.9), on evaluation, equal a correction term N_c given by

$$h_D = N_F + N_c \quad \ldots\ \ldots\ldots\ldots (11.10).$$

11.3 The evaluation of N_c .

Adopting the spherical approximation as previously done in section (5.1),

$$\frac{1}{\gamma}\left|\frac{\partial\gamma}{\partial h}\right| \frac{V_D}{r} = \frac{2\,V_D}{R\,r} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(11.11).$$

From equations (11.9), (11.10) and (11.11), using the symbol S for surface area,

$$N_c = -\frac{1}{2\pi\gamma}\iint_S \left\{\left[-\frac{2\,V_D}{R} + \Delta g_F\right]\frac{1}{r}\text{ versine }\beta + \right.$$

$$\left. + \gamma(\,\xi\tan\beta_1 + \eta\tan\beta_2\,)\frac{\cos\beta}{r}\right\} dS$$

$$= -\frac{1}{2\pi\gamma}\iint_S \left[f(a)\text{ versine }\beta + f(b)\cos\beta\right] dS$$

$$\ldots\ldots\ldots\ldots(11.12).$$

f(a) is evaluated using equations (5.99) and (5.32) as

$$f(a) = \left[\frac{2}{R}\sum_{i=2}^{\infty}\frac{A_i}{R^{i+1}} + \sum_{i=2}^{\infty}\frac{(i-1)\,A_i}{R^{i+2}}\right]\frac{1}{r} \;.$$

The use of equation (5.36) gives

$$f(a) = \sum_{n=2}^{\infty}\frac{n+1}{n-1}\,g_{nm}S_{nm}\,\frac{1}{r} \quad \ldots\ldots(11.13).$$

The expression of $\frac{1}{r}$ as a spherical harmonic using equations (5.23) through (5.25) gives

$$\frac{1}{r} = \frac{1}{R}\sum_{n=0}^{\infty}\left[\frac{R_m}{R}\right]^n P_n(\cos\psi\,) \quad \ldots\ldots\ldots(5.25).$$

The combination of equations (5.25) and (11.13) gives

$$\iint_S f(a) \text{ versine } \beta \, dS = \iint_S \sum_{n=2}^{\infty} \frac{n+1}{n-1} g_{nm} S_{nm} \frac{\text{versine } \beta}{R}$$

$$\sum_{j=0}^{\infty} \left[\frac{R_m}{R}\right]^j P_j(\cos \psi) \quad dS \; .$$

Proceeding on lines similar to that followed in equations (5.40) and (5.41), the above integral reduces to

$$\iint_S f(a) \text{ versine } \beta \, dS = \iint_S \Phi(\psi) \; \Delta g_F \text{versine } \beta \, dS$$

$$\ldots\ldots\ldots\ldots\ldots\ldots (11.14).$$

In this development it has been assumed that the relation between the disturbing potential and the gravity anomaly developed in section (5.2) will also apply to the two functions $V_D$versine β and Δg_Fversine β which should be capable of representation by spherical harmonics. This assumption has been considered to be reasonable. $\Phi(\psi)$ in equation (11.14) is given by

$$\Phi(\psi) = \lim_{R_m \to R} R_m \sum_{n=2}^{\infty} \frac{n+1}{n-1} \frac{R_m^n}{R^{n+1}} P_n(\cos \psi)$$

$$= \lim_{R_m \to R} R_m \left[\sum_{n=2}^{\infty} \frac{1}{R} \left[\frac{R_m}{R}\right]^n P_n(\cos\psi) \right.$$

$$\left. + \; 2 \sum_{n=2}^{\infty} \frac{R_m^n}{R^{n+1}} \frac{P_n(\cos \psi)}{n-1} \right] \ldots (11.15).$$

The two expressions on the right hand side of equation (11.15) are identical in form to equations (5.46) and (5.51) except for differences in the constant terms.

Evaluation on a similar basis gives

$$\Phi(\psi) = \lim_{R_m \to R} R_m \left[\left\{ \frac{1}{r} - \frac{1}{R} - \frac{R_m \cos\psi}{R^2} \right\} + \frac{2}{R^2} \left\{ -r - R_m \cos\psi \ \ln \frac{r + R - R_m \cos\psi}{2R} + R - R_m \cos\psi \right\} \right] .$$

In the limit, $R_m = R$ and $r = 2R \sin \frac{1}{2}\psi$. Thus

$$\Phi(\psi) = \tfrac{1}{2} \operatorname{cosec} \tfrac{1}{2}\psi - 1 - \cos\psi - 4 \sin \tfrac{1}{2}\psi -$$

$$- 2\cos\psi \ln\{\sin \tfrac{1}{2}\psi (1 + \sin \tfrac{1}{2}\psi)\} + 2 - 2\cos\psi$$

$$= \tfrac{1}{2} \operatorname{cosec} \tfrac{1}{2}\psi + 1 - 3\cos\psi - 4 \sin \tfrac{1}{2}\psi -$$

$$- 2\cos\psi \ \ln\{\sin \tfrac{1}{2}\psi (1 + \sin \tfrac{1}{2}\psi)\} \quad \ldots(11.16).$$

Thus N_c can be expressed as

$$- N_c = \frac{R_m}{2\pi\gamma_m} \int_0^{\sigma=4\pi} \Phi(\psi) \, \Delta g_F \operatorname{versine} \beta \, d\sigma +$$

$$+ \frac{R_m}{4\pi\gamma} \int_0^{\sigma=4\pi} \gamma(\xi \tan\beta_1 + \eta \tan\beta_2) \cos\beta \operatorname{cosec} \tfrac{1}{2}\psi \, d\sigma$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(11.17).$$

From fig (10.2) and equation (3.28), the deflections of the vertical at the surface are given by

$$\xi_{s_i} = -\left[\frac{\partial h_D}{\partial x_i}\right] - \frac{\Delta g_F}{\gamma_P} \tan\beta_i \ , \quad i=1,2 ,$$

where $\xi_{s_1} = \xi_s$ and $\xi_{s_2} = \eta_s$ ......(11.18)

and

$$\xi_{s_i} = \frac{\partial h_D}{R_m \partial \psi} \cos \alpha_i - \frac{\Delta g_F}{\gamma_P} \tan \beta_i \quad , \; i=1,2 \; ; \; \alpha_1 = A, \; \alpha_2 = \tfrac{1}{2}\pi - A \quad \ldots.(11.19).$$

This can be expressed with the aid of equation (11.10) as

$$\xi_{s_i} = \frac{\partial}{R_m \partial \psi} N_F \cos \alpha_i + \frac{\partial N_c}{R_m \partial \psi} \cos \alpha_i - \frac{\Delta g_F}{\gamma P} \tan \beta_i \; , \; i=1,2 \quad \ldots\ldots\ldots\ldots\ldots(11.20).$$

Thus equation (11.20) can be re-written as

$$\xi_{s_i} = \xi_{F_i} + \xi_{c_i} \quad , \; i=1,2 \quad \ldots\ldots\ldots(11.21),$$

where ξ_{F_i} (i=1,2) are the deflections of the vertical for the free air geoid. ξ_{c_i} (i=1,2) are given by

$$\xi_{c_i} = -\frac{206265 \text{ sec}}{2\pi\gamma_m} \int_0^{\sigma = 4\pi} \Delta g_F \text{ versine } \beta \frac{\partial}{\partial \psi} \Phi(\psi) \cos \alpha_i d\sigma$$

$$+ \frac{206265}{8\pi} \int_0^{\sigma = 4} (\xi \tan \beta_1 + \eta \tan \beta_2) \cos \beta \cos \alpha_i \operatorname{cosec} \tfrac{1}{2}\psi$$

$$\cot \tfrac{1}{2}\psi \, d\sigma - \frac{\Delta g_F}{\gamma_P} \tan \beta_i \quad , \; i=1,2 \quad \ldots.(11.22) \quad ,$$

where

$$\frac{\partial}{\partial \psi} \Phi(\psi) = -\frac{1}{4} \operatorname{cosec} \tfrac{1}{2}\psi \cot \tfrac{1}{2}\psi + 3 \sin \psi - 2 \cos \tfrac{1}{2}\psi +$$

$$+ 2 \sin\psi \; \ln\left\{ \sin \tfrac{1}{2}\psi (1 + \sin \tfrac{1}{2}\psi) \right\} -$$

$$- \cos \psi \frac{\cos \frac{1}{2}\psi (1 + 2 \sin \frac{1}{2}\psi)}{\sin \frac{1}{2}\psi \; (1 + \sin \frac{1}{2}\psi)} \quad \ldots..(11.23).$$

In the above equations, ψ has its usual definition. Equation (11.22) assumes the values of ξ and η to be given in radians. β_1 and β_2 are the ground slopes in the directions of the x_i (i=1, 2) axes which lie along the north and east directions respectively. β is defined by the use of equation (4.16). The nature of the distance functions in both surface integrals on the right hand side of equation (11.22) are more akin to the Vening Meinesz function defined in equation (10.30) than to the Stokes' function given in equation (5.53) as the term of controlling significance is $\mathrm{cosec}^2 \frac{1}{2}\psi$. Further, the gravity anomalies, which are of order 10^2 mgal or less are now multiplied by the versine or the cosine of the ground slope. The first of the surface integrals on the right hand side can be neglected in all but very mountainous regions where the near zone effects could be of significance. The second surface integral is about an order smaller than the contribution to the deflections of the vertical of the free air geoid using the Vening Meinesz integrals. Thus the effect of the distant zones can be assumed to be smaller than 0''.1.

Hence it should be possible to obtain adequate evaluation of surface deflections of the vertical which can be compared with astro-geodetic deflections by evaluating the integral set out in equation (11.22) for the region which lies within $\psi < 20^\circ$, noting that ocean regions do not contribute at all to this term. For any such calculations done on the Australian mainland region, it is only necessary to know deflections of the vertical over the mainland region to completely solve the problem with an accuracy equivalent to that obtainable from astro-geodesy. Thus any solution for the evaluation of surface deflections of the vertical from gravity anomalies has to be an iterative one, the first and major contribution being that due to the free air geoid. The deflections of the vertical for the free air geoid would provide adequate values of ξ and η for the evaluation of the second integral on the right hand side of equation (11.22), which on combination with the free air geoid terms and the ground gradient term give the complete solution for surface deflections of the vertical.

11.4 The comparison of surface deflections with those at the geoid.

From equations (5.60), (5.61), (10.26) and (10.31), the working formulae for deflections of the vertical at the geoid can be expressed as

$$\xi_i = \xi_{I_i} + \xi_{F_i} + \xi_{co_i} \quad , \; i=1,2 \quad ..(11.24),$$

where the normal Stokesian contribution has been separated into the component due to the free air geoid and that due to the reduction of observed gravity for the differential topographical effect to obtain the equivalent value on the geoid. The terms ξ_{F_i} (i=1, 2) are obtained by the use of free air anomalies in the Stokes and Vening Meinesz integrals, while ξ_{co_i} (i=1, 2) are obtained by the use of the differential topographical reduction (Δg_{co}) in these same integrals. If the deflections of the vertical at the surface (ξ_{s_i} , i=1, 2) are related to those at

$$\xi_{comp_i} = \xi_i - \xi_{s_i}$$

$$= \xi_{I_i} + \xi_{co_i} - \xi_{c_i} \quad i=1,2 \quad .....(11.25),$$

where ξ_{c_i} (i=1, 2) are defined by equation (11.22).

Working formulae for the three quantities on the right hand side of equation (11.25) are

$$\xi_{I_i}^{(sec)} = K_I \sum_j n_j^2 \sum_k T_{jk} \cos \tfrac{1}{2}\psi_{jk} \operatorname{cosec}^2 \tfrac{1}{2}\psi_{jk} \cos \alpha_{ijk} ,$$

$$i=1,2 \quad .....(10.32),$$

where K_I is given by equation (10.33) and T_{jk} by equation (10.27).

$$\xi_{co_i}^{(sec)} = K_s \sum_j n_j^2 \sum_k \Delta g_{co}^{(mgal)} \cos \phi_{jk} \left[\frac{\partial}{\partial \psi} f(\psi)\right]_{jk}$$

$$\cos \alpha_{ijk} \quad , \ i=1,2 \quad ..... (11.26),$$

where K_s is given by equation (10.34) and Δg_{co} by equations (6.36), (6.44) and (6.62). The expression for $\frac{\partial}{\partial \psi} f(\psi)$ is given by equation (10.30).

$$\xi_{c_i}^{(sec)} = K_c \sum_j n_j^2 \sum_k \Delta g_{F_{jk}}^{(mgal)} \text{ versine } \beta_{jk} \left[\frac{\partial}{\partial \psi} (\psi)\right]_{jk}$$

$$\cos \phi_{jk} \cos \alpha_{ijk} + K_d \sum_j n_j^2 \sum_k \Big[\xi_{jk}^{(sec)} \tan \beta_{1jk} +$$

$$+ \eta_{jk}^{(sec)} \tan \beta_{2jk} \Big] \cos \phi_{jk} \cos \tfrac{1}{2} \psi_{jk} \operatorname{cosec}^2 \tfrac{1}{2} \psi_{jk} \cos \alpha_{ijk}$$

$$\cos \beta_{jk} - 206265 \frac{\Delta g_F}{\gamma_P} \tan \beta_i \quad , \ i=1,2 \quad .....(11.27),$$

where

$$K_c = - \frac{206265 \pi \times 10^{-3}}{2\gamma_m^{(gal)} \times 180^2} \quad ....(11.28)$$

and

$$K_d = \frac{\pi}{8 \times 180^2} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (11.29).$$

$\partial/\partial\psi\{\phi(\psi)\}$ is given by equation (11.23). The use of the constants in the form given by equations (11.28) and (11.29) enable the results of equations (10.32), (11.26) and (11.27) to be added together directly to give ξ_{comp_i} (i=1, 2).

11.5 Conclusions.

Given a triangulation spheroid with the same dimensions as the reference spheroid, used in determining the gravity anomalies, with its centre at the centre of mass of the earth and correctly orientated in space, astro-geodetic deflections of the vertical, corrected for the divergence of the spheroid and associated spherop normals can be directly compared with gravimetric deflections of the vertical referred to the telluroid. If $W_o = U_o$, the latter are, to the first order, the deflections of the vertical obtained by the use of free air anomalies in the Vening Meinesz integrals, though considerable error could arise in adopting this approximation in regions of rugged topography. The relation between deflections of the vertical for the free air geoid and astro-geodetic values, on the same spheroid, are given by equations (11.8), (11.21), (11.22) and (11.23). The correction factors ξ_{c_i} (i=1, 2) are made up of three constituent terms each, one of which has a form similar to the Vening Meinesz integrals.

The term controlled by the function versine β , on the basis of ground slopes for $5^{\circ} \times 5^{\circ}$ square means being of the order of 1 part in 100 or less , will contribute an amount which should be about four orders smaller than that due to the Vening Meinesz integral and can hence, as pointed out earlier , be neglected for all points except those in mountainous regions when this effect will have to be considered for near zones. The effect of the second term has already been discussed. Two possibilities are available for the evaluation of this second term. Firstly, the problem could be treated as an iterative one, which would be convenient only if the calculations were regional in nature. In the case of point to point calculations where computations are not carried out at regular intervals, it could be feasible to obtain the values of the deflections of the vertical necessary for the evaluation of the second term from existing triangulation networks . The astro-geodetic deflections so obtained, after correction for dimension of the reference spheroid and any orientation error that may exist could be used to provide the interim values required. The latter effects are dealt with in section (13).

12. THE CALCULATIONS

12.1 Introduction.

The calculations involved in the mapping of the geoid are laborious, repetitive and, as such, most easily performed on an electronic computer. The final computations can be considered to consist of the following stages :-

(a) The calculation of N_F, ξ_F and η_F for the free air geoid.

(b) The calculation of the corrections N_{co}, ξ_{co} and η_{co} to the above quantities due to the fact that the anomaly to be used in that part of the solution due to the Stokes and Vening Meinesz integrals is not the free air anomaly at the surface of the earth but the free air anomaly at the geoid. These calculations are carried out using the same set of programs as used in section (a) above but using, instead, the topographical corrections (Δg_{co}) given by equations (5.59), (6.36) and (6.44).

(c) The evaluation of the corrections N_I, ξ_I and η_I to the Stokesian effect calculated under (a) and (b) above due to the topography exterior to the geoid. These terms are given by equations (6.66), (6.74), (10.32) and (10.33).

In carrying out the complete set of calculations at a given point it is necessary to evaluate two physical parameters of the earth at every point on its surface. These are

(a) elevations and

(b) free air anomalies.

A knowledge of the former enables the evaluation of both the correction Δg_{co} required in the calculation of the correction at (b) as well as the correction at (c). In the determination of elevations, it should be realised that only elevations above the geoid are necessary as the surface of the bounding equipotential coincides with the surface of the earth over oceans and no attempt is made to differentiate between different types of matter within this bounding equipotential. For continental areas, however, it is necessary to determine the elevation of the surface of the earth above the surface being mapped. This quantity is the orthometric height.

On a similar basis it is also necessary to know the gravity anomaly field at all points on the earth's surface, including ocean areas, as can be seen from the structure of equations (5.54) and (10.26). In practice, as has already been shown in sections (7.3) and (9), this extent of coverage is not available from land based gravity observations, especially in ocean areas. Under such circumstances any attempt at solution must be preceded by a field extension where values are predicted for the gravity anomaly field in unobserved regions. As such a solution cannot be expected to be a final solution, some attempt must be made to assess the accuracy of the final result obtained.

In assessing the accuracy of extended values , attention must be paid to the fact that the field is continuous and hence, correlation exists between consecutive values. Numerous methods are available for predicting values in a continuous field, the most popular of which are the use of Markov theory and auto-correlation analysis. These two methods are quite adequate but have two limitations. Firstly, the accuracy of any extended value will depend on the relevance of the sample to the locality in which the extension is being performed. Secondly, both methods involve considerable use of computer time if the first limitation is to be successfully overcome. A quicker method is the use of the trigonometrical series described in section (9) where a function is made to fit the ambient field as best as possible.

The use of the error of prediction to assess the accuracy of extension was shown to be of limited value under such circumstances as it was not possible to establish that there was any definite correlation between the error of prediction and the accuracy of interpolation in the case of the use of trigonometrical series. Adoption of any one of these extension techniques will eliminate the inconsistency which arises from the use of zero anomaly to represent unsurveyed areas as this could give rise to serious errors in the final result.

Comment is also necessary about the fact that the extended values in regions of sparse gravity data will be excessively biased towards the values of the available sample. This however, still provides a better value for the extension than the acceptance of the value of zero for all unsurveyed areas. In fact even the direct representation of limited areas by a a single local reading is a more acceptable alternative so long as this region is not too close to the computation point.

In addition to the programs listed under (a), (b) and (c) above, it is necessary to perform the following calculations :-

(i) The extension of the available gravity anomaly field to obtain estimates of the values for unsampled regions.

(ii) The assessment of the accuracy of these extensions.

(iii) The computation of the area means for both free air anomalies and elevations.

(iv) The computation of topographical corrections on a world wide basis.

These calculations have to precede those listed under (a), (b) and (c) above. Each of the sets of programs is dealt with under a separate section so that the link between the programs is readily seen.

12.2 Programs for sorting gravity data.

Gravity data can be obtained in one of two ways.

(a) The free air gravity anomaly can be calculated from the disturbing potential (V_D) by the use of equation (7.54), the latter being determined from an analysis of the orbital perturbations of artificial earth satellites as described in section(7). On expressing the disturbing potential in terms of spherical harmonics as shown in equation (7.59), the free air gravity anomalies can be calculated using equations (5.33), (5.36) and (5.38). As discussed in section (7.8 (iv) b), the use of a combination of zonal harmonics through to the fifth order along with a development of tesseral harmonics through to order (6, 6) can provide an adequate representation of the gravity anomaly field provided

( ) $30^{\circ} \times 30^{\circ}$ square means do not exhibit any correlation with adjacent square values ;

(ii) the tesseral harmonics can be adequately assessed.

(b) The free air anomaly field can also be determined from the results of ground surveys using either systems of pendulums or gravimeters. The computations carried out in the current investigation were in respect of data covering the Australian mainland region and the adjacent continental shelf. The gravity data was available in one of two forms

(i) gravity values based on a variety of datums;

(ii) Bouguer anomaly maps .

The latter were prepared either from close gravity surveys carried out for prospection purposes or from helicopter gravity reconnaisance surveys carried out by the Bureau of Mineral Resources. The maps were all converted to the "may 1965 Isogal" datum, defined in section (8.4) and these maps were read for values of Bouguer anomalies to represent the corners of $0.1^{\circ} \times 0.1^{\circ}$ squares. The actual gravity readings were sorted by the program given in appendix (5) and the results converted to free air anomalies. These anomalies were read into the computer, along with the Bouguer anomaly values and the entire set of data was processed by the program AN_COMP, written in PL/1 and listed in appendix (6).

In this program, the gravity data was in the form of Bouguer anomalies, with varying densities and computed on the "May 1965 Isogal" datum. In addition, the mean elevations of $0.1^{\circ} \times 0.1^{\circ}$ squares were also read in. Data for a $5^{\circ} \times 5^{\circ}$ square was processed at any one time, the program being capable of indefinite cycling. The Bouguer anomalies were converted to free air anomalies prior to storage by the following sequence of calculations.

(1) Establish the elevations of the corners of $0.1^{\circ} \times 0.1^{\circ}$ squares using the formula

$$\bar{h}_{ij} = \frac{1}{4} \sum_{u=i}^{i+1} \sum_{v=j}^{j+1} h_{uv} \quad \ldots\ldots\ldots\ldots(12.1),$$

where $\bar{h}_{ij}$ is the elevation of the location (i, j),

h_{uv} is the elevation representing the square (u, v), which has the location denoted by (u, v) as its south west corner.

(2) Compute the free air anomaly using the conventional relations (Heiskanen and Vening Meinesz, 1958, 153)

$$\Delta g_{F_{ij}}^{(mgal)} = \Delta g_{B_{ij}}^{(mgal)} + 0.1118\, \bar{h}_{ij}^{(met)} \quad \ldots.(12.2),$$

if $\rho = 2.67$ and $\bar{h}_{ij} > 0$, and

$$\Delta g_{F_{ij}}^{(mgal)} = \Delta g_{B_{ij}}^{(mgal)} + 0.4187\,(\rho - 1.03)\,\bar{h}_{ij}^{(met)} \quad \ldots\ldots\ldots.(12.3)$$

where ρ is the density used in the calculations and $\bar{h}_{ij}$ < 0. Δg_B is the Bouguer anomaly.

The free air anomalies so computed were chosen to represent the square corners. A general study of close contour gravity anomaly maps show that the error of representation of a $0.1^\circ \times 0.1^\circ$ square by a single reading is seldom in excess of $\pm$ 3 mgal and is very often much less. The error in each of the $0.1^\circ \times 0.1^\circ$ mean elevations is unlikely to exceed $\pm$ 30 metres. Further, the mean elevation of each $0.1^\circ \times 0.1^\circ$ square is independently assessed and the error in the value of $\bar{h}$ computed can be expected to be less than $\pm$ 20 metres.

The free air anomalies computed using the techniques adopted in the program AN_COMP are likely to have errors of representation seldom in excess of $\pm$ 3 mgal which is as precise as can be expected from any set of data. The following quantities were computed for each $5^\circ \times 5^\circ$ area over the Australian mainland :-

(i) <u>$\frac{1}{2}^\circ \times \frac{1}{2}^\circ$ free air anomaly means</u>

The program loads the following data in a single storage for which ten decimal digits are reserved. If these digits in a single storage are numbered from the right, a single unit contains the following information :-

Digits 1 to 4 = free air anomaly (unsigned) in tenth mgal ;

Digits 5 and 6 = number in sample (maximum value = 25) ;

Digits 7 and 8 = standard deviation of sample in mgal ;

Digit 9 = sign of the gravity anomaly.

The adoption of this procedure saved considerably on computer storage and enabled factors required in the weighting of observed c .tities to be loaded on the computer without any increase in the required storage. The value 0 was stored if there were no readings in the area. All data was punched on card, the values for five consecutive $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ squares a..ng a parallel of latitude appearing on a single card.

(ii) $1^{\circ} \times 1^{\circ}$ free air anomaly m...ns.

The ..alues of $1^{\circ} \times 1^{\circ}$ square means were stored in exactly the same manner as $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ square means. The only difference was that the maximum number of reading... in a $1^{\circ} \times 1^{\circ}$ square, being equal to 100, could not be fitted into the above format and was therefore represented by the number 99. The resulting error in calculating any weight coefficient ...ng equation (9.22) is negligible.

(iii) $5^{\circ} \times 5^{\circ}$ free air anomaly means.

(iv) $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ elevation means.

All $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ elevation means were of equal weight as all $0.1^{\circ} \times 0.1^{\circ}$ square means were available for the evaluation. Thus direct storage was used for each square mean, the data being arranged as described in section (i) above. As the estimated error of a single $0.1^{\circ} \times 0.1^{\circ}$ mean elevation was $\pm$ 30 metres, the estimated error of a

$\frac{1}{2}^{o} \times \frac{1}{2}^{o}$ elevation mean should be of the order of ± 5 metres. This figure would be valid if the contour maps from which the $0.1^{o} \times 0.1^{o}$ means were read were free from error. A more realistic figure for the error of a $\frac{1}{2}^{o} \times \frac{1}{2}^{o}$ elevation mean is the figure ± 10 metres.

(v) $1^{o} \times 1^{o}$ elevation means.

These were prepared in exactly the same manner as the $\frac{1}{2}^{o} \times \frac{1}{2}^{o}$ means. The estimated error of such a mean should be of the order of ± 5 metres.

In certain regions in Central Australia, the available elevation data is extremely sparse and it is likely that the estimates of $0.1^{o} \times 0.1^{o}$ means are correlated and in such regions it is better to use the figure of ± 7 metres for the error of a $1^{o} \times 1^{o}$ mean.

12.3 Gravity data for regions beyond the Australian mainland.

The required form of the gravity data has been set out in section (8.2). Areas exterior to the Australian mainland region fall within group (i) in that section. The paucity of gravity data over ocean areas makes it unsatisfactory to look for gravity data from earth based methods alone. Sections (7.5) and (7.6) show how it is possible to obtain values of gravity anomalies for $5^{o} \times 5^{o}$ areas by solving the equations for the perturbations of the orbital elements of artificial earth satellites for the harmonics up to order (6, 6) in tesserals and 5 in zonals to obtain adequate representation.

If the solution is further strengthened by making it fit the available gravity data on ground, it shoul: be possible to obtain an adeqi ate representation of the distant gravity field. This method of analysis has been fully developed by Kaula (Kaula et al, 1966, Sec II.D). His basic unit was the 300 nautical mile square. It was assumed that no correlation existed beyond 15° of each basic square. This is in keeping with the earlier assumption that $30^{\circ} \times 30^{\circ}$ square means are normally distributed without correlation. Commencing with all available $1^{\circ} \times 1^{\circ}$ square means for gravity anomalies, the mean gravity anomaly of each 300 n. mi. square was computed ·ng only such data as lay within the square. The auto- and cross-covariances of these 300 n. mi. were calculated and linear regression applied to extend the 300 n. mi. area means to 1200 n. mi. area means. The estimates based on the variation of the gravity field with topography were found to give unstable values and the extension of the gravity anomaly field was performed without taking the topography into account.

This ground gravity data was then combined with the low order harmonic coefficients of the gravity anomaly field obtained from the analysis of the orbital perturbations of artificial earth satellites, as described in section (7), in the following manner (Kaula, 1966a, 10).

Let the number of harmonic coefficients to order r available from the latter solution be k, where

$$k = \sum_{i=0}^{r} 2(i+1) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (12.4).$$

Let the gravimetric solution obtained by the method described previously be represented by a two-dimensional set of ℓ mean anomalies where $\ell > k$. Thus for $5^{o} \times 5^{o}$ squares, $\ell = 2592$. From equation (5.35), the form of the k observation equations is

$$(n-1)\left[S_x + dS_x \right] = \frac{1}{4\pi\gamma} \sum_{i=1}^{\ell} Q_{xi} (\Delta g_i + d\,\Delta g_i)\, d\sigma,$$

$$\ldots\ldots\ldots\ldots\ldots (12.5),$$

where S_x is one of the harmonic coefficients of order n obtained from the satellite solution and Q_{xi} is the value of the equivalent spherical harmonic of the same order for the unit of area i represented by a surface area $d\sigma$ on a unit sphere. The above observation equations are solved by imposing the least squares condition that

$$\sum_{i=1}^{\ell} w_{\Delta g_i} (d\,\Delta g_i)^2 + \sum_{j=1}^{k} w_{s_j} (dS_j)^2 = \text{MIN} \ldots (12.6).$$

In equation (12.6), w_{s_j} is the conventional term $1/(\text{standard deviation})^2$. Kaula estimates this quantity as the quartile range of a series of independent solutions from which the final values of the harmonic coefficients were adopted. The evaluation of $w_{\Delta g_i}$ is slightly more complex. It can be evaluated as set out in equation (9.27). Kaula (Kaula et al., 1966, II D-13) uses the form

$$w_{\Delta g_i} = a + b\, n_i \quad \ldots\ldots\ldots\ldots\ldots\ldots (12.7),$$

where n_i is the number in the sample used in the evaluation of Δg_i. Such a solution was obtained by Kaula (1966a) using the following determinations for the evaluations of the coefficients of the zonal and tesseral harmonics :-

(a) The Anderle set (Kaula, 1966a) ;

(b) The Gaposhkin set (ibid) .

(c) The Kaula solution (Kaula, 1966c).

(d) The Guier and Newton set (Guier and Newton, 1965).

Kaula meaned these sets arithmetically and used the mean values in equation (12.5) from which values were computed to represent the means of $5^\circ \times 5^\circ$ squares comprising a world wide set. This set of values, hereafter called the KAULA SET, was used to compute the distant zone effects for the terms involving the free air anomaly in the Stokes and Vening Meinesz integrals.

12.4 The computation of the differential topographical correction (Δg_{co}).

The structure of equation (5.54) and a study of equation (5.59) show that Δg_{co} must be evaluated on a world wide basis. This necessitates the determination of the following sets of values for this correction :-

(i) A set representing $0.1^\circ \times 0.1^\circ$ squares for regions within 1.5° of the computation point.

(ii) A set to represent $\frac{1}{2}^\circ \times \frac{1}{2}^\circ$ area means in the range

$1.5^\circ < \psi < 5^\circ$, where ψ has the usual definition.

(iii) A set to represent $1^\circ \times 1^\circ$ means for regions which lie in the range $5^\circ < \psi < 20^\circ$ of the computation point.

(iv) A set representing $5^\circ \times 5^\circ$ means for all areas not included in the above groups for regions which lie in the range $\psi > 20^\circ$ of the computation point.

Different techniques are used in each case for the calculation of Δg_{co}.

(i) <u>The calculation of values to represent $0.1^\circ \times 0.1^\circ$ squares within 1.5° of the computation point.</u>

This effect is computed using the program titled TOPNE (appendix (7)). From the development in section (6.2), it can be seen that it is necessary only to consider the topography within $1\frac{1}{2}^\circ$ of the point at which the differential topographical effect is to be computed to obtain the required correction. The basic equation is

$$\Delta g_{co} = C\, h_P^{(met)} \sum_i n_i^2 \sum_j \partial\rho_{ij} h_{ij}^{(met)} \cos \phi_{ij} \operatorname{cosec} \tfrac{1}{2}\psi \Big[3 - \operatorname{cosec}^2 \tfrac{1}{2}\psi + o\,\{f, f^2 \operatorname{cosec}^2 \tfrac{1}{2}\psi\} \Big] \quad \ldots\ldots (6.36),$$

where $\partial\rho$ is the density correction factor given by

$$\partial\rho_{ij} = \left[1 - \frac{h_{ij}^{(met)} \quad c_d}{21,000\ \rho} \right] c'_d \quad \ldots\ldots (12.7),$$

where $\rho = 2.77$ gm.cm^{-3} in accordance with equation (1.6). The quantities c_d and c'_d are given by the relations

$$\left.\begin{array}{l} c'_d = 1\ ,\ c_d = 1 \text{ if } h_{ij} < 2500 \text{ metres} \\ c'_d = \dfrac{2.67}{\rho}\ ,\ c_d = 0 \text{ if } h_{ij} > 2500 \text{ metres} \end{array}\right\} \quad \ldots(12.8).$$

In equation (6.36), C is given by

$$C = \frac{k\pi^2 \rho \times 10^7}{8 \times 180^2 R_m} = 1.1047 \times 10^{-13} \quad \ldots((12.9),$$

the [illegible] $-g_{co}$ being given in mgal., for $\rho = 2.77$.

Equation (6.36) breaks down at $\psi = 0$ and hence the four $0.1^{o} \times 0.1^{o}$ squares comprising the inner zone are replaced by a cylinder of height h_m equal to the mean elevation of the four squares omitted, the radius of the cylinder being given by equation (6.63). The topographical correction due to the inner zone (Δg^I_{co}) is given by

$$\Delta g^{I\ (mgal)}_{co} = C_1 h^{(met)}_m c'_d \left[1 - \frac{h^{(met)}_m c_d}{21,000}\right]\left[1 - \frac{h_P}{2r} + \frac{h^3_P}{8r^3} + o\{\frac{h^5}{r^5}\}\right] \quad \ldots\ldots.(6.43),$$

where $C_1 = -4\pi k \rho \times 10^5$ ....(12.10)

and c'_d ; c_d are defined in equation (12.8). C_1 takes the value -2.323×10^{-1} when $\rho = 2.77$. Thus the program TOPNE gives the quantity

$$\Delta g^{(mgal)}_{co} = \Delta g^{I\ (mgal)}_{co} + \Delta g^{E(mgal)}_{co} \quad \ldots..(12.11),$$

where Δg^I_{co} is given by equations (6.43) and (12.10).

Δg^E_{co} is calculated using a modified form of equation (6.36) which takes into account the fact that $\frac{1}{2}\psi$ does not exceed 1^o and also omits the four inner-most tenth degree squares. For very small angles it is in order to replace the trigonometrical functions by limited series as a final accuracy of five significant figures would suffice. This effects a considerable saving in computer time. The simplified form of equation (6.36) used in the calculations is

$$\Delta g^E_{co} = C\, h_P \sum_i n_i^2 \sum_j \partial\, \rho_{ij} h_{ij} \cos \phi_{ij}\, f_t(\psi) \quad \ldots (12.12),$$

where $f_t(\psi)$, for values of $\psi < 1\frac{1}{2}^o$ is given by

$$f_t(\psi) = \frac{8}{{}^3(1 - \frac{1}{8}\psi^2)} \; 3\left[\left\{\tfrac{1}{2}\psi - \frac{\psi^3}{48}\right\}^2 - 1\right]$$

$$= \frac{8}{\psi^3}\left[\frac{5}{8}\psi^2 - 1 + o\{\psi^4\}\right] \quad \ldots\ldots (12.13).$$

In practice, the value of Δg^E_{co} for the Australian mainland region is very small, seldom exceeding 2 to 3 mgal and usually being about 2 orders smaller than Δg^I_{co} which constitutes the major term.

The errors in the computed values of the differential topographical correction ($e_{\Delta g_{co}}$) arise from three sources. The first source of error is dependent on the deviation of the adopted model of the topography from the existent one. At present there is no method available for assessing the magnitude of this error and hence there is no alternative but to assume this source of error to be zero for the <u>adopted model of the topography exterior to the geoid.</u>

The error ($e_{\Delta g_{co}^E}$) in the calculation of Δg_{co}^E is one of the two sources which contribute to the second type of error due to errors in the elevations of the data used. If the set of elevations used has an estimated error e_h in the tenth degree square elevations,

$$e_{\Delta g_{co}^E} = \pm C \left\{ \sum_i \left\{ n_i^2 \right\}^2 \sum_j \left[\partial \rho_{ij} \cos \phi_{ij} \; f_t(\psi)_{ij} \; n \right]^2 h_P^2 + \left[\frac{\Delta g_{co}^E}{h_P} \, e_{h_P} \right]^2 \right\}^{\frac{1}{2}} \quad \ldots.(12.14).$$

Both terms are very small and the error due to these sources can be expected to be less than 1 mgal in all but a few exceptional cases. This estimate is much smaller than the error of representation.

The second contributor to second source of error is due to the computations for the inner zone ($e_{\Delta g_{co}^I}$). This error in the calculation of the major contributor to the effect is highly dependent on the accuracy of the mean elevation used in the calculation. As this quantity is unlikely to have an error in excess of $\pm$ 30 metres, the differential topographical correction is expected to have an error which approaches $\pm$ 2 mgal. In mountainous country, however, the error may be as large as $\pm$ 7 mgal. If greater precision is desired, it is necessary to carry out a closer survey of the topography in the near neighbourhood of the point investigated. If such a close contour survey were carried out within a fifteen mile radius of the computation point, it should be possible to obtain the mean elevation of the inner zone with errors smaller than $\pm$ 5 metres.

The third source of error is the result of considering only regions within the range of $\psi = 1.5^{\circ}$. As discussed in section (6.2), this effect is unlikely to exceed 0.5 mgal.

(ii) <u>The calculation of the mean topographical correction over $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ squares.</u>

Two methods are available for establishing $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ square means :-

(a) Adoption of a procedure similar to that used in the computation of $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ free air anomaly means in section (12.2). This would require the computation of the differential topographical correction at sufficient points on a regular grid system prior to the computation of the means. With point computations costing 5 c per station, the evaluation of the necessary data for the computation of a single $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ square would cost 1.25. As there are approximately 14,000 half degree squares over the Australian mainland, such computer time costs in adopting such a method would amount to approximately \$ 18,000.

(b) The second alternative is to use the fact that the regions outside the inner zone contribute about 1 per cent of the total magnitude of this correction. The structure of equation (6.43) shows that the adoption of a cylinder of radius of order 150 km as a model for the topography exterior to the geoid instead of the two-part model adopted in the previous case would give an adequate alternative. If this cylinder had the same height as the elevation of the station, the error in the correction so computed would be given by

$$[e_{\Delta g_{co}}]^{\frac{1}{2}^\circ} = \Delta g^{E}_{co} + 2\pi \, k \, \rho \, h^2_P \frac{1}{r} \, \ldots\ldots (12.15).$$

These two quantities are of opposite sign and hence the error in the computed quantity is not necessarily systematic. It is therefore possible to represent the mean topographical correction for a $\frac{1}{2}^\circ \times \frac{1}{2}^\circ$ square by a relation of the form

$$\overline{\Delta g_{co}} = -4\pi \, k \, \rho \, h_m \left[1 - \frac{h_m}{2r} + \frac{h_m^3}{8r^3} \right] \ldots\ldots (12.16),$$

where h_m is the mean elevation of the $\frac{1}{2}^\circ \times \frac{1}{2}^\circ$ square and r the radius of the cylinder which has been chosen in this set of calculations to coincide with the outer limit of the Hayford zone O of radius 166.7 km. The comparison of mean corrections using this technique with true means is given for ten randomly chosen half degree squares in table (12.2). The sample of hald degree squares was chosen at random.

The adoption of this technique enables the differential topographical correction means for half degree squares over the entire Australian mainland region to be computed for only $ 5.00. In a similar manner it was possible to compute the mean topographical corrections for both $1^\circ \times 1^\circ$ and $5^\circ \times 5^\circ$ squares. These computations were performed using the program TOPMNS given in appendix (8).

The accuracy of the means so calculated appear to have errors less than $\pm$ 5 mgal, the errors arising mainly in the assumption used in calculating the mean topographical effect. In the case of the computation of the mean topographical

$\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ square		Mean	Topographical corrn.	
Latitude ° '	Longitude ° '	Elevation metres	True mgal	From mean mgal
- 29 45	140 15	40	- 12	- 9
- 28 15	142 15	159	- 33	- 37
- 27 15	140 15	46	- 11	- 10
- 28 15	135 45	92	- 21	- 21
- 26 15	147 45	445	- 105	- 109
- 29 15	[illegible] 15	164	- 39	- 38
- 27 15	142 45	163	- 33	- 38
- 27 45	145 15	252	- 53	- 49
- 26 15	139 15	39	- 9	- 9
- 27 15	135 15	137	- 32	- 32

TABLE (12.1)

Comparison of true topographical correction means for $\frac{1}{2}^{\circ} \times \frac{1}{2}^{\circ}$ squares with those computed from the mean elevation.

corrections for $5^{\circ} \times 5^{\circ}$ squares on a world wide basis, the mean elevations used were those comprising the set known as the UCLA set (Kaula et al, 1966, II.A.-17). This set was assembled from the set of $1^{\circ} \times 1^{\circ}$ elevation means provided by W.H.K.Lee of the University of California and titled the UCLA 1S set. The comparison of the UCLA 1S set with the set of $1^{\circ} \times 1^{\circ}$ square means prepared using the techniques described in section (12.2 v) gave a comparison error of $\pm$ 83 metres, with no significant correlation. Thus the estimate of accuracy given by Lee (Kaula et al, 1966, II.A. 30)as $\pm$ 230 metres for this set is a conservative one.

On this basis, the accuracy of the $5^{o} \times 5^{o}$ elevation means in the UCLA set is estimated at $\pm$ 50 metres. The consequent error in the differential topographical correction computed from the mean elevation for $5^{o} \times 5^{o}$ squares is estimated as $\pm$ 10 mgal. It is expected that the actual values have errors smaller than this figure. Thus the use of the approximate relation for the computation of the square means provides a practical alternative for computing the required quantities with adequate precision to effect the relevant calculations.

12.5 The computation of the free air geoid.

The free air geoid is computed using equation (1.3) and the second term in the equation (10.26) with the anomaly Δg_o being replaced by Δg_F. The computation must be effected by quadratures over the entire surface of the earth. The discussion in section (8.1) shows that the computation has to be carried out in four stages, each of which is embodied in a different program capable of adaptation for use with both free air anomalies and differential topographical corrections.

(i) The computations for regions in the range $\psi > 20^{o}$ from the computation point.

The anomalies to be used in this region are $5^{o} \times 5^{o}$ means, which in the case of free air anomalies is provided by the Kaula set (see section (12.3)). This set was compared

Sample Size n	UNSW set - Kaula Set		Number in set
	Mean mgal.	Std. Dev. ± mgal.	
$n > 1000$	- 1.2	18	14
$300 < n < 1000$	- 1.6	14	16

TABLE (12.2)

Comparison of the Kaula set of free air anomaly means with the UNSW set

with the set of $5^{\circ} \times 5^{\circ}$ free air anomaly means prepared on the lines set out in section (12.2) nd the comparisons are set out in table (12.2). In attempting to assess Kaula's set from these comparisons it should be borne in mind that a considerable amount of land based data would have been included from the Australian region in evaluating Kaula's means. For the 30 comparisons almost entirely on land where the UNSW set could be considered to provide an adequate evaluation of the five degree square mean, the mean difference was - 1.4 mgal, the mean elevations of the regions considered being less than 1000 metres. In view of the magnitude of the standard deviation of the comparisons, this value could be considered as a negligible quantity. Comparison of the Kaula set with some poorly represented samples at sea gave a preponderance of positive comparisons. This would seem to indicate that the Kaula set underestimated the free air anomaly over elevated continental regions and overestimate it over ocean regions. A study of data on the correlation of gravity anomalies with elevation (Kaula, 1959, Table 2) indicates a definite correlation between free air anomalies and elevations. It was therefore decided to assign the value of the estimated error of a single value in the Kaula set according to the relation

$$e_{\Delta g} = \pm (a^2 + b^2 h^2)^{\frac{1}{2}} + ch \quad \ldots\ldots\ldots\ldots\ldots (12.14),$$

where h is the elevation of the station, negative if it is a depth and the quantities a, b and c are constants. In reality, the values of a, b and c should be different for continental and ocean areas but in this series of calculations which is essentially experimental in nature this was not done in view of the lack of sufficient information. It is likely that the value of a is $\pm$ 14 mgal and the magnitudes of b and c are sufficiently small to make Kaula's data set at least as accurate as a world wide representation of each five degree square by a single reading.

In addition to equation (12.14), the computation of which requires the loading of elevations along with free air anomalies, the basic equations programmed are derived from equations (5.54) and (10.26).

$$N_{F_5} = C_{N_5} \sum_i \cos \phi_i \sum_j \{ f(\psi) \}_{ij} \Delta g_{F_{ij}} \quad \ldots\ldots (12.15),$$

where $f(\psi)$ is given by equation (5.53). In equation (12.15), the suffix i denotes summation with respect to latitude and the suffix j with respect to longitude. The constant C_{N_5} is given by

$$C_{N_5} = \frac{R_m \pi \times 2.5 \times 10^{-2}}{4\gamma_m \times 180^2} ,$$

where R_m is in cm, γ_m in gal, the free air anomaly in mgal and the answer in cm. Numerically

$$C_{N_5} = 3.9408 \times 10^{-2}.$$

The deflections of the vertical are given by

$$\xi_{F_{5_i}} = C_{\xi_5} \sum_j \cos \phi_j \sum_k \left[\frac{\partial}{\partial \psi} f(\psi) \right]_{jk} \cos \alpha_{ijk} \; \Delta g_{F_{jk}}^{(mgal)}$$

$$i=1,2 \; \ldots\ldots (12.16),$$

where

$$C_{\xi_5} = \frac{\pi \times 206265 \times 2.5 \times 10^{-2}}{4\gamma_m \times 180^2}$$

$$= 1.276 \times 10^{-4} \; \ldots\ldots\ldots\ldots\ldots (12.17),$$

$$\alpha_1 = A \text{ and } \qquad \alpha_2 = \tfrac{1}{2}\pi - A \; \ldots\ldots (12.18).$$

The program also excludes those five degree squares so situated that the ψ value is less than 20°. To provide for uniformity of computation and to keep track of the areas excluded in each calculation, the following system was adopted instead when excluding regions from calculations. The area to be excluded was chosen to be symmetrical about the south west corner of the $5^{\circ} \times 5^{\circ}$ square in which the computation point was located, the five degree square being defined as a unit of surface area bounded by meridians and parallels the values of longitude and latitude which, on division by 5, yield integral answers without truncation. $\frac{\partial}{\partial \psi} f(\psi)$ is given by equation (10.30). ψ is calculated from the following equation obtained by a consideration of fig (5.1).

$$\psi_{ij} = \cos^{-1}(\sin \phi_o \sin \phi_i + \cos \phi_o \cos \phi_i \cos \Delta\lambda_j) \quad ..(12.19),$$

where the suffix $_o$ refers to the computation point.

As ψ is never zero under the conditions of the computation, it is adequate to compute A_{ij} by the relations

$$\cos A_{ij} = \frac{\sin \phi_i - \sin \phi_o \cos \psi_{ij}}{\cos \phi_o \sin \psi_{ij}} \quad .... (12.20)$$

if the computation point is not at the pole and

$$\sin A_{ij} = \sin \Delta\lambda_j \operatorname{cosec} \psi_{ij} \cos \phi_i \quad ....(12.21).$$

The error in the computed value due to errors in the adopted values of the free air anomalies is estimated using the relations

$$e_{N_{F_5}} = \pm C_{N_5} \left[\sum_j \cos^2 \phi_j \sum_k \left\{ [f(\psi)]_{jk} \, e_{\Delta g_{F_{jk}}} \right\}^2 \right]^{\frac{1}{2}} \quad ....(12.22)$$

and

$$e_{\xi_{F_{5_i}}} = \pm C_{\xi_5} \left[\sum_j \cos^2 \phi_j \sum_k \left\{ \left[\frac{\partial}{\partial \psi} f(\psi) \right]_{jk} \cos \alpha_{ijk} \, e_{\Delta g_{F_{jk}}} \right\}^2 \right]^{\frac{1}{2}}, \quad i=1,2 \;..(12.23).$$

In equations (12.22) and (12.23), $e_{\Delta g_{F_{jk}}}$ is the estimate of the error in $\Delta g_{F_{jk}}$ given, in the case of five degree square means by equation (12.14) . The cost of computation of this effect per point is \$ 0.13 using the program STOKUT given in appendix (9).

(ii) Computations for regions situated in the range $5^{\circ} < \psi < 20^{\circ}$.

The sequence of computation for this region is set out in the program titled STOKMD given in appendix (10) . The basic equations are essentially the same as those set set out in section (12.5(i)) with minor modifications in the nature of the areas excluded in the computations, the magnitude of the constant terms and the means used to evaluate the precision of the values of the free air means . The constants differ due to the change in the square size used, as discussed in section (8.1). The square size was $1^{\circ} \times 1^{\circ}$ and the amended values of the constants are

$$C_{N_1} = 1.576 \times 10^{-2} \quad \ldots\ldots\ldots\ldots\ldots\ldots(12.24)$$

and

$$C_{\xi_1} = 5.103 \times 10^{-6} \quad \ldots\ldots\ldots\ldots\ldots(12.25).$$

Computations in this program have to mesh with those covering areas outside its scope. These lie both exterior to and within the area covered in the present series of calculations. Continuing the convention adopted in effecting the exclusions in STOKUT, computations in STOKMD commence at the south west corner of the $45^{\circ} \times 45^{\circ}$ area excluded in the relevant calculation made using STOKUT. This

... is symmetrical about the $5^o \times 5^o$ area, defined in the previous sub-section, in which the computation point lies. Within this $45^o \times 45^o$ area, a $15^o \times 15^o$ area, also symmetrical about the $5^o \times 5^o$ square in which the computation point lies, is excluded. Thus, in incrementing the index, it is necessary to check two conditions for every calculation.

The input data must provide a gravity anomaly value for every $1^o \times 1^o$ square in the computation zone. This requires that all available data be extended on the lines set out in section (9) prior to loading. These field extensions are performed by the all-purpose program WLS, set out in appendix (11). The extensions so obtained were normalised manually and the data loaded using the technique described in section (12.2 (1)) with two variations in the case of values obtained by extension. Columns 5, 6, 9 and 10 contained data as follows for the different cases of actual readings, interpolations, interpolation/extrapolations, and extrapolations, all terms having the same definition as given in section (9.7).

Class	Values in column no s		
	5-6	9	10
Actual Readings	No. in sample	0 (or sign)	0
Interpolations	I_i	1	sign
Interpolation/ extrapolations	I_i	2	sign
Extrapolations	I_e	3	sign

TABLE (12.3)
Loading of data on $1^o \times 1^o$ anomaly data cards

The regeneration of the relevant data is effected by using certain properties of fixed point arithmetic on the computer and utilising the nature of the truncation which occurs. The values of the standard deviation (σ), number in sample (n) and the interpolation interval ($I_{i(e)}$) are used in assessing an estimate of the error of the value used to represent the mean square anomaly.

The estimates of the errors are used in calculating $e_{\Delta g_{F_1}}$ using the relation

$$e^2_{\Delta g_{F_1}} = \frac{1}{n}\left[\sigma^2 + \left(\frac{N-n}{N-1}\right)^2 E_s^2\right] \quad .....(9.23),$$

where N = maximum possible number comprising sample. This relation breaks down when n= 0, when the case which occurs will be one of either interpolation, interpolation/extrapolation or extrapolation. The program is designed to access values of the error of extension (e_{ext}) defined in section (9.7) which is stored as a (3, 6) array. In such cases $e_{\Delta g_{F_1}}$ is given by the equation

$$e^2_{\Delta g_{F_1}} = E_s^2 + e^2_{ext}(t, I_{i(e)}) \quad .... (12.26),$$

where E_s is the error of representation of a $1^\circ \times 1^\circ$ square by a single reading, estimated in table (9.1) as being $\pm$ 13.5 mgal for Australia. e_{ext} is obtained in a manner similar to that used in compiling table (9.3).

When n = 1, equation (9.23) reduces to

$$e_{\Delta g_{F_1}} = \pm E_s \quad \ldots\ldots\ldots\ldots\ldots\ldots (12.27).$$

The values of $e_{\Delta g_{F_1}}$ obtained by the use of equations (9.23), (12.26) and (12.27[1]) are then used to estimate the error in the values of N_1 and ξ_{1_i} (i=1, 2) using equations similar to (12.22) and (12.23) as modified by the use of equations (12.24) and (12.25).

It should be noted that the evaluation of estimates of error by the above method, while being adequate in the case of squares which have observations in them, will have a tendency to underestimate the error in the computed value due to a tendency for correlation to occur in the extended values. This is difficult to assess at present but it is hoped to study the nature of this effect when fresh data becomes available in 1968. For the present series of computations it has been assumed that the error of extended values has a systematic component $e_{\Delta g_s}$ in addition to the error which behaves as described above. If this is called $e_{\Delta g_a}$, the total error in N and ξ_i(i=1, 2) is estimated according to the relations

$$e_N^2 = C_N^2 \sum_j \cos^2 \phi_j \sum_k \left[f(\psi)\right]^2_{jk} \left[e_{\Delta g_{a_{jk}}}\right]^2 + \left[C_N \sum_j \cos \phi_j \sum_k \left[f(\psi)\right]_{jk} e_{\Delta g_{s_{jk}}}\right]^2 . \quad \ldots\ldots\ldots\ldots (12.28)$$

where $e_{\Delta g_{s_{jk}}}$ is given by

$$e_{\Delta g_s} = e_{\Delta g_s}(t, I_{i(e)})$$

t being the value given in column 9 as tabulated in table (12.3). This definition of t also holds for equation (12.26). Similar expressions hold for e_{ξ_i} (i=1, 2).

(iii) <u>Computations in the region lying in the range $1.5^\circ < \psi < 5^\circ$</u>

The anomalies used in these computations are $\frac{1}{2}^\circ \times \frac{1}{2}^\circ$ area means. The program is titled STOKNE and is set out in appendix (12). The computational techniques used are exactly the same as in STOKMD with the exception of the nature of the exclusions. In keeping with the pattern of exclusions established in the programs dealt with previously, all points of commencement and exclusion are referred to the south west corner of the $5^\circ \times 5^\circ$ and $1^\circ \times 1^\circ$ squares in which the point lies, the basic area considered being a $15^\circ \times 15^\circ$ area symmetrical about the $5^\circ \times 5^\circ$ area in which the computational point lies.

The area excluded is a $3^\circ \times 3^\circ$ block symmetrical about the $1^\circ \times 1^\circ$ square in which the computation point lies. If the latter has geographical coordinates which are integral degrees, the exclusion in such a case would commence at the next degree south (west) of the computation point, the area excluded still being a three degree square block. In this manner , the area excluded and hence computed by more detailed field representations is in excess of the minimum requirement set down in section (8.2). The error in the quantities computed is estimated as before.

(iv) <u>The computations for regions lying within the range $\psi < 1.5^{\circ}$.</u>

Computations in this inner zone are carried out in at least two stages. The first stage involves the calculation of the effect for all regions except theinnermost regions. The anomalies used in the calculation represent $0.1^{\circ} \times 0.1^{\circ}$ squares and the program, titled STOKIN and set out in appendix (13), considers the $3^{\circ} \times 3^{\circ}$ area excluded from the three previous programs, which is s. .nmetrical about the $1^{\circ} \times 1^{\circ}$ square in which the computation point lies. The program excludes four $0.1^{\circ} \times 0.1^{\circ}$ squares situated around the computation point. As the values of ψ in the range of this program are limited, a considerable saving of computer time is effected if ψ were computed as

$$\psi = \left[\{\Delta\phi\}^2 + \left[\cos\phi\,\Delta\lambda\right]^2\right]^{\frac{1}{2}} \quad \ldots\ldots(12.29)$$

$$\sin\psi = \psi + o\{10^{-6}\} \quad \ldots\ldots\ldots\ldots(12.30)$$

$$\cos\psi = 1 - \tfrac{1}{2}\psi^2 + o\{10^{-8}\} \quad \ldots\ldots\ldots\ldots(12.31).$$

The estimates of the error in Δg_F are given by equations (12.26) and (12.27) with $E_s = \pm 3$ mgal and the estimates of error in the final results by equations similar to (12.28), with the constants given by

$$C_{N_{0.1}} = 1.576 \times 10^{-4} \quad \ldots\ldots\ldots\ldots(12.\ 32)$$

and

$$C_{\xi_{0.1}} = 5.103 \times 10^{-8} \quad \ldots\ldots\ldots\ldots(12.33).$$

The second stage of the calculation covers the four $0.1^{\circ} \times 0.1^{\circ}$ squares excluded in the previous program, called STOKIN, can be computed using the formulae due to Sollins (1947, 282) for the deflections of the vertical. A complete set of formulae for the separation vector can be derived, using Sollins' technique as follows. If the innermost zone is a circle of radius r_o, where r_o could be given by equation (6.63), the value of ψ in this region (r_o < 15 km) will be small and the following replacements will be valid as $\psi \approx 2\mathrm{x}10^{-3}$.

$$\sin\psi = \psi + o\{10^{-9}\} \quad \ldots\ldots\ldots\ldots\ldots (12.34)$$

$$\cos\psi = 1 + o\{10^{-6}\} \quad \ldots\ldots\ldots\ldots\ldots (12.35).$$

Similar expressions will also hold for $\frac{1}{2}\psi$. Under these limiting conditions consider a unit of surface area $d\sigma$ at a distance s from the computation point and at an azimuth α.

is given by

$$\psi = \frac{s}{R_m} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (12.36).$$

In the Stokesian term for N, given in equation (10.26), the expression can be written as (see fig (12.1))

$$N_i = \frac{R_m}{4\pi\gamma_m} \int_0^{\psi = r_o/R_m} \int_0^{2\pi} \Delta g_o f(\psi) \sin\psi \, d\psi \, d\alpha \ldots$$

$$\ldots\ldots\ldots (12.37),$$

where the use of equation (5.54) gives

$$f(\psi)\sin\psi = 2\cos\tfrac{1}{2}\psi + \sin\psi - 6\sin\psi\sin\tfrac{1}{2}\psi -$$

$$- 3\sin\psi\cos\psi\ln\{\sin\tfrac{1}{2}\psi(1 + \sin\tfrac{1}{2}\psi)\} \quad \ldots (12.38).$$

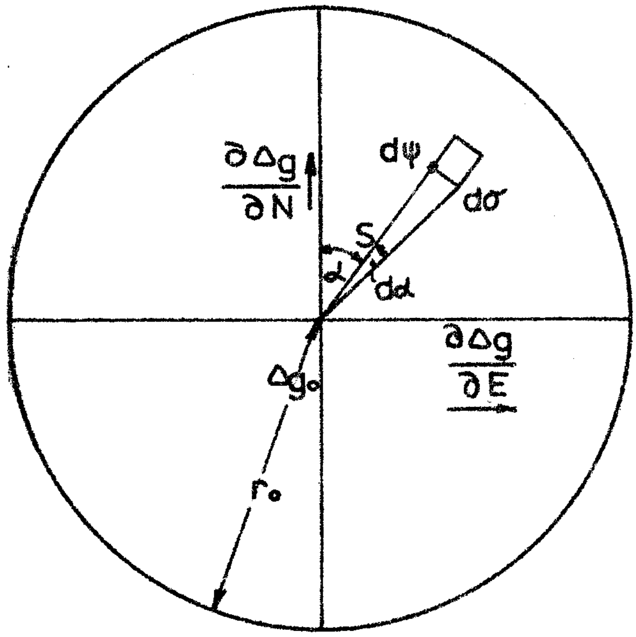


FIG. 12.1

For small values of ψ equation (12.38) reduces to

$$\lim_{\psi \to 0} \quad f(\psi)\sin\psi \;=\; 2 + \frac{s}{R_m} \qquad \ldots\ldots\ldots\ldots\ldots\ldots(12.39).$$

If the gravity anomaly gradients are linear within the circle of radius r_o, Δg_o can be expressed as a series of the form

$$\Delta g_o = \Delta g_{o_P} + s\cos\alpha\left[\frac{\partial\,\Delta g_o}{\partial N}\right] + s\sin\alpha\left[\frac{\partial\Delta\, g_o}{\partial E}\right] \qquad \ldots\ldots\ldots(12.40),$$

where Δg_{o_P} is the anomaly at the computation point P and the N and E axes are orientated north and east respectively. The substitution of equations (12.39) and (12.40) in equation (12.37) gives

$$N_i = \frac{1}{4\pi\gamma_m}\int_0^{r_o}\int_0^{2\pi}\left[\Delta g_{o_P} + s\sin\alpha\left[\frac{\partial\Delta g_o}{\partial E}\right] + s\cos\alpha\left[\frac{\partial\Delta g_o}{\partial N}\right]\right]\left(2+\frac{s}{R_m}\right)ds\,d\alpha \qquad \ldots\ldots\ldots(12.41).$$

As the integration of terms containing $\sin\alpha$ and $\cos\alpha$ between the limits 0 and 2π is zero, the above integral reduces to

$$N_i = \frac{\Delta g_o}{\gamma_m}\, r_o\left[1 + \frac{r_o}{4R_m}\right]$$

For computation purposes,

$$N_i^{(cm)} = \frac{\Delta g_{o_P}^{(mgal)}}{\gamma_m^{(gal)}} \times 10^2\, r_o^{(km)}\left[1 + \frac{r_o}{4R_m}\right] \ldots(12.42).$$

In a similar manner expressions can be derived for the deflections of the vertical, given by equation (10.33) and can, in the case of the inner zone be expressed as

$$\xi_i^{(sec)} = \frac{206265}{4\pi\gamma_m} \int_0^{\psi=\frac{r_o}{R_m}} \int_0^{2\pi} \Delta g_o \frac{\partial}{\partial\psi} f(\psi)$$

$$\qquad \qquad \qquad \qquad \qquad \qquad \qquad \qquad 3\alpha_i \sin\psi \, d\psi \, d\alpha \ , \quad i=1,2..(12.43),$$

where the indexes are defined in equations (10.18), (10.19) and (10.22). From equation (10.30),

$$\frac{\partial}{\partial\psi} f(\psi) \sin\psi = -\cos^2\tfrac{1}{2}\psi \operatorname{cosec}\tfrac{1}{2}\psi - 3\sin\psi\cos\tfrac{1}{2}\psi +$$

$$+ 5\sin^2\psi - 3\cos\psi\cos^2\tfrac{1}{2}\psi\frac{1+2\sin\tfrac{1}{2}\psi}{1+\sin\tfrac{1}{2}\psi} +$$

$$+ 3\sin^2\psi \ln \sin\tfrac{1}{2}\psi(1+\sin\tfrac{1}{2}\psi) \quad ..(12.44).$$

$$\lim_{\psi\to 0} = \frac{\partial}{\partial\psi} f(\psi) \sin\psi \, d\psi = -\frac{1}{R_m}\left[\frac{2R_m}{s} + 3 + o(10^{-3})\right]ds$$

$$............(12.45).$$

Evaluation of each of the components of the deflection of the vertical, in turn, together with the use of equation (12.40) gives

$$\xi^{(sec)} = -\frac{206265}{4\pi\gamma_m R_m} \int_0^{r_o} \int_0^{2\pi} \left[\Delta g_{o_P} \cos\alpha + \tfrac{1}{2}s(\cos 2\alpha + 1)\left[\frac{\partial \Delta g_o}{\partial N}\right]\right.$$

$$\left. + \tfrac{1}{2}s\sin 2\alpha\left[\frac{\partial \Delta g_o}{\partial E}\right]\right]\left[3 + \frac{2R_m}{s}\right]ds\, d\alpha$$

$$\xi^{(sec)} = - \frac{206265}{4\pi\gamma_m R_m} \int_0^{r_o} s \left[\frac{\partial \Delta g_o}{\partial N} \right] \left(3 + \frac{2R_m}{s}\right) ds \quad \ldots(12.46),$$

as $\int_0^{2\pi} \sin n\alpha \, d\alpha = \int_0^{2\pi} \cos n\alpha \, d\alpha = 0$.

Further integration and simplification gives

$$\xi^{(sec)} = - \frac{206265}{2\gamma_m} r_o \left[1 + \frac{3r_o}{4R_m} \right] \left[\frac{\partial \Delta g_o}{\partial N} \right] \quad \ldots(12.47).$$

On a similar basis,

$$\eta^{(sec)} = - \frac{206265}{2\gamma_m} r_o \left[1 + \frac{3r_o}{4R_m} \right] \left[\frac{\partial \Delta g_o}{\partial E} \right] \quad \ldots(12.48).$$

Equations (12.47) and (12.48) are Sollins' expressions and on combination with equation (12.42), give a definition for the contribution for the separation vector in the immediate vicinity of the computation point.

These three equations are only valid under conditions which permit the expression of the gravity anomaly in the inner zone in terms of equation (12.40) and hence imposes a restriction on the maximum size for the latter. Except in disturbed areas the gravity anomaly gradient seldom exceeds 3 mgal per 10 km. Thus the magnitude of r_o is critical in the solution only if the gravity anomaly gradient is non-linear over the distance r_o or is abnormally large. The contribution of the above average gradient to the deflection of the vertical is of the order of 0".3. For more disturbed regions or regions where the gravity anomaly gradient is not linear over the four innermost tenth degree squares excluded from the previous programs, the following procedure should be adopted.

(i) The innermost zone is replaced by a more accurate subdivision. The most convenient system already available is that of Rice's circular rings (Rice, 1952, 288). These are circular templates having a uniform angular aperture of 10°, constructed so that the radial deflection effect for unit anomaly is $0''.001$.

(ii) Calculations in the immediate vicinity of the computation point which is a circle of radius r_o are effected using equation (12.42) and Sollins' expressions, set out in equations (12.47) and (12.48). In certain calculations carried out by the U.S.Coast and Geodetic Survey, the magnitude of r_o was determined by

(a) the extent of gravity data available ;

(b) the irregularities of the anomaly field.

Unless the field was extremely disturbed the usual value adopted for r_o (Rice, 1952, 290) was 4.32 km. In cases where adequate field coverage was available and the field showed irregular variations, the value of r_o was reduced to 0.279 km.

The obvious criterion for assessing the minimum value admissible for r_o is the accuracy of the assumption embodied in equation (12.40). A more complete expression for Δg_o is given by

$$\Delta g_o = \Delta g_{o_P} + s \cos\alpha \left[\frac{\partial \Delta g}{\partial N}\right] + s \sin\alpha \left[\frac{\partial \Delta g_o}{\partial E}\right] + \tfrac{1}{2} s^2 \cos^2\alpha \ldots$$

$$\ldots \left[\frac{\partial^2 \Delta g_o}{\partial N^2}\right] + \tfrac{1}{2} s^2 \sin^2\alpha \left[\frac{\partial^2 \Delta g_o}{\partial E^2}\right] \quad \ldots .(12.49).$$

In the case of the expression for N_i, the initial integration will give rise to the following term which is independent of sin nα/cos nα

$$\Delta g_{o_P} + \frac{1}{4} s^2 \left(\left[\frac{\partial^2 \Delta g_o}{\partial N^2} \right] + \left[\frac{\partial^2 \Delta g_o}{\partial E^2} \right] \right) \quad \text{instead of } \Delta g_{o_P}.$$

This revised term becomes

$$\Delta g_{o_P} + \frac{1}{12} r_o^2 \left[\frac{\partial^2 \Delta g_o}{\partial N^2} + \frac{\partial^2 \Delta g_o}{\partial E^2} \right] + o\left\{ \frac{\partial^2 \Delta g_o}{\partial N(E)^2} \frac{r_o}{R_m} \right\}$$

in the final expression. Thus equation (12.42) can be replaced by

$$N_i^{(cm)} = \frac{\Delta g_{o_P}^{(mgal)}}{\gamma_m(gal)} \times 10^2 \, r_o^{(km)} \left\{ 1 + \frac{r_o^2}{12 \, g_{o_P}} \left(\left[\frac{\partial^2 \Delta g_o}{\partial N^2} \right] + \left[\frac{\partial^2 \Delta g_o}{\partial E^2} \right] \right) + \frac{r_o}{4R_m} \right\} \quad \ldots\ldots\ldots\ldots (12.50),$$

where the rate of change of the gravity anomaly gradient is expressed in mgal km^{-2}, r_o and R_m being expressed in km on the right hand side in the last two terms.

The derivatives of the gravity anomaly gradient have no effect on the deflections of the vertical as the terms involved in the initial integration are either $\sin\alpha \cos^2\alpha$ or $\sin\alpha \cos^2\alpha$, both of which integrate to zero between the limits 0 and 2π. Thus only odd order differential coefficients affect the deflections of the vertical. The basic term is

$$s^{2i-1} \cos(\sin)^{2i-1} \alpha \sin(\cos)\alpha.$$

When $i = 2$, the effective term is $f(\alpha)\, s^3 \left[\frac{\partial^3 \Delta g_o}{\partial N(E)^3}\right]$. The numerical coefficient of this term is $\frac{1}{48}$. Thus equations (12.47) and (12.48) are sufficient for most cases even when r_o is as large as 10 km if the field variations are orderly in a mathematical sense. In the case of slightly more variable fields it would appear prudent to use

$$\xi^{(sec)} = \frac{206265}{2\pi\gamma_m} r_o \left[1 + \frac{3r_o}{4R_m} \left[\frac{\partial \Delta g_o}{\partial N}\right] + \frac{r_o^3}{48}\left[\frac{\partial^3 \Delta g_o}{\partial N^3}\right]\right] \qquad \text{......... (12.51)},$$

with a similar expression for η. The possibility of evaluating the second differential coefficient of the gravity anomaly gradient with any degree of precision is debatable and, in the case of disturbed fields, there seems to be no alternative to adopting the procedure described previously for evaluating the effects in such cases.

Rice (1952, 290) suggests sampling the gradient by the establishment of 8 gravity stations in the vicinity of the computation point, as shown in fig (12.2). This would give three estimates in each case of the north-south and east-west gravity anomaly gradients, one of each of which pass through the computation point. The weighted mean was then used to determine the value of the gradients to be used in computations. Rice suggests unit weight for those gradients passing through the computation point and half this value for the outer ones.

In the case of the determination of N_i in the areas with disturbed fields, the same procedure can be applied using a set of templates constructed by Uotila (1960, 42).

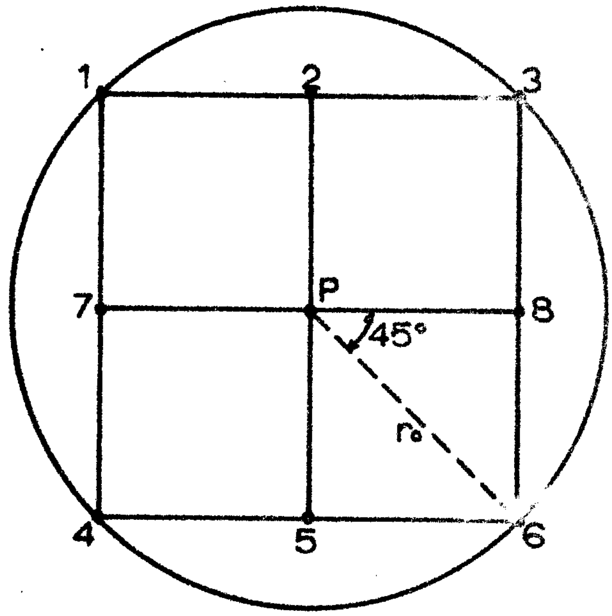


FIG. 12.2

SAMPLING ANOMALY GRADIENT IN INNER ZONE.

12.6 The computation of the indirect effect .

The indirect effect to the Stokesian effect for the geoid - spheroid system is given in equation (5.61) when free air anomalies are used. This has two constituent parts, one of which is dependent on the potential of matter exterior to the geoid at the computation point . The second is a Stokesian effect, computed using exactly the same formulae as before. The only difference is the use of differential topographical corrections (Δg_{co}) instead of free air anomalies (Δg_F) in the formulae listed in section (12.5). These topographical corrections are all negative, unlike the free air anomalies, which should, by virtue of the nature of the international gravity formula, be distributed about a zero mean. This particular proposition is examined in greater detail in section (13).

Consider the computation of the first of the terms composing the indirect effect . Using the suffix E to denote such terms, the basic equations used for the calculation of the relevant terms are given in equation (6.66) as modified by equations (10.26) and (10.33) as

$$N_{E_P}^{(cm)} = K_{E_N} \sum_i n_i^2 \sum_j T_{ij} \text{cosec} \tfrac{1}{2} \psi_{ij} \quad \ldots\ldots\ldots (12.52),$$

where

$$K_{E_N} = \frac{k R_m^2 \pi \rho \times 10^2}{\gamma_m 180^2} \quad \ldots\ldots\ldots\ldots\ldots (12.53)$$

and

$$T_{ij} = \partial\rho_{ij} h_{ij}^{(met)} \cos \phi_{ij} \left[1 + \tfrac{1}{2}\{ 3c_{ij} - c_P + \frac{3h_{ij}}{2R_m} \} \right]$$

........(12.54),

$\partial\rho_{ij}$ being given by equation (12.4), the quantities , c_d and c'_d having the same definition as in section (12.4(i)). K_{E_N} has the value 3.661 x 10^{-3} and is dimensionless. In this series of computations

(a) depth has no meaning as the quantity h is the elevation of topography exterior to the geoid which becomes zero over ocean areas ;

(b)the calculation thus covers continental areas only ;

(c) the effect is always positive. The contribution to the deflections of the vertical are given by

$$\xi_{E_i} = K_{E_\xi} \sum_j n_j^2 \sum_k T_{jk} \cos \tfrac{1}{2} \psi_{jk} \operatorname{cosec}^2 \tfrac{1}{2} \psi_{jk} \cos \alpha_{ijk} \quad i=1,2$$

.....(12.55),

where

$$K_{E_\xi} = - \frac{206265 \times 10^2 k \rho \pi^2}{2 \times 180^2 \gamma_m}$$

.............(12.66)

In the above equations all quantities have the same definition as in equation (12.52), α_{ijk}(i=1, 2) being defined by equation (10.22). In this case, the components of the deflection of the vertical are dependent on the disposition of the topography with respect to the computation point. The constant K_{E_ξ} evaluated under the same conditions used in the previous instance, has the value - 5.927 x 10^{-7} cm^{-1}, for values of h_{ij} in metres. The computation of the contribution of the external potential to the indirect effect was computed in three series of programs.

(i) For regions where $\psi > 20^\circ$.

In this case, n = 5 and the total number of quantities involved was approximately 2500. The values of h_{ij} used were $5^\circ \times 5^\circ$ mean elevations prepared by Lee (Kaula et al, 1966, IIA, 43 et seq). These elevations, for continental areas, are probably quite reliable as the set was constructed from $1^\circ \times 1^\circ$ mean elevations. As pointed out earlier, the comparison errors for the Australian mainland region were of the order of $\pm$ 80 metres and hence the error of any $5^\circ \times 5^\circ$ mean elevation for a continental area is unlikely to be in excess of $\pm$ 50 metres on the basis of Lee's tests (ibid, IIa, 30).

The accuracy of the quantity computed was evaluated according to the relations

$$e_{N_E}^{(cm)} = \pm K_{E_N} \times 25 \left\{ \sum_j \left[\frac{T_j^2}{h_j^2} \operatorname{cosec}^2 \tfrac{1}{2} \psi_j \right] \right\}^{\frac{1}{2}} \times 50$$

$$e_{\xi_{L_i}}^{(sec)} = \pm K_{E_\xi} \times 25 \left[\sum \left\{ \frac{T_j}{h_j} \cos \tfrac{1}{2} \psi_j \operatorname{cosec}^2 \tfrac{1}{2} \psi_j \ldots \right.\right.$$
$$\left.\left. \ldots \cos \alpha_{ij} \right\}^2 \right]^{\frac{1}{2}} 50 \quad , \; i=1,2 \ldots\ldots (12.58).$$

The program used to compute the above effect was coded INOUT and is set out in appendix (14). It was adapted for generalised computing by reading in all height data on a world wide basis and excluding a $45^\circ \times 45^\circ$ area symmetrical about the $5^\circ \times 5^\circ$ square in which the computation point was located.

(ii) For regions in the range $1.5^{\circ} < \psi < 20^{\circ}$.

The data used in this region were the set of $1^{\circ} \times 1^{\circ}$ elevation means computed in the program AN_COMP, described in section (12.2). The accuracy of these elevation means is hardly likely to exceed $\pm$ 7 metres. The equations programmed in this set of computations are embodied in the program INMID, set out in appendix (15), and are similar in nature to those described in section (12.6 (i)). A $45^{\circ} \times 45^{\circ}$ area, symmetrical about the $5^{\circ} \times 5^{\circ}$ square in which the computation point lies, is considered, an area of $3^{\circ} \times 3^{\circ}$, symmetrical about the $1^{\circ} \times 1^{\circ}$ square in which the computation point was situated, being excluded.

(iii) For regions in $0.1^{\circ} < \psi < 1.5^{\circ}$.

The indirect effect for this region was computed using the program titled INNEAR, given in appendix (16). The input data was in the form of $0.1^{\circ} \times 0.1^{\circ}$ elevation means for the $3^{\circ} \times 3^{\circ}$ area symmetrical about the $1^{\circ} \times 1^{\circ}$ square in which the computation point was situated. These elevations had an estimated error of $\pm$ 30 metres. The programs, similar in nature to those described previously, were amended slightly to take into account the limited range of ψ. The four innermost tenth degree squares were excluded and the effect of this zone was considered separately.

(iv) The region $\psi < 0.1^{\circ}$

The equations (12.52) to (12.58) fail in this innermost region due to the instability of $\operatorname{cosec} \frac{1}{2} \psi$. The separation in such a case is given by equation (6.74).

$$N_{E_i} = \frac{k\,\pi\rho h_P\, r}{\gamma m}\left(1 - \frac{h_P}{2r} + \ldots\ldots\right) \quad \ldots\ldots\ldots(6.74).$$

The contribution of a cylinder, constituting the inner zone, to the deflections of the vertical is zero, as can be seen from the consideration of the equations preceding equation (6.67) due to the integration

$$\int_0^{2\pi} \cos\alpha\, d\alpha = \int_0^{2\pi} \sin\alpha\, d\alpha = 0.$$

The effect of the departure of the actual topography from the cylindrical model has a negligible effect on the potential of matter exterior to the geoid at a point <u>on</u> the geoid, in view of the magnitude of the terms involved and can be ignored. Equation (6.74) is programmed for computations of the effect for varying heights and densities and is set out in appendix (17), titled ININ.

DEPARTMENT OF SURVEYING - UNIVERSITY OF NEW SOUTH WALES

Kensington, N.S.W. 2033.

Reports from the Department of Surveying, School of Civil Engineering.

1. The discrimination of radio time signals in Australia.
G.G. BENNETT (UNICIV Report No. D-1)

2. A comparator for the accurate measurement of differential barometric pressure.
J.S. ALLMAN (UNICIV Report No. D-3)

3. The establishment of geodetic gravity networks in South Australia.
R.S. MATHER (UNICIV Report No. R-17)

4. The extension of the gravity field in South Australia.
R.S. MATHER (UNICIV Report No. R-19)

UNISURV REPORTS.

5. An analysis of the reliability of Barometric elevations.
J.S. ALLMAN (UNISURV Report No. 5)

6. The free air geoid in South Australia and its relation to the equipotential surfaces of the earth's gravitational field.
R.S. MATHER (UNISURV Report No. 6)

7. Control for Mapping. (Proceedings of Conference, May 1967).
P.V. ANGUS-LEPPAN, Editor. (UNISURV Report No. 7)

8. The teaching of field astronomy.
G.G.BENNETT and J.G.FREISLICH (UNISURV Report No. 8)

9. Photogrammetric pointing accuracy as a function of properties of the visual image.
J.C. TRINDER (UNISURV Report No. 9)

10. An experimental determination of refraction over an Icefield.
P.V. ANGUS-LEPPAN (UNISURV Report No. 10)