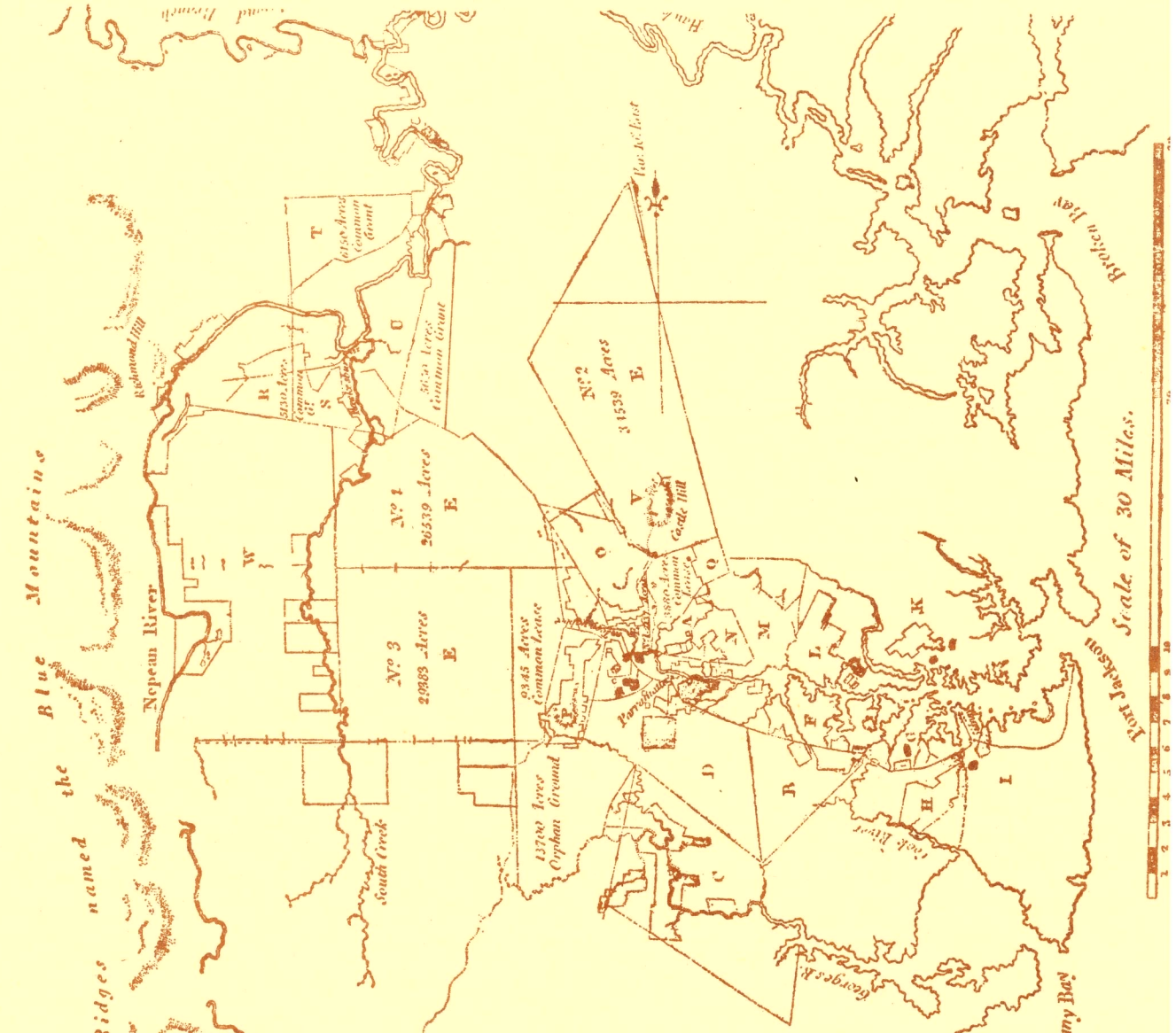

UNISURV REPORT No. S8

THREE-DIMENSIONAL CARTESIAN CO-ORDINATES OF PART OF THE AUSTRALIAN GEODETIC NETWORK BY THE USE OF LOCAL ASTRONOMIC VECTOR SYSTEMS.

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"Perhaps in another world we may gain other insights into the nature of space which at present are unattainable to us. Until then we must consider geometry of equal rank, not with arithmetic which is purely logical, but with mechanics which is empirical."

C. F. Gauss, 1817.

SUMMARY

The problem of terrestrial point co-ordination using the common geodetic measurements of directions, zenith distances, lengths, azimuths, latitudes, longitudes, levelled differences in elevation and gravity readings is analysed. Theoretical solutions are posed employing the data pertaining specifically to either the earth's gravity field or the atmosphere. However, because the curvature parameters defining the gravity field cannot at present be determined to sufficient accuracy and those defining the atmosphere cannot be measured at all, a combined model is formulated. Assuming the gravitational field to be suitably described by a Newtonian system and that adequate correction is possible for the measurements which are influenced by refraction, a Euclidean metric is selected for the complete definition of the solution. Within the proposed model, relative positions of points are defined by astronomic observations which refer essentially to the gravity field, and by the angular and linear data which pertain to the atmosphere. Absolute positions may be computed using world wide gravity data. Since the number of observations made in practice always exceeds the minimum required for a unique determination of the unknowns, adjustment is necessary. Formulae are given for a least squares solution, the variation of co-ordinates method being preferred on account of its suitability for programmed computation.

The model is applied to a significant portion of the network stations which were included in the 1966 Australian National Adjustment. A total of 263 points, covering an area of one million square kilometres, is considered. The computed Cartesian co-ordinates are compared to the corresponding ellipsoidal values obtained for the National Adjustment. The resulting discrepancies, which appear to be of a systematic nature, are attributed to:

(1) the computational method chosen for the National Adjustment being an approximation in comparison to a rigorous, one phase least squares solution;

(2) the test network being only a small portion of the National configuration; and

(3) the available data not being used in an identical manner.

Implementation of a full error analysis is estimated to require 100 hours of computer time. Only a component net comprising 63 stations is therefore examined.

Approximately 2 000 normal equations require solution and the coefficient matrix must be inverted to appraise the accuracy of the unknowns. Suitable algorithms for both of these tasks are presented, purely iterative techniques being rejected respectively in favour of the direct and partitioned methods. Nevertheless, for an accurate solution iterative refinement procedures are especially attractive and a method for improving the accuracy of the solution in this manner is given. The Cholesky approach to decomposition, which has all the virtues for the banded, positive definite and symmetric matrices that occur in practice, is recommended. Factorization by partitioning is only to be considered when the coefficient matrix cannot be wholly accommodated in the core of the available computer. Optimum scaling by equilibration is studied. The feasibility of evaluating the inverse using the FORTRAN H compiler on the IBM 360/50 computer is investigated. It is found that the full or partial inversion of matrices of order larger than 400 requires extensive amounts of computer time. An alternative for assessing the accuracy of the computed solution is thus offered.

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Part B: Adjustment.

Part C: The Solution of the Normal Equations and the Inversion of the Matrix of Coefficients.

Part D: The Test Network.

Part E: Computations, Results and Conclusions.

INTRODUCTION.

In the projective method of classical geodesy *(Wilcox 1963, 127-138)*, the problem of point co-ordination on the earth's surface is solved by:

(1) correcting the observations for curvature;

(2) projecting the 'corrected' measurements to a reference figure via the geoid and

(3) computing positions using the projected quantities.

Two co-ordinates of each station are computed on the reference ellipsoid or on a plane conformal transformation of it. The third dimension is usually provided by a combination of levelling and gravity data. At present, such a procedure is the most acceptable, in view of the uncertainty in the knowledge of the influence of refraction on vertical angle measurement. It is, however, only a practical solution to a complex physical problem.

Conventional geodetic measurements comprise directions, lengths, vertical angles, azimuths, latitudes, longitudes, levelled differences in elevation and gravity readings. Meteorological observations are also made in conjunction with the electromagnetically determined distances. It is hoped that these observations will provide sufficient information for the computation of the absolute spatial positions of points at which the measurements are made, the term absolute being used in the sense that the points are, in the ideal circumstance, to be related to known invariants - that is, the earth's mean centre of mass and mean rotational axis for some epoch - within the earth, via a reference frame.

The measurements are made through the atmosphere and with the exception of the distances are subject to the earth's gravitational field. Accordingly, an important aspect of terrestrial point co-ordination is the definition of the natural laws which influence the observations. These laws must in turn be related to an acceptable system of geometry that is suitable as the basis to co-ordinate computation.

If the problem of point co-ordination is considered as a problem in space, then three distinct sub-spaces come to mind:

(1) that of the earth's gravity field or Earth space *(Mather 1970a, 5)*;

(2) that of the atmosphere and

(3) that of the mathematical model.

Both Earth space and the atmosphere are a physical reality, whereas the mathematical model exists in the mind of the geodesist alone, hopefully to comply with physical reality.

The geometry of Earth space which may be considered to be governed by Newtonian gravitation over short intervals of time and after the appropriate reduction of observations, is three-dimensional Euclidean since the expression for force and potential require the distance between the test and massive particle to be finite and in a straight line. The field itself must extend to infinity and a Euclidean metric must therefore be chosen to describe it. The nature of the atmosphere is time dependent. Moreover, a geodesic within it is not, in general, a straight line. Hence, the geometry of the atmosphere dictates a Riemannian metric.

The geometry of the mathematical model must satisfy the geometry of its component spaces and will thus be non-Euclidean. However, since non-Euclidean space can always be mapped onto Euclidean space, Euclidean geometry can always be assumed to be the geometry of the adopted mathematical model, provided the requisite changes are made in the physical laws that determine it. It is purely a question of convenience whether it is preferred to have an easily intelligible geometry with complicated physical laws or a less intelligible geometry with simple physical laws. The consensus of opinion appears to be that geometry should be regarded as part of the physical situation and therefore the obvious system of geometry should be one in which the rest of the laws governing the mathematical model can be expressed as simply as possible. It is this consideration which would ultimately lead to a curved mathematical model.

The geodesist, who adopts Newtonian gravitation and Euclidean geometry as the basis to computations is thus handicapped, as he not only neglects the time dependence of his observations but also the possible incompatibility of reality and the mathematical model. Furthermore, when theories are compared with the observations and it is found that there is not an exact agreement, it is assumed that the results and not the mathematical model are in error, providing the discrepancy is within certain prescribed limits.

Euclidean geometry undoubtedly describes with close approximation the physical reality of geodetic observations; but whilst the effects of curvature and of time remain uncertain, any solution to the fundamental problem of geodetic point location must be considered as an approximation.

A temporary solution to the problem is one in which both the curvature of the mathematical model and the time dependence of the observations are neglected. A Euclidean metric is assumed and computations are effected using Cartesian co-ordinates. The basis of such a technique appears to have been first proposed by Bruns *(1878)*. An alternative approach was suggested by Marussi *(1949)* and was later formulated by Hotine *(1957, 1959, 1969)* who adopts as a foundation of the computations the three co-ordinate surfaces generated respectively by the astronomic observations of longitude and latitude, and the geopotential. Hotine *(1957, 7)* concludes that this type of solution cannot be put to practice because the higher derivatives of the parameters ϕ, ω, N respectively the astronomic latitude and longitude, and the geopotential, with respect to the element of arc joining adjacent points, cannot be determined from the measured quantities.

By excluding the third co-ordinate surface, classical geodesy has adopted the approach of projecting the observing station onto a

regular mathematical figure - more precisely an ellipsoid of revolution. Two co-ordinates of each station are computed on this ellipsoid or on a plane conformal transformation of it. The third geodetic co-ordinate of height above ellipsoid of the point in question is usually provided by levelling. Unfortunately, whereas transformed geodetic co-ordinates refer to an ellipsoid of revolution, the datum of height measurement is an equipotential surface of the earth's gravitational field. Thus within this method the reference surfaces for height and position are not identical, and even though computations are made in two dimensions, the method suffers from the additional disadvantage that computational formulae become more complicated.

Yet another approximate solution and one that is the subject of this dissertation is the method of astronomic vector systems *(Stolz 1970)*. The distinction between this technique and that treated by other authors *(Bruns 1878), (Dufour 1968), (Hotine 1969), (Mather 1969), (Molodenskii et al 1962), and (Wolf 1963b)* is that all preliminary computations are made in the natural reference frame of the observations.

The natural or astronomic reference frame, which comprises the directions of the astronomic meridian, the zenith and a third vector completing the triad* orthogonally and in a right-handed system, is defined under the assumption that the equipotential surface containing the observing station has no discontinuities at or in the vicinity of the spatial point, implying that a tangent plane to the equipotential surface exists at the point being considered. Within this system, the customary linear and angular measurements completely define the relative positions of points. Moreover, all observations are made at discrete points and all but the

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*The word triad is used throughout this dissertation to describe the unit vector triple defining the direction of the astronomic meridian, the astronomic zenith and a third vector completing the set in an orthogonal right-handed system.

measured distances are subject to the earth's gravitational field there. Thus, assuming that the field remains constant over the period of observations, all measurements are affected equally at a point and the earth's gravitational field does not affect the relative positions of points as defined. Orientation of the triads is achieved by astronomic azimuth observations. Naturally, the method does not give any information regarding the geocentricity* of the system as the potential is no longer a parameter of the solution.

The method is marked by simplicity of computation and, providing care is exercised in defining the direction parameters of the astronomic zenith, the measured quantities may be directly related to perpendicular components in the local astronomic triad which may then be rotated to provide co-ordinate increments between adjacent points.

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* The geocentre represents the Earth's centre of mass.

NOTATION.

Indices.

The index notation of tensor calculus *(McConnell 1957, 1)* offers many advantages and is used extensively throughout this dissertation. Lower case Roman letters i, j, k ... etc. will be used for indices, both as subscripts (a_i), indicating covariance and superscripts (a^i), indicating contravariance *(ibid 20 - 22)*. Thus the quantities a^1, a^2 and a^3 will be denoted by the symbol a^i. Symbolically this may be written -

$$a^i = (a^1, a^2, a^3)$$

and it must be remembered that a in this relation is a set of three quantities. No operation such as multiplication or division is implied. For i and j ranging from 1 to 3 independently, the symbols a^i_j, a_{ij} and a^{ij} represent nine quantities.

According to the summation convention, a summation is implied when an index is repeated. For example -

$$b_i c^i_j = b_1 c^1_j + b_2 c^2_j + b_3 c^3_j \quad (j = 1, 2, 3)$$

A repeated index is called a dummy index, because the value of the term does not depend on the symbol used. An index which is not repeated is called a free index; when a suffix occurs unrepeated in a term, it will be understood that it takes the values from the set (1, 2, 3). Evaluating $a_i = b_i$ for the range (1, 2, 3) gives

$$a_1 = b_1 \qquad a_2 = b_2 \qquad a_3 = b_3$$

For quantities pertaining to the origin of survey, the suffix o is employed. It is understood that this is not a free index. The following system of symbols and abbreviations is adopted throughout this dissertation: Only those symbols used frequently are given; minor use for some other quantity in only one context is not listed in the index, but is fully explained in the text.

Symbols.

A	Coefficient matrix of linear system $Ax = b$.
$\|\|A\|\|$	Norm of matrix A.
a_{ij}, $a_{i,j}$	General element of matrix A.
a	First co-ordinate in a general (a, b, C) co-ordinate system.
a, b, c	Cartesian components of a general unit vector.
b	Second co-ordinate in a general (a, b, C) co-ordinate system. Right-hand side vector of $Ax = b$.
C	Third co-ordinate in a general (a, b, C) co-ordinate system. C may be given various meanings in different contexts, for example, potential or geopotential in Chapter 3.
c	Velocity of light.
e	Eccentricity of meridian ellipse.
F_s	Force vector.
G	Matrix of weight coefficients.
g^{rs}, g_{rs}	Metric tensors in three-dimensions.
h	Orthometric elevation.
I	Unit matrix.
i_r, j_r, k_r	Fixed Cartesian axis vectors.
K	Gaussian curvature.
k	Refraction coefficient; Gravitational constant.
ℓ_r	Covariant vector (three-dimensions).
ℓ, m, n	Cartesian components of a general unit vector in local astronomic triad (λ_r, μ_r, ν_r)
m	Mass of a particle; Band-width of matrix A.
N	Geopotential; Normal equation matrix.
n	Magnitude of the gradient of geopotential; Order of matrix A.
O	Orientation constant.
Q	Latitude and Longitude matrix; Variance-covariance matrix of adjusted quantities.

q	Diagonal submatrix of variance-covariance matrix.
R, S	Matrices of components of λ^r, μ^r, ν^r and λ_r, μ_r, ν_r.
r_1, r_2	Curvature parameters. Normal curvature of N-surface in prime vertical (λ_r) and meridian (μ_r) directions respectively.
S	Optical path length.
t	Time.
t_1	Curvature parameter. Geodesic torsion of N-surface in prime vertical direction.
V	Attracting potential.
v	Accidental (normally distributed) error.
W	Weight matrix.
x	Solution vector of $Ax = b$.
$\|\|x\|\|$	Norm of vector x.
x, y, z	Cartesian co-ordinates in a geocentric equatorial right-handed system.
α	Azimuth.
β	Zenith distance.
Γ^r_{st}	Christoffel symbols.
γ_1, γ_2	Curvature parameters. Rate of change of (ℓn n) in prime vertical (λ^r) and meridian (μ^r) directions, respectively, in a (ω, ϕ, N) system.
Δ	Laplacian.
η	Prime vertical component of deflection.
λ_1, λ_n	Largest and smallest eigenvalues of matrix A.
λ_r, μ_r, ν_r	Local astronomic vector triad.
$\kappa(A)$	Spectral condition number of matrix A with respect to inversion.
μ	Refractive index.
ν, ρ	Principal radii of curvature of an ellipsoid in prime vertical and meridian directions.

ν_r	Unit normal.
ξ	Meridian component of deflection.
ρ^r, ρ_r	Position vector.
σ	Variance-covariance matrix of observed quantities.
σ_o^2	Variance factor.
ϕ	Latitude of gradient of N.
ω	Longitude of gradient of N.
$\bar{\omega}$	Angular velocity of earth's rotation.

Abbreviations.

AGD	Australian Geodetic Datum.
ppm	Parts per million.
O-C	Observed minus computed.

PART A: THE FUNDAMENTALS OF THREE-DIMENSIONAL COMPUTATIONS.

1. Definitions.

1.1 Spatial Co-ordinate Systems.

Several co-ordinate systems are used for the analysis of observational data within the context of this dissertation. They are briefly:

(1) a special but quite general co-ordinate system (a, b, C) *(Hotine 1969, 69)* generated by a continuously differentiable scalar function of position C in three-dimensional space;

(2) a global Cartesian System;

(3) a Celestial system for astronomical work and

(4) a local triad in which all the geodetic measurements are made.

These reference systems have to be carefully defined and their precise relationships established so that the spatial concepts involved may be fully exploited.

The (a, b, C) co-ordinate system is used in the analysis of the equipotential surfaces of the earth's gravitational field and in the formulation of the geometry of the wavefronts that arise when energy is transmitted through the atmosphere.

Theoretically, the global Cartesian system is defined by the mean terrestrial pole of a certain date and the mean meridian of Greenwich, and the Celestial system by the mean equator and ecliptic of a certain date. In practice however, both these systems are defined by a set of co-ordinates given to physical points, as the ideal global Cartesian system is replaced by a geodetic system that is parallel to the ideal frame and the Celestial system by one that is defined with reference to the mean places and proper motions given to stars in a particular catalogue. The two systems can be related when the position of the instantaneous pole, the sidereal time at Greenwich as determined by observation, and the precessional and nutational constants are known.

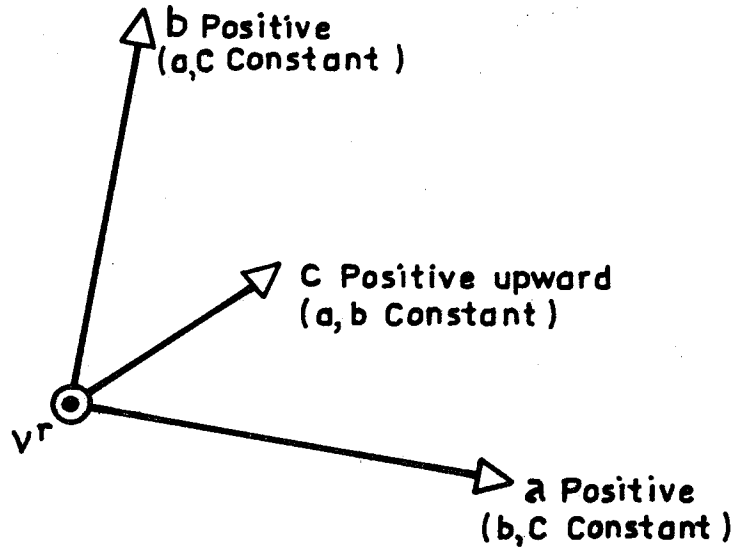
b Positive
(a,C Constant)
C Positive upward
(a, b Constant)
v^r
a Positive
(b,C Constant)

FIG. 1·1

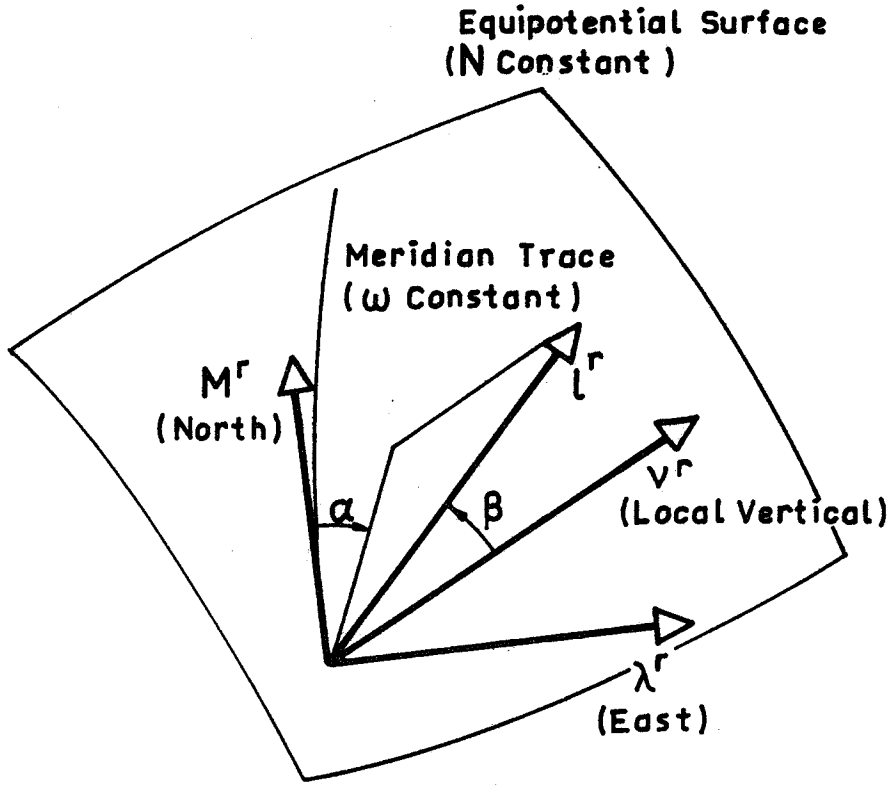
Equipotential Surface
(N Constant)
Meridian Trace
(ω Constant)
M^r
(North)
l^r
v^r
(Local Vertical)
α
β
λ^r
(East)

FIG. 1·3

In addition a system is needed to which the geodetic observations can be referred. For this purpose Cartesian components within a curvilinear reference frame will be used in preference to any other system as such a system has several advantages in analytical manipulation and can be transformed to any other by matrix algebra.

1.1.1 The (a, b, C) Co-ordinate System.

The (a, b, C) co-ordinate system is generated entirely by a continuously differentiable scalar function of position C in three-dimensional space. Points in space having a particular value of C will lie on a surface C = const. and for different values of this scalar, a family of surfaces is obtained. C may be taken as one of the co-ordinates of the system and if defined throughout some region of space, then so is the magnitude (n) and direction (ν_r) of its gradient (C_r) as by definition

$$C_r = n\nu_r \qquad \ldots\ldots \quad (1.1)$$

The direction ν_r in relation to three fixed Cartesian axes in flat space will define two independent scalars a, b which may be chosen as the other two co-ordinates of the space, and each generates a family of surfaces distinct from the C-surfaces and each other. Position in space can thus be described by the intersection of the three surfaces.

Each co-ordinate line, the line of intersection of two co-ordinate surfaces along which only the third co-ordinate varies, will in general be curved both on the C-surface and in space. Moreover, as well as using the scalars a, b as two of the space co-ordinates, it is possible to choose them as C-surface co-ordinates.

1.1.1.1 Sign Convention.

The (a, b, C) system is chosen right-handed in the same sense that (x, y, z) is conventionally right-handed (Fig. 1.1).

Looking along the positive direction of the C-co-ordinate line, the positive direction of the b-line is to the right of the positive direction of the a-line. A positive rotation about the C-line will be clockwise when looking along the positive direction of the C-co-ordinate line. Later on, when the C-co-ordinate is identified with the geopotential N, the positive direction of C as defined above, will make N negative in geodetic applications, but accords with the usual mathematical conventions of an outward drawn normal to a closed surface.

1.1.2 The Global Cartesian System.

The ideal Cartesian reference frame for all spatial points is an earth centred equatorial right-handed system x, y, z defined as follows: '*The origin is at the earth's centre of mass and the system is oriented so that the z-axis is directed towards the mean north pole as defined by the International Polar Motion Service. The xz-plane is parallel to the mean meridian of Greenwich as defined by the Bureau Internationale de l'Heure, but any other standard meridian is acceptable.*' This terrestrial co-ordinate system is fixed with respect to the earth's surface and the co-ordinates of any point do not change providing there is no crustal movement.

Unfortunately, the centre of mass of the earth is not known precisely and for this reason, the ideal system is replaced in practice by a rectangular geodetic system in which the axes are respectively parallel to those of the ideal system. Co-ordinates within this system thus remain relative to the origin of the geodetic system until the position of the geocentre with respect to a surface point is known. A translation will then produce geocentric co-ordinates.

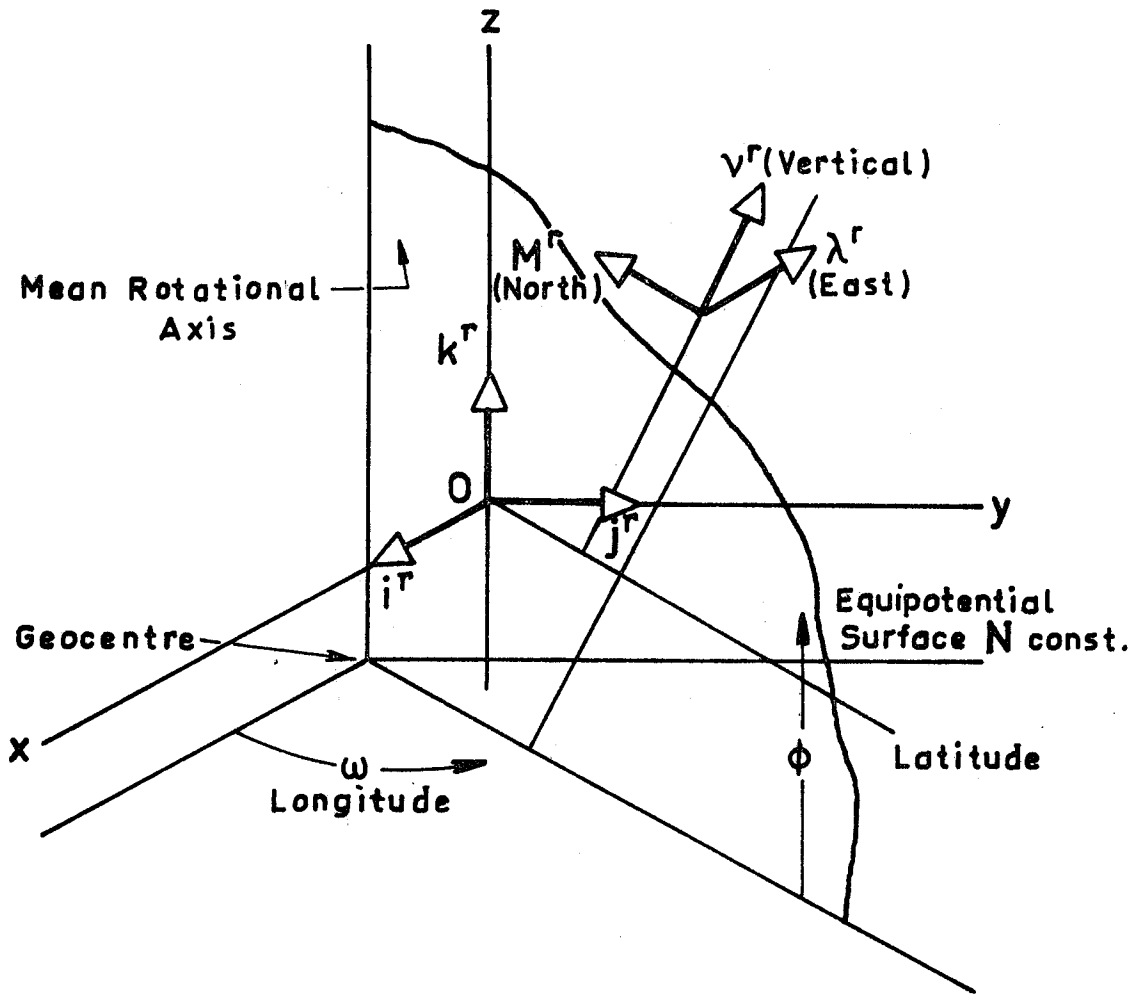
z
ν^r(Vertical)
M^r
(North)
λ^r
(East)
Mean Rotational
Axis
k^r
O
y
j^r
i^r
Equipotential
Surface N const.
Geocentre
x
ω
Longitude
Φ
Latitude

FIG. 1·2

1.1.3 The Celestial System.

It is necessary to define a Celestial system as within it the astronomic observations of latitude, longitude and azimuth are made. The precise aspects of this frame, with particular emphasis on permitting a transformation between the ideal Cartesian and Celestial systems have been investigated by Veis *(1963)*.

1.1.4 The Local System.

The local system is defined on the assumption that the equipotential surface containing a particular spatial point has no discontinuities at and in the vicinity of that point, which implies that a tangent plane to the equipotential surface exists at the point under consideration. A right-handed orthogonal vector triad λ^r, μ^r, ν^r at any spatial point is then defined as follows:

(1) λ^r is the direction of the prime vertical and lies in the plane tangent to the equipotential surface containing the spatial point;

(2) μ^r is the direction of the astronomic meridian and also lies in the tangent plane and

(3) ν^r is the direction of the astronomic zenith and is normal to the tangent plane.

All geodetic measurements are made within this local system.

In order to establish the precise relationships between the reference systems previously described as well as those between adjacent local systems it remains to define:

(1) a vector triad i^r, j^r, k^r;

(2) the astronomic latitude (ϕ) and longitude (ω) of a spatial point using the vectors i^r, j^r, k^r and λ^r, μ^r, ν^r and

(3) the aximuth α and zenith distance β of a spatial vector connecting two adjacent terrestrial points.

1.2 The Vector Triad (i^r, j^r, k^r).

The vector triad (i^r, j^r, k^r) (Fig. 1.2) is defined so that the i^r, j^r, k^r are unit axis vectors in the directions of the corresponding x, y, z axes.

If the Cartesian co-ordinate x is considered to be a scalar function of position, then its gradient is

$$x_r = i_r$$

a vector equation which is true in any co-ordinate system. Thus, using braces notation to denote column matrices

$$\{x_r, y_r, z_r\} = \{i_r, j_r, k_r\} \quad \ldots\ldots \quad (1.2)$$

In Cartesian co-ordinates the above components become

$$\{x_r, y_r, z_r\} = \{i_r, j_r, k_r\} = I \quad \ldots\ldots \quad (1.3)$$

where I is the unit matrix.

1.3 The Astronomic Latitude (ϕ) and Longitude (ω).

Astronomic latitude is taken as the angle between the vector ν^r and the xy-plane, positive north. The longitude is the angle between two planes both containing the mean axis of rotation one of which is parallel to ν^r, that is the vertical at the observation point whereas the other is parallel to the vertical at some defined point such as the site of the Greenwich transit telescope, or more precisely, as defined by Bomford *(1962, 86-7)*, positive east.

Longitude and latitude may thus be defined in terms of direction cosines of the unit normal ν^r by means of the following scalar products -

$$\begin{aligned} \cos\phi \cos\omega &= \nu_r i^r \\ \cos\phi \sin\omega &= \nu_r j^r \\ \sin\phi &= \nu_r k^r \end{aligned} \quad \ldots\ldots \quad (1.4)$$

These definitions accord with international practice and with astronomic convention for right ascension and local time (but not hour angle which is reckoned positive west). Longitude is thus made a positive rotation in the mathematical sense about the northward rotational axis of the earth.

From the previous definitions, it becomes clear that in the general case, the meridian will fail to pass through the ground point (Fig. 1.2). However, this is of no consequence as in the method of spatial point definition to be proposed, astronomic measurements of latitude and longitude are used only to define the direction of the vertical and not to define position. The relative positions of points are then given by the other linear and angular measurements. For the local astronomic triad this means, providing that an astronomic azimuth has been observed, that the directions of its unit vectors are uniquely defined by the astronomically observed values.

1.4 Azimuth (α) and Zenith Distance (β).

Azimuth is defined as a rotation about ν^r from μ^r to λ^r (Fig. 1.3). A unit vector ℓ^r which lies in the tangent plane to the equipotential surface with azimuth α is accordingly given by

$$\ell^r = \lambda^r \sin\alpha + \mu^r \cos\alpha \quad \ldots\ldots \quad (1.5)$$

A unit spatial vector in azimuth α and zenith distance β will be given by the vector equation -

$$\ell^r = \lambda^r \sin\alpha \sin\beta + \mu^r \cos\alpha \sin\beta + \nu^r \cos\beta \ldots\ldots \quad (1.6)$$

2. Physical Space.

The idea of space to which everyone is accustomed, was first imagined by Descartes. In this system, the properties of space were believed to be in accord with the geometry of Euclid.

The Greeks investigated very thoroughly and completely the mathematical properties associated with the experience of space. In the famous text-book of Euclid, the Elements of Geometry, these are presented as logical deductions from a set of axioms which are assumed without proof. Euclid's axioms are not, however, now considered as satisfactory, and it is not to be expected

that a single mathematical system of geometry, such as Euclid's, should furnish relations covering the whole aspect of reality. Nevertheless, Euclidean geometry undoubtedly describes with close approximation a large class of properties of the actual world.

The truth that destroyed truth, was clearly seen by C.F. Gauss. Before Gauss, the usual technique of investigating a curved surface was to operate with the three Cartesian co-ordinates x, y, z considering the curved surface as an entity embedded in the surrounding space of three dimensions. Gauss realised that Euclidean geometry may miss the mark and arrived at a new foundation of the geometry of the physical world which dispensed with the postulates in favour of certain fundamental measurements, thus making geometry a quantitative science. He discovered that geometry depended on the quantity K - the curvature. For example, if $K = 0$ at every point on the surface, the geometry becomes Euclidean, and therefore the geometry of the plane is contained in a single statement, that $K = 0$ at all points.

Riemann, ingenious pupil of Gauss, asked what would correspond to this quantity in more than two dimensions and, how one could tell from a given line element whether the geometry established by it was Euclidean or not *(Weber 1953, 272-288)*. He made the basic discovery that the decisive quantity is not a single number, but is in fact a tensor of fourth order. Euclid's flat geometry results if all the components of this tensor vanish. But even if one component happens to be non-zero, then the geometry is no longer Euclidean; a certain curvature exists.

The implications of non-Euclidean geometry are drastic. If both Euclidean and non-Euclidean geometry can represent physical space equally well, which is the truth about space and figures in space? The fact gradually forced upon mathematicians was that geometry is not the truth about physical space but the study about possible spaces.

2.1 Physical Space and Geometry.

In physics the idea of space and of situation in space, is derived by first introducing the concept of distance, which leads to the discovery of the empirical relations between the mutual distances of a number of particles. This introduces an ideal realm by the postulation of an exact mathematical form for these relations and the development of a theory of geometry. Having thus reached the concept of geometrical space, the geometrical space is finally projected back into the external world yielding the concept of physical situation. Accordingly, physical space is not an intuition, but a system.

It is important that geometrical space should be distinguished from physical space, as the former is nothing but an imagined substratum for the mathematical relations that are approximately valid there; it exists only in the mind of the mathematician. Nevertheless, geometrical space has played a great part in forming the concept of physical space: for having been originally derived from the comparison of measurements made in the actual world, it has been projected back, so to speak, onto nature and thus has furnished a generally acceptable picture of reality.

2.2 The Nature of Space from Measurements.

It is the characteristic of any branch of mathematics that the whole of it can be derived from a few definitions and assumptions specified at the beginning. The question therefore arises: What are the fundamental data or postulates from which the complete set of laws of the space that surrounds us can be deduced by pure mathematics? Moreover, are these data or postulates revealed by observation and experiment or are they self evident truths revealed and assured by intuition?

In order to propose a satisfactory answer to these questions, it is necessary to examine whether it is possible to determine by observation the system of geometry (Euclidean or non-Euclidean) which

corresponds to that of actual space. Many different systems may be chosen, any one of which will serve the purpose. Essentially, this space is a mere threefold continuity and nothing more. It would therefore seem to be impossible to tell by observation whether the space is Euclidean or non-Euclidean.

However, in studying space, one has to deal not only with threefold continuity but also with the notion of distance, which is purely physical. As soon as the distance between all pairs of points is defined there will be a relation between the 10 mutual distances of any 5 points *(Whittaker 1949, 7)*, and on this relation will depend whether the geometry of the space is Euclidean or not. Thus while there is no constraint, either observational or logical, which imposes on the pure mathematician either a Euclidean or non-Euclidean metric for the space he considers, the matter is not so for the physical scientist. If geometry is to be of use to him, he must define distance in accordance with the observationally determined properties of physical distance so that he has practically no freedom of choice for his quantitative study of physical space and therefore no choice whether it is to be Euclidean or not.

Since non-Euclidean space can always be mapped onto Euclidean space, however, Euclidean geometry can always be assumed to be the geometry of actual space, provided the requisite alterations are made in the physical laws of that space. It is purely a question of convenience whether it is preferred to have an easily intelligible geometry with complicated physical laws, or a less intelligible geometry with simple physical laws. The consensus of opinion is that geometry should be regarded as a part of physics, and therefore the obvious system of geometry should be one in which the rest of physics can be expressed as simply as possible. It is this consideration which ultimately leads to the curved space of general relativity.

2.3 The Geometry of Physical Space.

In Newtonian physics, space was like a huge empty box into which matter was placed from the outside. To boot, time existed in which physical action took place. Thus the three basic entities of space, time and matter appeared to be completely independent from another. For Einstein *(Lanczos 1965, 8)*, however, matter was not put from the outside but formed an integral part of the geometry of space. Contemporary opinion is therefore, that the geometry of space is four-dimensional and curved*, time having been absorbed as an added dimension and matter as a curvature property.

3. Earth Space.

The previous remarks apply to gravity in general. Newtonian gravitation, however, has received substantial support from observations on the outer planets of the solar system and, although the inverse square law has thus only been verified for the case of a few near-spherical bodies whose dimensions are small when compared with their distances apart, is adopted for the definition of Earth space.

3.1 Potential and Equipotential Surfaces.

3.1.1 Potential.

3.1.1.1 Potential of a Point Mass.

In a gravitational field set up by a single particle of mass m, the force of attraction on another particle of unit mass at a distance ℓ from the first particle is by Newton's law,

$$km/\ell^2 \qquad \ldots\ldots \quad (3.1)$$

in which k is the gravitational constant. The direction of the force

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* Curvature in the mathematical sense has nothing to do with the shape of the space, but is defined solely by the metric. What is meant, is that the relations between the mutual distances are different from those obtained in Euclidean geometry.

is toward the massive particle along the line joining the two particles. The particle of unit mass is usually known as the test particle.

The potential or the amount of work that must be done against the force of attraction to remove the test particle to an infinite distance, in a field set up by a single particle of mass m is accordingly

$$-\int_{\infty}^{\ell} km/\ell^2(-d\ell) \quad = \quad -km/\ell \qquad \ldots\ldots \quad (3.2)$$

3.1.1.2 Potential of a Rigid Rotating Body.

The attracting potential V in the field of a rotating continuous mass distribution is given by *(Heiskanen & Moritz 1967, 47)*

$$V \quad = \quad k\iiint dm/\ell + \tfrac{1}{2}\bar{\omega}^2 d^2 \qquad \ldots\ldots \quad (3.3)$$

where $\bar{\omega}$ is the angular velocity of the body, dm is an element of mass of the attracting body and d is the distance of the test particle from the rotational axis of the body.

3.1.2 Equipotential Surfaces.

Surfaces upon which the potential is constant are called equipotential surfaces.

3.1.2.1 The Central Field.

The equipotential surfaces in a central field set up by a single massive particle are spheres centred on the attracting particle and the outward unit normal to the equipotential surfaces is the gradient of ℓ, that is, ℓ_s. If N is considered as the potential, then by covariant differentiation

$$N_s \quad = \quad n\nu_s \quad = \quad km/\ell^2\ell_s \qquad \ldots\ldots \quad (3.4)$$

where n, the distance function of the family of equipotential surfaces, is seen to be the magnitude of the attracting force whose direction is $-\nu_s$.

If F_s is the force vector of magnitude km/ℓ^2 and direction $-\ell_s$ or ν_s then

$$F_s = -N_s = -n\nu_s \quad \ldots\ldots \quad (3.5)$$

which implies that the force vector is the gradient of the potential and for this reason the field is completely defined if the scalar potential N is known at each point of the field.

3.1.2.2 The Field of a Rigid Rotating Body.

Since the potential in the field of a rigid and rotating body no longer has the simple form $-km/\ell$ and the magnitude n of the resultant force is no longer km/ℓ^2, equipotential surfaces in such a field are no longer spheres.

The forces of attraction in the field formed by a number of massive particles have direction as well as magnitude and would have to be added vectorially. However, the vector equation $F_s = -N_s = -n\nu_s$ holds true between the gradient of the total potential and the vector sum of the individual force vectors. The direction ν_r of the gradient of the potential N is no longer the radial direction from a Cartesian origin but rather, is the unit normal to the equipotential surface containing the test particle.

3.2 The Earth.

For the earth it is customary to combine the gravitational effects of the rotational and attracting potential into a single force called gravity, since the two forces are usually indistinguishable. The scalar whose gradient is equivalent to the resultant force of gravity inclusive of the centrifugal force, is known as the geopotential.

Assuming that the earth rotates about a physical axis fixed in space, with a uniform angular velocity $\overline{\omega}$ and that the earth's centre of mass lies on this physical axis of rotation, then the geopotential N at a point distant d from the rotational axis is given by Equation (3.3).

It can therefore be said that the basic gradient equation $N_r = n\nu_r$ represents the Newtonian gravitational field of the rotating earth if N is the geopotential as previously defined, if n is gravity, and ν_r is the outward drawn normal to the N-surfaces, that is, the level surfaces of the combined attraction and rotation. The unit normal ν_r is accordingly the direction of the astronomic zenith.

3.3 The Geometry of Earth Space.

3.3.1 The Laplacian.

Newtonian gravitation in a region of space not occupied by matter is sufficiently expressed by the Laplacian of N(*ibid, 12 & 47)*

$$\Delta N = g^{rs}N_{rs} = -2\bar{\omega}^2 \qquad \ldots\ldots \quad (3.6)$$

where g^{rs} is the conjugate metric tensor.

3.3.2 The Newtonian System.

It is an essential part of the Newtonian system that the space to which it applies should be flat and unbounded, because the expressions of gravitational attraction and potential require ℓ to be finite and a distance in a straight line; the field itself must extend to infinity and satisfy Equation (3.2). Accordingly, simple Euclidean geometry may be chosen as a basis to computation and Cartesian co-ordinates may be used.

3.4 Point Co-ordination in Earth Space.

3.4.1 The Co-ordinate System.

The co-ordinate system best suited to the geometry of the equipotential surfaces of the earth's gravity field is of the (a, b, C) type, where (a, b) represent (ω, ϕ) the longitude and latitude - the direction parameters of the local vertical, or the normal to the equipotential surface, with respect to three Cartesian axes in Earth Space - and (C) represents the geopotential (N).

3.4.2 The Geometry of Equipotential Surfaces.

Now, any family of surfaces, with the exception of its position and orientation in space, is completely defined by five curvature parameters, respectively the normal curvatures r_1 and r_2 in any two perpendicular surface directions, the geodesic torsion t_1 in one of the previous directions and the curvatures γ_1 and γ_2 of the surface normal in the chosen directions *(Eisenhart 1960, 159)*. In the case of the equipotential surfaces of the earth's gravitational field, it is convenient to select the directions of the meridian and prime vertical as the two surface directions. Accordingly, the previous curvature parameters become the normal curvatures of the N-surface in the prime vertical and meridian directions, the geodesic torsion in the direction of the prime vertical and the curvature of the normal in the direction of the parallel and meridian.

3.4.3 The R and S Matrices.

Since the vectors λ^r and μ^r are essentially tangents to N-surface curves, and the vector ν^r represents the unit normal to the N-surface, it is possible to express them as functions of r_1, r_2, t_1, γ_1 and γ_2. The required relationships are *(Hotine 1969, 73-75)* depending on whether contravariant or covariant components are desired

$$R = \begin{pmatrix} \lambda^1 & \lambda^2 & \lambda^3 \\ \mu^1 & \mu^2 & \mu^3 \\ \nu^1 & \nu^2 & \nu^3 \end{pmatrix} = \begin{pmatrix} -r_1 \sec\phi & -t_1 & 0 \\ -t_1 \sec\phi & -r_2 & 0 \\ \gamma_1 \sec\phi & \gamma_2 & n \end{pmatrix} \quad \ldots\ldots \quad (3.7)$$

$$S = \begin{pmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{pmatrix} = \begin{pmatrix} -r_2 \cos\phi/K & t_1/K & \sec\phi\, \partial p/\partial\omega \\ t_2 \cos\phi/K & -r_1/K & \partial p/\partial\phi \\ 0 & 0 & p \end{pmatrix} \quad \ldots\ldots \quad (3.8)$$

where K is the Gaussian curvature of the N-surface at the point being considered and $p = \frac{1}{n}$.

3.4.4 Co-ordinate Transformations.

The vector set λ^r, μ^r, ν^r may be obtained from the i^r, j^r, k^r system by applying rotations:-

(1) $\frac{1}{2}\pi + \omega$ about k^r bringing i^r parallel to λ^r; and

(2) $\frac{1}{2}\pi - \phi$ about λ^r bringing k^r parallel to ν^r; or expressed in matrix notation

$$\{\lambda^r, \mu^r, \nu^r\} = \begin{pmatrix} -\sin\omega & \cos\omega & 0 \\ -\cos\omega & -\sin\omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \sin\phi & \cos\phi \\ 0 & -\cos\phi & \sin\phi \end{pmatrix} \{i^r, j^r, k^r\} \quad \ldots\ldots \quad (3.9)$$

$$= Q\{i^r, j^r, k^r\}$$

where

$$Q = \begin{pmatrix} -\sin\omega & \cos\omega & 0 \\ -\sin\phi\cos\omega & -\sin\phi\sin\omega & \cos\phi \\ \cos\phi\cos\omega & \cos\phi\sin\omega & \sin\phi \end{pmatrix}$$

Conversely

$$\{i^r, j^r, k^r\} = Q^{-1}\{\lambda^r, \mu^r, \nu^r\} \quad \ldots\ldots \quad (3.10)$$

or as Q is orthogonal

$$\{i^r, j^r, k^r\} = Q^T\{\lambda^r, \mu^r, \nu^r\} \quad \ldots\ldots \quad (3.11)$$

Alternatively, using the R and S matrices

$$\{i^r, j^r, k^r\} = Q^T R = Q^T S \quad \ldots\ldots \quad (3.12)$$

3.4.5 Point Co-ordination.

Since any family of surfaces, with the exception of its position and orientation in space, can be completely defined by the five curvature parameters r_1, r_2, t_1, γ_1 and γ_2, then, once a reference frame is established and the (ω, ϕ, N) system is related to this frame at one point,

point co-ordination can be achieved by measuring ω, ϕ and the above curvature parameters.

If

$$x_r = \{\frac{\partial x}{\partial \omega}, \frac{\partial x}{\partial \phi}, \frac{\partial x}{\partial N}\} \quad \ldots\ldots \quad (3.13)$$

and

$$x^r = \{\frac{\partial \omega}{\partial x} \quad \frac{\partial \phi}{\partial x} \quad \frac{\partial N}{\partial x}\} \quad \ldots\ldots \quad (3.14)$$

then by Equations (1.2) and (3.12)

$$\begin{pmatrix} \frac{\partial x}{\partial \omega} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial N} \\ \frac{\partial y}{\partial \omega} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial N} \\ \frac{\partial z}{\partial \omega} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial N} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \\ z_1 & z_2 & z_3 \end{pmatrix} = Q^T S \quad \ldots\ldots \quad (3.15)$$

The inverse transformation will be

$$\begin{pmatrix} \frac{\partial \omega}{\partial x} & \frac{\partial \phi}{\partial x} & \frac{\partial N}{\partial x} \\ \frac{\partial \omega}{\partial y} & \frac{\partial \phi}{\partial y} & \frac{\partial N}{\partial y} \\ \frac{\partial \omega}{\partial z} & \frac{\partial \phi}{\partial z} & \frac{\partial N}{\partial z} \end{pmatrix} = \begin{pmatrix} x^1 & x^2 & x^3 \\ y^1 & y^2 & y^3 \\ z^1 & z^2 & z^3 \end{pmatrix} = Q^T R \quad \ldots\ldots \quad (3.16)$$

It is seen that if r_1, r_2, t_1, γ_1 and γ_2 are measurable, the Equations (3.15) and (3.16) become functions of ω and ϕ alone and may therefore be integrated to provide either (x, y, z) or (ω, ϕ, N). Any subsequent computations can then be made in either system.

3.5 Measurement of r_1, r_2, t_1, γ_1 and γ_2.

The real problem of point co-ordination using Earth space alone is seen to be the determination of the five curvature parameters r_1, r_2, t_1, γ_1 and γ_2. In theory, a measurement of the vertical gradient of gravity

with a gravimeter leads to, via the law of gravity, $(r_1 + r_2)$, whereas torsion balance readings in several azimuths should provice $(r_1 - r_2)$, t_1, γ_1 and γ_2 *(ibid, 150-151)*. However, the direct measurement of the vertical gradient of gravity so far does not appear to have produced results which are comparable in accuracy with measurements made by torsion balance. Moreover, torsion balance measurements are extremely sensitive to the attraction of masses in the vicinity of measurement and are therefore, at present, not considered to give sufficiently representative results.

4. The Atmosphere.

4.1 The Nature of the Atmosphere.

The atmosphere has been likened to an onion, in that it is made up of many layers. A gross dissection shows the main layers to be, from the inside out:

(1) the troposphere, extending to an altitude of about 18 kilometres at the equator and about 8 kilometres at the poles;

(2) the stratosphere and

(3) the ionosphere.

Terrestrial geodetic observations are usually confined to the lowest part of the troposphere.

Throughout the troposphere the temperature of the air decreases with height at a rate, on the average, of about $0.16^{\circ}C$/30 metres. The actual rate at various levels and places naturely vary considerably from the average. It can be calculated that, if the atmosphere were well mixed and in hydrostatic equilibrium, a volume of air pushed upward becomes cooler (by expansion) at a constant rate of $0.3^{\circ}C$/30 metres *(Sutton 1964, 63)*. This rate of cooling, called the 'dry adiabatic lapse rate' is a theoretical standard that determines the stability or instability of the atmosphere. Accordingly, an atmosphere whose actual lapse rate is greater than the dry adiabatic rate must be stable in terms of vertical motions, because any volume of air that is displaced upwards will be less cold, less dense and thus lighter than its surroundings, and will therefore go on rising. The reverse applies in the case of downward displaced air.

All these arguments apply to moist air as well as to theoretically dry air. The moist air introduces a complication into any calculations made, but the conclusions remain essentially the same: any atmosphere, moist or dry, will be structurally stable, unstable or in neutral equilibrium according to whether its actual lapse rate is less than, equal to or greater than the adiabatic rate.

In these terms the earth's troposphere tends to be on the stable side; that is, the decline of temperature with height is on the average slightly less than the adiabatic lapse rate. Reversals of this situation are common. However, the significant point is, that in the troposphere as a whole, there are only isolated departures from the norm, usually associated with the movements of large air masses, the normal situation being a slightly subadiabatic lapse rate.

In the realm of micrometeorology - the first one hundred metres above the ground - the situation is quite different. There, large departures from the adiabatic lapse rate are a regular daily occurrence, particularly during periods of clear sunny weather. Consider what happens in the shallow layer near the ground during the 24 hours of a clear summer day. After sunrise the sun rapidly heats the surface of the ground to very high temperatures. From such hot surfaces the temperature decline in the air immediately above is quite sharp. In fact, the daytime temperature gradient near the ground frequently amounts to thousands of times the dry adiabatic lapse rate. Then as the sun sets, the picture changes rapidly. If there are clouds the ground radiates heat away freely and at a high rate. Because the air cools more slowly, an inversion occurs: the ground, having dropped to a lower temperature, takes the heat from the air immediately above it and thus the air temperature increases with height *(ibid, 64)*.

Winds, breezes and other factors, such as the nature of the soil, complicate the situation further.

4.2 A Model Atmosphere.

The problem of defining a model atmosphere is above all a problem in the analysis of the processes responsible for air movement in the atmosphere. Such an analysis points to some deep and difficult questions of fluid dynamics.

Now, there are two general types of fluid motion: laminar and turbulent. Nearly all motions of liquids and gases are turbulent. The detailed study of turbulence began in 1883 with laboratory experiments by O. Reynolds who found that the change from laminar to turbulent flow depended on three factors:

(1) the speed of flow;

(2) the diameter of the container and

(3) the viscosity of the fluid.

The product of the speed and the diameter divided by the kinematic viscosity, constitutes what is known as "Reynold's Number", and turbulence arises when this number exceeds a certain value.

However, Reynold's Number applies only to a fluid of uniform density throughout. The air of the lower atmosphere, of course, is not of uniform density, and in this case the transition from laminar flow to turbulence is determined primarily by the variation in density or, what amounts to the same thing, the rate of change of temperature with height.

Turbulent motion itself is still something of a mystery, not yet defined by a mathematical description such as has been developed for laminar motion. Nevertheless, meteorologists may before long be able to supply a sufficiently accurate model atmosphere of that existing at the time and in the locality of the observations performed in geodesy whence the accurate linearization of the data burdened with refraction will be possible.

4.3 Atmospheric Refraction.

The earth's atmosphere refracts the line of observation into a complicated space curve. Geodetic practise is to remove the effects of refraction by 'correcting' the observations so that the observed line becomes the straight line joining the two end points, a procedure which reduces the atmosphere to a space with a Euclidean metric. Any computations are then more readily performed in the latter space.

The corrections which are made to the observed ray in order to facilitate computations, ideally depend on the knowledge of the ray's curvature and torsion, which will be a function of the first and second covariant derivatives of the refractive index.

An approximation to a geodetic model atmosphere is one in which the torsion of the ray is ignored and in which the surfaces of equal refractive index are assumed gravitational potentials *(Hotine 1969, 214)*; the gradient of the refractive index is accordingly in the direction of the astronomic nadir. The geodesist is then left with the problem of measuring the magnitude of this gradient, which he does by sampling the temperature, pressure and humidity, but further assumptions are usually made before the magnitude can be established.

4.3.1 The Laws of Refraction.

The refractive index μ of a refracting medium is related to the velocity of light v in the medium by the equation

$$\mu = \frac{c}{v} \qquad \ldots\ldots \quad (4.1)$$

where c is the constant velocity of light in a vacuum. If ds is is an element of length along the path and t is the travel time, then

$$\mu\, ds = \frac{c}{v}\, ds = c\, dt \qquad \ldots\ldots \quad (4.2)$$

The optical path length or eikonal is defined as (ct) and denoted by S, so that

$$S = ct = \int \mu ds \qquad \ldots\ldots \quad (4.3)$$

According to Fermat's Principle, which states that light, for instance, will follow that path between two fixed points involving the least travel time, this integral has to be a minimum along the actual path, as compared with any other path joining the two fixed terminals.

4.4 Geometrical Wavefronts.

Consider a family of light rays emitted in all directions from a point source at the same instant. After a given time t, the

light will arrive at a surface known in physics as a geometrical wavefront and for different values of t, a family of surfaces S = constant results. The integral $\int \mu ds$ will have the same value over the actual path between the source and a given S-surface.

If it is assumed that the medium is isotropic so that μ is a point function (or scalar) having a definite value at each point of the space under consideration, then it can be visualised that the rays are normal to the wave-fronts and because of the minimum principle, the optical rays become geodesics of the surrounding space. Consequently, it may be said that the equation

$$S = \mu \ell_r \qquad \ldots\ldots \quad (4.4)$$

where ℓ_r is the unit tangent to the light ray or the unit normal to the wavefront, holds true in this space.

The expression of the space in Equation (4.4) by means of a single scalar S and the direction of its gradient can, as has been seen in Chapter 3, be made the basis of a general (a, b, C) type co-ordinate system. Contraction of Equation (4.4) with the expression $g^{rs}S_s = \mu g^{rs}\ell_s$ gives

$$S = g^{rs}S_s S_r = \mu^2 \qquad \ldots\ldots \quad (4.5)$$

and represents what is generally known as the Eikonal equation *(ibid, 210)*.

4.5 The Line of Observation.

The path in electronic distance measurement is generally curved slightly by atmospheric effects, the practise being to 'correct' the measurements to the direct ray in accordance with the best available refraction data before performing computations in a model space. Similarly, a theodolite is necessarily sighted along the tangent to an optical path curved by refraction and it again becomes inevitable to correct the measurement to the chord direction before proceeding further. The final results will be affected by any imperfections in the data used to correct

the measurements.

Now, the contravariant equation of a geodesic in three-dimensional space is given by *(ibid, 21)*

$$\ell^r{}_{,s}\ell_s = 0 \qquad \ldots\ldots \quad (4.6)$$

or

$$\frac{\partial \ell^r}{\partial s} + \Gamma^r_{st}\ell^s\ell^t = 0 \qquad \ldots\ldots \quad (4.7)$$

where ℓ^r is the unit tangent to the line and Γ^r_{st} are the Christoffel symbols. These three equations can be integrated in any co-ordinate system for which the Christoffel symbols are given, to provide the three contravariant components of the unit tangent in the atmosphere, and can be integrated further to provide co-ordinate changes along the line. In Cartesian co-ordinates, for example, the Christoffel symbols are zero *(Hawkins 1963, 135)* and Equation (4.7) indicates that all the three Cartesian components of the unit tangent are constant along the line, so that the changes in co-ordinates are proportional to the length of the line.

The fact that a solution to the problem exists in Cartesian co-ordinates for which the Γ^r_{st} are zero, indicates that the former must be a solution in vacuo. The Γ^r_{st} must therefore be determinable in terms of physical variables of the earth's atmosphere.

4.6 Point Co-ordination in the Atmosphere.

The geometry of the individual geometric wavefronts and their gradients is completely analogous to that of the (ω, ϕ, N) system described in Chapter 3, it only being necessary to change the notation from (N, n, ν_r) to (S, μ, ℓ_r) as required. Ideally, if the curvature parameters of the S-surfaces were known, that is, the normal curvatures in two perpendicular surface directions, the geodesic torsion in one of the former and the curvature of the normal (the line of sight) in the same two directions, then their geometry would be completely defined (Fig. 4.1).

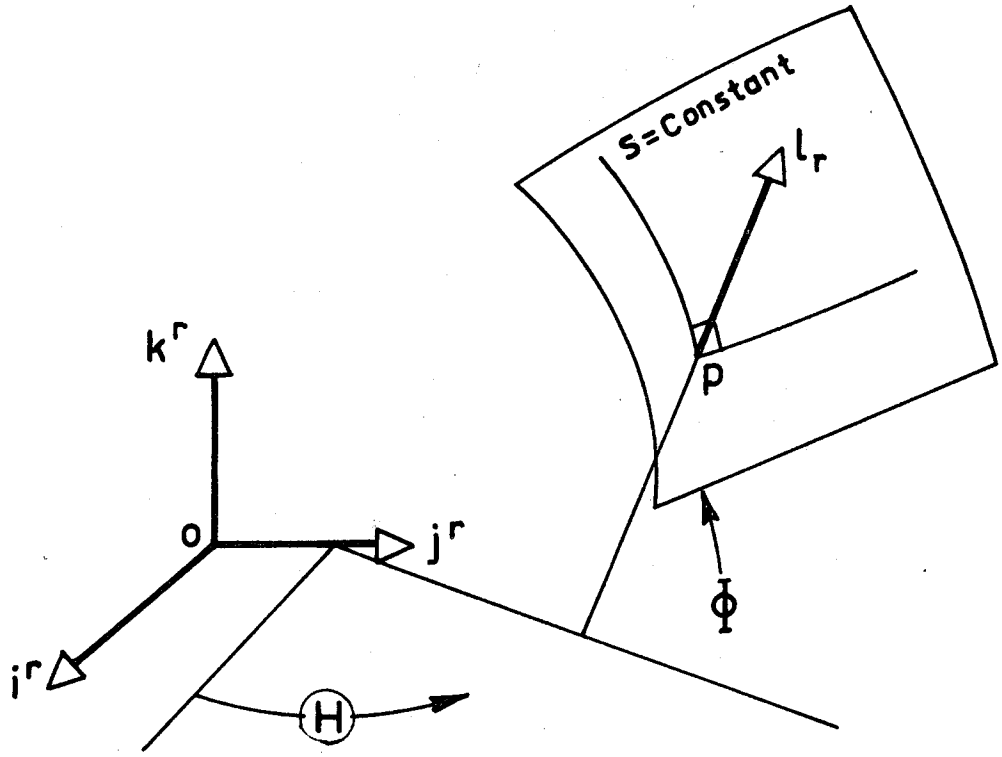

FIG. 4·1

Point co-ordination in the atmosphere alone becomes possible if as well as the five curvature parameters, the direction of the normal to the S-surface can be established in relation to the Cartesian axis vectors $\{i^r, j^r, k^r\}$. The solution to the problem is then completely analogous to that of Earth space and can be implemented by the use of Q, S and R type matrices.

5. The Mathematical Model.

Since the curvature parameters r_1, r_2, t_1, γ_1 and γ_2 cannot at present be measured to sufficient accuracy in the case of Earth space and in the case of the atmosphere cannot be measured at all, terrestrial point definition is not possible in either of the subspaces and the practice is to combine the two in order to effect a solution.

The geometry of Earth space is three-dimensional Euclidean, whereas that of the atmosphere, as evident from the fact that a geodesic within it is not a straight line, is Riemannian. However, the process of 'correcting' the ray path for curvature and for torsion will theoretically speaking reduce the latter to one with a Euclidean metric. The geometry of the combined space is thus also Euclidean and Cartesian co-ordinates may be chosen for point definition within it.

5.1 Point Co-ordination.

The azimuth (α) and zenith distance (β) define the direction of the S-surface normal (ℓ_r) relative to the vector triad (λ_r, μ_r, ν_r) (Fig. 5.1) and thus behave as the direction parameters of the normal to the geometrical wavefront in the sense (ω, ϕ) behaved as the direction parameters of the normal to the N-surface. The fact that the direction of ℓ_r is fixed relative to the N-surface normal ν_r, the direction of which is fixed relative to the Cartesian axis vectors $\{i_r, j_r, k_r\}$, can be used in conjunction with the measured distance s to provide Cartesian co-ordinate increments between connected points.

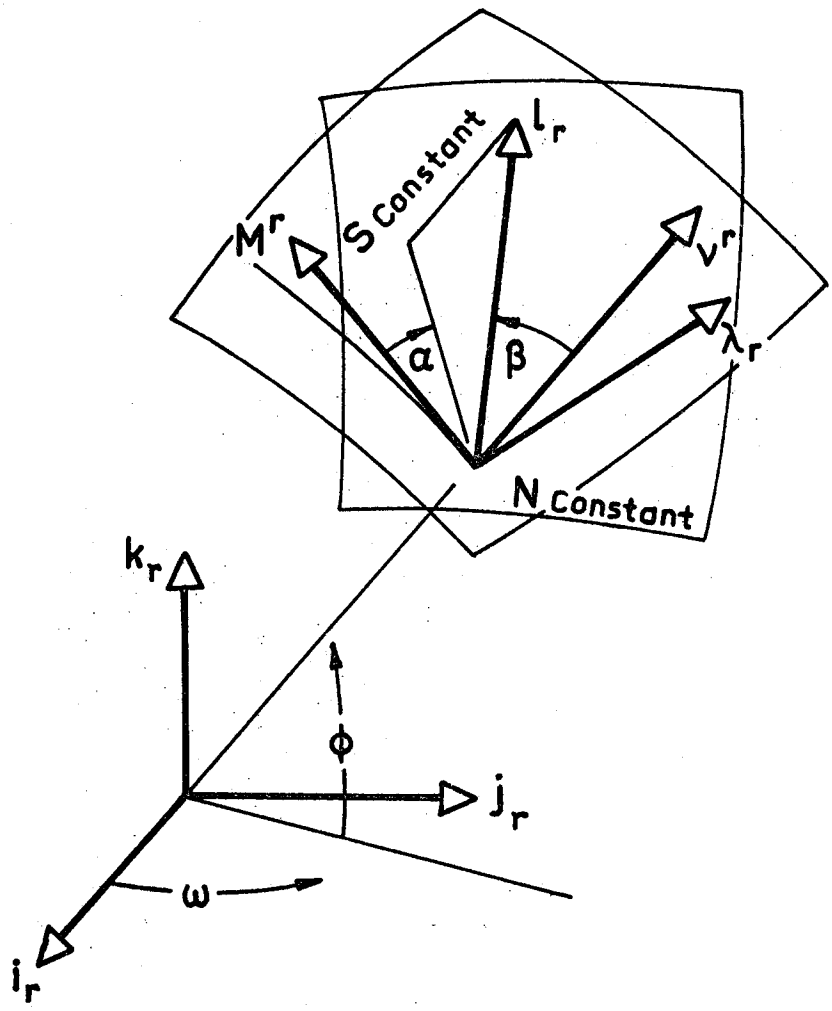

FIG. 5-1

5.1.1 The Co-ordinate System.

The co-ordinate system chosen to define terrestrial points is that described in 1.12.

5.1.2 Transformation Relationships.

The transformation formulae from the local $(\lambda_r, \mu_r, \nu_r)$ system to the geodetic Cartesian reference frame were found to be (Equation 3.9)

$$\{\lambda_r, \mu_r, \nu_r\} = Q\{i_r, j_r, k_r\}$$

the Celestial system (1.13) being implied within this transformation as the direction of the local vertical, as given by astronomic observations of latitude and longitude, is defined by it. By inversion, it was found (Equation 3.11) that

$$\{i_r, j_r, k_r\} = Q^T\{\lambda_r, \mu_r, \nu_r\}$$

If $\bar{\lambda}_r, \bar{\mu}_r, \bar{\nu}_r$ denote the unit vectors of the astronomic system at an adjacent point with astronomic co-ordinates $\bar{\phi}, \bar{\omega}$ then the previous relationships will be equally true for the new quantities. However, the vectors $\{i_r, j_r, k_r\}$ remain unchanged for both points, since their directions are fixed with respect to the rotating earth, i.e. they are independent of ϕ and ω.

Thus

$$\{\bar{\lambda}_r, \bar{\mu}_r, \bar{\nu}_r\} = \bar{Q}\{i_r, j_r, k_r\} = \bar{Q}\,Q^T\{i_r, j_r, k_r\}..... \quad (5.1)$$

where the elements of the matrix $\bar{Q}Q^T$ are given by *(Hotine 1959, 8)*.

$$q_{11} = \cos\delta\omega$$

$$q_{12} = \sin\phi\sin\delta\omega$$

$$q_{13} = \cos\phi\sin\delta\omega$$

$$q_{21} = \sin\phi\sin\delta\omega$$

$$q_{22} = \cos\bar{\phi}\cos\phi + \sin\bar{\phi}\sin\phi\cos\delta\omega$$

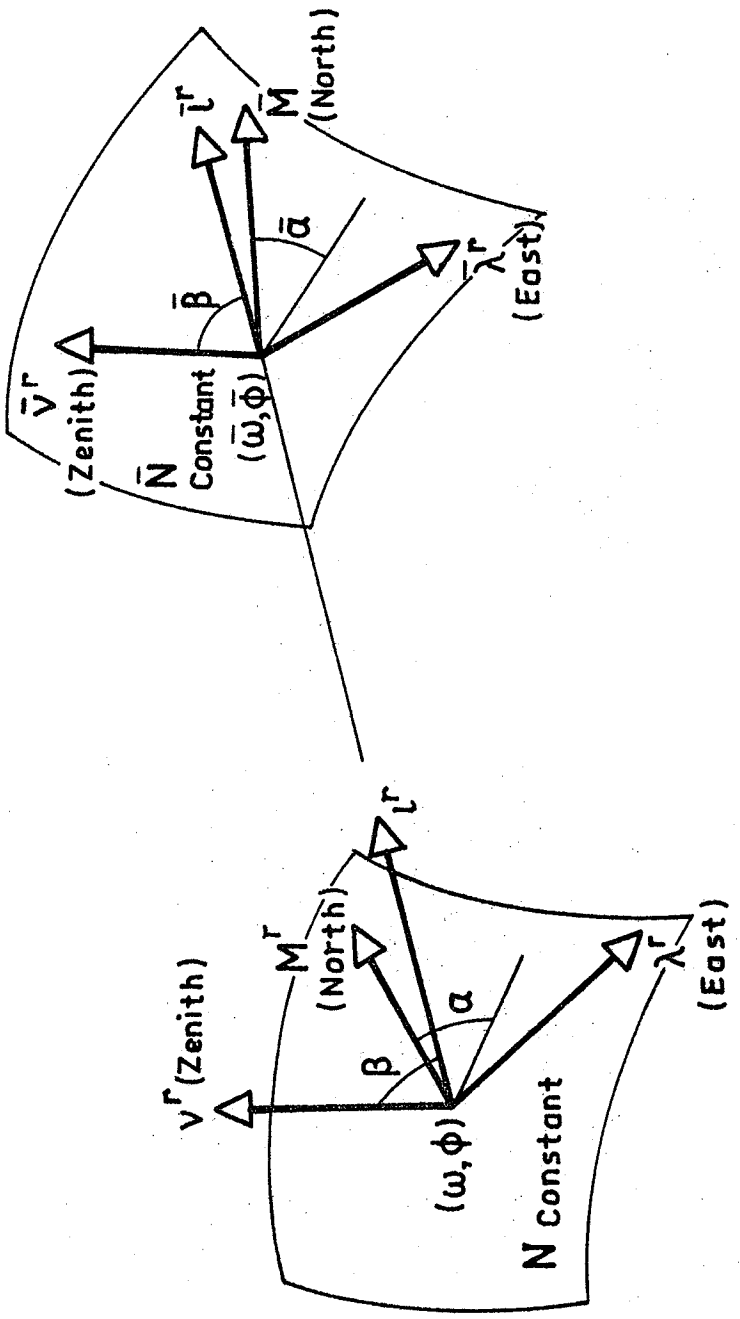
$\bar{v}^r$
(Zenith)
$\bar{N}$ Constant
$(\bar{\omega},\bar{\phi})$
$\bar{\beta}$
$\bar{\alpha}$
$\bar{l}^r$
$\bar{M}$
(North)
$\bar{\lambda}^r$
(East)
v^r (Zenith)
M^r
(North)
l^r
β
α
(ω,ϕ)
N Constant
λ^r
(East)

FIG. 5·2

$$q_{23} = \sin\bar{\phi}\sin\phi - \sin\bar{\phi}\sin\phi\cos\ \delta\omega$$

$$q_{31} = \cos\bar{\phi}\sin\ \delta\omega$$

$$q_{32} = \cos\phi\sin\bar{\phi} - \sin\phi\cos\bar{\phi}\cos\ \delta\omega$$

$$q_{33} = \sin\phi\sin\bar{\phi} + \cos\phi\cos\bar{\phi}\cos\ \delta\omega$$

and

$$\delta\omega = \bar{\omega} - \omega$$

They represent the transformation formulae between adjacent astronomic vector systems.

Previously the quantities $\{i_r, j_r, k_r\}$ were defined as unit axis vectors of the geodetic Cartesian reference frame x, y, z. In a similar manner it is possible to consider the elements $(\lambda_r, \mu_r, \nu_r)$ as unit axis vectors of a local Cartesian system. If as before ℓ_r is any unit vector fixed in space and (a, b, c) and (ℓ, m, n) are its respective components in the (x, y, z) and $(\lambda_r, \mu_r, \nu_r)$ systems, then the following transformation formulae exist between the component groups

$$\{\ell, m, n\} = Q^T\{a, b, c\} \quad \ldots\ldots \quad (5.2)$$

and conversely by inversion

$$\{a, b, c\} = Q\{\ell, m, n\} \quad \ldots\ldots \quad (5.3)$$

If the unit vector ℓ_r is now considered in the local co-ordinate system $(\lambda_r, \mu_r, \nu_r)$ then (Equation 1.6)

$$\{\ell, m, n\} = \{\sin\alpha\sin\beta, \cos\alpha\sin\beta, \cos\beta\} = Q^T\{a, b, c\} \quad \ldots\ldots \quad (5.4)$$

and by inversion and multiplication of matrices

$$\{a, b, c\} = \begin{pmatrix} -\sin\omega\sin\alpha\sin\beta - \sin\phi\cos\omega\cos\alpha\sin\beta + \cos\phi\cos\omega\cos\beta \\ \cos\omega\sin\alpha\sin\beta - \sin\phi\sin\omega\cos\alpha\sin\beta + \cos\phi\sin\omega\cos\beta \\ \cos\phi\cos\alpha\sin\beta + \sin\phi\cos\beta \end{pmatrix} \quad \ldots\ldots \quad (5.5)$$

The equation sets (5.4) and (5.5) are not independent, since from two equations

in each set the third follows, i.e.

$$\ell^{r}\ell_{r} = 1 \qquad \ldots\ldots \quad (5.6)$$

Alternatively if the same vector ℓ_r were to originate at an adjacent point with astronomical co-ordinates $(\bar{\omega}, \bar{\phi})$ and has there azimuth $(\bar{\alpha})$ and zenith distance $(\bar{\beta})$ then similar formulae will exist between these quantities. However, the components {a, b, c} will be the same in the relationships established for the new quantities as they refer to the same Cartesian system. Thus for this point

$$\{\sin\bar{\alpha}\sin\bar{\beta},\ \cos\bar{\alpha}\sin\bar{\beta},\ \cos\bar{\beta}\} = \bar{Q}^{T}\{a, b, c\} \qquad \ldots\ldots \quad (5.7)$$

and by substitution of Equation (5.5) into Equation (5.7) it is found, after some manipulation *(ibid, 9)* that

$$\{\sin\bar{\alpha}\sin\bar{\beta},\ \cos\bar{\alpha}\sin\bar{\beta},\ \cos\bar{\beta}\} = \begin{pmatrix} \ell\cos\delta\omega + m\sin\phi\sin\delta\omega - n\cos\phi\sin\delta\omega \\ -\ell\sin\bar{\phi}\sin\delta\omega + m(\cos\phi\cos\bar{\phi} + \sin\phi\sin\bar{\phi}\cos\delta\omega) + n(\sin\phi\cos\bar{\phi} - \cos\phi\sin\bar{\phi}\cos\delta\omega) \\ \ell\cos\bar{\phi}\sin\delta\omega + m(\cos\phi\sin\bar{\phi} - \sin\phi\cos\bar{\phi}\cos\delta\omega) + n(\sin\phi\sin\bar{\phi} + \cos\phi\cos\bar{\phi}\cos\delta\omega) \end{pmatrix}$$

$$\ldots\ldots \quad (5.8)$$

These three equations are again not independent, as from a pair the third follows. They give the transformation formulae for azimuth and zenith distance of any space vector from one local system into an adjacent one. Equation (5.8) gives the azimuth and zenith distance of the vector ℓ_r at the barred point as functions of the observed azimuth and zenith distance at the unbarred point. This is made possible by moving the vector ℓ_r parallel to itself into the barred system. It is important to note, however, that the quantities $(\bar{\alpha})$ and $(\bar{\beta})$ so obtained refer to the same sense of the $(\bar{\lambda}_r, \bar{\mu}_r, \bar{\nu}_r)$ vectors as do the quantities (α) and (β) (Fig. 5.2). Thus, to obtain back directions $(\bar{\beta})$ must be subtracted from 180° and 180° should be added to $(\bar{\alpha})$.

The difference in orientation between adjacent astronomic systems will usually be small and if

$$\{\bar{\phi}, \bar{\omega}, \bar{\alpha}, \bar{\beta}\} = \{\phi + \delta\phi, \omega + \delta\omega, \alpha + \delta\alpha, \beta + \delta\beta\} \qquad \ldots\ldots \quad (5.9)$$

then to the first order Equations (5.8) reduce to

$$\delta\alpha = \sin\phi\,\delta\omega + \cot\beta\,(\sin\alpha\delta\phi - \cos\alpha\cos\phi\delta\omega) \quad \ldots\ldots \quad (5.10)$$

$$\delta\beta = -\cos\phi\sin\alpha\delta\omega - \cos\alpha\delta\phi \quad \ldots\ldots \quad (5.11)$$

If in addition $\beta \simeq 90^{\circ}$, a legitimate assumption for terrestrial geodetic measurements, then Equation (5.10) becomes

$$\delta\alpha = \sin\phi\delta\omega \quad \ldots\ldots \quad (5.12)$$

This is the so-called Laplace azimuth equation which is used in the classical method to orient the reference ellipsoid.

5.1.3 Point Co-ordination.

Point co-ordination within the mathematical model follows from Equation (5.5), which provides Cartesian components of the unit vector ℓ_r emanating from a point of known position. Co-ordinate increments between connected points are given by multiplying these components by the measured distance s. Thus the co-ordinates of the forward point become

$$\{x, y, z\} = \{x_o, y_o, z_o\} + s\{a, b, c\} \quad \ldots\ldots \quad (5.13)$$

where (x_o, y_o, z_o) are the co-ordinates of the point of known position.

5.2 Origin Conditions.

5.2.1 Co-ordinates of the Datum Point.

As with all geodetic datums, the co-ordinates of the origin must be defined. Any arbitrary values of (x_o, y_o, z_o) may be chosen; however, if geocentric co-ordinates are desired, a global gravimetric analysis, in conjunction with astronomic and levelling data, or an analysis of the results from the geometrical use of satellites is necessary.

5.2.2 Orientation of the Survey Scheme.

The previous considerations only concern the positioning of the survey origin with respect to the geocentre. It remains to orient the scheme of the survey with respect to the Cartesian system. This may be achieved in two ways by the manipulation of the astronomic triad at the origin of survey.

In general, a co-ordinate system is said to be properly oriented in space when any one vector is uniquely defined within it and no rotation can be effected about this vector. Thus with respect to the

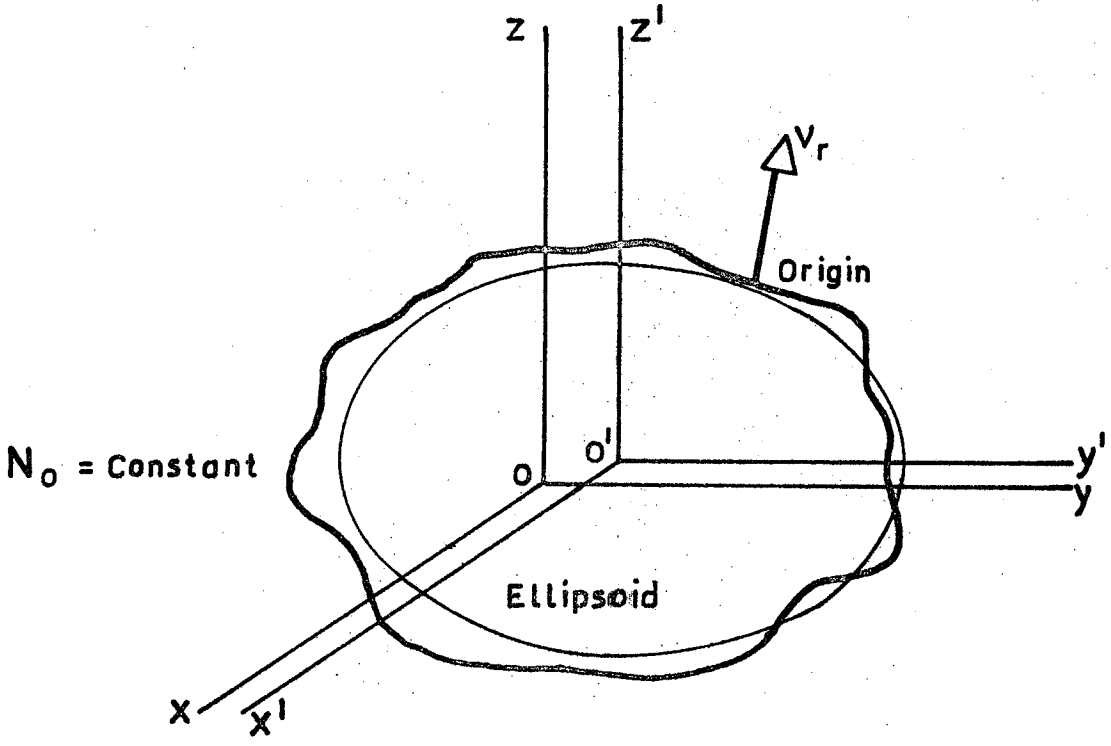

FIG. 5·3

Cartesian reference frame and the local astronomic triad at the initial point of a survey, two aspects must be considered:

(1) the direction of a vector in the astronomic triad must be fixed with respect to the Cartesian system and

(2) no rotation must be possible about the vector chosen in (1).

These aspects and their implication will be considered in sequence for the two separate methods that are available.

The natural vector to hold fixed in direction, with respect to the Cartesian system at the origin of survey would be the local vertical as its direction cosines are defined by astronomic latitude and longitude observations. Finally, the system would become completely rigid, when the direction of another vector within the local system is held fixed. The most convenient method by which to achieve this, is to observe the astronomic azimuth of a line emanating from the origin - a procedure which is customarily adopted.

The principles of an alternative method of orienting the local astronomic system have already been treated (*ibid, 10-12) (Heilbronner 1968, 46-48)* and will therefore be only considered briefly.

The parameters (α = azimuth and β = refraction free zenith distance) of the unit vector ℓ_r can be used to define the local co-ordinate system to a rotation about the vector ℓ_r. This degree of freedom may be eliminated by the fixation of a second direction $\bar{\ell}_r$ emanating from the same point. The method is best explained using geometric concepts. Thus when the astronomic triad is rotated about the vector ℓ_r (parameters α, β) the local vertical ν_r will generate a circular cone with apex at the origin. To spatially fix ν_r Hotine *(ibid)* allows this cone to be intersected or touched by another cone (parameter $\bar{\beta}$). The same effect is achieved when instead of a second cone, a plane (parameter $\bar{\alpha}$) that contains the apex of the previous cone and the vector ν_r is chosen. To avoid ambiguity Heilbronner *(1968)* emphasises that the difference between the azimuth α and $\bar{\alpha}$ should not be near or equal to 90°.

Thus in this latter method, it is possible to fix the unit vectors of the triad (λ_r, μ_r, ν_r) by means of the parameters α, β, $\bar{\beta}$, or α, β, $\bar{\alpha}$ of two independent directions in that triad.

As both the former methods lead to the same results, field conditions determine the method that is preferable. The refraction free zenith distance is not generally known and even if the levelled difference in elevation between the origin and an adjacent point has been determined, additional information such as astronomic position observations or gravity data, is required to calculate it. Overall therefore the initially described method is preferable.

5.2.3 Free-Net Adjustment.

All measurements are subject to error and statistically, it is not to be expected that a particular azimuth is better than another. Thus, since many azimuths are usually observed over a geodetic datum, it may be of advantage to keep the network free of azimuth constraints which, in a least squares adjustment procedure implies that the orientation of the survey scheme relative to the Cartesian system becomes such that the sum of the weighted squares of the residuals of the observed azimuths is a minimum.

5.3 Errors in Astronomic Data and their Effect Upon Orientation.

It is legitimate to assume that the astronomic measurements at the origin, even though of the highest precision, will contain errors.

5.3.1 Errors in (ω) and (ϕ).

In order to visualise the effect of errors in the observed longitude and latitude at the origin, it is convenient to introduce an ellipsoid of revolution whose principal axes are parallel to (i_r, j_r, k_r) and which is tangential, at the origin of survey, to the equipotential surface containing the survey origin (Fig. 5.3). It is emphasised that such an ellipsoid is not in general geocentric.

The rectangular Cartesian co-ordinates of the survey origin relative to the centre of this ellipsoid are then

$$\{x'_o, y'_o, z'_o\} = \nu\{\cos\phi \cos\omega, \cos\phi \sin\omega, (1-e^2) \sin\phi\} \quad \ldots\ldots \quad (5.14)$$

Thus if

$$\{\phi, \omega\} = \{\phi' + \delta\phi, \omega' + \delta\omega\}$$

and $\delta\phi$, $\delta\omega$ are small so that the terms containing the products of these

elements may be neglected, then to the first order

$$\{x'_o,\ y'_o,\ z'_o\} = \nu \begin{pmatrix} \cos\phi' \cos\omega' \\ \cos\phi' \sin\omega' \\ \overline{1-e^2} \quad \sin\phi' \end{pmatrix} + \nu \begin{pmatrix} -\sin\phi' \cos\omega', & -\sin\omega' \cos\phi' \\ -\sin\phi' \sin\omega', & \cos\omega' \cos\phi' \\ \overline{1-e^2} \quad \cos\phi', & 0 \end{pmatrix} \{\delta\phi,\ \delta\omega\}$$

..... (5.15)

Comparing Equations (5.14) and (5.15) it is seen that the ellipsoid centre is translated by amounts

$$\{\delta x,\ \delta y,\ \delta z\} = \nu \begin{pmatrix} -\sin\phi' \cos\omega', & -\cos\phi' \sin\omega' \\ -\sin\phi' \sin\omega', & \cos\phi' \cos\omega' \\ \overline{1-e^2} \quad \cos\phi', & 0 \end{pmatrix} \{\delta\phi,\ \delta\omega\} \quad ..... \quad (5.16)$$

The axes of the system and the ideal reference frame will, however, remain parallel.

5.3.2 Errors in (α).

The orientation of the scheme of survey is, as was previously mentioned, partly achieved by astronomic azimuth observations. Unfortunately, such observations are often severely influenced by systematic observational errors *(Bomford 1967b)*. The effect of introducing an erroneous azimuth at the datum point will be to rotate the local triad, and thus the system (x', y', z') (Fig. 5.3), about the local vertical at that point. This means that the system (x', y', z') is rotated so that its axes are no longer parallel to those of the (x, y, z) system. Thus, if an approximate azimuth α' is introduced at the datum point so that

$$\alpha = \alpha' + \delta\alpha$$

then the geocentric co-ordinates of the origin of survey, since it lies on the axis of rotation, will remain unchanged. However, co-ordinate changes will result in all other points of the datum by amounts

$$\{\delta x,\ \delta y,\ \delta z\} = R\{\bar{x} - x_o,\ \bar{y} - y_o,\ \bar{z} - z_o\} \quad ..... \quad (5.17)$$

where R is the rotational matrix resulting from the introduction of an erroneous azimuth at the datum point. The R matrix is given by

$$R = Q^T \begin{pmatrix} \cos\delta\alpha & \sin\delta\alpha & 0 \\ -\sin\delta\alpha & \cos\delta\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} Q - I$$

$$= \begin{pmatrix} 0 & \delta\alpha \sin\phi & -\delta\alpha \cos\phi \sin\omega \\ -\delta\alpha \sin\phi & 0 & \delta\alpha \cos\phi \cos\omega \\ \delta\alpha \cos\phi \sin\omega & -\delta\alpha \cos\phi \cos\omega & 0 \end{pmatrix} \quad \ldots\ldots \quad (5.18)$$

providing $\delta\alpha$ is small so that one may write

$$\{\cos\delta\alpha, \sin\delta\alpha\} = \{1, \delta\alpha\}$$

It must be remembered that a new rotational matrix R is introduced for every station where another azimuth is observed, the overall effect of which would be to produce a series of non-parallel reference frames (x', y', z').

5.4 Levelling.

In Fig. 5.4, S is the physical surface of the earth and p and q are arbitary points on S. ρ_r is the position vector joining the origin 0 of the (x, y, z) system and a surface point, whereas $qC = -dh$ is the infinitesimal levelling increment between p and q and is measured normal to the equipotential surface N = constant through p. Denoting the projection of $pq = s$ onto the surface ρ_r = constant by ds and the infinitesimal increase qD in the vector ρ_r as $-d\rho_r$ gives to the first order that $Cq = qE$, or

$$-|d\rho_r| + dh = \delta\, ds \quad \ldots\ldots \quad (5.19)$$

where δ is the component of the angle between the vectors ρ_r and $-\nu_r$ as measured in the azimuth α of the vertical plane containing p and q.

Thus

$$|d\rho_r| = dh - (\xi' \cos\alpha + \eta' \sin\alpha)ds \quad \ldots\ldots \quad (5.20)$$

where ξ' and η' are respectively the components of the angle δ in the plane of the meridian and of the prime vertical.

Equation (5.20) establishes the connection between an infinitesimal levelling increment and the increment in the vector ρ_r. The difference in

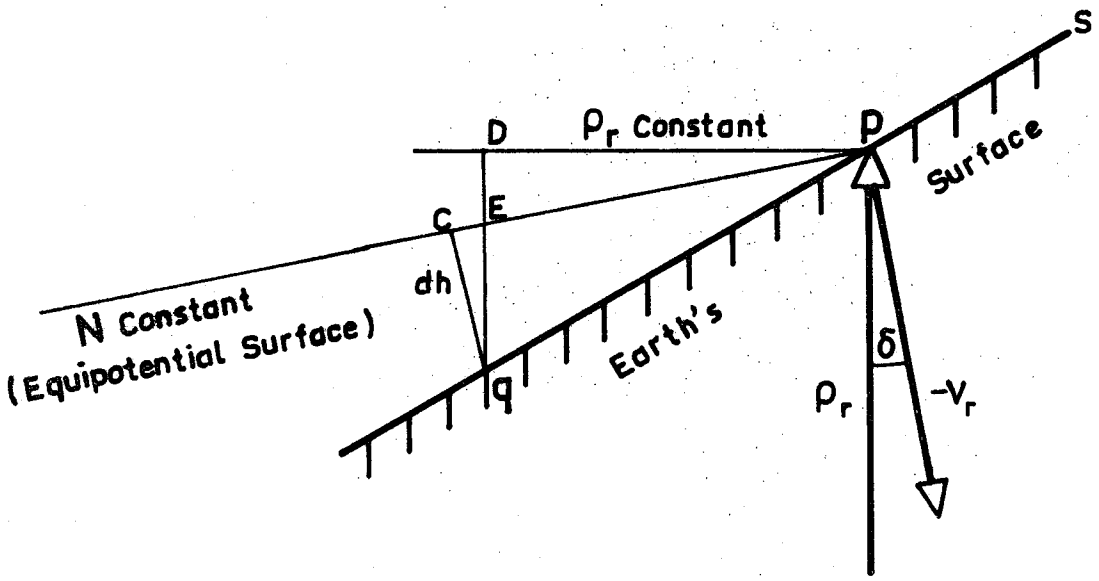
S
D
ρ_r Constant
P
Surface
C
E
dh
N Constant
(Equipotential Surface)
Earth's
q
δ
ρ_r
$-\nu_r$

FIG. 5·4

magnitude of the vectors ρ_r and $\bar{\rho}_r$ at p and q respectively, which are a finite distance apart is expressed by the equation

$$\bar{\rho}_r - \rho_r = \lambda = \int_p^q dh - \int_p^q (\xi' \cos\alpha + \eta' \sin\alpha)ds \quad \ldots\ldots \quad (5.21)$$

and since the integrals depend on the path of levelling, these must be calculated along the same path.

To determine ξ' and η' a knowledge is required of the direction of the vertical and the direction of the vector ρ_r at each point. The direction of the vertical may be established by astronomic latitude and longitude observations or may be interpolated by the use of gravity data, and the relative direction of the vector ρ_r may be obtained from Equation (5.5).

The problem of unifying the results of geometric levelling and all other geometric geodetic measurements thus reduces to the determination of the magnitude and direction of the vector $\bar{\rho}_r$. This vector is completely defined by the equation

$$\bar{\rho}_r = \rho_r + s\{a, b, c\}\{i_r, j_r, k_r\}^T \quad \ldots\ldots \quad (5.22)$$

where the $\{a, b, c\}$ are computed from Equation (5.5) and the results of levelling have not been included.

All the observations defining the $\{a, b, c\}$ are made at discrete points and all but the measured distances are subject to the earth's gravitational field at that point. However, assuming that the earth's gravitational field remains constant over the period of observation, all observations are equally affected at a point. For this reason Equation (5.22) completely defines the vector $\bar{\rho}_r$.

Considering the results of geometric levelling another vector $\bar{\rho}'_r$ may be obtained using the direction established by Equation (5.22) and the magnitude λ calculated in Equation (5.21) or $\bar{\rho}'_r = (|\rho_r| + \lambda)m^r$ where m^r is the unit vector in the direction of $\bar{\rho}_r$ and is defined by $m_r = \bar{\rho}_r/|\rho_r|$.

The position vector $\bar{\rho}'_r$ will give rise to another set of co-ordinate differences $s\{a', b', c'\}$ where in general $\{a', b', c'\} \neq \{a, b, c\}$ and since the direction of the two position vectors $\bar{\rho}'_r$ and $\bar{\rho}_r$ are identical, this additional set of co-ordinate

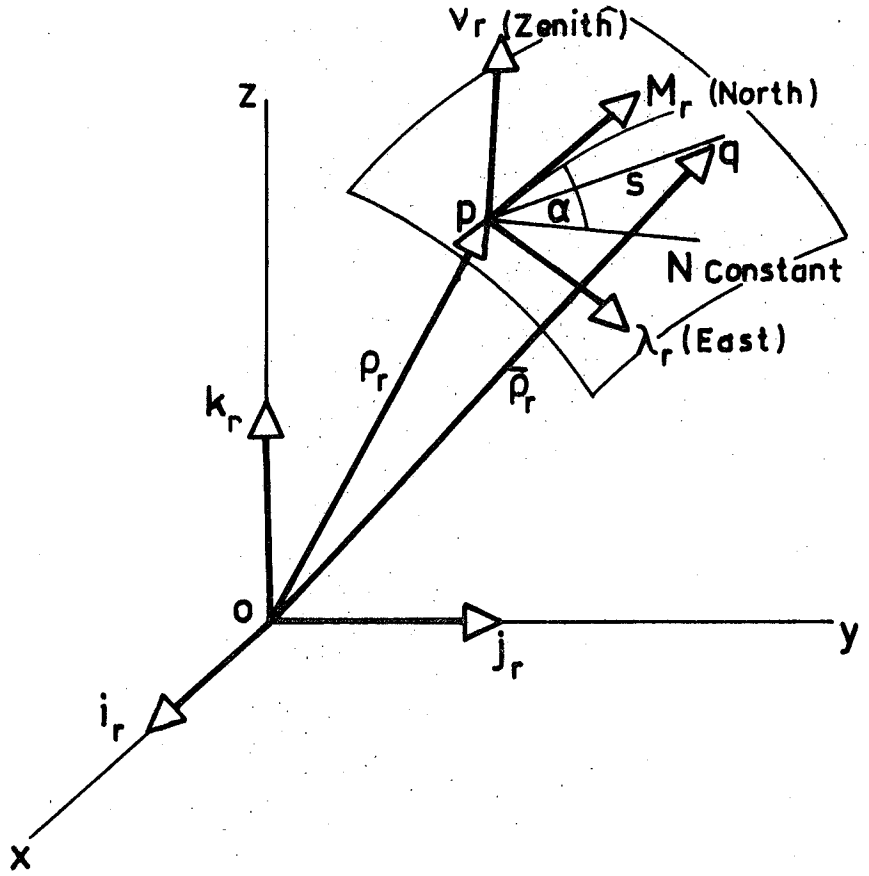
v_r (Zenith)
M_r (North)
q
s
α
p
N Constant
λ_r (East)
z
ρ_r
$\bar{\rho}_r$
k_r
o
y
j_r
i_r
x

FIG. 5·5

differences $s\{a', b', c'\}$ gives rise to another zenith distance (refraction-free) β between connected points. This new zenith distance is defined by $\cos\beta = \nu_r \bar{\ell}^r$ where $\bar{\ell}_r$ is the unit vector in the direction of pq as obtainable from $\bar{\ell}_r = (\rho_r - \bar{\rho}'_r)/(|\rho_r - \bar{\rho}'_r|)$.

The relationship between this computed zenith distance and the observed zenith distance remains somewhat obscure and the question arises whether it is possible to do without observed zenith distances when all or some points of a spatial geodetic network have been connected by geometric levelling. Previously it was stated that Equation (5.22) completely defined the vector $\bar{\rho}_r$ when the results of geometric levelling were not considered and that an additional refraction-free zenith distance could only be calculated using the direction as established by this equation and the magnitude $|\rho_r| + \lambda$ as given by levelling. This gives the impression that the two are algebraically related. It can be shown however, that in order to determine the $\{a, b, c\}$ it is not a prerequisite to know the observed zenith distance when the points p and q have been connected by levelling and that therefore the two quantities are distinct.

From Fig. (5.5) it can be seen that the point q is uniquely defined by the intersection of the spatial segments pq, Oq with the vertical plane containing the line segment pq (parameter α). Two sets of solutions $\{a, b, c\}$ are obtained from the three conditions that are to be satisfied, i.e.

$$\begin{aligned} \ell_r \ell^r &= 1 \\ m_r m^r &= -\bar{\ell}_r \bar{\ell}^r \qquad \ldots\ldots \quad (5.23) \\ \tan\alpha &= \frac{b\cos\omega - a\sin\omega}{c\cos\phi - a\sin\phi\cos\omega - b\sin\phi\sin\omega} \end{aligned}$$

The required solution may then be extracted, as the direction of the line segment pq is known and it can be seen that the measured and calculated zenith distances are independent.

5.5 The Solution.

The ideal theoretical requirements of the method of point co-ordination presented in this chapter are:

(1) astronomic determinations of latitude and longitude of every station

as well as the observed azimuth of one line in each local triad and

(2) the connection of adjacent stations by spirit levelling.

Astronomic position determinations are necessary to define the direction of the local vertical and an azimuth is needed to orient the astronomic triad. The connection of trigonometric stations by levelling is desirable because the measured zenith distances are known to be strongly influenced by uncertainties in refraction for average terrestrial networks and thus the precision of the vertical angle measurement is not compatible with that of other geodetically determined quantities.

Apart from giving theoretically sound results, compliance with the abovementioned conditions will, in conjunction with other geodetic measurements, ensure a high degree of over-determination and thus a good estimate of the precision of the results.

Unfortunately, departures from such an ideal system are the rule rather than the exception. However, most modern terrestrial networks, of which the AGD is an example, comprise a fairly regular pattern of Laplace stations and trigonometric points where both the latitude and longitude have been observed. Levelling connections are also made where practicable and at fairly regular intervals and apart from the abundance of geometric data, a good coverage of gravity data is commonly available.

Thus, providing a gravity survey round the station exists, the problem of the deficiency of astronomic position determinations may be overcome by gravimetric interpolations of the deviations of the vertical. A precision of ± 0".5 is possible by this method and is approximately the order of accuracy of the astronomic observations.

One possibility of reducing the effect of refraction on measured zenith distances would be to adjust the level net separately and to recompute 'new' zenith distances from the trigonometric heights derived from an additional adjustment in which the points connected by levelling have been held fixed. A better method is to compute the zenith distances between points connected by levelling according to the proposals within 5.4 and to incorporate this computed zenith distance into the adjustment with its appropriate weight coefficient.

PART B: ADJUSTMENT.

The formulae derived in Chapter 5 are those that are required for the computation of provisional co-ordinates from observed quantities which are all subject to observational error. The observed values will be used in this section to set up observation equations from which corrections to the provisional co-ordinates can be deduced in order to obtain estimates of the most probable values of these quantities.

In modern geodetic work, it is customary to measure directions, distances, vertical angles, azimuths, latitudes, longitudes, levelled differences in elevation and gravity. The number of observations made always exceeds the minimum required for the unique determination of the unknown quantities. These redundant observations serve to guard against blunders and to obtain a statistically more precise estimate of the unknowns.

One way to deal with the problem of adjustment is to form condition equations, one for each redundant observation, with the corrections to the observed quantities as unknowns. Assuming a normal distribution of errors, the most probable set of corrections is given, according to the principle of least squares, by that set which minimises the sum of the weighted squares of the residuals, while satisfying the conditions equations. A different approach is to assume initial values for all the unknowns and to express the effect of small changes in the observed quantities upon these values. The most probable corrections to the initial estimate of the unknowns are then those which make the sum of the weighted squares of the changes in the observed quantities a minimum. As opposed to the adjustment by 'conditions' this method represents an adjustment by 'parameters or differential displacements.'

On account of its suitability for programmed computation, only the method of differential displacements is considered throughout this dissertation. Moreover, the least squares procedures and error analysis are quicker.

The procedure is commonly commenced by assuming approximations for the required unknowns. If the displacements resulting from the

adjustment are too large, then the corrected values of the unknowns may be used as second approximations and the process reapplied.

The main disadvantage of the method is the inability of the procedure to detect mistakes, either in the data or the results, at the completion of one cycle of the adjustment.

When the unknowns are the geographical or Cartesian co-ordinates, the differential displacement method is known by the name of 'variation of co-ordinates.'

6. Adjustment.

6.1 Differential Displacements.

It is now proposed to find the changes in length, azimuth and zenith angle resulting from a change δx, δy, δz in the end co-ordinates of a line segment with unit vector ℓ_r.

Multiplying both sides of Equation (5.5) by s and remembering that

$$s\{a, b, c\} = \{\bar{x} - x, \bar{y} - y, \bar{z} - z\}$$

where the $\{\bar{x} - x, \bar{y} - y, \bar{z} - z\}$ represent the changes in Cartesian co-ordinates between adjacent surface points, then

$$s\{\sin\alpha \sin\beta, \cos\alpha \sin\beta, \cos\beta\} = Q^T\{\bar{x} - x, \bar{y} - y, \bar{z} - z\} \quad \ldots \quad (6.1)$$

these relationships can be used to calculate the measured values s, α and β, i.e.

$$s = \sqrt{(\bar{x} - x)^2 + (\bar{y} - y)^2 + (\bar{z} - z)^2} \quad \ldots\ldots \quad (6.2)$$

$$\tan\alpha = \frac{\ell}{m} = \frac{(\bar{y} - y)\cos\omega - (\bar{x} - x)\sin\omega}{(\bar{z} - z)\cos\phi - (\bar{x} - x)\sin\phi\cos\omega - (\bar{y} - y)\sin\phi\sin\omega} \quad \ldots\ldots \quad (6.3)$$

$$\cos\beta = \frac{n}{s} = \frac{1}{s}\left|(\bar{x} - x)\cos\phi\cos\omega + (\bar{y} - y)\cos\phi\sin\omega + (\bar{z} - z)\sin\phi\right| \quad \ldots\ldots \quad (6.4)$$

Upon differentiating Equations (6.2), (6.3) and (6.4) one obtains *(Wolf 1963b)*

$$ds = \frac{1}{s}\left|(\bar{x} - x)(\delta\bar{x} - \delta x) + (\bar{y} - y)(\delta\bar{y} - \delta y) + (\bar{z} - z)(\delta\bar{z} - \delta z)\right| \quad \ldots\ldots \quad (6.5)$$

$$d\beta = \frac{(\bar{x} - x)\cos\beta - s\cos\phi\sin\omega}{s^2 \sin\beta}(\delta\bar{x} - \delta x) +$$

$$\frac{(\bar{y} - y)\cos\beta - s\cos\phi\sin\omega}{s^2 \sin\beta}(\delta\bar{y} - \delta y) +$$

$$\frac{(\bar{z} - z)\cos\beta - s\sin\phi}{s^2 \sin\beta}(\delta\bar{z} - \delta z) + \quad \ldots\ldots \quad (6.6)$$

$$\cos\alpha\,\delta\phi - \cos\phi\sin\alpha\,\delta\omega$$

$$d\alpha = \frac{\sin\phi\cos\omega\sin\alpha - \sin\omega\cos\alpha}{s\sin\beta}(\delta\bar{x} - \delta x) +$$

$$\frac{\sin\phi\sin\omega\sin\alpha + \cos\omega\cos\alpha}{s\sin\beta}(\delta\bar{y} - \delta y) +$$

$$\frac{\cos\phi\sin\alpha}{s\sin\beta}(\delta\bar{z} - \delta z) + \quad \ldots\ldots \quad (6.7)$$

$$\cos\beta\sin\alpha\,\delta\phi + (\sin\phi - \cos\alpha\cos\phi\cos\beta)\,\delta\omega$$

which represent the sought expressions.

6.2 Variation of Co-ordinates.

The method of variation of co-ordinates is readily adaptable to the simultaneous adjustment of a spatial network on a digital computer. The required provisional co-ordinates may either be obtained by a 'forward computation' or from the geographical co-ordinates of a prior adjustment and may then be used in conjunction with the iterative procedure previously mentioned.

The unknowns of the observation equations are $\delta x, \delta y, \delta z$, the corrections to the provisional Cartesian co-ordinates x, y, z of the network stations and $\delta\phi, \delta\omega$ the corrections to astronomic latitude and longitude respectively, it being necessary to introduce ϕ and ω as unknowns because all observations with the exception of the measured distances are subject to the direction of gravity.

As is customary, the approximations -

$$\{x, y, z, \phi, \omega, 0, k\} \qquad \ldots\ldots \quad (6.8)$$

$$= \{x' + \delta x, y' + \delta y, z' + \delta z, \phi' + \delta\phi, \omega' + \delta\omega, 0' + \delta 0, k' + \delta k\}$$

are introduced, where $\delta 0$ is the unknown optimum correction to the round of theodolite directions at the observing station and δk is the correction to be applied to the assumed value of the coefficient of refraction.

Denoting the unknowns δx, δy, δz, $\delta\phi$ and $\delta\omega$ of the observation equations by the vector δY, the changes $\delta\alpha$ in the azimuth, $\delta\beta$ in zenith angle and δs in distance - resulting from displacements δx, $\delta\bar{x}$, δy, $\delta\bar{y}$, δz, $\delta\bar{z}$ in the provisional co-ordinates of the two end stations and changes $\delta\omega, \delta\bar{\omega}, \delta\phi, \delta\bar{\phi}$ in the direction of the verticals - are given by equations of the form

$$\begin{aligned} \delta\alpha &= a\{\delta Y - \delta\bar{Y}\} \\ \delta\beta &= b\{\delta Y - \delta\bar{Y}\} \\ \delta s &= c\{\delta Y - \delta\bar{Y}\} \end{aligned} \qquad \ldots\ldots \quad (6.9)$$

The interim result will be the matrix of coefficients A and the vector of right-hand sides b of the observation equations

$$AY = b + v \qquad \ldots\ldots \quad (6.10)$$

The equations that must be satisfied by the unknowns, subject to $v^T Wv$ being a minimum, may be of the following types:-

(i) Directions (α)

$$v_\alpha = -\delta 0 + a\{\delta Y - \delta\bar{Y}\} + (\alpha' - \alpha + 0') \qquad \ldots\ldots \quad (6.11)$$

where

$$\tan\alpha' = \frac{(\bar{y}' - y')\cos\omega' - (\bar{x}' - x')\sin\omega'}{(\bar{z} - z')\cos\phi' - (\bar{x}' - x')\sin\phi'\cos\omega' - (\bar{y}' - y')\sin\phi'\sin\omega'} \qquad .. \quad (6.12)$$

$$\delta\bar{\phi} = \delta\bar{\omega} = 0 \qquad \ldots\ldots \quad (6.13)$$

and

$$\begin{aligned} a_1 &= (\sin\phi'\cos\omega'\sin\alpha' - \sin\omega'\cos\alpha')/s'\ \sin\beta' \\ a_2 &= (\sin\phi'\sin\omega'\sin\alpha' + \cos\omega'\cos\alpha')/s'\ \sin\beta' \\ a_3 &= -\cos\phi'\sin\alpha'/s'\ \sin\beta' \\ a_4 &= \cot\beta'\sin\alpha' \\ a_5 &= \sin\phi' - \cos\alpha'\cos\phi'\cot\beta' \end{aligned} \qquad \ldots\ldots \quad (6.14)$$

(ii) Zenith angles (β).

$$v_\beta = -s\delta k + b\{\delta Y - \delta\bar{Y}\} + (\beta' - \beta + sk') \quad \ldots\ldots (6.15)$$

where

$$\cos\beta' = \frac{1}{s'}\left|(\bar{x}' - x')\cos\phi'\cos\omega' + (\bar{y}' - y')\cos\phi'\sin\omega' + (\bar{z}' - z')\sin\phi'\right| \quad \ldots\ldots (6.16)$$

$$\delta\bar{\phi} = \delta\bar{\omega} = 0 \quad \ldots\ldots (6.17)$$

$$s' = \sqrt{(\bar{x}' - x')^2 + (\bar{y}' - y')^2 + (\bar{z}' - z')^2} \quad \ldots\ldots (6.18)$$

and

$$b_1 = \frac{(\bar{x}' - x')\cos\beta' - s'\cos\phi'\cos\omega'}{s^2\sin\beta'}$$

$$b_2 = \frac{(\bar{y}' - y')\cos\beta' - s'\cos\phi'\sin\omega'}{s^2\sin\beta'}$$

$$b_3 = \frac{(\bar{z}' - z')\cos\beta' - s'\sin\phi'}{s^2\sin\beta'} \quad \ldots\ldots (6.19)$$

$$b_4 = -\cos\alpha'$$

$$b_5 = -\cos\phi'\sin\alpha'$$

(iii) Distances (s).

$$v_s = c\{\delta Y - \delta\bar{Y}\} + (s' - s) \quad \ldots\ldots (6.20)$$

where

$$\delta\phi = \delta\bar{\phi} = \delta\bar{\omega} = \delta\omega = 0 \quad \ldots\ldots (6.21)$$

and

$$\{c_1, c_2, c_3\} = \left\{\frac{(\bar{x}' - x')}{s'}, \frac{(\bar{y}' - y')}{s'}, \frac{(\bar{z}' - z')}{s'}\right\} \quad \ldots\ldots (6.22)$$

(iv) Astronomic azimuths (α^*).

$$v_{\alpha^*} = a\{\delta Y - \delta\bar{Y}\} + (\alpha' - \alpha^*) \quad \ldots\ldots (6.23)$$

where

$$\delta\bar{\phi} = \delta\bar{\omega} = 0 \quad \ldots\ldots (6.24)$$

and the coefficients a were defined in Equation (6.14).

(v) Astronomic latitudes (ϕ).

$$v_\phi = \delta\phi + (\phi' - \phi) \quad \ldots\ldots \quad (6.25)$$

(vi) Astronomic longitudes (ω).

$$v_\omega = \delta\omega + (\omega' - \omega) \quad \ldots\ldots \quad (6.26)$$

where in all cases the dashed quantities pertain to the approximately known values and the asterisk is used to distinguish directions from astronomic azimuths.

6.3 The Normal Equations.

The observation equation set $AY = b + v$, when subjected to the least squares principle where v^TWv is to be minimised and W is the weight matrix, become the normal equations *(Thompson 1962)*

$$A^TWAY = A^TWb \quad \ldots\ldots \quad (6.27)$$

or, adopting a shorter notation

$$NY = d \quad \ldots\ldots \quad (6.28)$$

A solution to the normal equations will provide the vector of unknowns Y.

6.4 Dissimilar Quantities.

Many authors *(Gale 1958)*, *(Murphy 1958)*, *(Rainsford 1968)* to mention only some, have considered the problem of combined adjustment of angular and linear data. The apparent difficulty is best overcome by the use of σ, the variance-covariance matrix of the observations. However, in many practical situations, the actual variances and covariances of the data are not known and for this reason, quantities are introduced which are proportional to the variances and covariances *(Tienstra 1956, 108)*. These quantities are usually called weight coefficients *(Richardus 1966, 57)*. If G is the matrix of weight coefficients, then the relationship between σ and G is given by *(ibid)*

$$\sigma = \sigma_o^2 G \quad \ldots\ldots \quad (6.29)$$

where σ_o^2, the constant of proportion, is the variance factor. The variance factor has no dimensions, as the elements of G are assigned the same units

as the variances and covariances and may be chosen at will *(ibid, 59)*. It thus becomes mere routine to compare observed linear and angular data.

The weight matrix W is defined as *(ibid)* $W = G^{-1}$, that is, it is the inverse of the matrix of weight coefficients. W is diagonal when the observations are not correlated.

7. Error Analysis.

7.1 Variance-Covariance Matrix.

Assuming normally distributed observations and using the principle of least squares, the inverse of the coefficient matrix of the normal equations in the differential displacement technique of adjustment, gives an estimate of the precisions of the unknown quantities. The matrix N^{-1} (Equation 6.28) whose diagonal and non-diagonal elements when multiplied by σ_o^2, the variance factor, are respectively the variances and co-variances of the unknowns *(Ashkenazi 1965b, 78-80)* is known as the variance-covariance matrix of the unknowns. In the case of the differential displacement technique of adjustment it is denoted by

$$Q_{YY} = \sigma_o^2 N^{-1} \quad \ldots\ldots \quad (7.1)$$

For a three-dimensional space, where the unknowns are the Cartesian co-ordinate sets (x, y, z) the diagonal sub-matrices q_{xx} of Q_{YY} comprising the error squares in the direction of the co-ordinate axes and the error rectangles that describe the mutual correlation of the derived co-ordinates, give an indication of the precision of the point definition.

The variance-covariance matrix is also used to calculate the estimated precision of any quantity which is derived from the adjusted co-ordinates *(ibid, 80-82)*.

7.2 Error Ellipsoid.

The geodesist requires a mathematical model to represent the uncertainties in the point definition. An error ellipsoid is customarily adopted for this purpose, which, in the context of spatial point definition, may be considered as a surface of equal probabilities of the definition of the point in three-dimensional space, the probabilities mentioned being according to the law of normal distribution.

The analysis of the error ellipsoid facilitates the calculation of the standard deviation in any arbitrary direction, the weakest and the strongest being two cases of special interest. Moreover, for an overall indication of the precision of the point definition, the mean radius of the error ellipsoid may be computed.

It can be shown *(Malhotra 1969)* that the spatial orientations and magnitudes of the principal axes of the error ellipsoids may be found by determining the eigenvectors and eigenvalues of the diagonal sub-matrices q_{xx}. Thus, once the inverse matrix N^{-1} has been calculated, standard mathematical techniques may be employed in order to derive the spatial orientation and magnitude of the individual ellipsoids. The problem of finding the precision in any arbitrary direction is not as simple and the Mohr circle technique illustrated by Grafarend *(1967, 162)* may be appropriate.

PART C: THE SOLUTION OF THE NORMAL EQUATIONS AND THE INVERSION OF THE MATRIX OF COEFFICIENTS.

It is well known that as long as the matrix of coefficients of a set of equations can be stored simultaneously in the core of a computer, either fully or in band form* and there is sufficient room for the implementation of the reducing algorithm, the direct solution by decomposition is the best method. A suitable alternative, when insufficient working space is available is to resort to a partitioning scheme, which however, may introduce the secondary problem of generating the coefficient matrix in partitioned form if the partitions are of core size.

Otherwise, iterative methods *(Varga 1962)* appear attractive, that is, provided the systems considered are not excessively ill-conditioned. This is so since much less storage space is required and round-off errors do not accumulate. However, despite these apparent advantages the use of iterative procedures raises many questions about their convergence. Particular problems are:-

(1) the existence of convergence;

(2) the rate of convergence and

(3) possibly ways of accelerating convergence.

The answers to these questions are determined almost entirely by the eigenvalues of the coefficient matrix, the calculation of which for matrices of large order, is as complicated a problem as finding the solution *(Wilkinson 1969)*.

An optimum accelerating factor for the Successive Over-Relaxation (SOR) technique can be calculated for matrices that possess property A *(Martin & Tee 1961, 244)*. Matrices with property A occur in levelling network adjustment problems *(Ashkenazi 1965b)* but not in geodetic spatial networks. Davis *(1967)* and Brown et al *(1964)* speak favourably of the application of SOR to large photogrammetric problems, although the method by which the accelerating factor is determined is not elaborated. Again

---

* A matrix A is said to be a band matrix if $a_{ij} = 0$ for $|i-j|>m$, because the non-zero elements form a band along the main diagonal. Band matrices possess the advantage that in their triangular decomposition on a computer, only the elements within the band need to be stored (see 8.21).

Brown *(1968)* seems to have abandoned iterative techniques of the solution of the previous problem for a direct solution.

Accordingly, in view of these considerations, iterative techniques of the solution of normal equations were rejected, respectively in favour of:

(1) a direct solution and

(2) a partitioned solution.

Moreover, geodetic normal equation matrices are always symmetric and positive definite*. For this reason the Cholesky method is favoured for the decomposing algorithm since it is not necessary to pivot, a procedure which usually increases the band-width and thus reduces the size of the matrix that can be stored in the core of a computer.

In geodetic adjustment problems the inverse matrix of coefficients of the normal equations gives an indication of the precision of the unknowns. Thus, although experts *(Forsythe & Moler 1967, 79)* advise against its computation, the calculation of the inverse cannot be avoided for geodetic point definition. Two basic techniques which depend on the size of the problem are suitable.

If the original matrix N (Equation 6.28) is banded and can be accommodated in core then the corresponding N^{-1} is usually dense and cannot, a fact which limits the full inversion to matrices of order 150 *(ibid, 14)* on medium capacity computers. Fortunately, in an error analysis of geodetic point computation, only some of the elements of N^{-1} are required. Consequently, if the upper half of the banded matrix N can be stored in core, a method of partial inversion is suitable. Otherwise, if it cannot be stored simultaneously, a partitioning scheme is required with the disadvantage of the requirement of a substantial amount of backing storage.

The solution and inversion processes involve many arithmetic operations which, because computer representation is limited, introduce rounding errors. It is thus necessary to investigate whether the calculated quantities are sufficiently good approximations of the true solution and inverse, and if they are not, to explore means of improving their accuracy.

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* A matrix A is positive definite if its quadratic form $x^TAx>0$, for any real non-zero vector x. Matrices of coefficients of normal equations are always symmetric and positive definite *(Fox 1964, 81)*.

In order to gain an insight into both of these problems, the factors which determine the sensitivity of a system of equations $Ax = b$ with respect to small changes in the matrix A and the right-side b need to be considered.

8. The Solution of the Normal Equations. Inversion of the Normal Equation Matrix.

8.1 The Condition of a Computing Problem.

A computing problem is described as ill-conditioned if the values to be computed are very sensitive to small changes in the data. Thus a system of linear equations $Ax = b$ is said to be ill-conditioned if small changes in the coefficients of the unknowns or the numerical terms of the equations produce relatively large changes in the unknowns.*

8.1.1 Measure of Condition.

It is convenient to have some number which defines the condition of a matrix with respect to a computing problem and to call such a number a condition number. Ideally, this number should give some overall assessment of the rate of change of the solution with respect to changes in the coefficients and should therefore be proportional to this rate of change.

8.1.2 Condition Number.

In order to characterise and motivate the definition of the condition number to be used, the sensitivity of x in

$$Ax = b \qquad \ldots\ldots \quad (8.1)$$

where A is a square matrix of order n, with respect to small changes δA in A and δb in b, is considered. With respect to only small perturbations δb in b, the new solution $x + \delta x$ of (8.1) is defined by

$$A(x + \delta x) = b + \delta b \qquad \ldots\ldots \quad (8.2)$$

---

* In geodetic networks, adjusted by the variation of coordinates method and using the Least Squares principle, ill-conditioning is generally associated with the amount of change introduced into the unknown quantities by small changes in the values of the observed quantities.

Subtraction of (8.1) and (8.2) yields

$$A\delta x = \delta b \qquad \ldots\ldots \quad (8.3)$$

that is

$$\delta x = A^{-1}\delta b \qquad \ldots\ldots \quad (8.4)$$

and hence (see Fox *(1964, 48-51)* for a discussion of matrix norms)

$$||\delta x|| = ||A^{-1}\delta b|| < ||A^{-1}|| \; ||\delta b|| \qquad \ldots\ldots \quad (8.5)$$

using the inequality

$$||b|| < ||A|| \; ||x||$$

it follows from (8.1) that the relative error $||\delta x||/||x||$, in the solution satisfies

$$||\delta x||/||x|| < ||A^{-1}|| \; ||\delta b||/||A||^{-1}||b||$$

$$= ||A|| \; ||A^{-1}|| \; ||\delta b||/||b|| \qquad \ldots\ldots \quad (8.6)$$

when $||A|| \; ||A^{-1}||$ is very large the solution x will be very pessimistic for most right-hand sides b and perturbations δb.

Thus, since the size of $||A|| \; ||A^{-1}||$ provides an indication of the sensitivity of the solution of (8.1) to perturbations, it is taken as a condition number. An additional justification for the use of this definition, based on the effect of perturbations in A may be found in Wilkinson *(1969, 190)*.

The condition number in most common use, the *spectral condition number of* A *with respect to inversion,* is *(ibid)*

$$\kappa(A) = ||A||_2 ||A^{-1}||_2 \qquad \ldots\ldots \quad (8.7)$$

Two important properties of the spectral condition number κ are:-

(i) For any non-zero scalar, c

$$\kappa(cA) = \kappa(A) \qquad \ldots\ldots \quad (8.8)$$

(ii) If A is symmetric

$$\kappa(A) = |\lambda_1|/|\lambda_n| \qquad \ldots\ldots \quad (8.9)$$

where λ_1 and λ_n are the eigen values of largest and smallest modulus. Allowing $||A||_2 = 1$, which can always be achieved by multiplying (8.1) by

a suitable constant, then

$$\kappa(A) = ||A||_2 ||A^{-1}||_2 = \frac{1}{|\lambda_n|}$$

if A symmetric. A symmetric normalized matrix A is therefore ill-conditioned only if $||A^{-1}||_2$ is large or, λ_n is small. In using κ as a condition number, the implication is merely that a normalized matrix is ill-conditioned if its inverse is 'large', which is consistent with the standard concept of ill-conditioning since the inference is that a small change in some of the right-hand sides will result in large changes in the solution.

The following observations are made from Equations (8.8) and (8.9) respectively:-

(a) the condition of a system of equations is not affected by multiplying both sides by a constant;

(b) in order to identify and ill-conditioned matrix, $||A||$ and $||A^{-1}||$ or λ_1 and λ_n must generally be computed, the calculation of either of which for matrices of large orders is not trivial.

8.1.3 Normal Equations.

The spectral condition number for the normal equations is *(Taussky 1950, 111-112)*

$$\kappa(A^T WA) = \kappa^2(L) \qquad \ldots\ldots \quad (8.10)$$

where

$$L = W^{\frac{1}{2}}A$$

Consequently, if an ill-conditioning problem already exists in the reduced observation equations, then the normal equation matrix will be significantly more ill-conditioned.

8.2 The Solution of the Normal Equations.

8.2.1 Direct Solution.

If the band-width m of a positive definite, symmetric matrix A is defined as the largest number of elements in any one of its rows, from the rightmost non-zero element inclusive, then it may be decomposed on a computer by the Cholesky method into the product

$$A = GG^T \qquad \ldots\ldots \quad (8.11)$$

where:

(1) G is a lower triangular matrix with positive diagonal elements;

(2) the band-width of the triangular matrix G^T does not exceed that of A;

(3) at no stage of the decomposition process is required information located beyond the mth column and

(4) as soon as a particular element of G^T is computed the corresponding element of A is no longer needed and may therefore cede its storage location to that of G^T *(Ashkenazi 1965b, 26-27)*.

The triangularization of a positive definite symmetric matrix by the Cholesky method has all the virtues. No interchanges are necessary to ensure the existence and accuracy of the decomposition *(Forsythe & Moler 1967, 114)*. Moreover, ignoring rounding errors,

$$A = GG^T \quad \text{and} \quad A^{-1} = (G^T)^{-1} G^{-1}$$

giving *(Wilkinson 1969, 245)*

$$||A||_2 = ||G||_2^2 \quad \text{and} \quad ||A^{-1}||_2 = ||G^{-1}||_2^2$$

which is most satisfactory, since the spectral condition number equals that of the square root of A. Perhaps the only adverse comment which might be made is that n-1 square roots must be calculated, whereas none is needed in ordinary Gaussian elimination. Only the Cholesky technique is therefore considered as a means of providing a direct solution to the normal equation system.

If $a_{i,j}$ and $g_{i,j}$ are elements of the A and G matrices,then for the diagonal coefficients of A

$$a_{j,j} = g_{j,1}^2 + g_{j,2}^2 + \ldots\ldots + g_{j,j}^2 \qquad \ldots\ldots \quad (8.12)$$

and for non-diagonal terms

$$a_{i,j} = g_{i,1}g_{j,1} + g_{i,2}g_{j,2} + \ldots\ldots + g_{i,j}g_{j,i} \quad (j<i) \qquad \ldots\ldots \quad (8.13)$$

Used in correct sequence, these equations determine the elements of G, and it will be noticed that the elements of A above the diagonal are not involved because of symmetry.

The solution is obtained by a forward elimination ($Gy = b$) and back-substitution ($G^Tx = y$) process.

8.2.1.1 Storage Requirements.

Because of the symmetry of A it is necessary to store only $\frac{1}{2}n(n + 1)$ or slightly over half of its elements resulting in an important saving of computer storage for large matrices. If in addition the matrix is banded, further space is saved. Space is saved because only the non-zero part of the matrix needs to be stored and such storaged arrangement requires $(m + 1)\frac{1}{2}(2n - m)$ locations in comparison to the previous.

However, the concept of banding is useful only if m is appreciably smaller than n. After all, any matrix is a band matrix with $m = n - 1$, and thus storage space is only saved if $m < (n - 1)$.

It is emphasised that no intermediate storage is needed since the elements of G are created at the same time the elements of A are replaced and that almost all the computer time required to solve $Ax = b$ is spent finding G; the computations actually involving b are relatively short. Hence, if at a later stage it is required to solve another system with the same matrix of coefficients as is the case with iterative improvement or if the inverse is desired, there is every reason to retain G and thereby avoid repeating the triangular decomposition.

8.2.2 Partitioned Solution.

In the method of partitioning, the coefficient matrix is subdivided into convenient parts. Fig. 8.1 demonstrates an established partitioning scheme for banded symmetric matrices, the reasons that the band-width is chosen as a criterion for division being that:

(1) storage requirements are optimised and

(2) programming the formation of the partitions is simplified.

The method upon which the solution algorithm is based *(Zienkiwicz & Cheung 1968, 243)* accounts for the symmetry and sparseness of the matrix and the fact that non-zero elements lie close to the diagonal by proper ordering of nodes. In a tridiagonalised manner, the matrix may be written

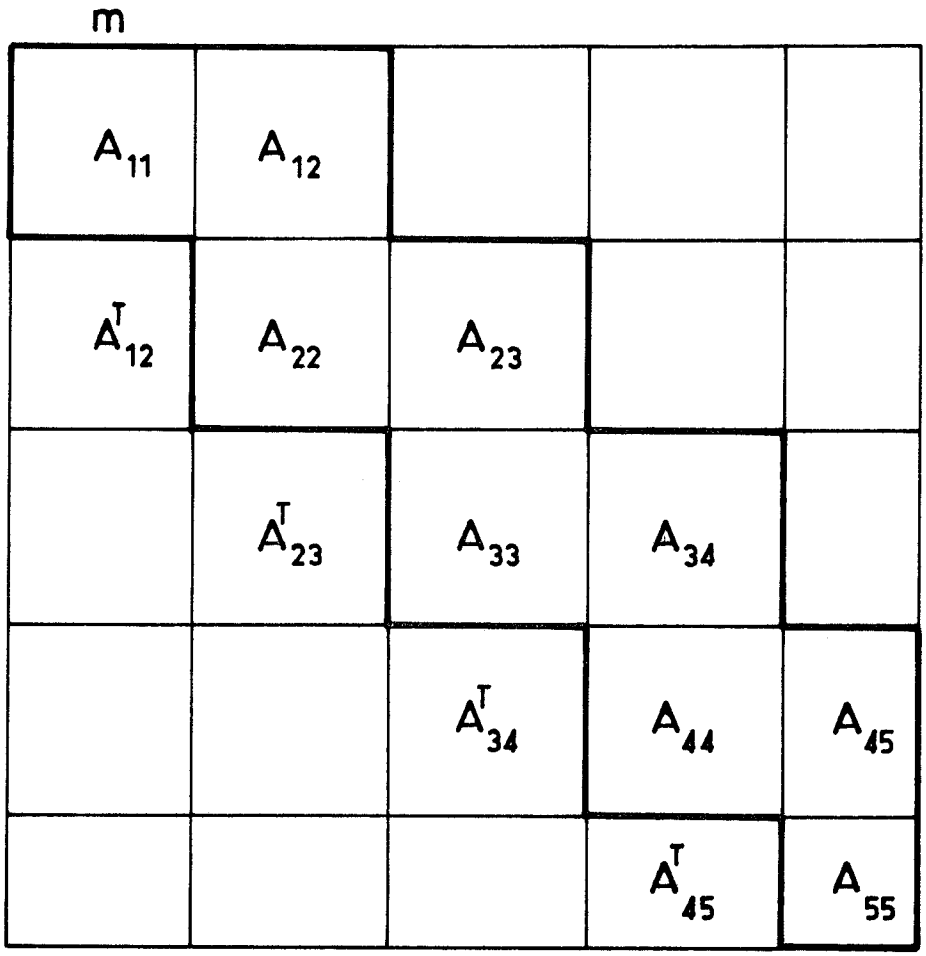

FIG. 8·1

$$\begin{pmatrix} A_{11} & A_{12} & 0 & 0 & . & . & 0 & 0 \\ A_{12}^T & A_{22} & A_{23} & 0 & . & . & 0 & 0 \\ 0 & A_{23}^T & A_{33} & A_{34} & . & . & . & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & A_{(n-1)(n-1)} & A_{(n-1)n} \\ 0 & 0 & 0 & 0 & . & . & A_{(n-1)n}^T & A_{nn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ . \\ . \\ x_{n-1} \\ x_n \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ b_3 \\ . \\ . \\ b_{n-1} \\ b_n \end{pmatrix} \quad \ldots\ldots \quad (8.14)$$

The first two matrix equations can be written in full as

$$A_{11}x_1 + A_{12}x_2 = b_1 \quad \ldots\ldots \quad (8.15)$$

$$A_{12}^T x_1 + A_{22}x_2 + A_{23}x_3 = b_2 \quad \ldots\ldots \quad (8.16$$

a solution of which will yield for x_1

$$x_1 = A_{11}^{-1}b_1 - A_{11}^{-1}A_{12}x_2 \quad \ldots\ldots \quad (8.17)$$

Equation (8.17) upon substitution into Equation (8.16) gives

$$(A_{22} - A_{12}^T A_{11}^{-1}A_{12})x_2 + A_{23}x_3 = b_2 - A_{12}^T A_{11}^{-1}b_1 \quad \ldots\ldots \quad (8.18)$$

which by introducing the symbols

$$\bar{A}_{22} - (A_{22} - A_{12}^T A_{11}^{-1}A_{12})$$

$$\bar{b}_2 = b_2 - A_{12}^T A_{11}^{-1}b_1$$

may be written

$$\bar{A}_{22}x_2 + A_{23}x_3 = \bar{b}_2 \quad \ldots\ldots \quad (8.19)$$

x_2 may be extracted from Equation (8.19) as before and may then be substituted into the new row equation to give modified $\bar{A}_{33}$ and $\bar{b}_3$. Such a process of substitution and elimination is possible until the last row equation is reached, i.e.

$$\bar{A}_n x_n = \bar{b}_n \quad \ldots\ldots \quad (8.20)$$

the direct inversion of which will yield x_n. If the process is reversed

and the known values are back-substituted into equations of type (8.19) then all the unknowns may be found.

It is seen that, since in this type of solution the matrices are eliminated in blocks, a substantial amount of backing storage is needed for the A_{ij} and for making the intermediate calculations. Thus, if the band-width is very large, it is advisable to look for an alternative solution.

8.3 Inversion of the Coefficient Matrix.

8.3.1 Partial Inversion.

The inverse A^{-1} of a symmetric and positive definite matrix A which has been decomposed by the Cholesky method is given by

$$A^{-1} = (G\,G^T)^{-1} = (G^T)^{-1}G^{-1} = S^T S \qquad \ldots\ldots \quad (8.21)$$

Let $G^{-1} = (s_{i,j})$ with $s_{i,j} = 0$ for $i<j$ and let G_i and S_j be, respectively, the ith column of G and the jth row of G^{-1}. Then for $1 < k < n$

$$S_k G_k = 1 = s_{k,k} \qquad \ldots\ldots \quad (8.22)$$

$$S_k G_j = 0 = \sum_{i=j}^{k} s_{k,i} g_{i,j} \quad j = k-1, \ldots, 1 \qquad \ldots\ldots \quad (8.23)$$

from which $s_{k,i}(i = k, k-1, \ldots, 1)$ may be calculated for any k, thereby obtaining all the elements of G^{-1}. The desired elements of A^{-1} are then given by Equation (8.21).

8.3.2 Partitioned Inversion.

If a large matrix A is divided into four partitions:-

$$A = \begin{pmatrix} W & X \\ Y & Z \end{pmatrix} \qquad \ldots\ldots \quad (8.24)$$

where

A is a n x n matrix
W p x p
X p x q
Y q x p

the inverse A^{-1}, if partitioned in a similar manner is given by *(Ayres 1962, 57)*

$$A^{-1} = \begin{pmatrix} K & L \\ M & N \end{pmatrix} \qquad \ldots\ldots \quad (8.25)$$

where

$$\begin{aligned} N &= (Z - YW^{-1}X)^{-1} \\ M &= (-N\ YW^{-1}) \\ L &= -W^{-1}XN \\ K &= W^{-1} - W^{-1}XM \end{aligned} \qquad \ldots\ldots \quad (8.26)$$

Theoretically, the technique can be expanded to invert a matrix containing any number of partitions by inductively moving out from the top left hand corner partition *(Knight 1965)*.

8.4 Error Analysis.

The solution of the simultaneous equations $Ax = b$ comprises the following computational steps:

(a) decomposition of the coefficient matrix according to the Cholesky method ($A = G\ G^T$) ;

(b) forward elimination ($Gy = b$); and

(c) back substitution ($G^T x = y$).

An additional back-substitution ($A^{-1} = (G^T)^{-1}G^{-1}$) is necessary to compute the inverse matrix. Thus, apart from the initial rounding errors occurring in the formation of A and b, those occurring in steps (a), (b), (c) and the inversion will affect the quality of the results.

Computations are to be made exclusively using a digital computer and since there are two main modes of operation on automatic computers, a decision is required regarding the most advantageous of the two for the problem to be solved. The first mode is called fixed-point computation. In this mode the computation must be framed so that every computed number x satisfies certain inequalities such as $-1 \leq x \leq 1$. In general, each number will be allowed a fixed quantity t, of binary (decimal) digits for its representation and it is said that the computer works with words of t binary (decimal) digits. If it is necessary to work to a higher precision than 1 part in 2^t then numbers which are represented by a multiple of t binary digits may be employed. The other mode of computation is called floating-point computation. In this mode each number x is represented by an ordered pair a and b such that $x = \beta^b(a)$. Here β is the base of the floating-point

computation, b is an integer, positive or negative, and a is a number satisfying $-\frac{1}{2} \geq a \geq -1$ or $\frac{1}{2} \leq a \leq 1$. The number b is commonly called the exponent and the number a is called the mantissae or the fractional part.

The number of digits allocated to a and b together is quite commonly the same as that available for a fixed-point number. A typical division of the word (IBM 360/50) is 8 binary digits in b and 24 binary digits in a for a computer with a word of 32 binary digits. This means that fixed-point computation is usually capable of giving a higher precision, though often, in order to ensure that numbers do not grow out of the permissible range, it is convenient to work for most of the time with numbers which are appreciably smaller than unity. Some of the potential advantage of fixed-point computation is then forfeited because of the detailed analysis which would be necessary.

It is seen that for a given number of digits in the word, fixed-point is potentially more precise than floating-point because of the number of digits allotted to the exponent in the latter. However, much more careful preliminary analysis is needed for fixed-point computation than floating-point to ensure that the numbers do not pass outside the permitted range and the natural flow of the computations must often be modified in fixed-point by the introduction of appropriate scale factors. Accordingly, in order to avoid an excessive use of scale of factors, numbers in fixed-point computation are scaled so that they are much smaller than unity. In this way more digits may be sacrificed than would be required by the exponent in floating-point computation. If scale factors have to be introduced at frequent intervals in fixed-point computation then it is little more than an ad hoc floating-point computation.

In accumulating an inner product in fixed-point, the danger of the sum exceeding capacity must be borne in mind constantly. The provision of the accumulation facility in connection with floating-point computation avoids this difficulty which is always present in fixed-point work. Further, it may happen that a partial sum exceeds capacity but when the later terms are added it returns to the permissible range. If it were known in advance that the final sum would be in the permitted range then the spill-over could be ignored because the final result would be correct, but unfortunately this is not usually the position. These points are covered automatically in floating-point computation, and although in general some rounding errors are inevitable, assuming double word accumulation, they will be in the digit at the end of

the second half of the register and therefore involve 2^{-2t} instead of 2^{-t}.

Accordingly, in view of these considerations floating-point arithmetic is chosen as the basis to the computations within the context of this dissertation.

8.4.1 Limitations Imposed by t-Digit Representation.

Since the A and b of $Ax = b$ generally require more than t-digits for their correct representation, it is necessary to work with the approximations A' and b'. Thus,

$$A' = A + \delta A, \quad b' = b + \delta b \qquad \ldots\ldots \quad (8.27)$$

where for floating point computations and binary arithmetic *(Wilkinson 1969, 197)*

$$|\delta A_{ij}| \leq 2^{-t}|A_{ij}|, |\delta b_i| \leq 2^{-t}|b_i| \qquad \ldots\ldots \quad (8.28)$$

so that

$$||\delta A||_2 \leq 2^{-t} n^{\frac{1}{2}} ||A||_2, \; ||\delta b||_2 \leq 2^{-t} ||b||_2 \qquad \ldots\ldots \quad (8.29)$$

Wilkinson *(1964, 93)* shows that when working with t-digit floating-point arithmetic, it is not even possible to obtain an approximate solution to a set of equations for which $\kappa(A) \geq n^{-\frac{1}{2}} 2^t$.

Concerning the effect of rounding errors made during the course of the computations, Wilkinson *(1969, 198)* concludes that when the original elements are not exactly representable by t-binary digits, the error resulting from any initial roundings that were necessary are as serious as those arising from all steps in the solution.

8.4.2 Cholesky Decomposition.

The computed G matrix satisfies

$$G^T G = A + \delta A$$

there being a danger of rounding errors destroying positive definiteness when $||A^{-1}||_2 < 2^{t+1} / (n + 1)$ *(Wilkinson 1964, 117)*. However, if accumulation of inner products is possible, the bound for δA using a given precision of computation is as small as possible for any method.

8.4.3 Forward Elimination and Back-Substitution.

The forward elimination $(Gy = b)$ and the back-substitution $(G^Tx = y)$ processes constitute the solution of two sets of equations with a lower and upper triangular matrix of coefficients respectively. The computed vectors y and x satisfy

$$(G + \delta G)y = b \quad \text{and} \quad (G + \delta G)^T x = y \qquad \ldots\ldots \quad (8.30)$$

δG being some function of b *(Wilkinson 1969, 248)*. Residuals corresponding to the solution of (8.30) are smaller than those of the correctly rounded solution, even when the matrix G is very ill-conditioned *(ibid, 247-251)*.

8.4.4 Inversion.

An indication of the relative error in the computed inverse B of A is given by the expression *(ibid, 253)*

$$||B - A^{-1}||_\infty / ||A^{-1}||_\infty \leq (3.1)n2^{-t}||A^{-1}||_\infty / \{1-(3.1)n2^{-t}||A^{-1}||_\infty\}. \qquad (8.31)$$

from which it may be concluded that for a normalized matrix A, the computed inverse B has a low relative error if $(3.1)n2^{-t}||A^{-1}||_\infty$ is small.

8.5 Accuracy of Computed Results.

A small residual $(AB - I)$ necessarily implies that B is a good inverse, whereas a small residual vector $(Ax - b)$ does not necessarily imply that x is a good solution *(ibid)*. Thus when an inverse B has been computed, $(AB - I)$ may be calculated, thereby obtaining a reliable bound for the error in B. However, in order to be able to recognise the accuracy of the solution x of $(Ax - b)$, an accurate approximation of the inverse B is required as well as a reasonably sharp bound for $||B - A^{-1}||$.

8.6 Matrix Scaling.

Suppose that x_j in Equation (8.1) is replaced by $d_j^{(2)} x_j'$ for $j = 1, \ldots\ldots, n$. The substitution take the form

$$x = D_2 x' \qquad \ldots\ldots \quad (8.32)$$

where D is a non-singular diagonal matrix defined by

$$D_2 = \begin{pmatrix} d_1^{(2)} & & 0 \\ & \ddots & \\ & & \ddots \\ 0 & & d_n^{(2)} \end{pmatrix} \quad \ldots\ldots \quad (8.33)$$

Similarly, if

$$D_1 = \begin{pmatrix} d_1^{(1)} & & 0 \\ & \ddots & \\ & & \ddots \\ 0 & & d_n^{(1)} \end{pmatrix} \quad \ldots\ldots \quad (8.34)$$

is a second non-singular matrix and the substitutions

$$b = D_1 b' \quad \ldots\ldots \quad (8.35)$$

are made for the right-hand sides of $Ax = b$ then

$$A D_2 x' = D_1 b' \quad \ldots\ldots \quad (8.36)$$

or

$$D_i^{-1} A D_2 x' = b' \quad \ldots\ldots \quad (8.37)$$

It is seen that the changes of variable (8.32) and (8.35) produce a new linear system with coefficient matrix $A' = D_1^{-1} A D_2$ and right-hand side $b' = D_1^{-1} b$.

The transformed matrix A' is said to be diagonally equivalent to A if non-singular diagonal matrices D_1 and D_2 exist so that $A' = D_1^{-1} A D_2$. Moreover, since $A D_2$ is the matrix A with its columns multiplied by the factors $d_1^{(2)} \ldots\ldots d_n^{(2)}$ and $D_1^{-1} A$ is the matrix A with its rows multiplied by the factors $1/d_1^{(1)} \ldots\ldots 1/d_n^{(1)}$, $A D_2$ and $D_1^{-1} A$ are referred to as the column and row scaled equivalents of A respectively, whereas $D_1^{-1} A D_2$ is spoken of as the scaled equivalent of A. In floating-point computations with base β, the $d_i^{(1)}$'s are usually selected to be integer powers of the base β, thus altering the exponents of the floating-point numbers but leaving their fractional parts unchanged. No rounding-errors are then introduced by Equation (8.37).

The subject of how scaling affects the solution of a linear system is not very well understood *(Forsythe & Moler 1967, 38)*, although, since the critical factor which determines the convergence of the iterative refinement method for solving simultaneous equations (See Chapter 9) is that the spectral

condition number should be sufficiently small, it is generally agreed, at least in theory, that the best way to scale matrix is to make the condition number of its scaled equivalent as small as possible. The problem then becomes that of finding the diagonal matrices D_1 and D_2 which yield the smallest condition number $\kappa(A')$. The minimizing D_1 and D_2, for a general matrix A, depend essentially on its inverse A^{-1} which will certainly not be known as the object in scaling A is to help determine $A^{-1}b$. The calculation of eigenvalues for symmetric matrices A' of large order is again not a practical proposition. Wilkinson's *(1961)* approach to the scaling problem is therefore to insist that a matrix A be equilibrated* before a linear-system solver is applied, the matrix being roughly equilibrated if all its rows and columns have approximately the same length in some norm *(Wilkinson 1969, 192)*. Regarding rounding-errors, the most satisfactory results are obtained for the solution of $Ax = b$ when the matrix A is equilibrated because then a small perturbation of one row (or column) of A is of the same magnitude as that of any other row (or column) of A. Unfortunately, there is no unique equilibrated form of a matrix and different equilibration algorithms will change it into different matrices which can vary considerably in their condition and scaling *(Forsythe & Moler 1967, 45)*.

Note that if the matrix A is symmetric, it is of advantage to preserve symmetry in the scaling operation. Moreover, since $x = D_2x'$, it is necessary to multiply x' by D_2 to obtain the correct solution.

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* A matrix A is: row equilibrated (relative to the norm $||x||_\infty$) if for each row index i, $\beta^{-1} \leq \max |a_{ij}| \leq 1 (1 \leq i \leq n)$; column equilibrated (relative to the norm $||x||_\infty$) if for each column index j, $\beta^{-1} \leq \max |a_{ij}| \leq 1 (1 \leq j \leq n)$; and equilibrated if it is both row and column equilibrated. The use of β permits the matrix to be equilibrated by changes of exponent only.

9. Iterative Refinement of Approximate Solutions.

9.1 Iterative Refinement of the Solution.

For the accurate solution of linear equations iterative refinement procedures are very attractive since high accuracy solutions x can be obtained to most linear systems $Ax = b$ with only a modest increase in time over obtaining the first solution. Moreover, for large systems a single precision factorization plus iterative refinement has the advantage over double-precision computation without refinement that it requires less storage (only about half), is faster and gives useful indication about the condition of the coefficient matrix. It possesses the disadvantage that if the coefficient matrix is too ill-conditioned, iterative refinement will not work whereas double-precision computation might well give an answer of acceptable (although unknown) accuracy.

The key to improving the accuracy of the first solution $x^{(1)}$ is a double precision computation of the residual $r^{(1)} = b - Ax^{(1)}$. Knowing $r^{(1)}$, it is then necessary to solve the system $A\delta x^{(1)} = r^{(1)}$. Thus, if $\delta x^{(1)}$ were known precisely, then $x^{(2)} = x^{(1)} + \delta x^{(1)}$ would accurately solve the system $Ax = b$ because

$$Ax^{(2)} = A(x^{(1)} + \delta x^{(1)}) = Ax^{(1)} + A\delta x^{(1)} = Ax^{(1)} + r^{(1)} = b \quad \ldots\ldots \quad (9.1)$$

Such a procedure should furnish a second solution $x^{(2)}$ which is more accurate than $x^{(1)}$. Forming $r^{(2)} = b - Ax^{(2)}$, the technique may be repeated until the required precision is achieved. The previous is contained in the iterative scheme

$$x^{(k+1)} = x^{(k)} + B^{-1}(b - Ax^{(k)}), \quad x^{(o)} = B^{-1}b \quad \ldots\ldots \quad (9.2)$$

where B^{-1} is an approximation to A^{-1} obtained implicitly during the solving process. Successive applications give

$$\begin{aligned} x^{(o)} &= B^{-1}b \\ x^{(1)} &= (I + C)B^{-1}b \\ x^{(2)} &= (I + C + C^2)B^{-1}b \qquad \ldots\ldots \quad (9.3) \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ x^{(k)} &= (I + C + C^2 + \ldots + C^k)B^{-1}b \end{aligned}$$

where

$$C = I - B^{-1}A$$

It will be noted that each system (9.3) employs the same matrix B^{-1} and is the reason that iterative improvement of a first solution adds only a moderate amount to the computational time of the algorithm. However, it is essential that the residual $r^{(k)}$ be computed with a higher precision than the remainder of the calculations.

9.1.1 Convergence.

The series $I + C + C^2 + \ldots + C^n \ldots$ converges to $(I - C)^{-1}$ for $||C|| < 1$ *(Fox 1964, 144)*. Thus when $||C|| < 1$ which is satisfied if $||A^{-1}||\ ||\delta A|| < \frac{1}{2}$ *(Wilkinson 1969, 255)*, $x^{(k)}$ converges to

$$(I-C)^{-1}\ B^{-1}b = A^{-1}b \qquad \ldots\ldots \quad (9.4)$$

9.1.2 Effect of Rounding Errors.

It is plausible to propose that if the iteration is performed using t-digit arithmetic, then the accuracy of $x^{(k)}$ cannot increase indefinitely with k since only t-digits are being used to represent the components. Furthermore, for floating-point arithmetic the residual corresponding to the first solution will most likely be of the same order of magnitude as that corresponding to the correctly rounded solution and it can therefore hardly be expected that the residual should diminish by the factor 2^{-p} with each iteration. Nevertheless, this is precisely what does happen *(ibid, 251)*.

9.1.3 Accuracy.

The iterative process gives a virtual guarantee that the final vector is the correctly rounded solution. However if $n2^{-t}||A^{-1}|| > \frac{1}{2}$, then it will not in general converge and none of the computed solution will have any correct figures.

9.1.4 Limitations.

If the iterative technique does not converge, then it is said that the 'matrix is too ill-conditioned to be solved by the method in question' unless higher arithmetic is used. The possibility remains however, that despite convergence, the wrong answers are given. On this matter Wilkinson *(ibid, 262)* reports that pivots* of order of magnitude $n2^{-t}$ or solutions of order of magnitude $2^{-t}/n$ are a sure sign that the condition of a matrix is such that the convergence of the iterates in the unlikely event

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* Corresponding to the diagonal elements of the G matrix in the Cholesky approach.

of their occurrence are not to be taken as guaranteeing a correct solution.

9.2 Least Squares Solutions.

The previous remarks apply to the solution of the general equation system $Ax = b$. In the standard least squares problem however, the vector x is required for which $||b - Ax||_2$ is a minimum *(Golub 1965, 206)*. A being an mxn matrix $(m \geq n)$ and b and x vectors. In practice, whether or not $m = n$, rounding errors contaminate the solution. If $x^{(1)}$ is the computed solution and $\delta x^{(1)}$ is determined so that $||(b - Ax^{(1)} - A\delta x^{(1)})||_2$ is a minimum, then since this implies that $||b - A(x^{(1)} + \delta x^{(1)})||_2$ is a minimum, $x^{(1)} + \delta x^{(1)}$ is the least squares solution of the original problem *(Golub & Wilkinson 1966, 139)*. Hence if $r^{(1)}$ is the residual vector defined by

$$r^{(1)} = b - Ax^{(1)} \qquad \ldots\ldots \quad (9.5)$$

the required correction is the least squares solution with matrix A and right-hand side $r^{(1)}$.

9.2.1 Linear Equation Case.

When $m = n$, then the computed solution satisfies the equation

$$(A + \delta A^{(1)})\, x^{(1)} = b + \delta b \qquad \ldots\ldots \quad (9.6)$$

where if A is very ill-conditioned, $A + \delta A^{(1)}$ could be singular. Moreover, if *(ibid, 141)*

$$27n^{\frac{3}{2}}\, 2^{-t}||A||_2||A^{-1}||_2 < 2^{-p} (p \geq 0) \qquad \ldots\ldots \quad (9.7)$$

then

$$||x - x^{(1)}||_2/||x||_2 < 2^{-p}/(1 - 2^{-p-1}) \qquad \ldots\ldots \quad (9.8)$$

where $x = A^{-1}b$ is the true solution. Experience indicates that (9.8) is satisfied when p is defined by some relation such as

$$n^{\frac{1}{2}}2^{-t}||A||_2||A^{-1}||_2 = 2^{-p} \qquad \ldots\ldots \quad (9.9)$$

Thus unless A is too ill-conditioned the iterative refinement procedure is certain to work with any right-hand side provided the errors made in computing the residuals are unimportant. On the average $x^{(s)}$ should gain roughly the same number of figures in each iteration until it is correct to working accuracy and provided $\kappa(A)$ satisfies the requisite

bound, and $x^{(s)}$ is ultimately attained for which

$$||x - x^{(s)}||/||x|| \leq 2^{-t} \qquad \ldots\ldots \quad (9.10)$$

at which stage x is "correct to working accuracy."

9.2.2 The Least Squares Case With $m > n$.

An analogous result to (9.10) cannot be obtained for the least squares case, because if $A^TWb = 0$, the correct solution is $x = 0$. Nevertheless, it is reasonable to expect that $||x - x^{(s)}||_2$ will show a progressive decrease in magnitude if A satisfies a bound of the form (9.7). However, since the condition number for the normal equations $A^TWAx = A^TWb$ is given by the expression

$$\kappa(A^TWA) = \kappa^2(L) \quad \text{(See Equation 8.10)} \qquad \ldots\ldots \quad (9.11)$$

then if A has a condition number of order $2^{\frac{1}{2}t}$, A^TWA has a condition number of order 2^t and it will not be possible to solve the normal equations using t-digit representation. Thus in striking contrast with the linear equation case there will be right-hand sides for which iterative refinement will never give solutions which are correct to working accuracy, whatever precision of computation is used.

9.3 Iterative Refinement of an Approximate Inverse.

Methods similar to those for linear equations can be applied to improve the accuracy of an approximate inverse. If B_o^{-1} is an approximation to A^{-1}, the iterative scheme

$$B_{k+1}^{-1} = B_k(2I - AB_k^{-1}) \qquad \ldots\ldots \quad (9.12)$$

will give a sequence of matrices B_k^{-1}, which converge under certain conditions to A^{-1}. Thus *(Fox 1964, 156)*

$$
\begin{aligned}
B_1^{-1} &= B_o^{-1}(I + C) \\
B_2^{-1} &= B_o^{-1}(I + C + C^2 + C^3) \\
B_3^{-1} &= B_o^{-1}(I + C + C^2 + C^3 + C^4 + C^5 + C^6 + C^7) \\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \qquad \cdots\cdots\cdots \quad (9.13) \\
B_k^{-1} &= B_o^{-1}(I + C + C^2 + \ldots + C^{2k-1})
\end{aligned}
$$

where

$$C = I - A\,B_o^{-1}$$

9.3.1 Convergence.

The above process will converge to $A^{-1} = B_o^{-1}(I - C)^{-1}$ if $||C|| < 1$ and it is seen that the matrix C is not exactly that of the linear equation case, which was $I - B_o^{-1}A$, but since the norm cannot exceed $||B_o^{-1}||\ ||B_o - A||$ in either case, the criterion of convergence will be the same. Thus if B_o^{-1} is suitable for the iterative refinement of linear equations, it is also suitable for iteration to produce an accurate inverse. The rate of convergence for the iterative refinement of the inverse is however considerably better than that for the linear equations case *(ibid)*.

PART D: THE TEST NETWORK.

10. The Network.

The network (Fig. 10.1) chosen to test the local astronomic vector technique of computing three-dimensional Cartesian coordinates of terrestrial points, is located predominantly in the State of South Australia, covering approximately two-thirds of its area and containing also that ratio of the State's total number of first-order traverse stations, (in all 263 points were selected) but comprises parts of the Northern Territory and Queensland networks. It is part of the AGD which was established to provide mapping control for Australia and New Guinea. However, it must be emphasised that the accuracies obtained from the survey methods used was much greater than that required for mapping purposes.

10.1 Size and Location.

The State of South Australia has the ocean to the south and is flanked on the other sides by land. It lies south of the 26th parallel of south latitude and has as a western boundary the 129th meridian of east longitude. The eastern boundary, north of the River Murray, corresponds with the 141st meridian, while to the south, between the river and the sea, the boundary lies approximately 3.2 kilometres to the west of this meridian.

The State is approximately 1 200 kilometres from east to west at the northern boundary and 1 150 kilometres at the head of the Great Australian Bight; from north to south it varies from 630 kilometres near the western extremety to approximately 1 330 kilometres near the eastern boundary; its coastline, excluding islands, measures approximately 3 400 kilometres. South Australia covers a total area of approximately one million square kilometres (one-eighth of the area of the Australian continent).

10.1.1 Physical Features.

South Australia is a land of generally low relief, the inland area being largely covered by featureless plains, or sand and gibber deserts. Approximately 50 per cent of the State is less than 150 metres above sea level and over 80 per cent is less than 300 metres above sea level.

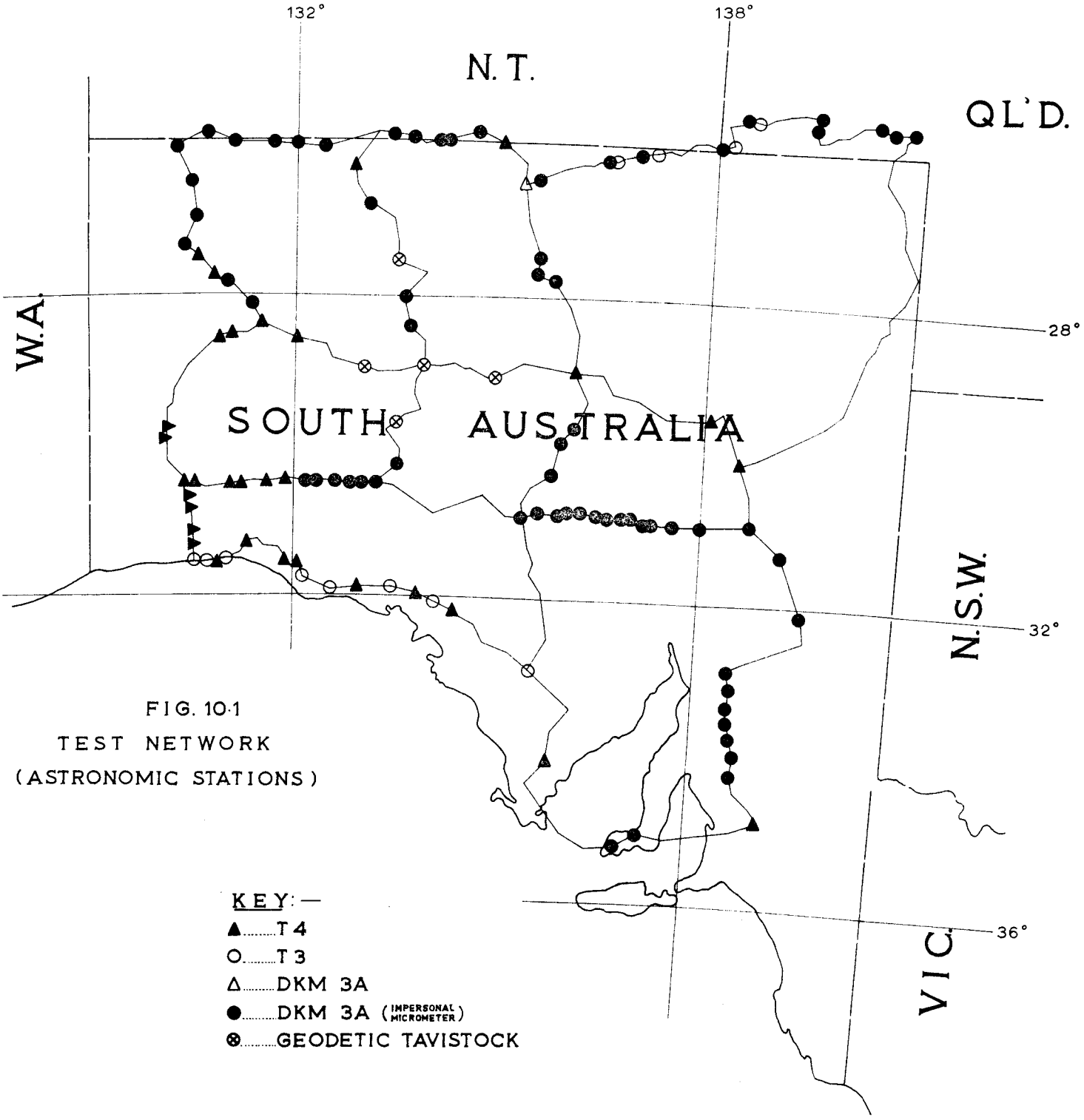

FIG. 10·1
TEST NETWORK
(ASTRONOMIC STATIONS)

10.2 Data.

The observational material for the test network, completed between 1957 and 1971, was extracted from the field records of the Division of National Mapping, Canberra; the South Australian Lands Department, Adelaide; the Department of the Interior, Woomera and the Royal Australian Survey Corps, Keswick.* Instrumentation and observing methods varied somewhat with technological developments and experience. For this reason observing methods are summarised and major problems are presented with a view of providing a foundation to the observational accuracies quoted in Section 10.3, which are to be used in the derivation of a set of weight coefficients for an adjustment procedure by least squares.

10.2.1 Astronomic.

Prior to 1945, astronomic observations in Australia were made with geodetic theodolites of English manufacture (e.g. the Geodetic Tavistock). The methods employed were not of an impersonal character. It was only with the introduction of the modern universal instrument (T4 and DKM3a) and its associated equipment that a high precision was possible.

Between 1957 and 1962 the T4 astronomic theodolite was used extensively at Laplace stations. The Khancoban tests *(Leppert 1963)* gave the impression that there was little difference in accuracy between azimuths observed with astronomic theodolites (Type T4) and a geodetic theodolite (Type T3). However, as the sample size used to arrive at this conclusion was quite small, it must be appreciated that these indications were not necessarily conclusive.

The results of the tests had a critical bearing on the observing techniques of the Division of National Mapping, which at the time was concerned about a significant disagreement between

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* Gravimetric data *(Mather 1970a, 71-92)* was used as well, to interpolate the astronomic latitude and longitude where these had not been observed.

simultaneous and reciprocal* azimuth observations of a line *(Johnson 1962)*, the disagreement being attributed to lateral refraction errors. It was hoped to minimise these discrepancies, particularly on the coastal traverses, by double ended reciprocal azimuth observations. Moreover, astronomical theodolites require observing tents and expensive equipment, as well as experienced observers. This was not the case with geodetic theodolites. The previous considerations in conjunction with the conclusions drawn from the Khancoban azimuth tests, dominated the decision by the Division of National Mapping, to observe T3 and T4 double ended azimuths (Fig. 10.1).

Observations for latitude and longitude made at stations in order to determine the geoid, were usually performed with the DKM3a theodolite and proved less troublesome than the azimuth observations.

10.2.1.1 Azimuths.

Since σ-Octantis is used for azimuth observations in Australia, and unlike Polaris cannot be observed during twilight, when shimmer is reduced and the temperature gradient changes sign, observations were commenced the instant the star could be seen in the telescope and were stopped within an hour or so.

The system of double pointing was used on both T3 and DKM3a theodolites, the aim being to observe 12 zeros of double pointing from each end of the line on each of two evenings; with additional sets of 6 zeros for each evening a survey team was at a station for more than two evenings. For single ended observations the minimum number of zeros was increased to 18.

Lateral refraction errors were hoped to be eliminated by considering the mean of reciprocal azimuth observations made simultaneously from both ends of a line *(Bomford 1967b, 55)* even though the two often differed by as much as one second of arc *(Bomford et al 1970, 2)*.

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* The term 'reciprocal' is used here analogous to its use in trigonometric heighting *(Bomford 1962, 204)*

ASTRONOMIC DATA - SUMMARY OF OBSERVING METHODS

INSTRUMENT	LATITUDE	LONGITUDE	AZIMUTH	TIME
WILD T4	HT	MT		C
WILD T3		A(30°)	SIGMA	SW
DKM3a(I)	CMA	A(35°)	OCTANTIS	C
TAVISTOCK		A(30°)		

TABLE 10.1

KEY:

HT	Horrebow-Talcott pairs.
CMA	Circum-meridian altitude pairs.
A	Almucantar altitude pairs.
I	Impersonal micrometer.
C	Chronometer - Chronograph
SW	Split hand stop watch
MT	Meridian Transit sets.

10.2.1.2 Latitudes.

Meridian or Circum-meridian altitudes of FK4 stars were usually observed with the geodetic theodolites, whereas the Talcott method has many advantages and is theoretically more precise than the Circum-meridian altitude technique. However, it requires an extensive catalogue in order to obtain the pairing of the stars. For this purpose the FK4 does not suffice, and a lengthy prediction analysis is required using the Boss catalogue. This in itself is not inconvenient provided the prediction has been programmed on a computer. However, an analysis of the Boss catalogue by the Washington Observatory has detected systematic errors which for the southern hemisphere are reputed to exceed 1" *(Bomford 1965, 36)* and consequently, the theoretically higher precision of the Horrebow-Talcott method is somewhat diminished by errors in the catalogue.

On the whole, Circum-meridian latitudes of FK4 stars were found thoroughly satisfactory for Australian conditions. Twelve pairs were stipulated as the minimum for a first-order station; preferably 16 pairs. Observations on a single night were acceptable, although 8 pairs observed on each of two nights were preferred. For geoidal section stations and at points where latitudes were required only for azimuth computation, 6 pairs on one night were usually considered adequate *(ibid, 36)*.

10.2.1.3 Longitudes.

The method of almucantars (equal altitudes) was preferred for longitude observations (Table 10.1), although the meridian transit technique was used extensively earlier in conjunction with the T4 astronomic theodolite. Both stop watch and impersonal almucantars were observed, the former with geodetic theodolites and the latter with the DKM3a. The resulting standard deviation of the stop watch method was often no larger than that of observations made with the impersonal micrometer. However, the results are burdened with the personal equation of the observer which was usually determined by observing stop watch longitudes at stations where the longitude had already been determined impersonally. Nevertheless, stop watch longitudes were found unsatisfactory and after 1966, only impersonal observations were made *(Bomford et al 1970, 1)*.

ANGULAR & LINEAR DATA - INSTRUMENTATION & OBSERVING TECHNIQUES

AUTHORITY	DIRECTIONS		VERTICAL ANGLES					DISTANCES
	No. of Arcs	Inst.	Inst.	Type	No.of Arcs	No.of sets	Time	
LANDS DEPARTMENT	12 - 36	T3	T3	SR	3	2 - 4	10-15	MRA1 & 2
NATIONAL MAPPING	> 36	T3	T3	SR	3		13-15	MRA1, 2, 4
INTERIOR DEPARTMENT				SR^{+}	4	2	$19^{o}22$	MRA1 & 3
SURVEY CORPS(ARMY)	> 36	GT	GT	SR^{+}	4		10-15	MRA1 & 2

TABLE 10.2

KEY:

SR Simultaneous & Reciprocal observations.

SR^{+} Usually simultaneous & Reciprocal Observations.

o Observations on two separate nights.

Laplace stations were customarily observed in pairs, with impersonal observations at one end of a line and stop watch observations at the other.

For Laplace determinations, whether by stop watch or impersonal methods, the aim was to measure 16 pairs, that is, 8 on each of two nights. The minimum was 12 pairs, with not less than 6 pairs on each of two nights. For geoidal determinations 8 pairs of stars on one evening were usually observed.

10.2.2 Linear.

10.2.2.1 Distances.

Network distances were measured with the tellurometer. Model MRA1 was used in the early stages of measurement, however, the MRA2 appears to have been used extensively later (Table 10.2), and some remeasurement was made with the MRA4. Individual lines were measured from both ends, 36 or more fine readings being taken. The mean value of the two determinations was then adopted. On some lines towers were installed in order to raise the line of sight, and thus reduce ground swing.

10.2.2.2 Levelling.

South Australia is covered by an extensive third-order levelling network (Fig. 10.2). Levelling lines were usually confined to access routes which on the average follow the tellurometer traverse. The data was obtained from the South Australian Lands Department, the Division of National Mapping and the Department of the Interior.

Generally speaking, the observations were made according to the specifications stipulated by the Division of National Mapping *(Leppert 1965, 599-600)*, the levelling procedure adopted being one of dividing the separate loops into sections of length 1.6 to 8 kilometres. The sections were then forward and reverse levelled.

The Zeiss Ni2 and Watts Autoset levels appear to have been used extensively. Approximately 50 connections were made to the horizontal control network (Fig. 10.2).

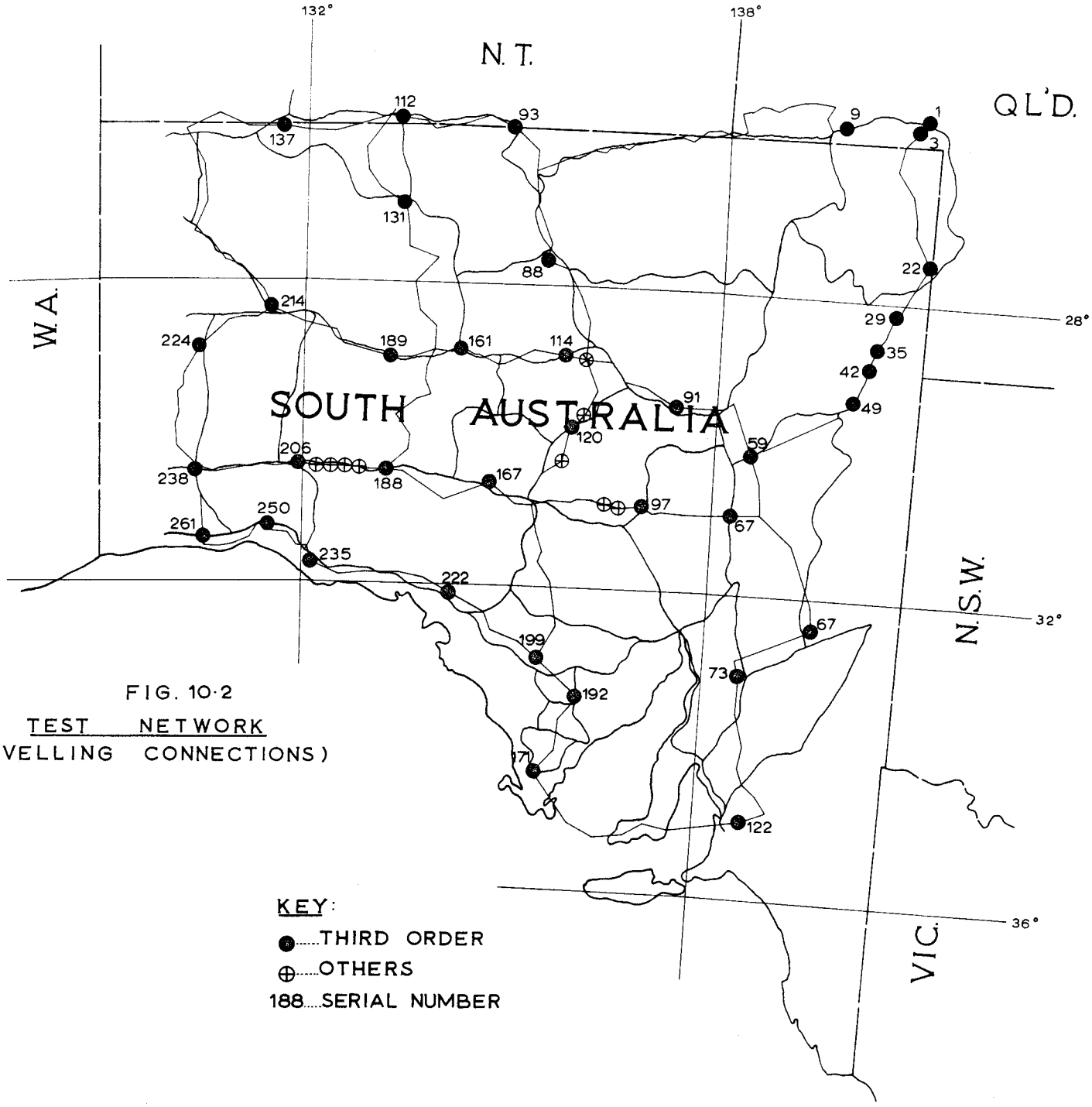


FIG. 10·2

TEST NETWORK

(LEVELLING CONNECTIONS)

AZIMUTH: REPEATED OBSERVATIONS

Station Name	Year Obs'd	Instrument	Obs'd Value (secs)	No. of Obs.	Standard Deviation (secs)	Standard Deviation of Mean (secs)
	1957	T4	19.95	23	1.01	0.21
Attraction	1957 1960	T3	23.36	41	-	-
Bates	1961	T4	59.75	27	1.40	0.27
	1962	T3	0.11	24	2.62	0.39
NM/E/18	1960	T4	31.17	24	1.22	0.25
	1961	T3	27.25	6	-	-
	1962	T3	27.46	24	1.64	0.32
Colona	1962	T4	10.49	24	1.49	0.30
	1961	T3	7.18	6	-	-
	1962	T3	9.06	24	1.64	0.33
Coppudurba	1960	T4	20.52	27	0.62	0.12
	-	T3	20.45	12	-	-
Cooper	1958	T4	43.98	30	1.04	0.19
	-	T3	41.89	-	-	-
Kingoonya	1961	T4	11.93	29	0.97	0.18
	1962	T3	11.81	24	1.34	0.28
NM/E/89	1960	T4	38.13	24	0.78	0.16
	1963	T3	37.93	12	1.49	0.59
NM/E/61	1965	DKM3a	14.39	12	2.10	0.61
	-	T3	15.29	24	-	-
NM/E/178	1965	DKM3a	23.20	12	2.35	0.68
	-	T3	25.52	23	-	-
Oak	1958	T4	36.93	28	1.48	0.28
	1961	T3	35.45	12	1.64	0.47

TABLE 10.3

10.2.3 Angular.

10.2.3.1 Directions.

Prior to 1957 normal geodetic triangulation methods were implemented. In 1957, the tellurometer arrived in Australia and changed geodetic techniques, since it was no longer necessary to use only the most elevated terrain, but rather was cheaper, easier and of greater practical value to confine trigonometrical stations to near access routes. Control could thus be extended by traversing.

Directions were usually observed when atmospheric conditions stabilised, that is, approximately one to two hours before sunset, but depending on the time of the year. Triangle and figure 'closures' so obtained were of a high quality and were well within the precision expected of first-order angular work.

Some noticeable trouble, in the angular work, was experienced at Boola Boola, which although not in South Australia, could be considered as an extremum of the occurence of such an event. *Johnson (1962, 219-220)* reports a swing of 13" within a period of one hour and 19" overall - not in what he calls a wild zero, but in each set of 12 pointings. Double ended reciprocal astronomic azimuth observations indicated that the value of lateral refraction at this station was approximately 43".2. Therefore, when difficulty was expected, the practice was, in the hope of minimising these effects, to observe reciprocal astronomic azimuths

From Table 10.2 it is seen that the majority of horizontal angular control was observed with the T3 theodolite. In most cases, more than 36 arcs were observed, an arc consisting of a double pointing to an object with a micrometer reading for each pointing, followed by a double pointing to another object with a micrometer reading. However, in some of the work completed by the South Australian Lands Department, the records indicate that prior to 1965, only 12 arcs were observed. Otherwise and in general, more than 36 arcs were observed.

Where angular observations were proving troublesome, a larger number of arcs were measured and Laplace station spacing was close.

LATITUDE: REPEATED OBSERVATIONS

Station Name	Year Obs'd	Instrument	Obs'd value (secs)	No. of Obs.	Standard Deviation (secs)	Standard Deviation of Mean (secs)
Bates	1961	T4	49.83	18	1.10	0.25
	1967	DKM3a	49.34	20	0.59	0.13
Hearne	1958	T4	4.15	20	0.71	0.16
	1967	DKM3a	4.40	18	0.60	0.14
Kingoonya	1961	T4	46.63	20	1.10	0.30
	1962	T3	46.39	71	0.70	0.08
	1967	DKM3a	46.23	19	0.35	0.08
Low Cliff	1957	T4	50.21	17	0.52	0.12
	1967	DKM3a	50.70	30	0.52	0.09
Macaw	1965	DKM3a	57.27	13	0.76	0.21
	1967	DKM3a	57.44	18	0.43	0.10
Stoney Range	1959	T4	5.35	16	0.86	0.22
	1968	DKM3a	4.44	20	0.49	0.11
Wild	1961	T4	16.38	21	0.59	0.12
	1967	DKM3a	17.94	20	0.54	0.12
NM/E/89	1960	T4	59.66	25	0.61	0.12
	1963	T3	59.20	5	0.34	0.35
NM/E/178	1962	T4	4.79	15	0.72	0.19
	1965	DKM3a	4.37	16	0.64	0.16
Oak	1963	T3	58.28	4	0.78	0.39
	1963	T3	56.68	4	0.57	0.23
	1958	T4	57.98	28	0.87	0.16

TABLE 10.4

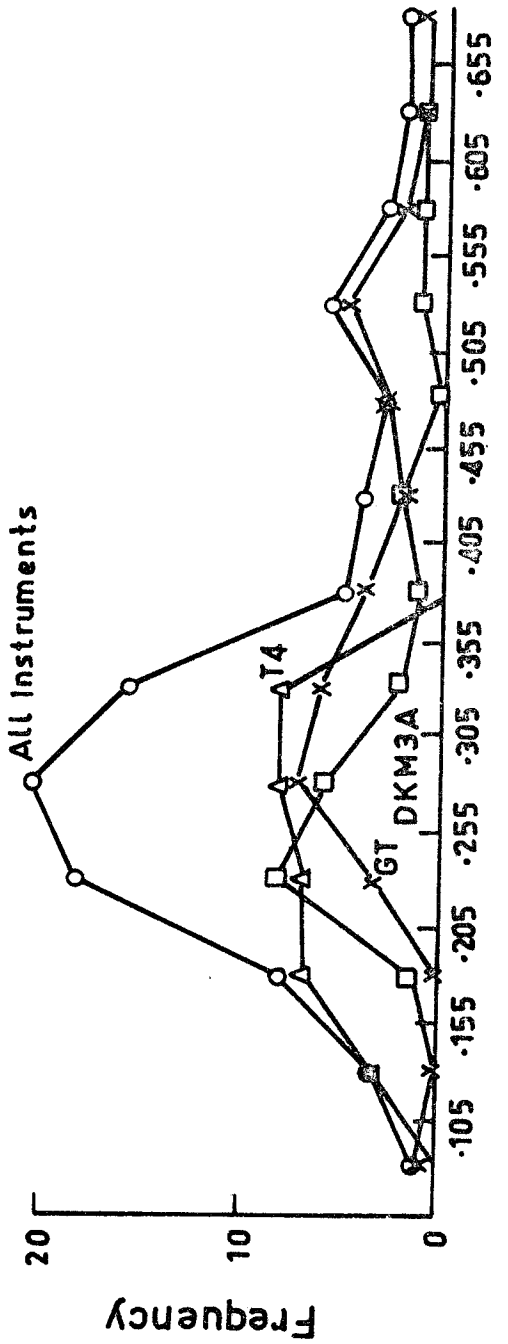


FIG. 10·3: FREQUENCY DISTRIBUTION – AZIMUTHS.

10.2.3.2 Zenith Distances.

Zenith distances are known to be severely influenced by atmospheric refraction. Observational procedure varies as apparent from Table 10.2, common practice being to observe from 2 to 4 sets of simultaneous reciprocal zenith distances, 3 to 4 arcs comprising a set. Observing times were, unfortunately, not always optimised.

It is known *(e.g. Brown 1967, 4)* that the effect of refraction on vertical angle measurement is least around or within a few hours of noon. Accordingly, most observations were made between 10.00 and 15.00 hours. However, the Department of the Interior completed its observations during the hours 19.00 - 22.00, when the effect of refraction upon measurement is largest *(ibid, 4)*. Instrumentation is shown in Table 10.2. The T3 appears to have been used extensively.

Frequently, as for linear measurement, towers were used in order to raise the line of sight above the terrain.

10.2.4 Gravity.

A detailed description of the gravity data which was used to interpolate deflections of the vertical at points where the astronomic latitude and longitude had not been observed, is given by Mather *(1970a, 73)* and by Barlow *(1970)*.

10.3 Accuracies.

10.3.1 Azimuths.

Standard deviations of the mean of 92 distinct azimuth observations within the network are represented by Fig. 10.3. It is seen that a precision of ± 1".0 is always attainable.

Observations repeated at different times, with different instruments and by different observers are given in Table 10.3. The differences obtained are usually larger than ± 1".0. Moreover, Bomford et al *(1970, 2)* found that the average 'misclosure' of 136 reciprocal Laplace azimuths was 1".93 but magnitudes of up to 8" were sometimes obtained, with no reason to suppose that any blunders were made.

LONGITUDE: REPEATED OBSERVATIONS

Station Name	Year Obs'd	Instrument	Obs'd value (secs)	No. of Obs.	Standard Deviation (secs)	Standard Deviation of Mean (secs)
Bates	1961	T4	31.19	7	-	0.13
	1962	T3	30.17	13	0.24	0.08
	1967	DKM3a	31.35	19	1.09	0.25
Black	1960	T4	56.82	6	-	0.13
	1962	T3	56.86	12	0.24	0.08
Colona	1962	T4	9.54	5	-	0.13
	1962	T3	9.03	13	0.24	0.08
Hearne	1958	T4	21.14	4	-	0.09
	1967	DKM3a	22.50	18	0.85	0.20
Kingoonya	1961	T4	53.26	7	-	0.17
	1962	T3	53.45	93	0.60	0.06
	1967	DKM3a	53.14	24	1.16	0.29
Low Cliff	1957	T4	21.51	4	-	0.07
	1967	DKM3a	22.06	29	1.19	0.22
Macaw	1965	DKM3a	26.36	15	1.01	0.26
	1967	DKM3a	26.37	20	1.16	0.26
Stoney Range	1959	T4	33.00	7	-	0.19
	1968	DKM3a	33.30	19	0.92	0.21
Wild	1961	T4	15.12	6	-	0.16
	1967	DKM3a	15.48	19	0.66	0.15
Wingilpin	1962	T3	11.08	18	1.37	-
	1967	DKM3a	10.68	22	0.84	0.18
NM/E/89	1960	T4	6.35	8	-	0.17
	1963	T3	6.75	8	0.23	0.08
Oak	1958	T4	36.92	7	0.22	0.13
	1963	T3	36.92	8	-	0.08
	1963	T3	36.92	8	0.23	0.08

TABLE 10.5

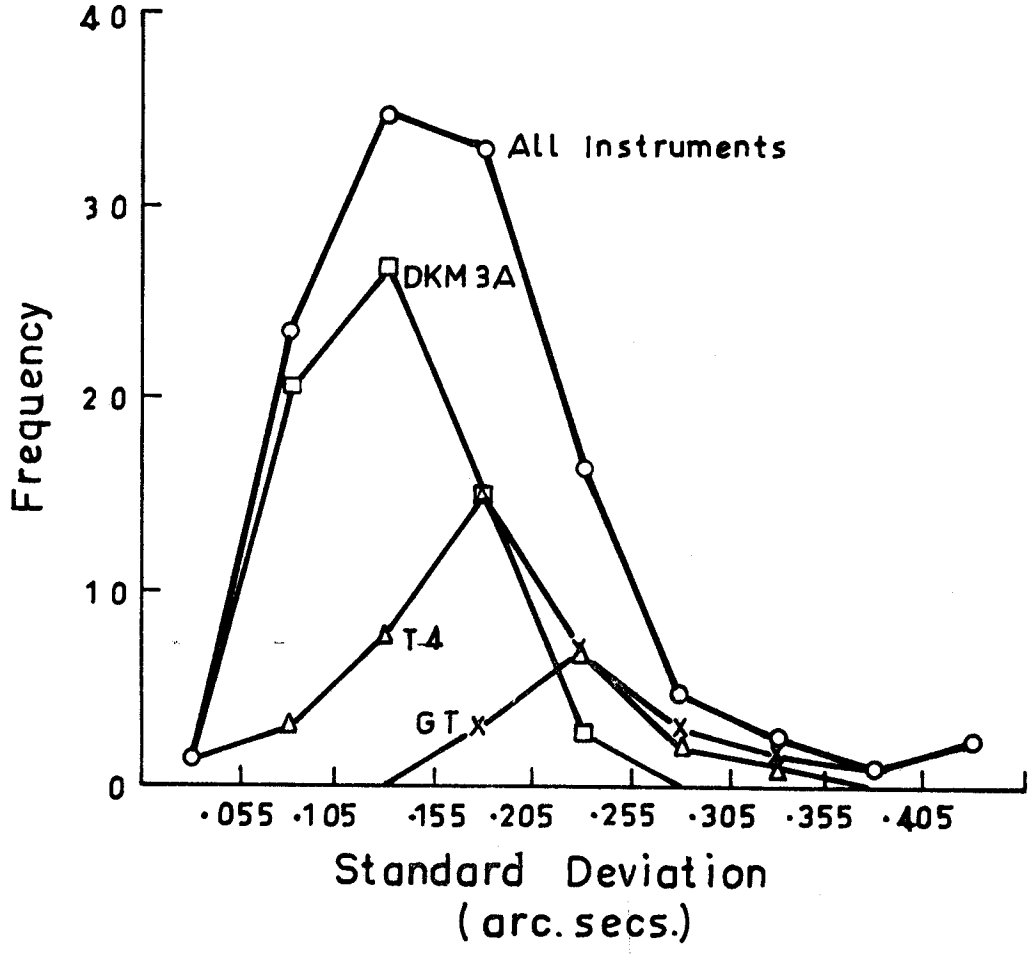


FIG. 10·4: FREQUENCY DISTRIBUTION-LATITUDES

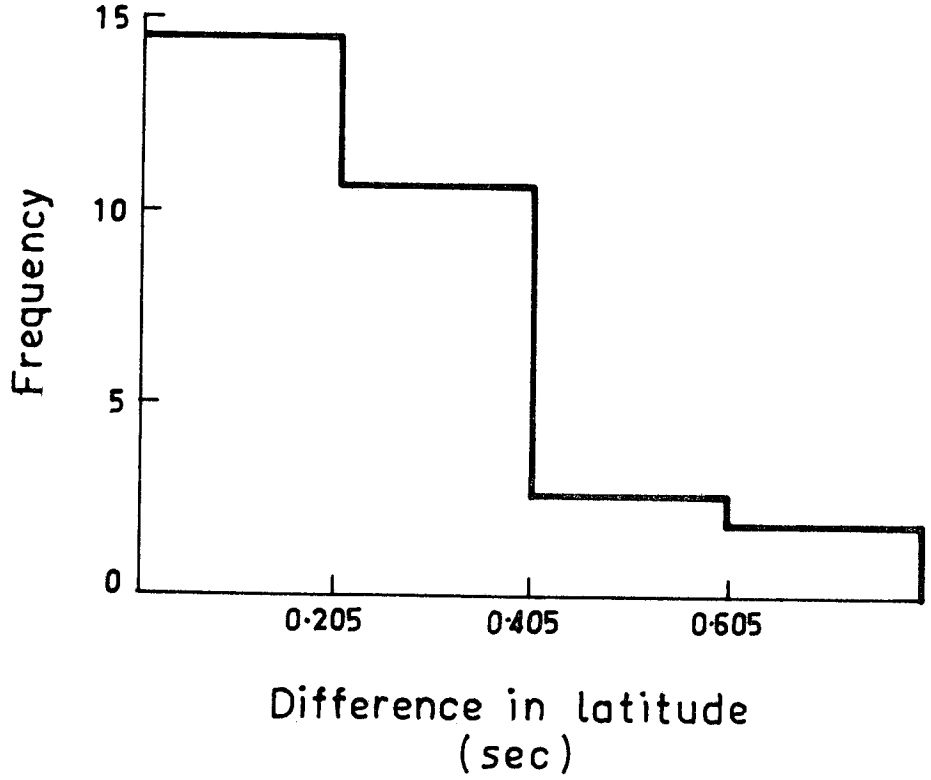

15
10
5
0
Frequency
0·205
0·405
0·605
Difference in latitude
(sec)

FIG. 10·5

10.3.2 Latitudes.

Fig. 10.4 shows the standard deviation of the mean of 124 latitude observations made over the test network. Values for DKM3a observations lie generally between 0".1 and 0".15, whereas observations made with the T4 are grouped between 0".0 and 0".35. For the geodetic theodolites values seem to fluctuate between 0".15 and 0".4.

Recent *(1967)* DKM3a determinations (Fig. 10.5) indicate an excellent agreement between the means of observations made on separate nights. Of 30 points analysed, observations at 25 differ by not more than 0".4 and the remainder do not exceed 0".7. Bomford et al *(ibid, 1)* quote similar magnitudes.

Observations repeated at different times, with different instruments and by different observers are analysed in Table 10.4. The average difference between set means is 0".6. Bomford et al *(ibid)* derive a value of 0".4 for a sample space of 110 re-observed latitudes.

10.3.3 Longitudes.

Of 125 longitudes considered (Fig. 10.6) most standard deviations of the mean were found to be less than 0".35, although some (T3 observations using a stop watch) were as large as 0".8.

Observations made with the DKM3a on two separate nights are shown in Fig. 10.7. Agreement is not good but averages 0".5.

Repeated observations are represented in Table 10.5. It is apparent that impersonal observations and observations in which the T3 was used, compare favourably.

10.3.4 Distances.

Errors in electromagnetic distance measurement comprise systematic (of an instrumental nature) and random (varying atmospheric conditions affecting the velocity of the wave propagation) components. Instrumental errors may be expressed as ± a units of measurement and remain unaltered irrespective of the length of the line. The random atmospheric components b are conveniently expressed in ppm and occur because the meteorological conditions along the line, used to calculate the refractive

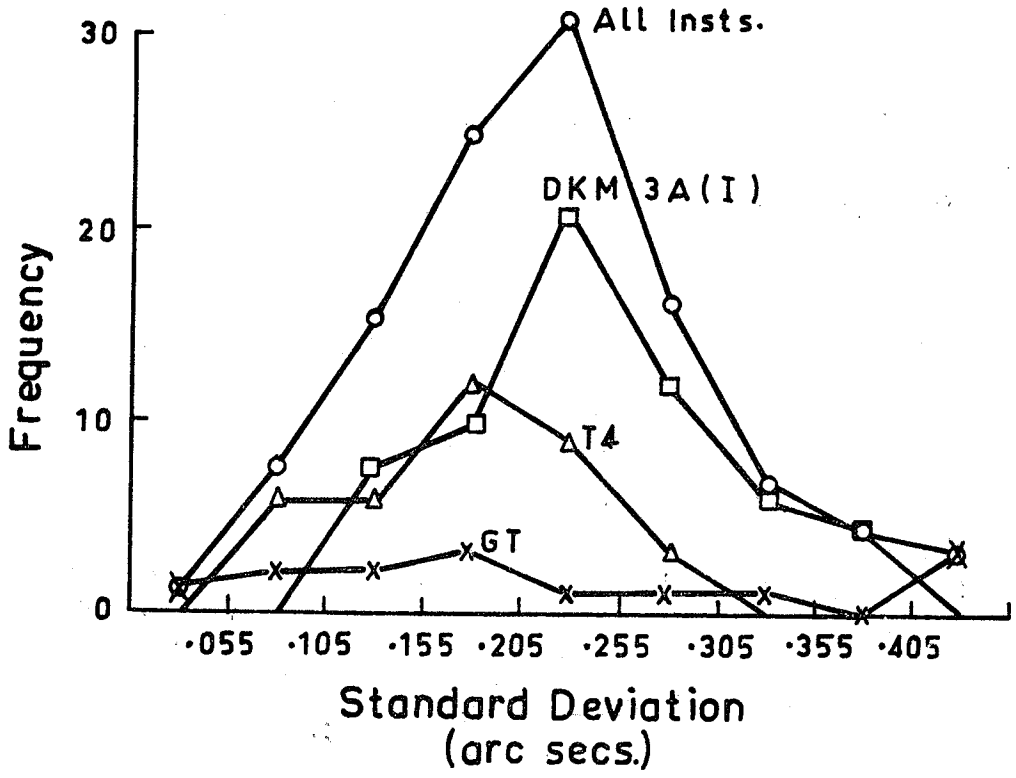


FIG. 10·6: FREQUENCY DISTRIBUTION LONGITUDES.

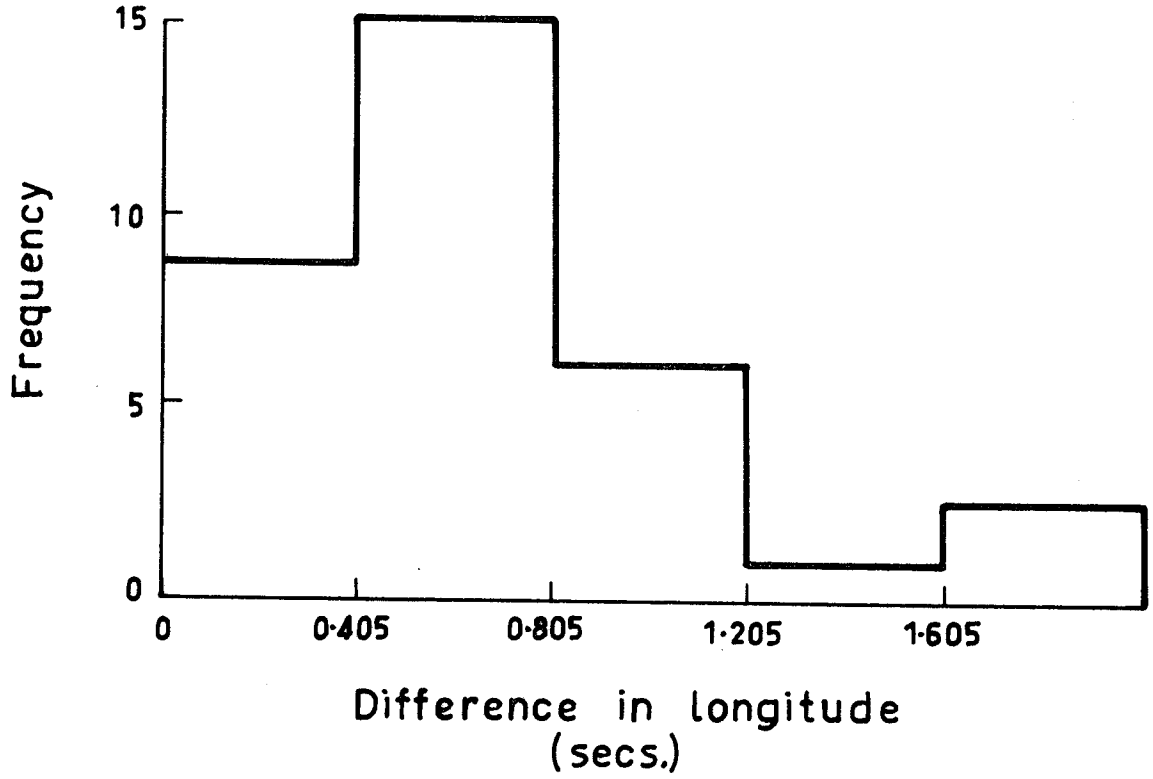

15
10
5
0
Frequency
0
0·405
0·805
1·205
1·605
Difference in longitude
(secs.)

FIG. 10·7

ESTIMATES OF SYSTEMATIC AND RANDOM ERRORS IN ELECTRONIC DISTANCE MEASURING DEVICES FOR USE IN AUSTRALIA.

INSTRUMENT	a (± cms)	b (±ppm)
Tellurometer:		
MRA 1	7	6
MRA 2	5	6
MRA 101	2.5	6
MRA 3 (101)	2.5	6
MRA 4	0.3*	6
Laser Geodimeter:		
A.G.A. Model 8	0.6	1+

TABLE 10.6

KEY:

* Read out range.

\+ Manufacturers specifications.

index of the atmosphere, have been inadequately sampled. The total error e in a measurement is thus given by

$$e = \pm(a + bs\ 10^{-6})$$

where s is the measured distance. Estimates for a and b for the various instruments used over the network are presented in Table 10.6 and are based on investigations by Robinson *(1968)*.

10.3.5 Levelling.

In Australia, the accuracy of geometric levelling is assessed either from the observed differences of independently double levelled sections or from the loop closures obtained by single or independent double levelling *(Leppert 1965, 595)*. Moreover, since it is suspected that the levelled difference in elevation depends almost entirely on the length of the levelled route *(ibid)*, the accuracy is judged by the standard deviation of the difference in elevation of two hypothetical terminals 1.6 kilometres (1 mile) apart.

Leppert *(ibid, 605)* for an analysis of 95 loops and 6 513 sections (length 1.6 to 8 kilometres) gives mean standard deviations of ± 11 mm and ± 3 mm respectively, the disagreement in values being attributed to systematic effects.

Most of the South Australian levelling has been completed since the publication of the above results. For the 25 loops considered (Fig. 10.2), the standard deviation of the mean of 1.6 kilometres of double levelling is ± 11 mm. An analysis of individual sections was not made by the author.

10.3.6 Directions.

The quality of triangulation methods is usually judged by the overall triangular misclosure that was obtained. With tellurometer traverses, however, this technique is not always feasible.

Johnson *(1962, 219)* states that the average triangular misclosure to 1962 was 0".75, which indicated good work. Bomford *(1967b, 55)* quotes the standard deviation of a direction as 0".5, being equivalent to an average triangular misclosure of 1".0 *(Bomford 1962, 144)*, but emphasises that much of the triangulation in Australia is of better quality. He

ZENITH DISTANCE

Individual Set Pointings - With Towers

Station	Set 1	Set 2	Set 3	Set 4	Set 1	Set 2	Set 3	Set 4
	Face		Left		Face		Right	
		89°	56'			90°	02'	
	58.2	57.0	58.3	56.6	04.9	05.7	06.9	04.9
T2/654	57.8	57.3	58.2	57.0	04.3	04.8	05.0	02.9
	59.1	57.2	56.9	58.2	05.1	04.0	05.2	03.6
		89°	56'			90°	02'	
	43.4	47.9	44.2	43.0	15.8	10.9	13.5	12.0
T2/654	43.1	48.1	44.1	46.7	17.3	12.3	13.3	15.9
	43.9	46.1	46.5	46.5	17.5	12.8	16.5	11.9

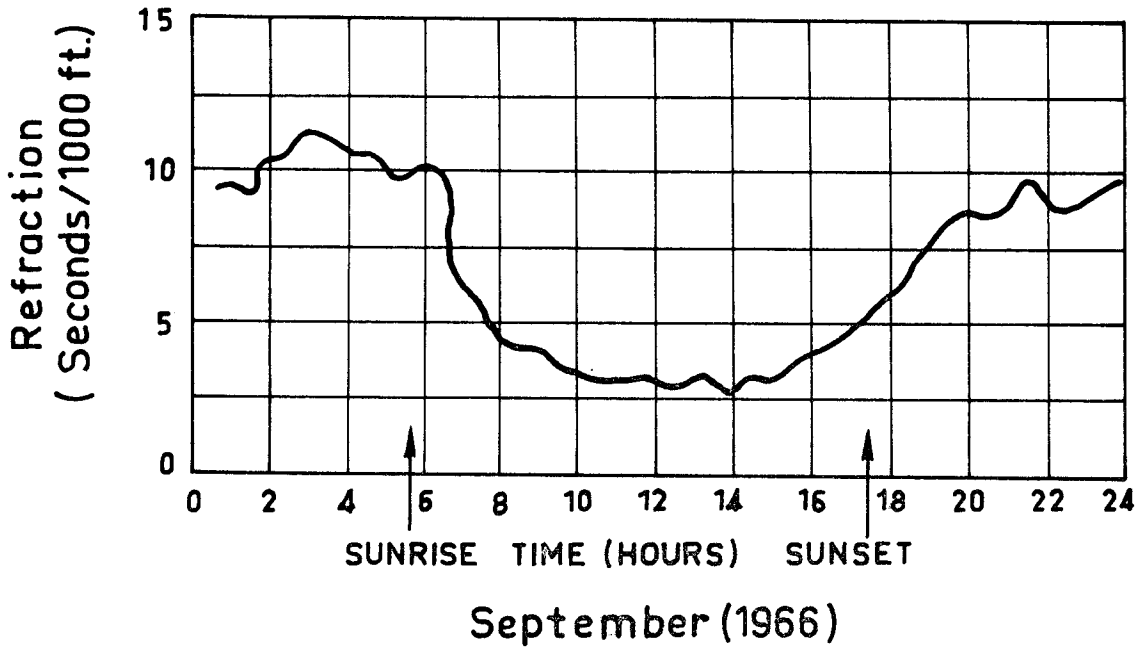
TABLE 10.7

ZENITH DISTANCE

Individual Set Pointings - Without Towers

Station	Set 1	Set 2	Set 3	Set 4	Set 1	Set 2	Set 3	Set 4
	Face		Left		Face		Right	
		90°	20'			89°	36'	
	56.6	56.8	57.5	57.3	57.1	56.5	58.8	57.3
Michael	56.5	56.1	57.5	57.0	57.2	57.9	58.2	57.5
	57.6	56.7	57.8	57.1	56.7	57.9	57.4	57.8
		89°	58'			90°	00'	
	07.7	10.8	11.6	13.3	54.1	49.6	49.2	47.3
Hardy	06.5	09.1	11.6	13.2	53.4	50.6	50.7	46.6
	07.7	10.1	12.3	13.7	52.1	48.6	49.4	47.4

TABLE 10.8



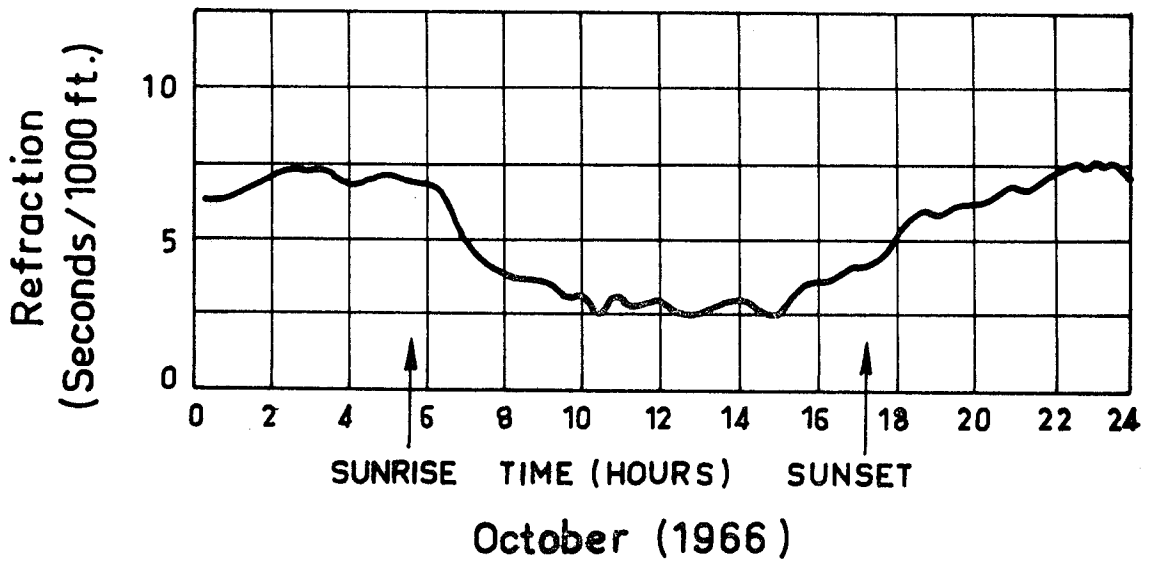


FIG.10·8: DIURNAL VARIATION ON REFRACTION.

continues to say, that on traverses there is not the same continual check and that some of it may be less good.

On the basis of this information, an individual South Australian analysis of angular work was not made by the author, the figures quoted by Bomford *(1967b, 55)* being accepted.

10.3.7 Zenith Distances.

Representative sample zenith distances are shown in Tables 10.9 and 10.10. They are taken from the field records of the South Australian Lands Department. It is seen (Tables 10.7 and 10.8) that a high precision is possible, although in general, individual set means cannot be reproduced (Tables 10.9 and 10.10). The fact is, that zenith distances are burdened with refraction.

Experiments made by Brown *(1968)*, in a region which with the exception of the coast, is typical of the topography and climate of the test network *(Aitchinson 1969, 1-34)*, indicate that the angle of refraction displays a remarkably similar pattern throughout the year (Fig.10.8).

Thus, providing the coefficient of refraction can be accurately calculated, the inclusion of zenith distances into the network computations appear feasible.

10.3.8 Gravimetric Deflections.

The basis to calculating gravimetric deflections of the vertical are the Vening-Meinesz integrals *(Heiskanen & Moritz 1967, 114)* which depend on Stokes' function $S(\psi)$.

The structure of Stokes' function in which ψ is the element of solid angle about the computation point, and its derivative $dS(\psi)/d\psi$, indicates that the magnitude of the function as ψ approaches 0^{o}, depends on $\operatorname{cosec} \psi/2$ which is very unstable in this range of values. Accordingly, the accuracy of the gravity anomaly used is more critical for the areas closer to the computation point and these regions require more accurate field determinations. On the basis that $S(\psi)$ and $dS(\psi)/d\psi$ are required to be linear for the elements of surface area over which $\{\xi, \eta\}$ are being evaluated, it is generally agreed*(Mather 1968, 3)* that division of the earth's surface into meridian and parallel squares as shown in Table 10.11 is adequate.

ZENITH DISTANCE

Comparison of Set Means - With Towers

Station		Zenith Distance	Time
At	To	90° 04' 12".8	10.15
T2/654	T1/654	15.2	11.00
(74m)	(79m)	15.8	11.35
		13.0	11.50
		90° 05' 06".8	11.45
T2/654	T3/654	04' 49.2	12.35
	(74m)	59.0	13.25
		55.8	14.20

TABLE 10.9

ZENITH DISTANCE

Comparison of Set Means - Without Towers

Station		Zenith Distance	Time
At	To	89° 16' 00".4	13.15
		01.8	14.00
Michael	Patawarta	01.0	14.50
(67m)	(1 008m)	00.8	15.20
		90° 03' 31".8	15.10
		19.2	15.25
Hardy	Norman	16.0	15.40
(56m)	(56m)	07.4	15.55

TABLE 10.10

Typical contributions of the various zones to $\{\xi, \eta\}$ *(Mather 1970b)* are also shown in Table 10.11.

For the test network sufficient data is available to accurately calculate the contributions to $\{\xi, \eta\}$ of regions where ψ is greater than a tenth degree. This is not so, however, for the four one tenth degree squares surrounding the computation point. An accurate evaluation of the contribution to $\{\xi, \eta\}$ of these compartments requires a closely spaced gravity survey *(Rice 1952, 285-312)* in order to test whether the gravity anomaly gradient is linear over the region.

Table 10.11 indicates that without a close gravity survey about the computation point, calculated deflections cannot be expected to be better than ± 1".0. Heilbronner *(1968, 41)* also quotes this value.

11. A System of Standard Deviations for Weighting.

Standard deviations of observed quantities suggested by various authors are shown in Table 11.1. The values that are to be adopted in the calculation of a set of weight coefficients for adjustment purposes are also presented (Table 11.1) and are based on the analysis made in Chapter 10.

It should be noted that the effect of measuring distances from both ends of a line and taking the mean of the observed values is to reduce the standard deviation by a factor of the square root of two.

GRAVIMETRIC DEFLECTIONS

Range of ψ	Representation Adopted	Typical Contribution to $\{\xi, \eta\}$
$\psi < 0^\circ.1$	Individual readings	0.4 seconds
$0^\circ.1 < \psi < 1^\circ.5$	$0^\circ.1 \times 0^\circ.1$ sq. values	4.4
$1^\circ.5 < \psi < 5^\circ$	$0^\circ.5 \times 0^\circ.5$ sq. means	-3.4
$5^\circ < \psi < 20^\circ$	$1^\circ \times 1^\circ$ sq. means	-1.2
$\psi < 20^\circ$	$5^\circ \times 5^\circ$ sq. means	0.1

TABLE 10.11

STANDARD DEVIATIONS

Reference	Azimuth (sec)	Direction (sec)	Distance (cm)	Latitude (sec)	Longitude (sec)	Zenith Distance (sec)	Levelling (cm)
Bomford (1967)	1.0	0.5	3 + 3ppm	0.2	0.2	-	-
Heilbronner (1968)	0.5	0.1-0.15	-	0.2-0.3	0.2-0.3	-	-
Ramsayer (1969)	-	0.5	2 + ½dKm/Km	0.2	0.23	$1.5\sqrt[4]{\frac{10}{n}}$	-
Mather (1969)	-	-	-	0.3-0.5	0.3-0.5	0.6	-
Stolz (Chapter 10)	1.0	0.5	3 + 3ppm	0.4* 1.0@	0.6* 1.0@	1.5	$1.5\sqrt{L}$

TABLE 11.1

KEY:

*Observed @Interpolated.

n = number of measurements.

PART E: COMPUTATIONS, RESULTS AND CONCLUSIONS.

Previously, the problem of point definition in Earth space was considered in theory, that is:

(1) the fundamentals of the problem were examined;

(2) the problems that are likely to arise regarding the solution of the large linear systems encountered and the inversion of the coefficient matrix were investigated; and

(3) a test network was proposed.

Computations are to be made exclusively using a digital computer and superficially it seems that all which remains is to program the proposed model, punch the data, load both into a computer and, wait for the results to appear. Unfortunately, matters are not as simple. There will be obstacles of a computational nature that need to be overcome. In any practical application for example, the size of the problem which can be solved is limited by the storage capacity of the available computer and larger than capacity problems require special programming methods. Moreover, an appropriate method of numbering the network stations will ensure a narrow band-width of the normal equation matrix and thus may determine whether a linear system can be solved directly or the partitioned approach must be adopted.

The data because it is copied will be subject to transcription errors and since computer time is costly, careful preliminary analysis is necessary before extensive computations are made. In the variation of co-ordinates method loop or figure closures are commonly used for this purpose with the disadvantage that, although indicating that something may be wrong, they do not tell where in the loop the error lies. A direct comparison of measurements may be a suitable alternative when the computations have already been performed in another system. Thus, recalling that the test network is part of the AGD which was adjusted on the spheroid in 1966 *(Bomford 1967b)*, the directions, azimuths and distances can be compared. Zenith distances however, because they are generally only used to obtain geoidal heights and astronomic position determinations which are implicit in the Laplace equation, require special treatment to see if they are in error.

Practical and economic considerations dictate that levelling connections to the trigonometrical network be made at a limited number of points. The inclusion of levelling in the computations will thus pose computer storage problems, since widely separated connections imply a large band-width in the normal equation matrix. As an alternative to a direct solution, the partitioned approach may have to be rejected on account of the time required to solve the system generated.

At network stations where the latitude has not been observed, the diagonal coefficient of latitude in the normal equation matrix, since it receives its dominant contribution from the zenith distances which are of a lower precision than the other geodetic measurements, are quite small. An attempt at solving such an equation system will exhibit all the symptoms of an ill-conditioning problem and, while it is possible to avoid this difficulty by proper scaling, it will be found more satisfactory, when sufficient data is available, to interpolate the vertical gravimetrically.

The solution of the large number of normal equations and the inversion of the coefficient matrix is not mere routine, a major problem being the choice of method. The method chosen, direct or partitioned, is usually governed by the amount of computer core space available, by the time required to implement the adopted technique and, to a degree by the condition of the linear system which is to be solved. Accordingly, if the coefficient matrix is not excessively ill-conditioned and can be stored in the core of the computer, then the direct method with iterative improvement is generally preferred, not only because it is faster but also because the inverse may be obtained from the factorization with only a modest increase in storage requirements. Other reasons why the partitioned approach should be avoided if possible, are that the inversion by partitioning is not simply an extension of the decomposition process and that substantial amounts of backing storage are necessary.

Finally, because numerous arithmetic operations are involved in the full or the partial inversion of matrices of large orders, extensive computation times are envisaged and it may happen that costs prohibit the inversion of the system under consideration. The solution will then be of unknown accuracy - although providing successive iterates are converging it may still be correctly rounded - and a smaller problem needs to be

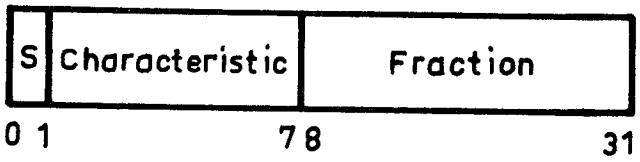


FIG. 12·1a

Short – Floating – point Number (One Word)

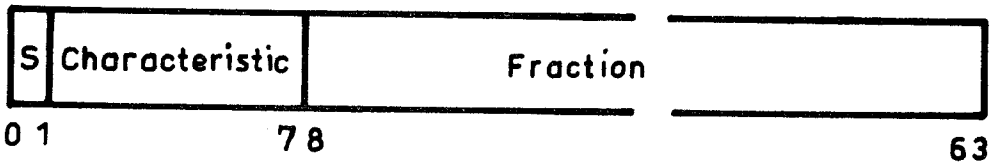


FIG. 12·1b

Short – Floating – point Number (Double Word)

examined in order to arrive at some conclusions regarding the accuracy of the point location.

12. Preliminary Considerations.

12.1 Digital Computations.

Computations are to be made exclusively using a digital computer. Regarding the computer available, the following are of interest:

(1) storage capacity;

(2) the word structure; and

(3) the precision of the arithmetic.

Storage capacity is important since the size of the problem that can be solved, is limited by it - whereas the word structure and the precision of the arithmetic are of value in assessing the effect of rounding errors on the solution to the problem.

The times required to perform the various arithmetic operations, which depend essentially on the storage locations of the numbers concerned, are also useful knowledge, since they may be extrapolated to give an indication of the total computation time for a particular task. A lengthy analysis will however normally be necessary using this approach and other methods which are however less accurate are generally preferred.

12.1.1 Computing Machinery.

An IBM 360/50 electronic computer was used in the network computations.

12.1.2 Storage Capacity.

Approximately one million bytes* of core storage were provided. In addition, five disc drives (providing approximately seven million bytes of backing storage each), two of which are in constant use by systems utility programs and two magnetic tape units (providing approximately 24 million bytes of backing storage each) were accessible.

12.1.3 Computer Word Structure (Floating-point)

Floating-point numbers on the IBM 360/50 which occur

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* One byte is equivalent to eight binary bits.

in either short or long formats, differ only in the length of their fractions (Fig. 12.1a and 12.1b). It is seen that the floating-point numbers are either 32 or 64 binary bits long. The short length, which is equivalent to seven significant decimal places permits a maximum number of operands to be placed in storage and gives the shortest execution times. The long length gives up to 17 significant figures and is used when a higher precision of computation is required.

The fraction of a floating-point number is expressed in hexadecimal (base 16) digits, each consisting of four binary bits. In the short format, the fraction consists of six hexadecimal digits occupying bits 8-31 and in the long format the fraction has 14 hexadecimal digits occupying bits 8-63. To provide the proper magnitude for the floating-point number, the fraction is considered to be multiplied by a power of 16, the characteristic portion (bits 1 - 7) being used to indicate this power.

Bit position 0 in either format is taken up by the sign (S) of the fraction.

12.1.4 Floating-point Registers.

Four floating-point registers which are two words in length and can contain either a short (one word) or long (two word) floating-point operand are available for floating point computation. The second part of the register is ignored and remains unaltered during short-precision arithmetic.

12.1.5 Computer Arithmetic.

Although the final results in short-precision arithmetic have six fraction digits, intermediate results in addition, subtraction and division usually extend to seven fraction digits. Intermediate results in long precision computations however, do not exceed 14 fraction digits.

12.1.6 Number Representation.

The range covered by the magnitude (M) of a normalized*

---

* The process of normalization consists of shifting the fraction left until the high order hexadecimal digit (bit positions 8, 9, 10 and 11) is non-zero, and reducing the characteristic by the number of hexadecimal digits shifted.

floating-point number is *(IBM 1966a, 41)*:

(a) in short precision,

$$16^{-65} \leq M \leq (1 - 16^{-6})16^{63},$$

(b) in long precision,

$$16^{-65} \leq M \leq (1 - 16^{-14})16^{33},$$

or approximately

$$5.4 \times 10^{-79} \leq M \leq 7.2 \times 10^{75} \quad \text{in either precision.}$$

12.2 Mesh-point Numbering.

In the context of a geodetic adjustment problem which is to be solved by the differential displacement technique, the band-width is given by the expression $2(\ell - 1)d + 1$, where ℓ is the number of unknowns per station and d is the maximum difference in serial numbers between adjacent network points.

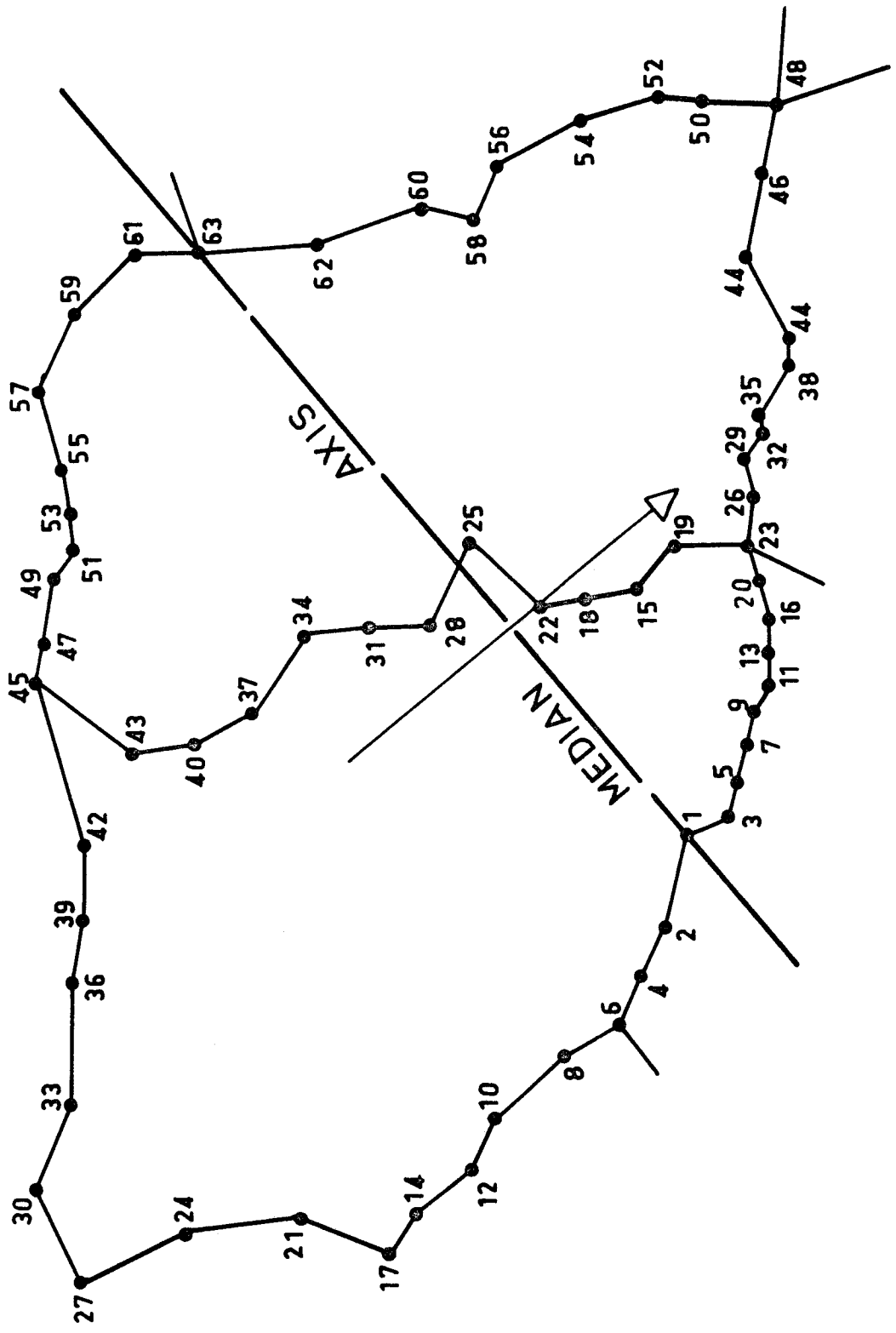
An appropriate method of numbering the network stations will ensure a narrow band-width of the normal equation matrix and may determine whether a generated equation system can be solved directly or in partitioned form.

A situation in which the band-width is actually a minimum, can only be achieved for idealised network configurations, such as open traverses, closed loops or simple triangulation chains. Idealised network configurations do not, however, commonly occur in practice and for the general network encountered, it is at best only possible to accomplish an approximate minimum by trial and error.

The general principles of a trial and error approach which may serve as a guide to treating more complex layouts are as follows *(Ashkenazi 1965b, 30-32)*:

(1) a median axis (Fig. 12.2) is drawn to bisect the network and
(2) numbering of the network stations now proceeds in a direction at right-angles to this median axis, the serial numbers being increased as shown.

Serial numbers were allocated to the test network in this manner


AXIS
MEDIAN
1
2
3
4
5
6
7
8
9
10
11
12
13
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15
16
17
18
19
20
21
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63

FIG. 12·2

and a value of seven was obtained for the maximum difference in serials between adjacent network points.

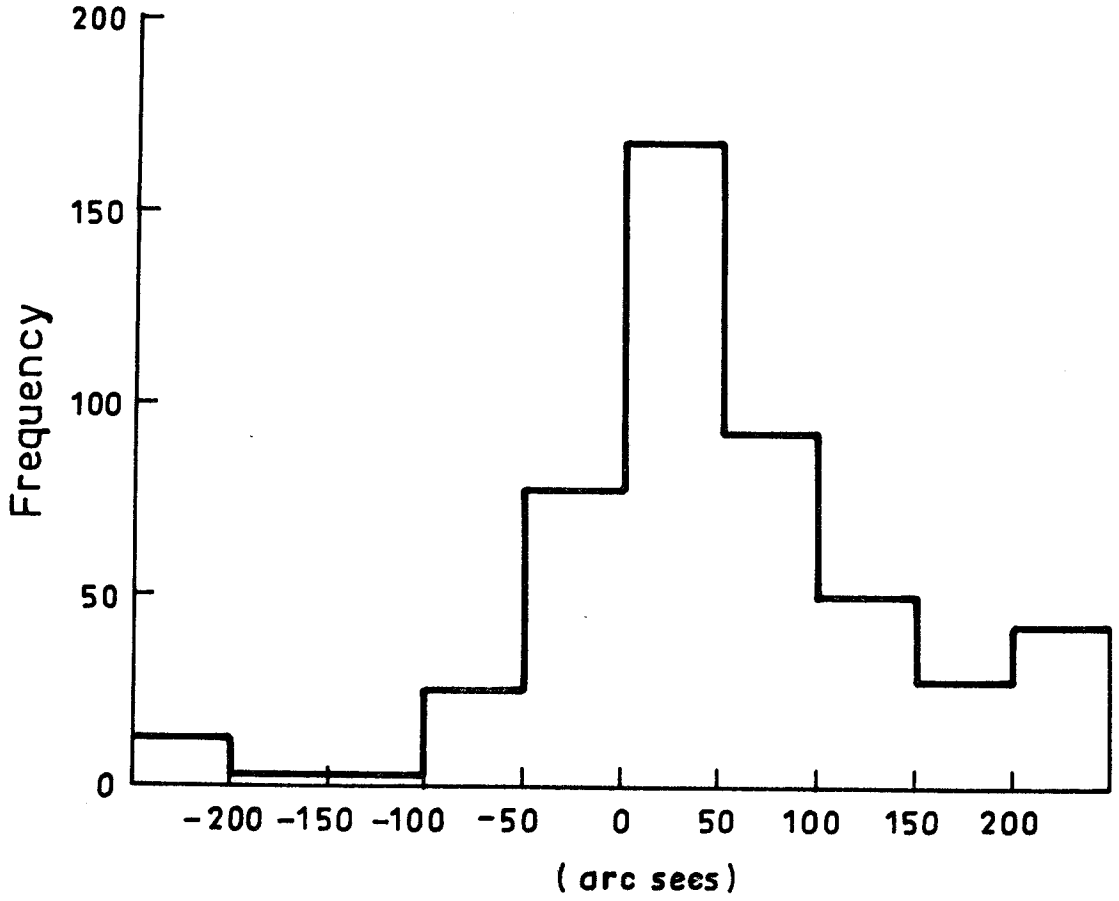
In mesh configurations of the test network type (a series of closed loops) experience confirms that it is possible to obtain a maximum difference in serial numbers of the mesh stations equal to the maximum number of traverse lines cut by any perpendicular bisector of the median axis, plus one.

12.3 Levelling.

The maximum difference in serial numbers between points connected, en route, by levelling is 77 and occurs between serials 214 and 137 (Fig. 10.2). A serial difference of 77 will give a band-width of 924 [2x(7-1)x77+1] and 1270950 [½(924+1)(2x1836-924)] elements within the upper half of the band. If the matrix were stored in short format (Four byte words) then 5 083 800 bytes are necessary, only about one fifth of this storage requirement being however permissible on the IBM 360/50. If the partitioned approach to the solution of the normal equations is adopted and the partitioning scheme of Fig. 8.1 is implemented, then three partitions of order 924x924, 924x912 and 912x912 respectively are created, which when stored in short length computer words require 10 112 832 bytes of backing storage. Disc storage is preferable in the generation of the partitioned normal equation matrix (See 13.1.3) and, since a single disc on the IBM 360/50 provides only seven million bytes of backing storage, the three partitions would pose some programming problems. Apart from the programming aspects, which may be overcome, inversion of a full matrix of order 924x924 is likely to demand large amounts of computer time. Even if a more efficient partitioning scheme were chosen, it is quite likely that the solution time for 1 836 simultaneous equations with a band-width of 924, using the standard programming languages, will be large. Refraction-free zenith distances as calculated from the combination of astronomic observations and levelling data (Section 5.4) were therefore excluded from the computations.

12.4 Zenith Distances.

The main disadvantage of the variation of co-ordinates method of adjustment is that it is difficult to detect mistakes, either in the data or the results, at the completion of one cycle of the adjustment process.



Observed - Computed Zenith Distances

Fig. 12·2 a

An established technique of checking the data is to compute loop closures, which, although indicating that something is wrong, do not reveal where in the loop the error lies. The test network is however part of the AGD, which was adjusted on the spheroid, by the variation of co-ordinates technique, in 1966 *(Bomford 1967b)*. The computer output for this adjustment *(Lambert 1969)* gives adjusted and observed measurements. The observations shown could accordingly be projected back into space and there be compared with the available data. Any differences which were considered excessive could then be investigated.

The previous approach is useful for the measured distances, directions and azimuths. Zenith distances however, because they are generally only used to obtain geoidal heights and, astronomic co-ordinates, which are implicit in the Laplace equation, require special treatment to see whether they are in error. Transcription errors in the astronomical observations will be difficult to check in a pre-adjustment analysis, but large corrections during adjustment should be regarded with suspicion. A method of checking the measured zenith distances is to compare them with those computed from measured distances, astronomic co-ordinates and spheroidal heights *(Wolf 1963a, 112)*. Geoidal undulations for the region of the test network when referred to the Australian National Spheroid are always less than 10 metres *(Mather et al 1971, Fig. 3.1)*, and therefore, assuming an average distance of 30 kilometres, rarely affect a computed zenith distance by more than ± 60 arc secs. For this reason and the fact that zenith distances are burdened with refraction, geoidal heights were used in the calculations. Observed minus computed values are shown in Fig. 12.2a by a frequency histogram. Only about 73% of the observations lie within the range $-100'' < O-C < 100''$ which is considered acceptable in virtue of the neglected geoidal undulations and the effect of refraction. 11% of the differences are in excess of ± 200". That all is not well with the observed zenith distances is confirmed by Fig. 12.2b which shows the coefficients of refraction at network stations as calculated from simultaneous reciprocal determinations, the formulae quoted in Clark *(1957, 426)* being used in the calculations. About 37% of the calculated values appear within the range 0 - 0.105.

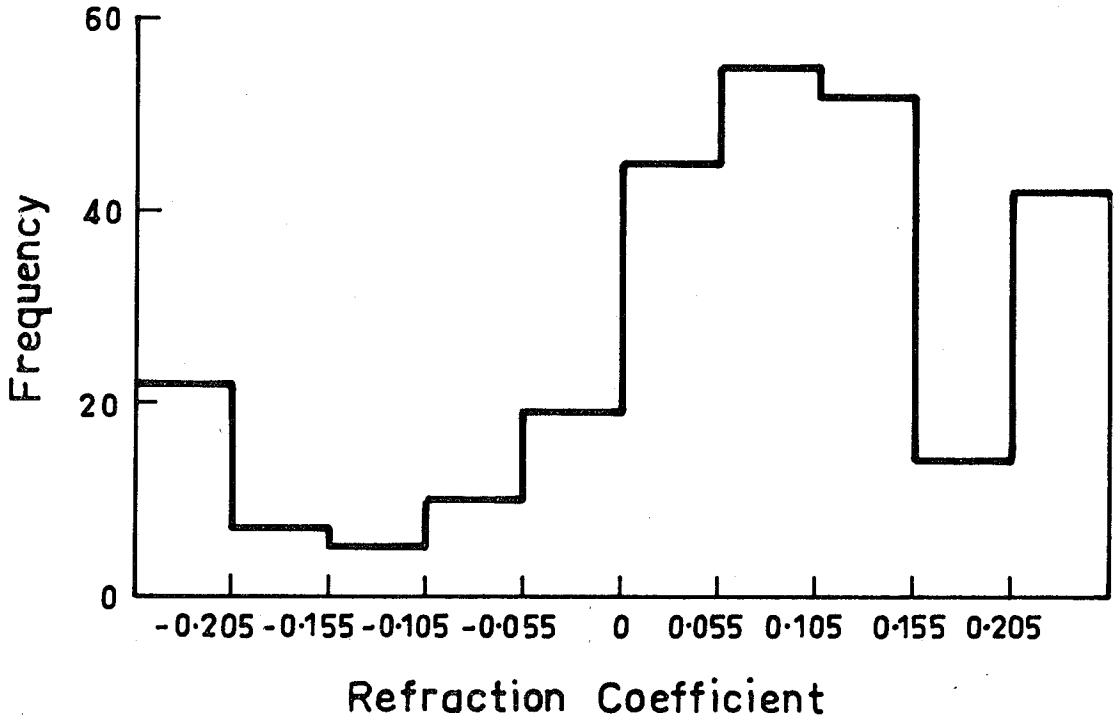

60
40
20
0
Frequency
-0·205 -0·155 -0·105 -0·055 0 0·055 0·105 0·155 0·205
Refraction Coefficient

FIG. 12·2b

Reconsultation of the field records would undoubtedly have resolved some although not all of the discrepancies revealed by Figs. 12.2a and 12.2b. Instead, it was decided to compute zenith distances from a model topography in the following manner:

(1) a topographic model based on the heights that were used in the projection of the observations for the 1966 AGD adjustment *(Lambert 1969)* to the spheroid, was chosen; and

(2) zenith distances were now calculated from formula (6.4) using observed or gravimetrically interpolated astronomic co-ordinates, the rectangular Cartesian co-ordinates as computed from the adjusted 1966 spheroidal co-ordinates and the assumed topographic model.

A small random refraction effect of approximate magnitude ± 1.5 arc secs was also introduced by means of truncation.

12.4.1 Derived Weight Coefficients.

An error in the calculated zenith distances resulting from errors in the model parameters is given by Equation 6.6. The rectangular Cartesian co-ordinates used in the calculations are adjusted quantities and some of the astronomic latitudes and longitudes are gravimetrically interpolated. Assuming the calculated Cartesian co-ordinates to be in error by not more than ± 1 metre (See Fig. 15.10) and considering the standard deviations shown in Table 11.1, it is seen that the errors in the calculated zenith distances will be dominated by those in the latitudes and longitudes, i.e.

$$\sigma_\beta^2 = \cos^2\alpha\ \sigma_\phi^2 - \sin 2\alpha \cos\phi\ \sigma_\phi\sigma_\omega + \cos^2\phi \sin^2\alpha\ \sigma_\omega^2 \quad \ldots\ldots \quad (12.1)$$

The above expression may be suitably programmed using the figures quoted in Table 11.1, but it will be found simpler, in view of the assumption already made, to work with an average value. Accordingly, based on the figures shown in Table 12.1 and the fact that Equation (12.1) will give an underestimate for σ_β^2, the value 2.25 arc sec^2 was adopted as the variance of all computed zenith distances.

COMPUTED ZENITH DISTANCES

(Estimated Standard Deviations)

$\phi = 30^{\circ}$

AZIMUTH	OBSERVED ϕ, ω	INTERPOLATED ϕ ω
90° or 270°	± 0".5	± 0".9
0° or 180°	± 0".4	± 1".0

TABLE 12.1

12.5 Origin Conditions.

12.5.1 Choice of Origin.

Trigonometrical station GRUNDY (Serial No. 98, Fig.10.2), with geodetic latitude -25° 54' 11".0780, geodetic longitude 134° 32' 46".4570 and height 397.5 metres *(Lambert 1969)*, the point which was held fixed in the 1966 Australian National Adjustment *(Bomford 1967b, 62)*, was selected as the datum point of the test network.

12.5.2 Co-ordinates of the Datum Point.

The rectangular Cartesian co-ordinates of the datum point were calculated from the geodetic co-ordinates assigned to GRUNDY, the parameters of the Australian National Spheroid, that is, a = 6 378 160 metres and f = 1:298.25, and Formulae 3.14. They were found to be

$$\{x_o, y_o, z_o\} = \begin{Bmatrix} -4\ 027\ 503.765 \\ 4\ 091\ 807.129 \\ -2\ 769\ 590.118 \end{Bmatrix} \text{ metres} \qquad \ldots\ldots \quad (12.2)$$

It is emphasized here that the reference spheroid for the AGD is not geocentric *(Mather & Fryer 1970, 6)* and thus the coordinates given in Equation (12.2) will not refer to a geocentric reference frame. AGD co-ordinates may however, if desired, be converted to a geocentric system by applying at the Johnston Origin *(Lambert 1968)* the three orientation corrections

ΔN_o = 7.2m ± 0".2m, $\Delta\xi_o$ = -4".2 ±0".2, $\Delta\eta_o$ = -4".5 ± 0".2 *(Mather 1970c, 77)* to the geoid-spheroid separation (N) and the components (ξ and η) of the deflection of the vertical in the meridian and prime vertical respectively.

12.5.3 Constraints.

At GRUNDY the direction of the vertical as defined astronomically was assumed free of error.

12.5.4 Orientation of the Survey Scheme.

The test network was permitted to rotate about the local vertical at the datum point in accordance with the principles of a free net adjustment discussed in section 5.2.3.

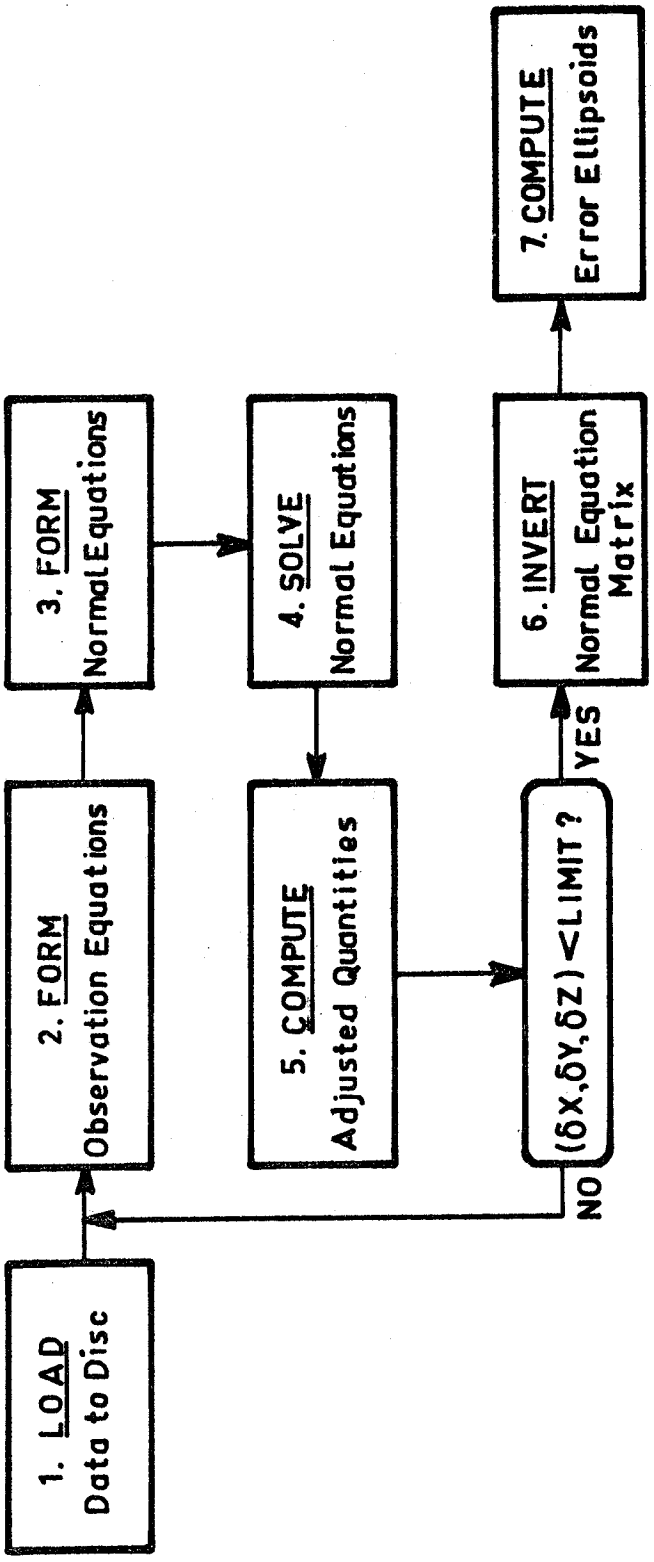

1. LOAD
Data to Disc
2. FORM
Observation Equations
3. FORM
Normal Equations
4. SOLVE
Normal Equations
5. COMPUTE
Adjusted Quantities
(δX,δY,δZ) <LIMIT ?
YES
NO
6. INVERT
Normal Equation
Matrix
7. COMPUTE
Error Ellipsoids

Fig. 13·1

13. Adjustment.

13.1 Programming.

The following computational steps are necessary in order to effect a solution to a defined adjustment problem using the differential displacement technique and the least squares principle:-

(1) the formation of the observation equations;
(2) the formation of the normal equations;
(3) the solution of the normal equations and
(4) the calculation of the adjusted variables.

Several iterations of the above procedure are generally required before the desired or possible precision is attained. The inverse of the normal equation matrix may then be calculated and standard eigenvalue strategy *(Malhotra 1969)* will provide the error quadrics of the particular adjustment problem.

Although it is not a difficult programming task to automate the entire adjustment procedure on an electronic computer, it is customary, since computer time is costly, to peruse the results after each iterative cycle and to continue with the next step only if successive iterations are converging and there are no errors in the data, or if sufficient iterative cycles have been completed. The programming principles involved are described by Bomford *(1967a)* and by McLellan et al *(1970)*.

Any of the standard languages, such as FORTRAN or PL/1, appears suitable for adjustment problems involving less than say 200 normal equations.

13.1.1 Problems associated with Large Adjustments.

13.1.1.1 Storage.

For very large adjustment problems, the storage requirements of the normal equation matrix by far exceed those of all other variables and program statements combined. It is therefore of advantage, in order to capitalize on the available core space, to divide the whole adjustment computation into distinct steps as shown in Fig. 13.1. However, since each step is then processed separately, a considerable amount of backing storage is required for information which is used in more than one step (Figs. 13.2, 13.3, 13.4, 13.5, 13.6 and 13.7).

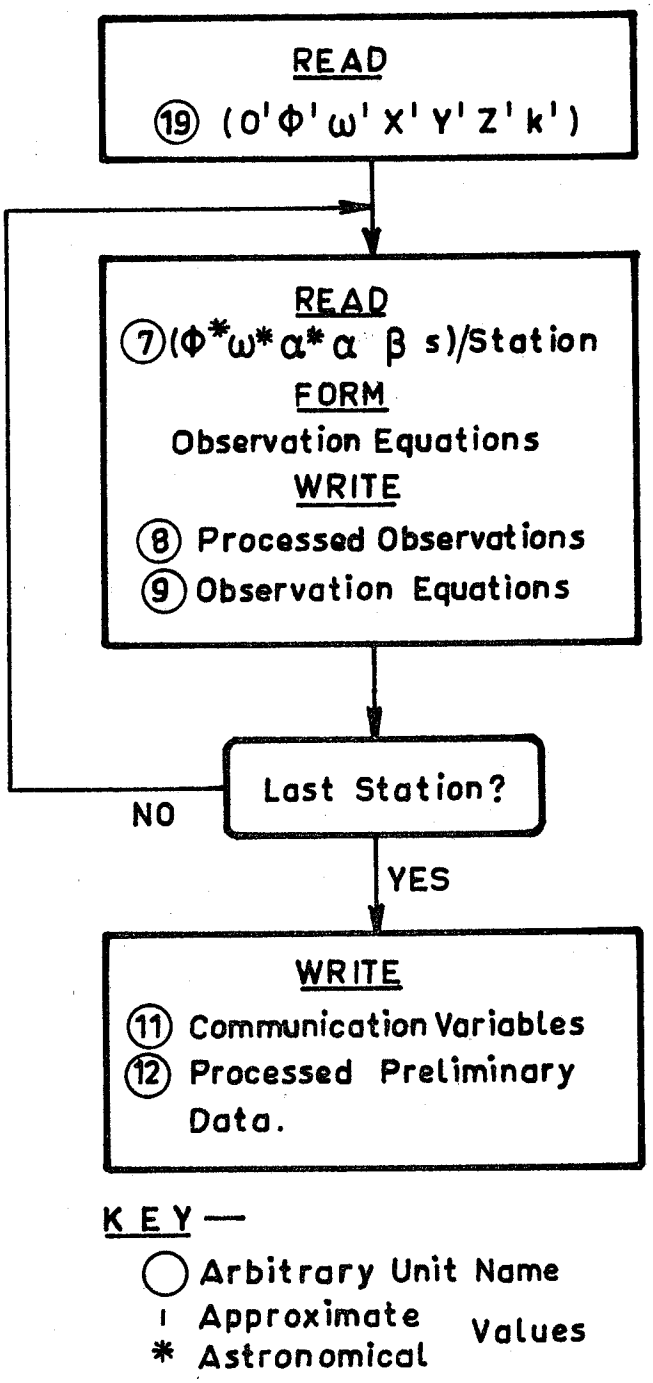


FIG. 13·2. FORMATION OF OBSERVATION EQUATIONS.

13.1.1.2 Programming Principles.

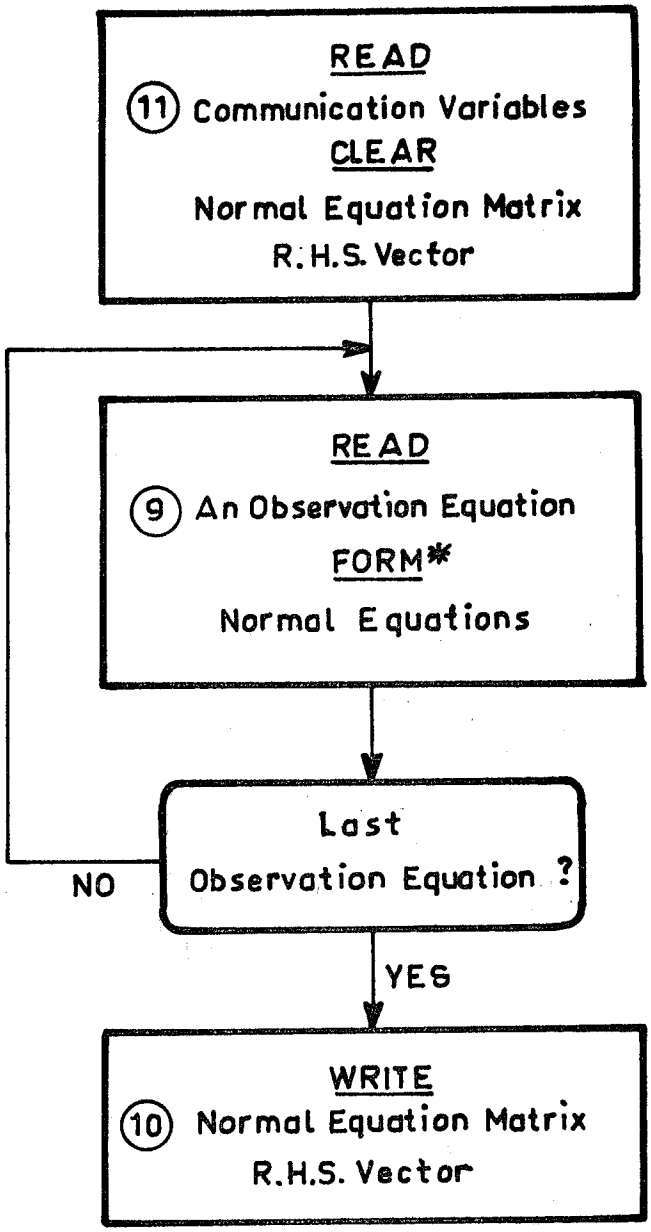
The programming fundamentals which are necessary to implement a three-dimensional adjustment in accordance with the abovementioned divisions are summarised in Figs. 13.2, 13.3, 13.4, 13.5, 13.6 and 13.7 . Standard programming languages are inefficient and no longer suffice to decompose or invert matrices of large order. Machine language is recommended in programming these steps. Moreover, since systems failure is imminent during large computation times, it becomes necessary to periodically dump the intermediate results together with the information that is required to enable recommencement of the calculations at the point of interruption, onto backing storage. The latter applies in particular to the inversion program (Fig. 13.6).

13.1.2 Disc Loading of Programs and Data.

Considerable amounts of computer time are saved if, once the programs have been finalised and any errors in the data have been corrected, both are disc loaded.

13.1.3 Partitioned Normal Equation Matrix.

If all the elements of the normal equation matrix within the band cannot be accommodated in the computer core simultaneously, then the matrix may be partitioned. In practice, however, observations are customarily processed station by station and in the case of the geodetic traverses, the network stations are treated not in numerical sequence but in sections between junction points, within which, since the band-width should be as small as possible, the numbering of nodes is not consecutive. Thus if the observation equations are processed in the order in which they were formed, accumulation of coefficients becomes rather disorderly. If many partitions exist, then the processing of the observation equations in the order in which they were generated (Fig. 13.2) can be time consuming, since the movement between partitions implies INPUT/OUTPUT operations from backing storage, the time for which greatly exceeds that required for the arithmetic operations (on the IBM 360/50 it takes approximately three seconds to locate and read/write a partition of 10 000 elements, but only 0.14 seconds to process an observation equation).



KEY:

* A fixed point requires special treatment

◯ Arbitrary Unit Name.

FIG. 13·3: FORMATION OF NORMAL EQUATIONS

OBSERVATIONS

PROPOSED WEIGHTING SCHEME

OBSERVATION	WEIGHT	UNITS
AZIMUTH	1	Arc Seconds^{-2}
DIRECTION	4	
LATITUDE @	6.25	
LATITUDE +	1	
LONGITUDE @	2.78	
LONGITUDE +	1	
ZENITH DISTANCE	0.44	
DISTANCE	p	Decimetres^{-2}

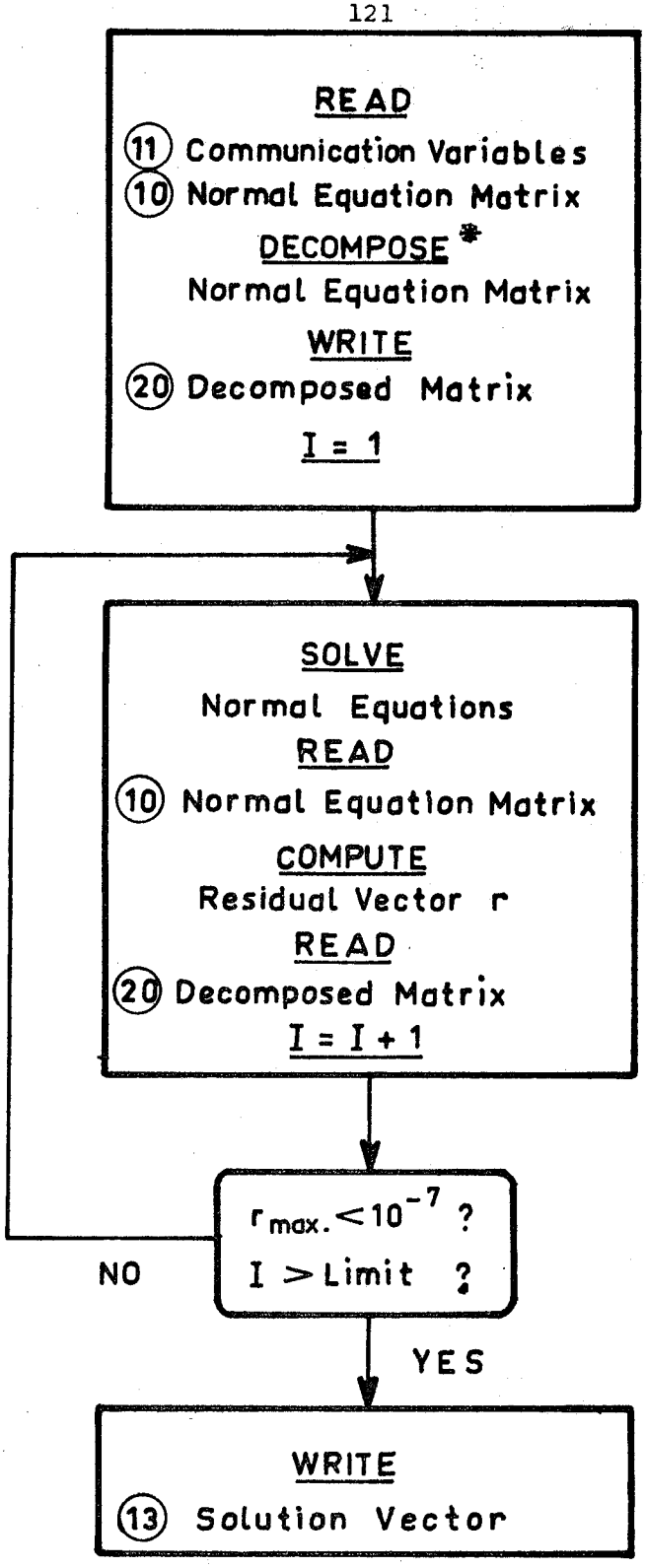
TABLE 13.1

KEY:

@ OBSERVED

\+ INTERPOLATED

p $(0.3 + 0.3s10^{-6})^{-2}$: s being in metres



KEY: * Decomposed matrix replaces the normal equation matrix in the computer core.
◯ Arbitrary Unit Name.

FIG. 13 4: SOLUTION OF NORMAL EQUATIONS.

INPUT/OUTPUT transfers may be reduced, if it is realised that each observation connects only the unknowns defining two points. Therefore, considering the partitioning scheme shown in Fig. 8.1, the coefficients contributed to by a particular observation equation will lie within the four consecutive partitions $A_{i,i}$, $A_{i,i+1}$, $A^{T}_{i,i+1}$ and $A_{i+1,i+1}$ or since only the upper symmetric partitions need to be stored, in the three adjacent partitions $A_{i,i}$, $A_{i,i+1}$ and $A_{i+1,i+1}$.

If all the observation equations pertaining to m consecutive rows of the normal equation matrix, where m is the band-width, are treated in sequence, INPUT/OUTPUT transfers are minimised. This condition may be enforced by sorting the observation equations prior to the formation of the normal equations. Since the partitioned information is frequently updated during the generation of the normal equation matrix however, Direct Access programming techniques *(IBM 1966b, 62)* are recommended.

13.2 Initial Co-ordinates.

Formulae 5.14, the parameters of the Australian National Spheroid and the corresponding geodetic co-ordinates of the 1966 Australian National Adjustment *(Lambert 1969)* were used to calculate the initial rectangular Cartesian co-ordinates of all network points.

13.3 The Variance Factor.

Since the choice of variance factor is arbitrary, the value unity was selected.

13.4 Weights.

The weights of the observations adopted for adjustment purposes are shown in Table 13.1 and are based on a variance factor of unity and the standard deviations quoted in Table 11.1. Observations were assumed to be uncorrelated giving a diagonal weight matrix W.

13.5 The Normal Equations.

13.5.1 Basic Matrix Properties.

13.5.1.1 Order.

The test network contains 262 variable points each with seven degrees of freedom. Two additional degrees of freedom are needed at the survey origin to account for uncertainties in the approximate orientation and the refraction coefficient, yielding 1 836 (262 x 7 + 2) normal equations.

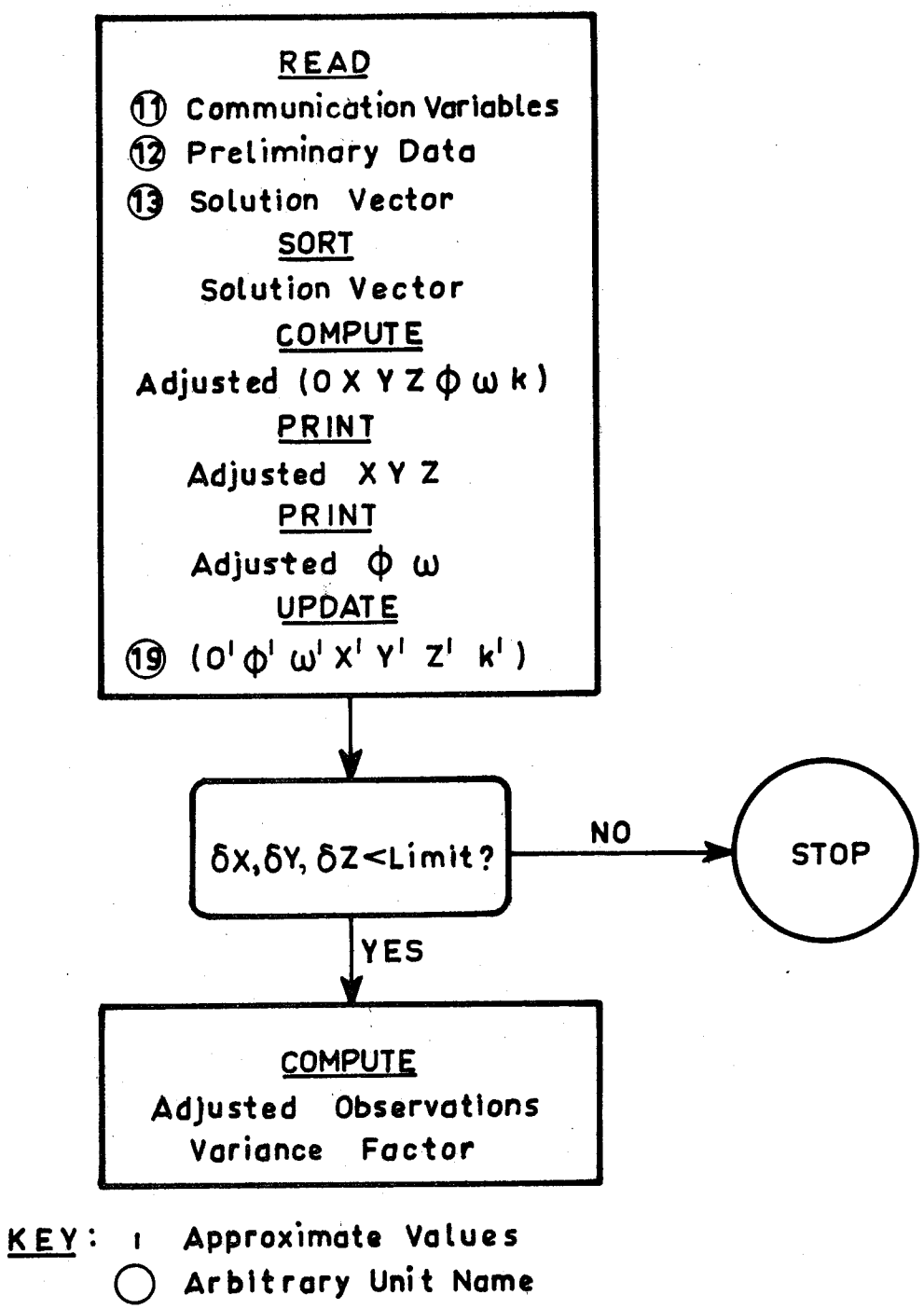


FIG. 13·5 COMPUTATION OF ADJUSTED QUANTITIES.

13.5.1.2 Band-Width.

A maximum difference of 7 in serial numbers between adjacent network stations will give a band-width in the normal equation matrix of 85 [2 x (7 - 1) x 7 + 1].

13.5.1.3 Storage Requirements.

The total number of elements in the normal equation matrix required in order to effect a direct solution and partial inversion was thus 154 241 [½(85 + 1)(2 x 1 836 - 85)], being equivalent to 616 964 (154 241 x 4) bytes of storage if working in single length computer words or 1 233 928 (154 241 x 8) bytes of storage if working in double length computer words.

In order to effect a solution by partitioning using double length computer words, approximately 150 cylinders of disc storage (200 Cylinders to a disc) were found necessary whilst the inversion required roughly three magnetic tape units.

13.5.1.4 The Diagonal Coefficient of $\delta\phi$.

The diagonal coefficient of $\delta\phi$ in the normal equation matrix receives contributions as indicated in Table 13.2. At stations where the latitude has not been observed, this diagonal element will be dependent solely upon the contributions from the observed azimuths, directions and zenith distances emanating from the point under consideration. Contributions from observed azimuths and directions are quite small, since the zenith distances for average terrestrial geodetic networks approximate to 90^{o}. The dominant contribution will thus be received from the observed zenith distance. Zenith distances are, however, commonly of a lower precision than the other geodetic measurements and their contributions to a particular diagonal element will be less than those of the other measurements. It is also not uncommon for a measured line to be of azimuth near 90^{o} or 270^{o}, for which $\cos^2 \alpha$ is significantly zero. Moreover, in the formation of the normal equations, if the coefficient of the unknown in the observation equation is of order 10^{-t}, then the contribution of the diagonal coefficient of the unknown in the normal equation matrix will be of order 10^{-2t}. The diagonal coefficient of $\delta\phi$, as well as off-diagonal terms in the row and column pertaining to $\delta\phi$ in the normal equation matrix will thus be significantly

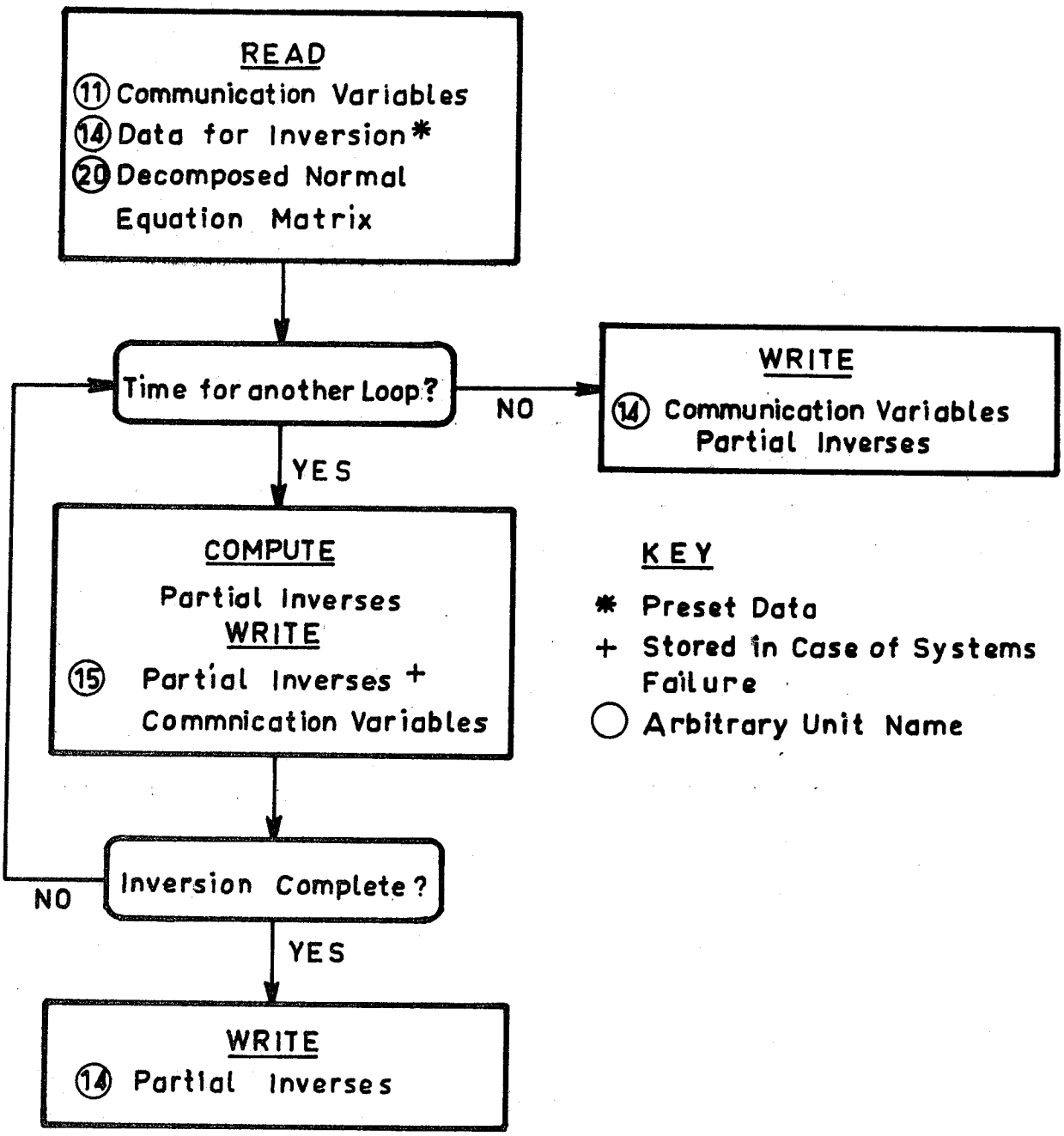


FIG. 13·6. PARTIAL INVERSION.

NORMAL EQUATIONS

DIAGONAL COEFFICIENT OF $\delta\phi$

OBSERVATION	CONTRIBUTION
AZIMUTH	$\cos^2\beta \sin^2\alpha$
DIRECTION	$4 \cos^2\beta \sin^2\alpha$
LATITUDE @	6.25
LATITUDE +	1
ZENITH DISTANCE	$0.44 \cos^2\alpha$

TABLE 13.2

KEY:

@ OBSERVED

\+ INTERPOLATED

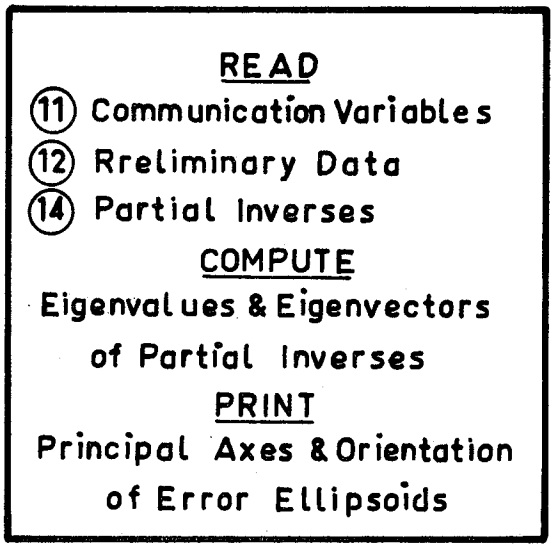


FIG. 13·7. ERROR ANALYSIS.

zero at stations where the latitude has not been observed astronomically In the evaluation of the determinant of N, a near zero row or column implies a near zero determinant $|N|$, meaning that the matrix is almost singular.

An attempt at solving such an equation system will exhibit all the symptoms of an ill-conditioning problem. These are, for a solution by Gaussian elimination with partial pivoting *(Wilkinson 1969, 254)* the emergence of:

(1) 'small' pivotal elements;

(2) a 'large' computed solution; and

(3) a 'large' residual vector.

Experience confirmed these symptoms. If the Cholesky approach is adopted, then the diagonal elements corresponding to the relevant coefficient of $\delta\phi$ tend to become negative during the triangularization algorithm indicating that rounding errors have destroyed positive definiteness or that $||N^{-1}||_2 > 3.7 \times 10^7 (16^8/1\ 837)$ (See 8.4.2).

If sufficient gravimetric data is available then this difficulty may be overcome by interpolating the latitude gravimetrically. As an alternative, when gravimetric data is not available it is possible to ignore the relevant $\delta\phi$ row and column and to solve only for the other unknowns at the points in question. This proposal will however be a departure from reality.

13.5.1.4.1 Interpolated Deflections of the Vertical.

The programs and the gravity data necessary to compute deflections of the vertical on the AGD, at stations where the astronomic latitude had not been observed, were available to the author *(Mather 1970a)*. The calculated deflections shown in Table 13.3 were used to deduce the 'astronomic' latitude and longitude from the formulae *(Heiskanen & Moritz 1967, 187)*

$$\phi^* = \phi_g + \xi \qquad \ldots\ldots \quad (13.1)$$

$$\omega^* = \omega_g + \eta \sec\phi$$

where ϕ_g and ω_g are respectively the geodetic latitude and longitude on the AGD.

INTERPOLATED DEFLECTIONS OF THE VERTICAL

S	ξ"	η"	S	ξ"	η"	S	ξ"	η"	S	ξ"	η"	S	ξ"	η"
3	0.00	-1.21	40	2.99	-1.39	101	1.23	-0.55	164	0.32	1.61	216	8.90	-5.63
5	-0.66	-1.09	41	-3.10	-0.42	105	6.35	-1.86	166	2.72	2.02	218	6.73	-0.58
7	-2.33	-0.93	42	3.97	-1.23	106	0.00	0.67	167	-0.49	1.26	220	3.67	-1.42
8	-2.77	1.87	43	-0.32	-0.95	113	3.16	0.67	168	2.87	-2.33	226	-0.03	3.76
9	-2.66	0.54	44	4.34	-1.03	114	3.15	0.21	169	2.78	-2.66	228	0.14	3.04
10	-1.61	0.47	45	-2.48	0.75	115	1.19	2.59	170	2.31	-2.88	229	1.18	2.47
11	-2.33	-0.66	47	2.87	0.10	116	9.84	-5.86	171	4.20	-5.80	230	-0.86	-4.22
12	0.11	-0.31	49	-0.12	-4.22	117	-0.70	2.11	173	2.48	-4.43	232	1.13	0.30
13	-2.11	1.65	50	-1.37	-0.41	121	2.35	0.69	174	-2.81	-3.52	237	2.36	1.71
14	2.05	-1.31	51	7.71	3.99	122	4.00	-3.71	175	0.53	-2.30	240	0.84	-1.74
16	1.39	-0.57	52	-1.36	-0.32	124	-6.39	-5.34	177	0.95	-0.10	241	0.42	-1.71
18	-3.36	0.85	53	8.09	0.01	129	0.83	-0.53	180	2.71	-1.31	242	-0.31	-2.64
19	-0.45	-0.11	54	-1.65	0.44	131	-7.43	-7.36	181	-3.14	-3.02	246	-0.88	-5.55
20	-1.80	0.81	56	4.94	-4.61	133	5.76	0.45	182	2.78	-2.52	247	-1.93	-3.36
21	-0.26	-0.52	58	-0.88	0.09	135	7.40	-12.67	183	-1.27	-4.54	248	-0.52	-4.03
22	-2.55	-1.14	60	3.73	-7.02	140	3.98	-0.59	185	7.05	-1.75	250	-3.06	-6.22
23	-0.03	-0.99	61	-1.35	-0.29	141	3.00	-3.04	187	2.75	-1.96	253	0.26	-0.80
24	4.94	-1.81	66	-1.40	-0.31	142	3.49	-2.92	189	1.60	-2.48	254	-2.27	-6.46
25	0.29	-3.32	67	2.78	1.34	145	3.32	0.38	192	4.10	-4.35	255	-0.28	-1.97
26	1.65	0.57	68	-3.42	-3.23	147	0.32	-0.68	194	1.28	-1.29	256	-2.60	-5.37
27	2.23	-2.78	70	4.59	0.29	149	2.62	0.69	195	1.90	-2.25	261	-1.14	-1.38
29	2.18	-1.64	72	2.08	-1.03	150	4.08	-1.71	197	1.44	-2.00			
31	3.75	-2.12	84	0.35	0.93	152	-0.03	0.62	198	1.70	-2.55			
32	-1.74	-0.55	86	-0.47	-3.53	155	1.72	0.08	199	-	-4.56			
34	-3.34	-1.63	89	-1.18	0.32	157	3.85	-2.00	203	-	-3.50	S = Serial Number		
35	3.30	-2.62	91	2.75	-1.77	159	2.57	-2.74	205	0.88	-4.44			
38	2.97	-2.24	95	2.75	0.72	162	0.70	-3.14	208	7.04	-5.99			
39	-1.43	-0.79	100	1.71	0.20	163	-4.87	-4.33	209	2.50	-6.15			

TABLE 13.3

13.5.2 Scaling.

It is convenient to preserve symmetry and positive definiteness in scaling a matrix, if these properties were existent in the original. The product of a matrix and its transpose will always be symmetric and positive definite and, in the least squares problem, scaling is therefore best effected once the observation equation has been formed. Consider the observation equations

$$Ax = b + v = b' \quad \ldots\ldots \quad (13.2)$$

in which the dimensions of a particular left-hand side Ax are the same as those of the corresponding right-hand side b'. Scaling of the x_j and b'_i should thus be preceded by rendering the observation equations dimensionless and is usually achieved by multiplication with $W^{\frac{1}{2}}$, the square root of the weight matrix *(Grossmann 1961, 107)*, that is,

$$W^{\frac{1}{2}}Ax = W^{\frac{1}{2}}b' \quad \ldots\ldots \quad (13.3)$$

Unknowns x_j or right-hand sides b'_i may now be scaled at will.

The coefficients of the normal equations given by $(W^{\frac{1}{2}}A)^T(W^{\frac{1}{2}}A)$ are essentially a series of summed products, the equilibration of which implies that none of their moduli should exceed unity or be less than 16^{-1}. It is however quite unclear how to program such an equilibrating algorithm *(Forsythe & Moler 1967, 45)*. Inspection revealed that, for the case of the test network the moduli of the scaled coefficients $(D_2W)^{\frac{1}{2}}A$ [See Section 8.6] could be transformed to roughly within the range 0.1 and unity, by adopting the scaling scheme shown in Table 13.4. While the normal equation matrix is then unlikely to be equilibrated in the sense of the definition, coefficients within the separate rows (columns) will be of approximately the same order of magnitude. Consequently, a small perturbation within any one of the rows (columns) of the coefficient matrix is of the same magnitude as that of any other row (column) and a linear equation solver can be applied.

14. Solution of the Normal Equations, Inversion of the Coefficient Matrix. Error Analysis.

14.1 Solution of the Normal Equations.

Since only 960K bytes of computer storage were available for use and the banded normal equation matrix required 1 233 928 bytes in

OBSERVATION EQUATIONS: CHOICE OF UNITS

VARIABLE	UNIT OF MEASUREMENT	SCALED UNIT
δo, $\delta\phi$, $\delta\omega$	arc seconds	1
δk	dimensionless	100
δX, δY, δZ	metres	$\frac{1}{10}$

TABLE 13.4

double precision computations but only 616 964 bytes in single precision computations, it was necessary to decide whether to perform:

(1) a partitioned solution using a double precision matrix of coefficients, without iterative improvement; or

(2) a direct solution using a single precision normal equation matrix, plus iterative improvement.

The partitioned matrix, which requires almost a disc to itself for backing storage, has first to be generated in partitioned form. On the IBM 360/50 this will mean an additional 45 minutes over and above the solution time, the latter being about the same for both methods. The normal equations if generated in single precision words take approximately 3 minutes of computation time. Moreover, once the decomposition has been accomplished using the direct approach, only an additional back-solution is needed to calculate the inverse - whereas the calculation of the inverse by partitioning necessitates a completely new approach and thus extensive additional computer time. Considerable amounts of intermediate storage are also necessary in the calculation of the inverse by partitioning *(Knight 1965)*. Accordingly, it was decided to store the normal equation matrix in short length computer words and to use the direct method of solution with iterative refinement.

14.1.1 Algorithm.

The following algorithm is commonly used to factorize the coefficient matrix A of the linear system $Ax = b$ by the Cholesky method:

$$g_{11} = (a_{11})^{\frac{1}{2}}; \quad g_{1j} = a_{ij}/g_{11} \quad (i = 1) \qquad \ldots\ldots \qquad (14.1)$$

$$g_{ii} = (a_{ii} - \sum_{\ell=1}^{i-1} g_{\ell i}^2)^{\frac{1}{2}} \quad (i > 1) \qquad \ldots\ldots \qquad (14.2)$$

$$g_{ij} = (a_{ij} - \sum_{\ell=1}^{i-1} g_{\ell i} g_{\ell j})/g_{ii} \quad (j > i) \qquad \ldots\ldots \qquad (14.3)$$

A diagonal element g_{ii} is thus computed from the square root of the difference between the corresponding diagonal element a_{ii}, and the sum of the squares of all the previously computed elements g in the same column. A non-diagonal element g_{ij} is obtained by

subtracting from the corresponding element a_{ij} the sum of the products, row by row, of the corresponding elements g taken from the columns i and j. This difference is then divided by the diagonal element g_{ii} for the row. It is seen that once a particular g_{ij} has been found, a_{ij} is not again required in the factorization; so g_{ij} may temporarily replace a_{ij} in the computer storage.

The vector y may be found from the formulae:

$$y_1 = b_1/g_{11} \qquad (i = 1) \qquad \ldots\ldots \qquad (14.4)$$

$$y_i = (b_i - \sum_{\ell=1}^{i-1} g_{\ell i} y_\ell)/g_{ii} \qquad (i < 1) \qquad \ldots\ldots \qquad (14.5)$$

Each element y_i is found in turn by subtracting from the corresponding element b_i the sum of the products, row by row, of the elements in column i and the previously found y_i. The difference is then divided by the diagonal term. Unless iterative improvement is to be effected, the b_i is no longer needed once the y_i has been found so that y may replace b in storage.

Similarly, but by working in reverse order, the unknowns x are found from the equations:

$$x_n = y_n/g_{nn} \qquad \ldots\ldots \qquad (14.6)$$

$$x_i = (y_i - \sum_{\ell=i+1}^{n} g_{i\ell} x_\ell)/g_{ii} \qquad (i < n) \qquad \ldots\ldots \qquad (14.7)$$

and x may replace y in storage.

Note that because A is banded, the limits ℓ are reduced in accordance with the band-width.

14.1.1.1 Inner Products.

Because of storage limitations a_{ij} and g_{ij} will be represented in short length computer words. In order to minimise the effect of rounding errors the inner products $\sum g_{\ell i}^2$, $\sum g_{\ell i} g_{\ell j}$, $\sum g_{\ell i} y_\ell$ and $\sum g_{i\ell} x_\ell$ of equations (14.2), (14.3) (14.5) and (14.7) should however be computed using double precision arithmetic.

14.1.1.2 Negative Square Roots.

If any computed randicand $g_{kk}^2 (k = 1, \ldots, n)$ is not positive, then the normal equation matrix is not positive definite, possibly due to round-off errors. While this predicament is not difficult to overcome by working in complex arithmetic *(Fox 1964, 108-109)* any results beyond the kth row will generally be found meaningless and in the implementation of any computer program it is best to terminate the decomposition when a diagonal element g_{kk} becomes negative.

14.1.2 Computer Representation of Matrix Elements.

The scaled observation equations were stored in double length words. The normal equation matrix stored in single length words was generated from the observation equations as a series of accumulated inner products and since the general element of N is a single length word, the separate products that are to be accumulated will be truncated to single length after their multiplication. The vector of right-hand sides was formed in a similar manner but computations were made throughout in double precision arithmetic. The elements N and d of the normal equations $Nx = d$ will thus be correct to (See Equation 8.28)

$$|\delta N_{ij}| \leq f_1 16^{-7} |N_{ij}| \quad \ldots\ldots \quad (14.8)$$

$$|\delta d_i| \leq f_2 16^{-14} |d_i| \quad \ldots\ldots \quad (14.9)$$

f_1 and f_2 depending essentially on the number of accumulations in the inner products *(Osborne 1961, 629)*. The roundings $|\delta N_{ij}|$ and $|\delta d_i|$ are seen to be minute for a well scaled matrix N and right-hand side d.

14.1.3 Computation Times.

14.1.3.1 Measure of Work.

Neglecting for the moment that the matrix A in $Ax = b$ is banded, the total arithmetic for the computation of G in $A = G G^T$ is given by *(Fox 1964, 184)* $\frac{1}{6} n^3 + \frac{1}{2} n^2 - \frac{2}{3} n$ multiplications, n root reciprocals, $\frac{1}{6} n^3 - \frac{1}{6} n$ additions. In solving linear equations from the formulae $Gy = b$, $G^T x = y$, each takes $\frac{1}{2} n(n + 1)$ multiplications and $\frac{1}{2} n (n - 1)$ additions, so that the full total of arithmetic is $\frac{1}{6} n^3 + \frac{3}{2} n^2 + \frac{1}{3} n$ multiplications, n root reciprocals, $\frac{1}{6} n^3 + n^2 - \frac{7}{6} n$ additions. Iterative refinement constitutes the solution of the two triangular systems $G\delta y = \delta b$ and $G^T \delta x = \delta y$, and each will require a

further $\frac{1}{2}n(n + 1)$ multiplications and $\frac{1}{2}n(n - 1)$ additions per iterative cycle.

In the solution of the system $Ax = b$ with full and symmetric matrix of coefficients of order 1 836, 1 036 551 132 multiplications, 1 836 reciprocal roots and 1 034 862 930 additions are seen to be required, that is, a total of 2 071 415 898 arithmetic operations, not considering iterative improvement. Since the normal equation matrix is banded however and the zero coefficients outside the band play no part in the solution only $(m + 1)(2n - m)/n(n + 1)$ [154 241/1 686 366] or about 9.15% of the previous,need actually be performed.

14.1.3.2 Solution Time on the IBM 360/50.

The normal equations, including two cycles of iterative refinement, took approximately 82 minutes to solve using the FORTRAN H compiler on the IBM 360/50, a dissection of the times for the component steps of the solution process being shown in Table 14.1.

SOLUTION OF THE NORMAL EQUATIONS

(Computation Times)

	COMPONENT STEP	TIME (Minutes)
(1)	DECOMPOSITION	65
(2)	FORWARD ELIMINATION & BACK-SUBSTITUTION	9
(3)	ITERATIVE REFINEMENT (Two Cycles)	8

TABLE 14.1

14.2 Inversion of the Coefficient Matrix

The method of partial inversion being an extension of the Cholesky factorization was adopted to partially invert the normal equation matrix.

14.2.1 Measure of Work.

The inversion of the G matrix involves $\frac{1}{6}n^3 + \frac{1}{2}n^2 - \frac{2}{3}n$ multiplications *(ibid, 84)*, an additional $\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$ multiplications being necessary to evaluate the upper triangular part of A^{-1}. While the G^{-1} matrix is required in full, only about 4n elements of A^{-1} need to be calculated, since the method of partial inversion was chosen. Thus, once G^{-1} has been computed, another $(\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n)\ \frac{4n}{\frac{1}{2}n(n+1)}$ multiplications will determine the partial inverses sought.

14.2.2 Algorithm.

The elements s_{ik} of G^{-1} (see Section 8.3.1) are computed using the following recursive formulae:-

$$s_{ik} = -\frac{1}{g_{ii}} \left(\sum_{m=i+1}^{k} g_{im} s_{mk} \right) \quad (i < k) \qquad \ldots\ldots \qquad (14.10)$$

$$s_{ik} = \frac{1}{g_{ii}} \qquad (i = k) \qquad \ldots\ldots \qquad (14.11)$$

$$s_{ik} = 0 \qquad (i > k) \qquad \ldots\ldots \qquad (14.12)$$

The evaluation of the elements of G^{-1} proceeds column by column and in reverse order, the individual partial inverses being symmetric and of order seven - though not at the survey origin where the inverse is of order two. Thus, once seven (two) corresponding columns of G^{-1} have been generated, their contributions towards the partial inverses may be computed and since these columns then play no further role in the calculations, they may cede their storage locations to the next seven (two) columns of G^{-1}. In this way only seven (two) column vectors (order n) of temporary storage are required over and above the storage requirements of the G matrix.

The above scheme may be conveniently programmed. Moreover, to minimize the effects of rounding errors, the accumulations

			Matrix Order	
			436	1 836
(1)	Number of Elements in G^{-1}	$\frac{1}{2}n(n+1)$	95 266	1 686 366
(2)	Required Number of Elements of A^{-1}	$4n$	1 744	7 344
(3)	(2)/(1) Expressed as Percentage		1.83	0.44
(4)	Multiplications Necessary to Evaluate G^{-1}	$\frac{1}{6}n^3 + \frac{1}{2}n^2 - \frac{2}{3}n$	13 908 400	1 033 178 400
(5)	Multiplications Necessary to Evaluate A^{-1}	$\frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$	13 908 836	1 033 180 236
(6)	Multiplications in Partial Inversion	$(4) + (5) \times \frac{(3)}{100}$	14 162 932	1 037 724 393

TABLE 14.2

(14.10) and divisions (14.11) are best performed using double precision arithmetic.

14.2.3 Computation Times.

The computation time for the full or partial inversion of a matrix of order 1 836 using a particular computer and compiler, is likely to be large. An acceptable method of estimating the time required to invert such a matrix is to invert one of smaller order (say 400). From the time taken to invert this matrix and the number of arithmetic operations in both the smaller and actual matrices, it is possible, by extrapolation, to arrive at a rough estimate of the time required to invert the larger matrix.

The network from which the smaller matrix was derived is shown in Fig. 12.2 and corresponds to the north-western portion of the test network. It consists of 62 variable points and one fixed station, giving a normal equation matrix of order 436, a band-width of 49 being obtained for a maximum difference in station serials of 4. The inversion of the G matrix and calculation of the 63 partial inverses took 82 minutes.

The number of multiplications and additions necessary to evaluate the partial inverses from the G matrices of both networks are shown in Table 14.2. An estimate of time for the inversion of the 263 partial inverses of the test network is accordingly, 100 hours $(\frac{1\ 037\ 714\ 393}{14\ 162\ 937} \times \frac{82}{60})$. Neither the time nor the money for such an extensive computation was available and regarding the error analysis to be made, it was therefore decided to draw conclusions from the results obtained for the smaller network.

14.2.4 Size of the Elements of N^{-1}.

The size of the elements of N^{-1} are of interest since they give a useful indication of the condition of the matrix N. The moduli of the diagonal elements of the inverse normal equation matrix are shown in Table 14.3 Off-diagonal terms are not shown but are commonly smaller in magnitude than the diagonal elements. In this sense, the elements of N^{-1} are not large and it is therefore quite likely that N is not excessively ill-conditioned. However, it should be noted that the above observation

is made solely from the elements of the inverse which were calculated - i.e., only a small percentage of the total inverse.

INVERSE NORMAL EQUATION MATRIX

(Range of diagonal Elements)

Modulus

SCALED UNKNOWN	RANGE OF DIAGONAL COEFFICIENT
$\delta 0$	1 - 2
δx	50 - 300
δy	50 - 300
δz	25 - 300
$\delta \phi$	0.2 - 1
$\delta \omega$	0.3 - 1
δk	0.2 - 1

TABLE 14.3

14.3 Iterative Refinement.

14.3.1 The Solution.

14.3.1.1 Algorithm.

Since computer storage is limited, the input matrix A of the linear systems $Ax = b$ will generally have to be destroyed to make room for its factorization. The matrix A will again be required however to evaluate the residual vector $r = b - Ax$ and the matrix G will again be required to solve the system $A\delta x = \delta b$. It is thus necessary to store both A and G on backing storage, from which each may then be recalled at will. Once this has been programmed (Fig.13.4), the residual vector r may be calculated from the recalled matrix A and the computed solution x. Note that r must be evaluated using double precision arithmetic.

14.3.1.2 Results.

The maximum element $|r|_{max}$ of the residual vector r in the solution of the normal equation system is shown in Table 14.4, for four successive iterations.

SOLUTION OF THE NORMAL EQUATIONS

(Iterative Refinement)

ITERATION	$\|r\|_{max}$
1	4.6×10^{-4}
2	2.8×10^{-5}
3	1.5×10^{-6}
4	0.7×10^{-7}

TABLE 14.4

The element $|r|_{max}$ is seen to converge and decrease in magnitude by roughly 10^{-1} digits per iterative cycle. The elements of the vector x were also found to decrease, by about $10^{-2}x$; per iterative cycle.

14.3.1.3 Implications of Convergence.

The steady convergence rate of the elements of r and x indicates that $||I - B^{-1}A|| < 1$ (See 9.11) where B^{-1} is the approximate inverse of A implicit in the solution process. Thus $||A^{-1}||\ ||\delta A|| < \frac{1}{2}$, or the matrix A is not too ill-conditioned and the vector x is the correctly rounded solution.

14.3.1.4 Fixed Observations.

In a geodetic adjustment problem, fixed observations are commonly assigned a small variance *(Bomford 1967a, 11)*. Small variances will however cause correspondingly large elements within the normal equation matrix and thus give rise to large norms. One might expect the condition of the system to deteriorate. Iterative refinement of the solution however, with two fixed observations (a distance and an

azimuth emanating from the survey origin) of weight 10^6 still converged at a similar rate to that of a free net, although $|r|_{max}$, which occurred in the row containing the large matrix elements tended to be several orders larger.

14.3.2 The Inverse.

The approximate full inverse is required for its iterative improvement. Since only partial inverses were calculated however, iterative improvement of the inverse was not attempted.

14.4 Accuracy of the Computed Results.

Previously (Section 8.5), it was noted that a small residual $(AB^{-1} - I)$ implied that B was a good inverse but that a small residual $(Ax - b)$ did not necessarily imply that x was a good solution. Moreover, in order to be able to recognise the accuracy of x, it was disclosed that an accurate inverse B was necessary as well as a reasonably sharp bound for $||B - A^{-1}||$. The full inverse B was not calculated and the solution x of $Ax = b$ will thus be of unknown accuracy.

14.5 Error Analysis.

Programs for the analysis of error ellipsoids were taken from Malhotra *(1969, 38-47)*, the calculation of the eigenvalues and eigenvectors for the real and symmetric submatrices q_{xx} being based on the diagonalization method of Jacobi *(Wilkinson 1969, 266)*. The algorithm *(IBM 1968, 164)* is general and is designed for a real and symmetric matrix of order n. It is not as direct a method as that described by Körner *(1968, 40)*. Nevertheless, the total computation for the 62 free points in question took only 23 seconds.

15. Three-Dimensional Cartesian Co-ordinates of Test Network Points.

15.1 The Adjusted Unknowns.

Corrections to approximate orientation, observed or gravimetrically interpolated astronomical longitude and assumed refraction coefficient are shown in Figs. 15.1, 15.3 and 15.4 respectively, in the form of frequency histograms. Corrections to astronomic latitudes are generally within the range of ±0.01 arc seconds and are not shown.

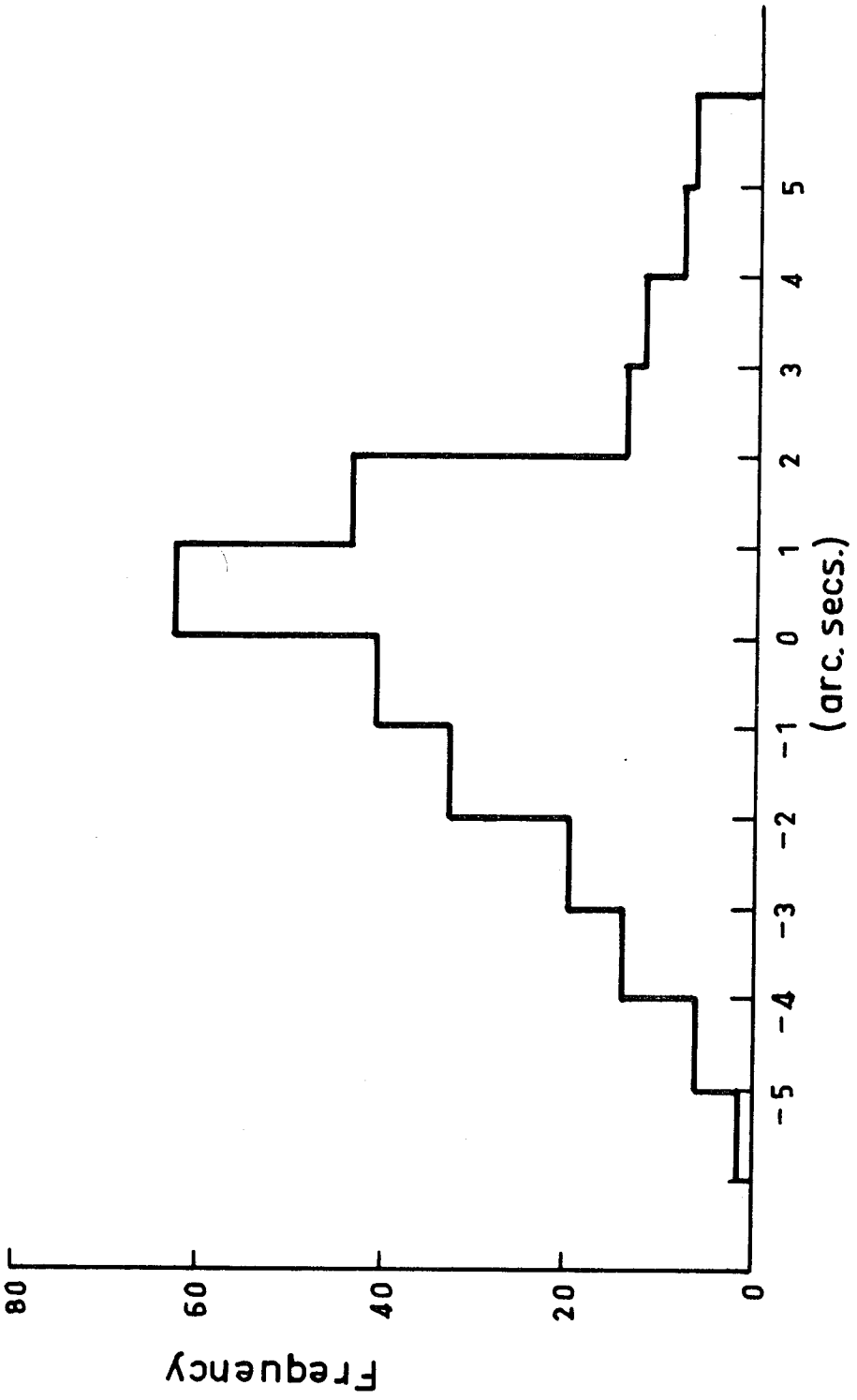


FIG. 15·1: ORIENTATION CONSTANT.

As a method of terrestrial point co-ordination the classical approach of reduction to the ellipsoid and computation by means of ellipsoidal co-ordinates can be shown to be formally equivalent to the spatial technique proposed in this dissertation *(Levallois 1963)*. Accordingly , since the topographical model used to calculate model zenith distances was identical to that used in projecting the observations of the 1966 Australian National Adjustment to the Australian National Spheroid both methods should give the same results.

In order to facilitate a comparison with the ellipsoidal co-ordinates calculated in the 1966 Australian National Adjustment, the corrections (δx, δy, δz) to the provisional Cartesian co-ordinates were transformed to changes in geodetic co-ordinates before presentation. Changes in geodetic latitude and longitude are shown in Fig. 15.2, while the resulting difference in height co-ordinates are presented in Fig. 15.5.

15.1.1 Cartesian Co-ordinates.

Departures from the final co-ordinates of the 1966 Australian National Adjustment as a result of a one phase computation using the local astronomic vector method are represented in Fig. 15.2 as vector quantities of magnitude $(\delta\phi^2 + \delta\lambda^2)^{\frac{1}{2}}$ and direction $\tan^{-1} \frac{\delta\lambda}{\delta\phi}$. Co-ordinate differences shown are not random in nature nor do they appear to be, with the possible exception of the northern and north-eastern portions of the test network, the result of a rotation about the datum point. The magnitude of the vector increases gradually with distance from the origin to a maximum of five metres in the south-eastern sector, while its azimuth is predominantly north-easterly, suggesting model differences between the two systems which are of systemmatic character. The height changes given in Fig. 15.5 are generally in the range $-20 \text{ cm} < \delta h < 20 \text{ cm}$.

15.1.1.1 Conversion Formulae.

Small changes δx, δy and δz in rectangular Cartesian co-ordinates (x, y, z) may be converted to corresponding

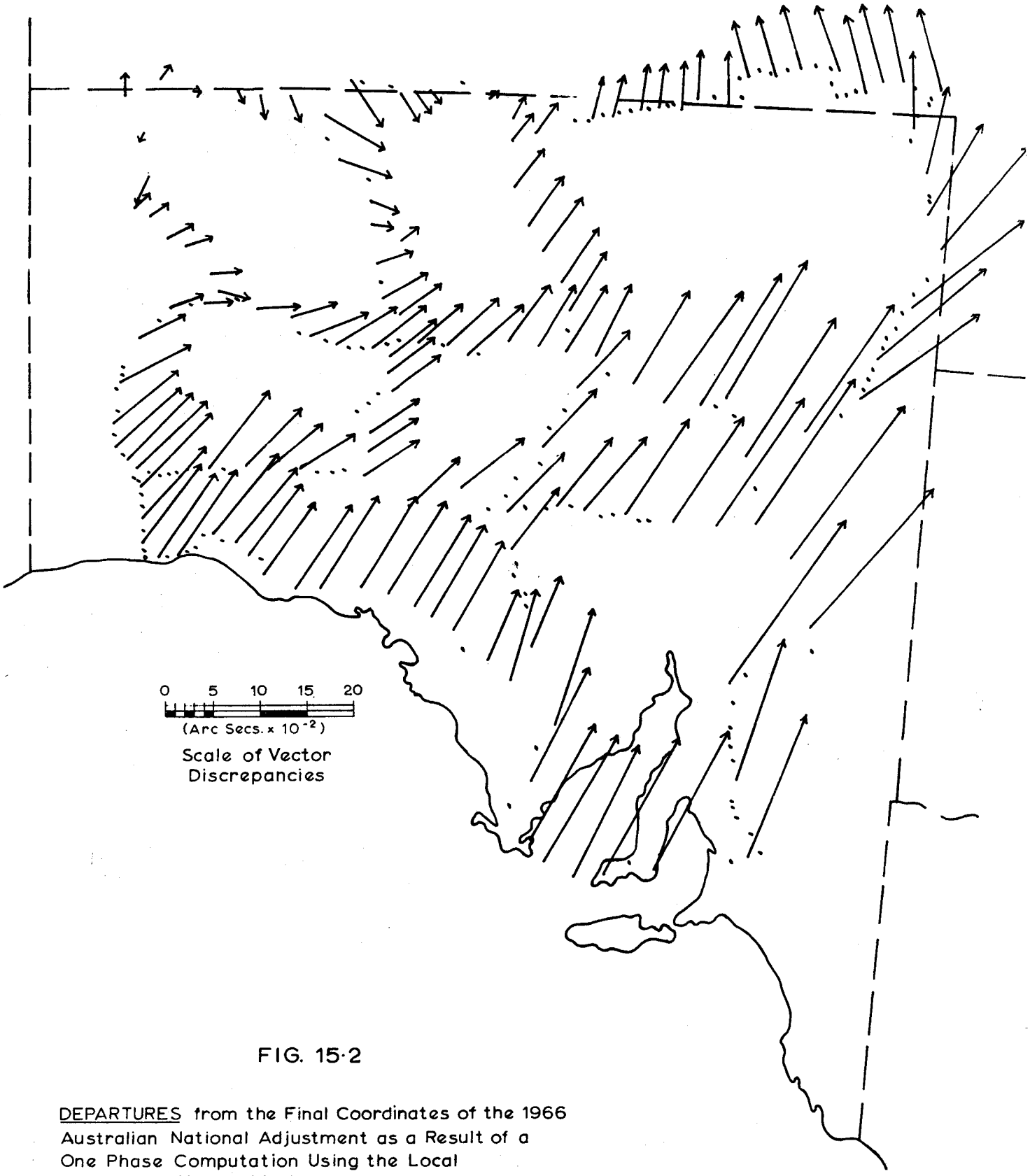


FIG. 15·2

DEPARTURES from the Final Coordinates of the 1966 Australian National Adjustment as a Result of a One Phase Computation Using the Local Astronomic Vector Method.

increments $\delta\phi$, $\delta\lambda$ and δh in Ellipsoidal co-ordinates using the following first order approximations:

$$\{\delta\phi,\ \delta\lambda,\ \delta h\} = \begin{pmatrix} \frac{1}{\rho + h}\,\delta s \cos\alpha \sin\beta \\ \frac{1}{\nu + h}\,\delta s \sin\alpha \sin\beta \\ \delta s \cos\beta \end{pmatrix} \quad \ldots\ldots \quad (15.1)$$

where

$$\{\delta s, \tan\alpha, \delta s \cos\beta\} = \begin{pmatrix} \sqrt{\delta x^2 + \delta y^2 + \delta z^2} \\ \dfrac{\delta y \cos\lambda - \delta x \sin\lambda}{\delta z \cos\phi - \delta x \sin\phi \cos\lambda - \delta y \sin\phi \sin\lambda} \\ \delta x \cos\phi \cos\lambda + \delta y \cos\phi \sin\lambda + \delta z \sin\phi \end{pmatrix} \quad \ldots. \quad (15.2)$$

and ϕ and λ are the ellipsoidal co-ordinates of the initial point. Equations (15.1) and (15.2) are seen to be applications of formulae (5.4). Alternative formulae for $\delta\phi$ and $\delta\lambda$ which may be used as a check on the computations are given by:

$$\{\delta\lambda, \delta\phi\} = \begin{pmatrix} \left(\dfrac{\cos\lambda}{x}\right)^2 (x\delta y - y\delta x) \\ \dfrac{-\delta x \sin\phi - \delta z \cos\phi \cos\lambda + \frac{1}{2}(\nu + h)\cos^2\phi \sin\lambda\, \delta\lambda}{(\nu + h - e^2\cos^2\phi)\cos\lambda} \end{pmatrix} \quad \ldots. \quad (15.3)$$

and can be derived by differentiating and manipulating the expressions

$$\{x, y, z\} = \begin{pmatrix} (\nu + h)\cos\phi \cos\lambda \\ (\nu + h)\cos\phi \sin\lambda \\ \overline{\nu(1 - e^2)} + h \sin\phi \end{pmatrix} \qquad \text{(See Equation 5.14)}$$

Equations (15.1), (15.2) and (15.3) may be conveniently programmed, the data input being the co-ordinate increments δx, δy, δz, the ellipsoidal co-ordinates ϕ and λ and the relevant spheroid parameters.

15.1.1.2 The 1966 Australian National Adjustment.

The geodetic survey of Australia at the completion of the 1965 field season comprised 2 506 network stations *(Bomford 1967b, 52)*. In a rigorous least squares adjustment of the network using directions and the variation of co-ordinates method (i.e. three unknowns per station, 7 516 (3 x 2 506-2) normal equations require solution. The band-width is likely to be large. Extensive computation times can thus be expected in solving the generated normal equation system and special programming considerations (e.g. partitioning) will be necessary in view of computer storage limitations. The previous complications can be avoided, unfortunately at the expense of rigor *(Wolf 1968, 378)* by employing the Bowie technique *(Adams 1930)* in which the survey to be computed is divided into sections between junction points. The method of computation is briefly: each component section between junction points is adjusted independently using the least squares principle; the azimuths and distances between the junction points of the independently computed sections are obtained and in turn are placed into a single least squares adjustment to provide the co-ordinates of the junction points; and finally each component section is re-adjusted holding fixed the co-ordinates of the junction points.

In the original Bowie method of computation as described by Adams *(1930)*, loop configurations were adjusted separately in latitude and longitude, and no allowance was made for the fact that the component sections could differ in precision, refinements which were however incorporated in the 1966 Australian National Adjustment. Various approximations were made throughout the computations. The method of calculation as adopted for the 1966 Australian National Adjustment is therefore briefly elucidated, the reader being referred to Bomford *(1967a, 1967b)* for a full treatment of the subject. Constitutent steps of the computations are discussed under the following headings:-

(1) the Free Adjustment;
(2) the Inverse Computations;
(3) the Rod Adjustment and
(4) the Forced Adjustment.

15.1.1.2.1 Free Adjustment.

The intention in adjusting independently each component section between junction points, using the

least squares principle and the variation of co-ordinates method was to produce the most probable values for the azimuths and distances between junction points. Overlaps beyond junction points were common and any observation which was thought to affect the azimuth and distance between one junction point and another was included in the computations. At Laplace stations the Laplace condition was maintained automatically as the geodetic longitudes were modified during the adjustment process.

15.1.1.2.2 Inverse Computations.

The geodesic azimuth and distance between junction points was determined from the adjusted latitudes and longitudes of the free adjustment once the loop closures indicated that the constituent sections of the loop were free from gross errors, the formulae given by Robbins *(1962)* being employed in the computations.

15.1.1.2.3 Rod Adjustment.

Geodesic azimuths and distances between connected points were now adjusted in a single phase least squares calculation, azimuths and distances being weighted inversely as the length of each section. Since nearly all sections comprised a combination of tellurometer traverse and triangulation, their linear accuracy in ppm was not thought to vary significantly. However, in view of the fact that in some sections only single-ended astronomical azimuths had been observed at approximately every eighth station, while in others simultaneous reciprocal azimuths were determined for every line, their angular accuracy was expected to differ considerably. A suitable, although somewhat empirical weighting scheme was therefore devised *(Bomford 1967b, 62-66)*. The Laplace condition was again maintained throughout the computations.

15.1.1.2.4 Forced Adjustment.

In the forced adjustment, each section was re-adjusted holding fixed the co-ordinates of the junction points as obtained in the rod adjustment, the aim being to produce a unique set of co-ordinates for each of the 2 506 network stations, together with a unique set of adjusted observations. Overlaps were common and more than one set of co-ordinates would normally be obtained for stations adjacent to the junction points. Accordingly, since the intention was to produce a unique set of co-ordinates and observations, the azimuth and length of

one line at each junction point was determined by selecting the weighted mean of the azimuths and lengths obtained in the free adjustment. However, before this task could be implemented, each azimuth taken from the free adjustment had to be modified slightly to maintain the Laplace condition, the longitude of the junction point having been altered in the rod adjustment. The co-ordinates of the far end of the line were then computed and were subsequently held fixed in all sections emanating from the junction point in question. Moreover, since only one set of co-ordinates was desired for any point, overlaps included in the free adjustment were generally removed in the forced computations.

15.1.1.2.5 Error Analysis.

The precision of the point location was not evaluated in the 1966 Australian National Adjustment, probably because the interpretation of the results would have been difficult in the light of the computational procedure adopted.

15.1.1.3 Point Co-ordination Using the Modified Bowie Method.

Although an improvement on the original method described by Adams *(1930)*, insofar that the latitudes and longitudes are adjusted simultaneously and that the varying precision of angular work within component sections is accounted for, the modified Bowie technique of point co-ordination as adopted in the 1966 Australian National Adjustment must be considered an approximation when compared to a rigorous one phase least squares computation *(Wolf 1968, 378)*. Distortions will be introduced, mainly due to:

(1) the assumption that Laplace azimuths are error free;

(2) the oversimplified weighting scheme adopted for the derived lengths and azimuths of the Rod Adjustment; and

(3) the exclusion of the overlaps during the Forced Adjustment.

If the discrepancies shown in Figs. 15.2 and 15.5 could be attributed entirely to the effects of the modified Bowie technique - and unfortunately they cannot, since for example the data sample spaces involved differ significantly - then, as the magnitude of the resultant vector increases steadily with distance from the origin and its direction is predominantly northerly, it would appear that the distortions produced are systemmatic

in character. Irregular distortions occur in the north-western portion of the network however, where the adjustments to the AGD co-ordinates "were larger than could happily be attributed to the accumulation of random errors" *(Bomford 1967b, 69)*.

At the south-eastern extremety of the test network the magnitude of the vector differences between the two systems is alarmingly large. While it is not suggested that the approximations within the Bowie method of computation are the sole contributors to these discrepancies, a specific study into their possible effects would certainly be beneficial, particularly since the intention is to recompute AGD co-ordinates with improved height data during 1972.

15.1.1.4 The Co-ordinate Differences.

Previously it was established that a significant contribution towards the vector discrepancies shown in Fig. 15.2 could be expected in consequence of approximations within the modified Bowie technique. The height co-ordinate which was permitted to vary in the present solution but not for that of the 1966 National Adjustment represents a change along the geodetic normal for which both the geodetic latitude and longitude are constant and thus should not affect the results given in Fig. 15.2. Nevertheless, as a method of terrestrial point co-ordination the classical approach of reduction to the ellipsoid and computation by means of ellipsoidal co-ordinates is formally equivalent to the local astronomic vector technique *(Heiskanen & Moritz 1967, 224)* and since the spheroidal heights used in the calculation of model zenith distances were identical to those used in the projection of the observations of the 1966 Australian National Adjustment to the Australian National Spheroid then, if the data and network configurations are also identical, both methods should theoretically give the same results. But the data sample spaces and the network configurations used, as can be visualized by comparing Fig. 10.1 and Bomford *(1967b, Fig. 1)*, were not the same for the two configurations, nor was the available data used in an identical manner. For example, single ended azimuths were chosen in preference to the mean of two simultaneous reciprocal values, while the distance, measured from both ends of the line, was averaged and included as two separate observations. Contributions towards the co-ordinate

differences can also be expected as a result of the fact that in the local astronomic vector method of computation the survey scheme, in accordance with the least squares principle, is rotated about the local vertical at the datum point, whereas in the ellipsoidal approach the rotation takes place about the corresponding ellipsoidal normal.

The configuration used in the 1966 Australian National Adjustment covers all of Australia and consists of triangulation as well as tellurometer traverses. The test network, however, contains only tellurometer traverses and roughly 10% of the total number of stations. The results could therefore not be identical even if both networks had been adjusted rigorously in a single phase least squares computation. Moreover, since the national network is more extensive, a greater number and better distribution of azimuths will determine its orientation. In fact, 275 well distributed simultaneous reciprocal azimuths were used in the orientation of the national net *(ibid, 57)* while the test network was oriented using only 92 single ended observations a significant proportion of which were located in the Great Australian Bight region where lateral refraction anomalies were large *(Johnson 1962)*. Hence it is conceivable that the two networks should differ somewhat in orientation, although the results obtained cannot be attributed to a rotation alone.

There is no reason, logical or otherwise, why lateral refraction should be eliminated in the mean of two corresponding simultaneous reciprocal azimuths. Single ended values were therefore preferred in the local astronomic vector technique. However, as lateral refraction was not accounted for in the model it is to be expected that the results, rather than departing from those of the National Adjustment, should depart from reality.

Distances were measured independently from both ends of the line and may thus be included in the computations as two separate measurements. The differences, even for the longer lines, rarely exceed a few decimetres in magnitude and adopting the mean should not significantly bias the results. The impact of additional observations in the adjustment is to enhance the degree of overdetermination and to produce larger coefficients in the normal equation matrix, the latter implying, on

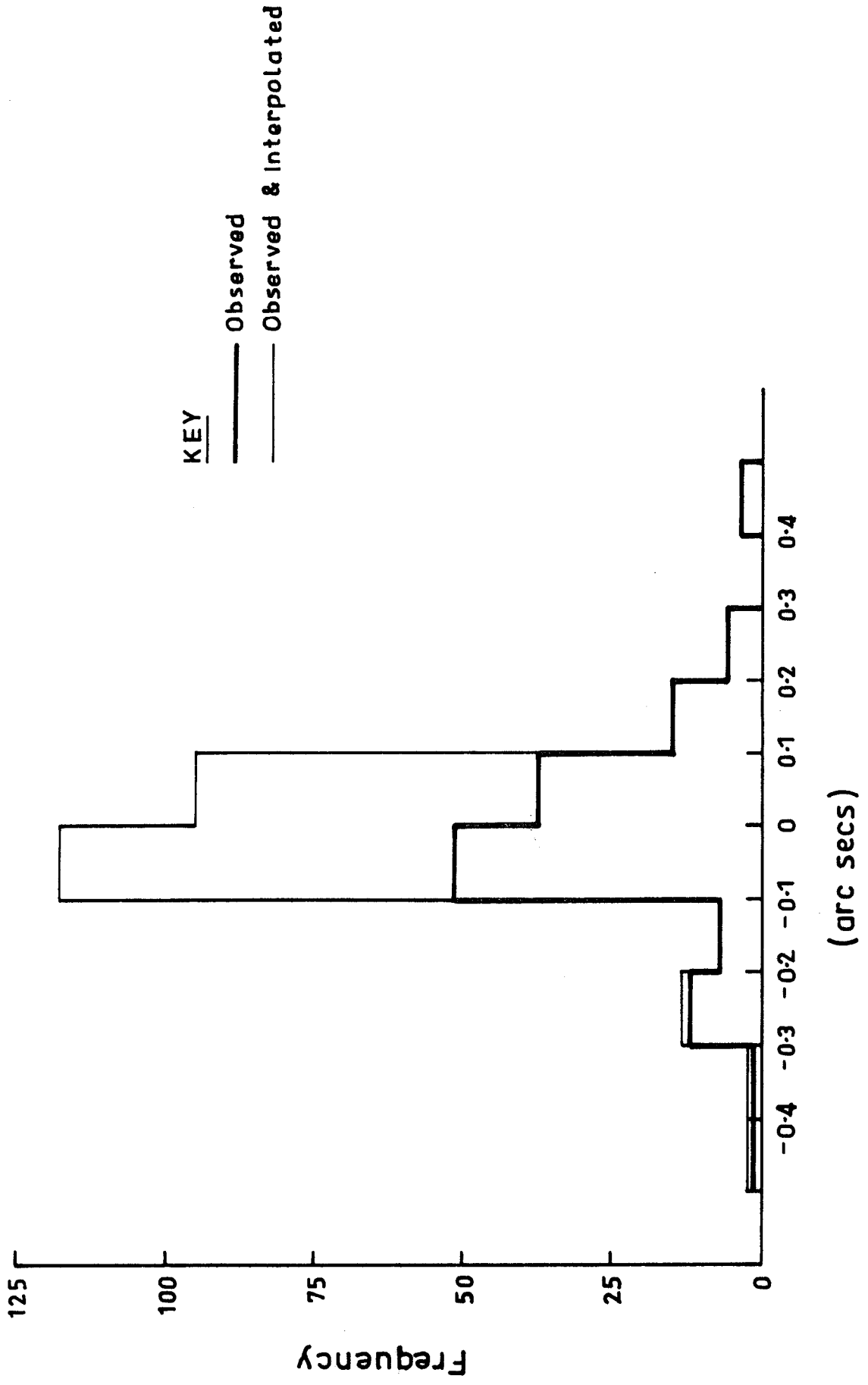


FIG. 15·3: ADJUSTMENT TO LONGITUDES.

the assumption that the magnitudes of the unknowns as well as those of the right-hand side vector are roughly equal, a smaller solution.

The local vertical and the geodetic normal at the datum point do not coincide, the two differing by 0.12 arc seconds in latitude and 0.55 seconds in longitude. Co-ordinate increments resulting from a rotation about any axis through the origin are defined by Equation (5.18). Thus if (ω, ϕ) and $(\omega + \delta\omega, \phi + \delta\phi)$ are the direction parameters of the normal and vertical respectively, then the co-ordinate differences produced in rotating the same survey scheme by amounts $\delta\alpha_1$ and $\delta\alpha_2$ respectively about these directions will be given by

$$\{\delta x, \delta y, \delta z\} = \delta R \{x-x_o, y-y_o, z-z_o\} \quad \ldots\ldots \quad (15.4)$$

where δR is a square matrix of order three, and

$$\delta R_{11} = \delta R_{22} = \delta R_{33} = 0$$

$$\delta R_{12} = (\delta\alpha_1 - \delta\alpha_2) \sin\phi - \delta\alpha_2 \, \delta\phi \cos\phi$$

$$\delta R_{13} = -(\delta\alpha_1 - \delta\alpha_2) \cos\phi \sin\omega + \delta\alpha_2 \, (\cos\phi \cos\omega\delta\omega - \sin\phi \sin\omega\delta\phi - \sin\phi \cos\omega\delta\phi\delta\omega)$$

$$\delta R_{23} = (\delta\alpha_1 - \delta\alpha_2) \cos\phi \cos\omega + \delta\alpha_2 \, (\cos\phi \sin\omega\delta\omega + \sin\phi \cos\omega\delta\phi - \sin\phi \sin\omega\delta\phi\delta\omega)$$

Since $\{x-x_o, y-y_o, z-z_o\}_{max} = 10^7\{0.5, 0.05, 0.6\}$ metres for the test network, it is seen that second and third order terms in Equation (15.4) are negligible and that therefore $\delta R = (\delta\alpha_1 - \delta\alpha_2)R$. Accordingly, if $\delta\alpha_1 = \delta\alpha_2$, no significant co-ordinate increments are generated.

15.1.2 Orientation.

The correction to the assumed orientation at network stations (Fig. 15.1) generally lies within the range ±2 arc seconds, although 31.4% of the values lie outside this range.

15.1.3 Astronomic Latitude and Longitude.

Adjustments to observed longitude are shown in Fig. 15.3. 81.4% of the corrections are seen to be in the range ±0.1 arc seconds. The corresponding adjustments to observed latitude rarely exceed

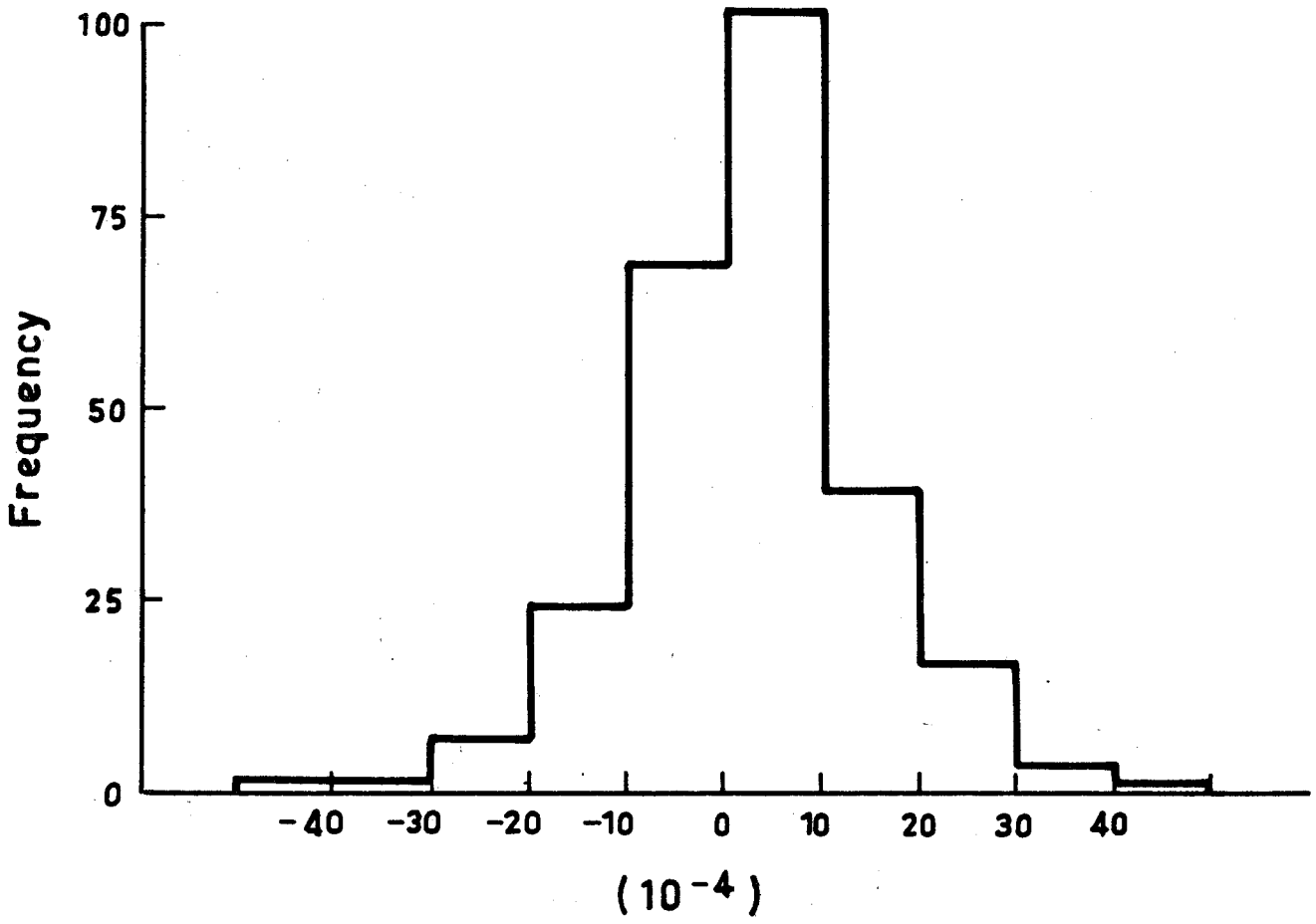


FIG. 15·4: FICTITIOUS REFRACTION COEFFICIENT

±0.01 arc seconds and for this reason are not shown. Two aspects of these adjustments require analysis, namely, that -

(1) the correction to latitude should be one order of magnitude smaller than the correction to longitude; and

(2) the correction to astronomically determined longitude should be larger than its gravimetrically interpolated counterpart (Fig. 15.3).

The fact that the correction to latitude is less than the correction to longitude is partly explained by their respective weight ratio which is 2:1 (Table 13.1). At the first application of the least squares process however, ϕ and ϕ' in the latitude equation (Equation 6.25) were set equal and the contribution of this equation to the right-hand side terms in the normal equation system will consequently be zero. Moreover, the rows and columns of the normal equation matrix for which the coefficient of $\delta\phi$ is the diagonal elements, are characterized by very small off-diagonal terms (13.5.1.4) while the dominant contributions to the corresponding right-hand side constant are received from the zenith distances *(ibid)*. Zenith distances are however commonly less accurate and are therefore assigned a smaller weight. Accordingly, since for the topographical model the (O-C) term in the zenith distance equation was of the same magnitude as that of the remaining observation equations, the previous right-hand side elements will be smaller than those pertaining to the other variables. Hence assuming the scaled unknowns to be of approximately equal size, and the actual solution confirms that they are, the correction to latitude will also be smaller than the remaining unknowns. The latitudes thus fit the mathematical model more closely than do the longitudes.

That the corrections to the astronomically observed longitudes should be considerably larger than the corrections to the gravimetrically interpolated quantities is perplexing, particularly since their weight ratio is 3:1 (Table 13.1). The apparent inversion can possibly be attributed to the fact that the gravimetrically interpolated longitudes are of better quality than initially estimated, or that the astronomic longitudes are not as accurate as they are thought to be.

15.1.4 Refraction Coefficient.

The value zero was adopted as the approximate refraction coefficient k' in equation (6.15). 89% of the corrections δk to k' lie

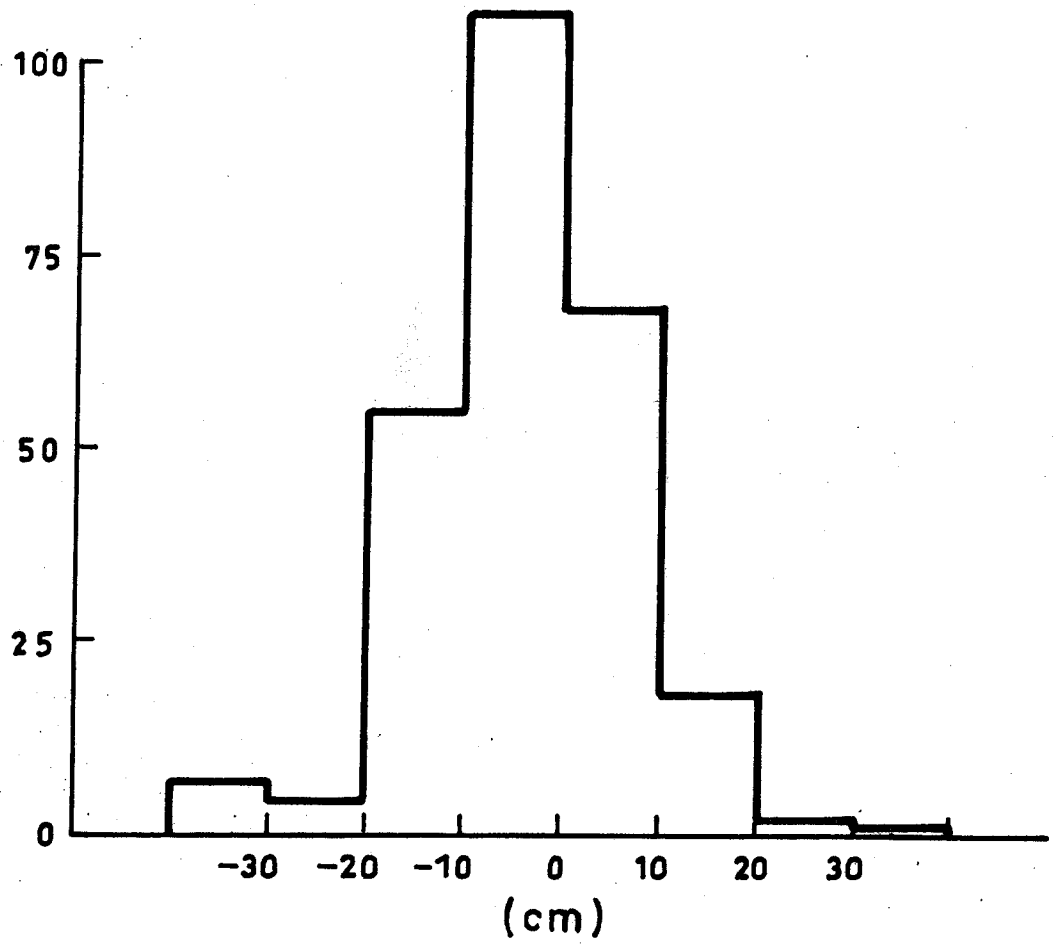


FIG. 15·5: ADJUSTMENT TO HEIGHT.

within the range ±0.002 (Fig. 15.4) corresponding to an angular displacement $\left(\frac{s\delta k}{2R}\rho''\right)$ of ±1.94 arc seconds in a distance of 30km, the average network distance. The δk are in agreement with the amount of refraction introduced into the model zenith distances (See 12.4) and, since the corrections to the model zenith distances due to adjustment are always smaller than 0.05 arc seconds in modulus (See 15.2.4) appear to account for the height variations shown in Fig. 15.5.

15.2 The Adjusted Observations.

The adjustments to the observations indicate the fit of the measured data to the adopted mathematical model, small corrections for example revealing a good fit. A comparison with the corresponding corrections obtained in the 1966 National Adjustment, although of interest, was not made on account of the distortions introduced during the forced adjustment.

Adjustments to the astronomic azimuths, the directions and the distances of the test network are presented in Figs. 15.6, 15.7 and 15.8 respectively. The corrections to the distances, expressed in ppm are given as well (Fig. 15.9), while those pertaining to the model zenith distances, since they are always less than 0.05 arc seconds in modulus, are not shown.

15.2.1 Astronomic Azimuths.

Of the 92 azimuths within the test network, 64.5% (59) received adjustments of ±1 arc second (the 'a priori' standard deviation) or less. The size of the correction varies somewhat, there being no general pattern to suggest that the adjustments to the observations made with the astronomic theodolites (T4 and DKM3a) are smaller than those made with the geodetic theodolites (T3 and Geodetic Tavistock), nor did the coastal azimuths, for which large refraction anomalies were expected, receive larger corrections.

15.2.2 Directions.

From Fig. 15.7 it is seen that 95.4% of the corrections to the observed directions lie within the range ±0.5 arc seconds, that is,

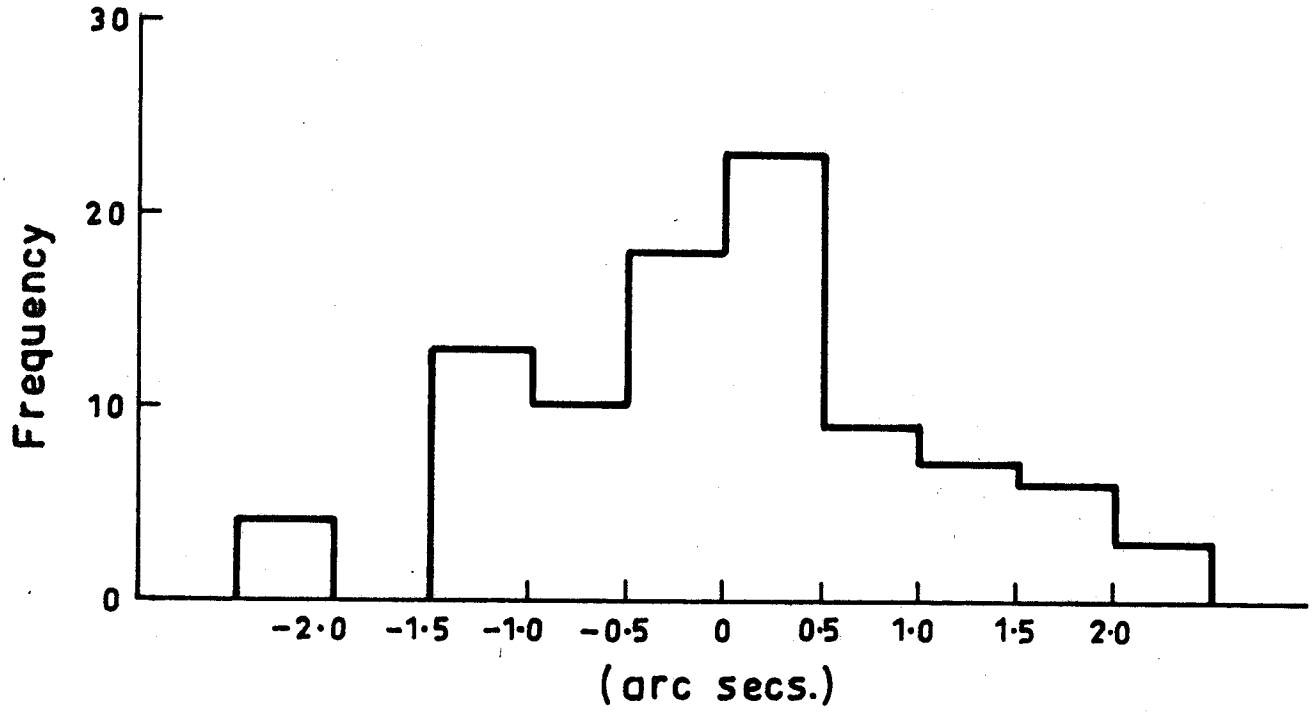


FIG. 15·6: ADJUSTMENT TO ASTRONOMIC AZIMUTHS.

the 'a priori' standard deviation. The values which exceed 0.5 arc seconds in modulus generally do so by not more than 0.1 arc seconds, although several directions on the coastal traverse received corrections as large as 0.9 arc seconds in modulus.

An observation with a residual which is larger than three times the 'a priori' standard deviation is commonly rejected. None of the corrections to the measured directions exceed 1.5 arc seconds in modulus and there is thus no statistical reason to suspect poor quality data. The larger adjustment, however, occur solely in the coastal segments of the network and could possibly be construed to signify the presence of lateral refraction.

15.2.3 Lengths.

A significant proportion of the corrections to the measured distances are smaller in modulus than 4 ppm (Fig. 15.9), that is, the average 'standard deviation' assuming a mean network distance of 30 km. The segments of the test network corresponding to loop 45 and part of loop 64 in the National Adjustment *(Bomford 1967b, Fig. 1)*, however, are characterized by adjustments which often exceed 10 ppm. The MRA2 tellurometer, for which errors in crystal calibration were increasingly noted *(ibid 69)*, was used to measure many of these lengths. Moreover, loops 45 and 64 traverse the Musgrave and Everard Ranges of Central Australia as well as parts of the Great Victoria Desert. The Great Victoria Desert is distinguished by low terrain and sand ridges, while very hot temperatures can be encountered in any of the above regions, even during the winter months, that is, the field season. Bomford *(ibid)* feels that, on the average, the recorded temperature may be a little high, an error of 0.5^{o}C affecting the distance by roughly 1 ppm. However, it is difficult, though not impossible, to imagine the temperature to be in error by as much as 5^{o}C and only component of the test network to be influenced by crystal calibration errors. Nor is it logical to expect the meteorological conditions over one region, the topography, vegetation and climatic aspects of which are essentially the same as elsewhere *(Aitchinson 1969)* to be less accurately sampled. Accordingly, as already noted by

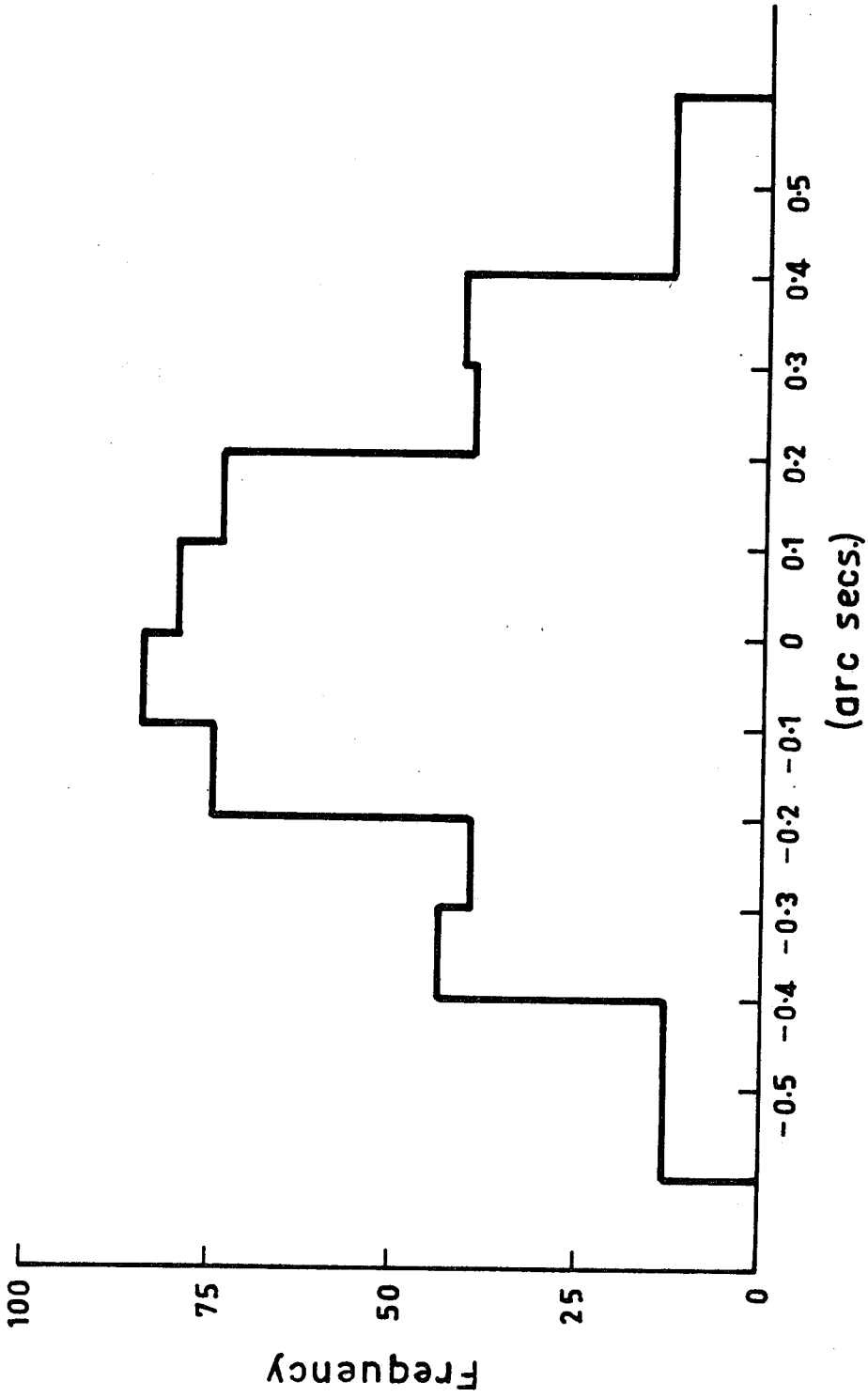


FIG.15·7: ADJUSTMENT TO DIRECTIONS

STANDARD DEVIATIONS OF UNKNOWNS. CORRECTIONS RESULTING FROM ADJUSTMENT.

Serial	δx	$\pm\sigma_x$	δy	$\pm\sigma_y$	δz	$\pm\sigma_z$	$\delta\phi$	$\pm\sigma_\phi$	$\delta\omega$	$\pm\sigma_\omega$	Interpolated or Observed
1	0.53	0.98	0.40	1.10	-0.22	0.94	0.00	0.63	-0.01	0.62	I
2	0.32	1.01	0.52	1.12	0.13	0.95	-0.01	0.63	0.01	0.61	I
3	0.75	0.98	0.55	1.09	-0.34	0.93	-0.01	0.63	0.01	0.61	I
4	-0.01	1.07	0.44	1.18	0.66	0.99	0.00	0.25	-0.20	0.36	0
5	0.88	0.98	0.49	1.09	-0.53	0.93	0.00	0.63	0.00	0.61	I
6	0.04	1.05	0.43	1.17	0.58	0.98	0.00	0.25	-0.15	0.37	0
7	0.97	0.97	0.45	1.08	-0.78	0.93	-0.01	0.63	0.02	0.59	I
8	0.09	1.09	0.28	1.22	0.43	1.03	0.00	0.25	0.01	0.37	0
9	1.01	0.96	0.35	1.07	-0.91	0.92	0.00	0.25	0.42	0.37	0
10	-0.04	1.05	0.21	1.19	0.18	1.01	0.00	0.25	0.00	0.38	0
11	1.07	0.96	0.32	1.06	-0.97	0.92	0.00	0.63	0.02	0.60	I
12	0.04	1.09	0.09	1.22	0.12	1.04	0.00	0.25	-0.23	0.37	0
13	0.98	0.94	0.24	1.04	-1.04	0.91	0.00	0.63	0.00	0.60	I
14	-0.19	1.07	-0.04	1.20	0.14	1.03	0.00	0.25	0.20	0.37	0
15	1.04	0.90	0.32	1.01	-0.70	0.91	0.00	0.25	0.00	0.38	0
16	0.95	0.94	0.17	1.03	-1.06	0.91	0.00	0.63	0.01	0.60	I
17	-0.23	1.09	-0.10	1.23	0.13	1.06	0.00	0.25	0.00	0.38	0
18	0.79	0.92	0.41	1.02	-0.51	0.91	0.02	0.63	0.00	0.63	I
19	0.76	0.92	0.23	1.03	-0.71	0.92	0.01	0.63	-0.01	0.62	I
20	0.84	0.92	0.12	1.03	-0.99	0.91	0.00	0.63	0.00	0.60	I
21	-1.09	1.08	-0.14	1.19	1.14	1.05	0.00	0.25	0.00	0.38	0
22	0.69	0.90	0.48	0.99	-0.29	0.90	0.00	0.25	0.00	0.38	0
23	0.73	0.91	0.10	1.02	-0.89	0.90	0.00	0.25	-0.19	0.37	0
24	-0.80	1.16	-0.38	1.26	0.33	1.08	0.00	0.25	0.00	0.38	0
25	0.33	0.97	0.09	1.06	-0.22	0.94	0.01	0.63	0.00	0.63	I
26	0.73	0.92	0.16	1.03	-0.85	0.91	0.00	0.63	0.00	0.62	I
27	-0.45	1.14	-0.62	1.13	-0.37	0.99	0.00	0.25	0.00	0.38	0
28	0.23	0.86	0.34	0.92	0.11	0.88	0.00	0.25	-0.10	0.37	0
29	0.79	0.93	0.18	1.03	-0.87	0.90	0.00	0.63	0.00	0.61	I
30	-0.24	1.14	-0.30	1.13	-0.22	0.90	0.00	0.25	0.00	0.37	0
31	0.34	0.86	0.56	0.92	0.32	0.86	0.01	0.63	0.00	0.63	I

TABLE 15.1

STANDARD DEVIATIONS OF UNKNOWNS. CORRECTIONS RESULTING FROM ADJUSTMENT.

Serial	δx	$\pm\sigma_x$	δy	$\pm\sigma_y$	δz	$\pm\sigma_z$	$\delta\phi$	$\pm\sigma_\phi$	$\delta\omega$	$\pm\sigma_\omega$	Interpolated or Observed
32	0.83	0.93	0.20	1.03	-0.93	0.90	0.00	0.63	0.00	0.58	I
33	-0.35	1.10	-0.21	1.07	0.12	0.79	0.00	0.25	0.00	0.38	0
34	0.49	0.86	0.97	0.89	0.65	0.80	0.01	0.63	0.00	0.62	I
35	0.87	0.94	0.18	1.03	-0.98	0.91	0.00	0.63	0.00	0.60	I
36	-0.35	1.11	-0.04	1.11	0.36	0.76	0.00	0.25	-0.01	0.36	0
37	0.76	0.83	1.08	0.83	0.49	0.78	0.00	0.25	0.00	0.38	0
38	0.99	0.93	0.23	1.02	-1.10	0.90	0.00	0.63	0.00	0.61	I
39	-0.45	1.02	0.10	0.97	0.67	0.63	0.00	0.25	0.00	0.38	0
40	0.89	0.82	1.34	0.76	0.62	0.73	0.01	0.63	0.00	0.63	I
41	1.03	0.94	0.18	1.03	-1.19	0.90	0.00	0.25	0.07	0.37	0
42	-0.39	1.03	0.32	1.00	1.01	0.64	0.00	0.25	-0.01	0.37	0
43	0.85	0.83	1.38	0.75	9,84	0.67	0.00	0.25	0.18	0.37	0
44	0.93	1.02	0.02	1.10	-1.26	0.95	0.00	0.63	0.01	0.63	I
45	0.06	0.76	1.02	0.77	1.20	0.50	0.00	0.63	0.01	0.61	I
46	0.91	0.95	-0.09	1.01	-1.39	0.92	0.00	0.63	0.00	0.63	I
47	0.09	0.76	0.99	0.77	1.10	0.49	0.00	0.25	0.00	0.38	0
48	0.93	1.06	-0.16	1.09	-1.53	0.99	0.00	0.25	0.00	0.38	0
49	0.02	0.56	0.55	0.52	0.70	0.36	0.00	0.25	0.00	0.40	0
50	0.88	0.93	-0.13	0.97	-1.37	0.96	0.00	0.25	0.00	0.38	0
51	0.01	0.56	0.52	0.50	0.66	0.33	0.00	0.63	-0.01	0.62	I
52	0.86	0.91	-0.08	0.96	-1.32	0.97	0.00	0.25	0.00	0.38	0
53	0.03	0.53	0.46	0.47	0.56	0.29	0.00	0.25	0.00	0.43	0
54	0.90	0.86	-0.06	0.93	-1.19	0.95	0.00	0.26	0.00	0.38	0
55	0.04	0.55	0.31	0.50	0.30	0.32	0.00	0.25	0.00	0.46	0
56	0.89	0.80	0.11	0.90	-1.21	0.92	0.00	0.26	0.00	0.38	0
58	0.94	0.69	0.08	0.83	-1.22	0.87	0.00	0.25	0.00	0.38	0
59	0.06	0.47	0.09	0.55	-0.07	0.34	0.00	0.25	0.00	0.38	0
60	0.86	0.70	0.14	0.84	-1.10	0.88	0.00	0.25	0.00	0.38	0
61	0.43	0.42	0.01	0.60	-0.45	0.48	0.00	0.63	0.00	0.63	I
62	0.80	0.70	0.04	0.80	-0.75	0.80	0.00	0.63	0.00	0.63	I
63	0.80	0.70	0.04	0.80	-0.75	0.80	0.00	0.63	0.00	0.63	I

TABLE 15.1 (contd.)

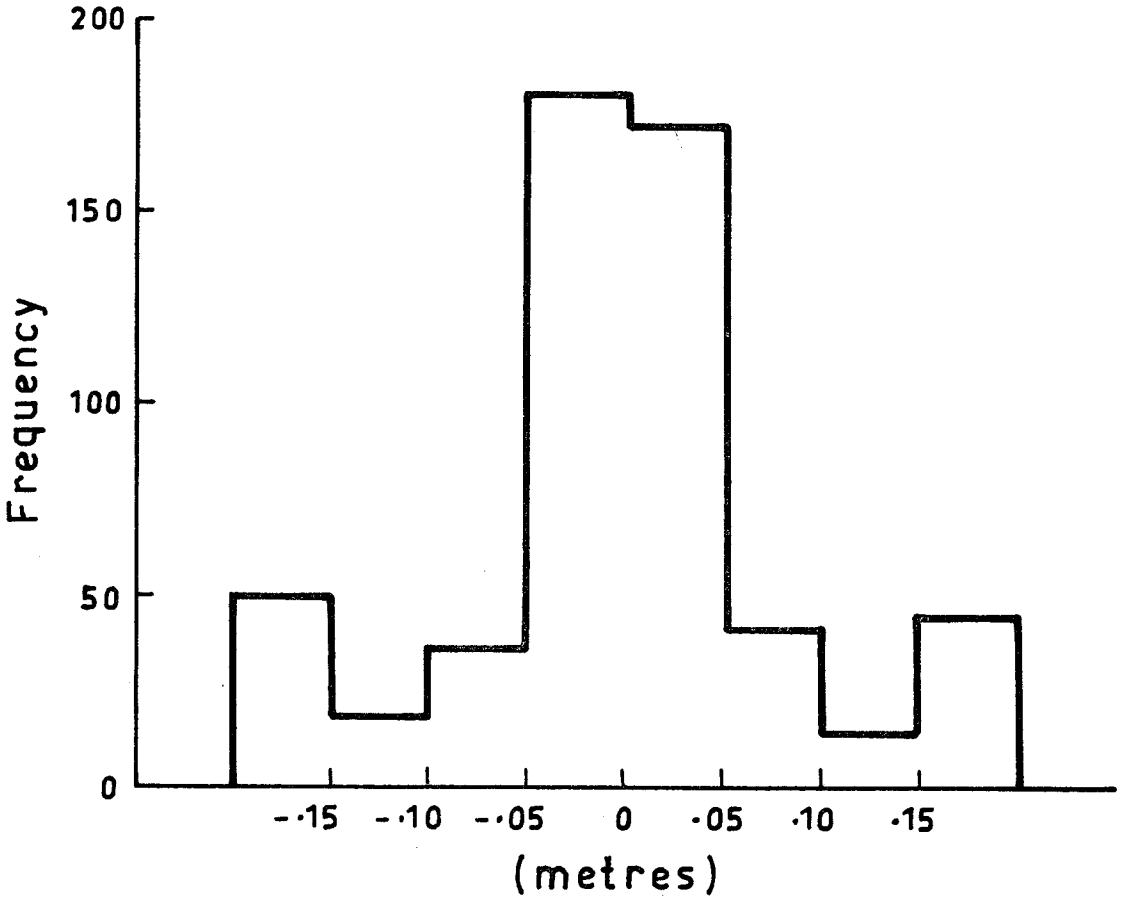


FIG.15·8: ADJUSTMENT TO DISTANCES

Bomford *(1967b, 69)*, it is more likely for an undetected gross error to exist within either or both of these loops.

15.2.4 Zenith Distances.

Despite the fact that the model zenith distances possess the smallest weights, their corrections arising from the adjustment are always less than 0.05 arc seconds in modulus, that is, about the size of the adjustments to the observed latitudes. Height variations δh, resulting from the computations, were almost totally absorbed by the correction δk to the assumed refraction coefficient (See 15.1.4) and the adopted variance of 2.25 arc seconds^2 for the derived zenith distances thus appears to be a considerable under-estimate of their precision.

15.3 Error Analysis.

Since the implementation of a full error analysis was estimated to require approximately 100 hours of computer time (See 14.2.3), only the sample variance factor and the 'a posteriori' standard deviations of the observations were calculated for the test network. However, in order to evaluate the precision of the spatial point definition, a smaller component net (Fig. 12.2) comprising 62 free points was fully analysed. The standard deviations of the unknowns, together with the corresponding corrections resulting from adjustment are shown in Tables 15.1 and 15.2. Typical error ellipsoids are presented in Figs. 15.11a and 15.11b whereas the mean radii (Fig. 15.10) of the error ellipsoids are used to give a comprehensive representation of precision.

15.3.1 The Variance Factor.

The sample variance S^2, an estimator of the variance factor σ_0^2, may be calculated from the expression

$$S^2 = \frac{v^T W v}{r} \qquad \ldots\ldots \quad (15.5)$$

where r is the number of degrees of freedom, v is the vector of residuals and W is the weight matrix. The network adjustment comprising 401

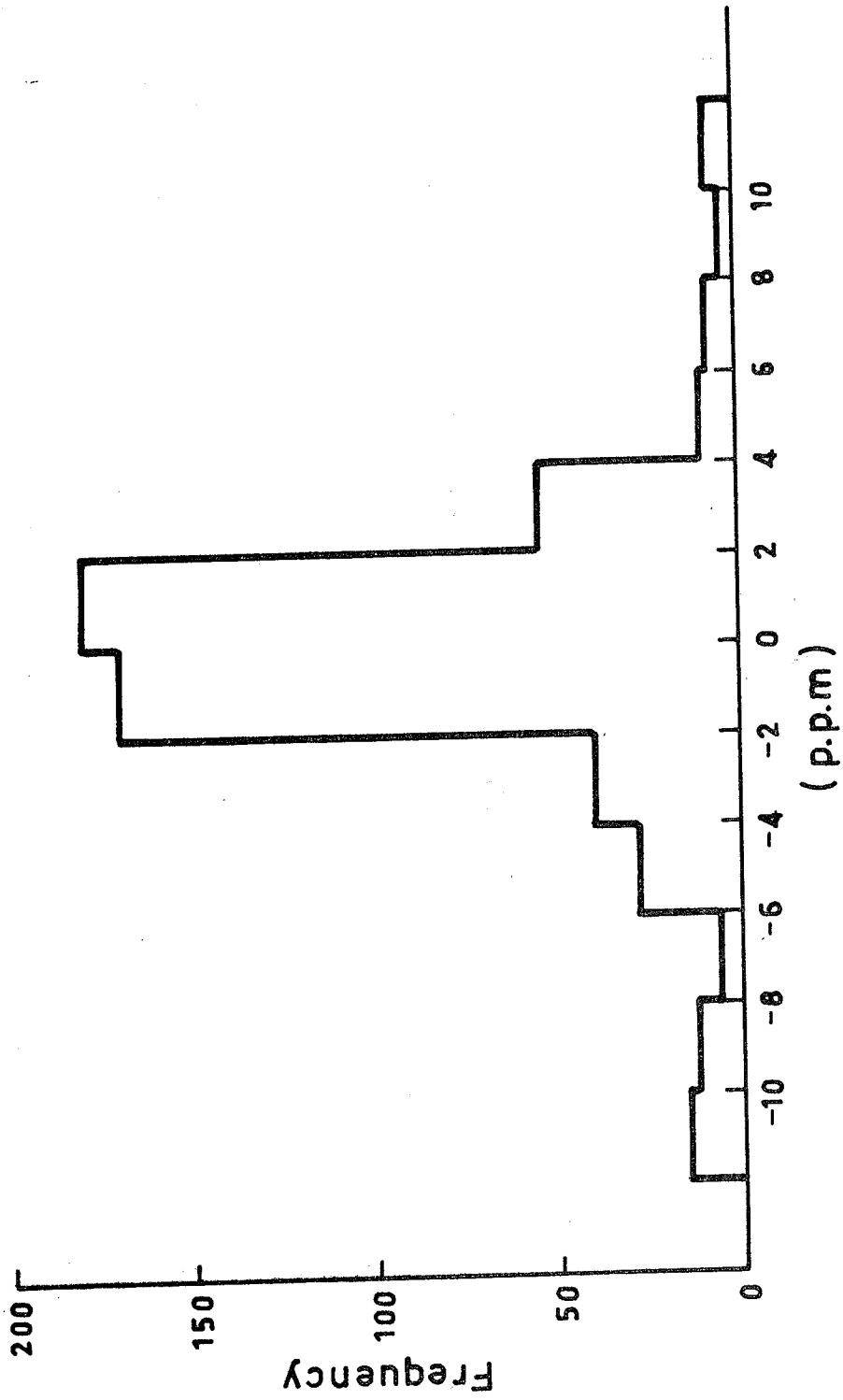


FIG.15·9: ADJUSTMENT TO LENGTHS

redundancies yielded a figure of 0.617 7 for the sample variance.

15.3.2 Standard Deviations of the Observations.

The 'a posteriori' standard deviations of the observations are defined by

$$s_i = \frac{S}{\sqrt{w_i}} \qquad \ldots\ldots \quad (15.6)$$

where S is the square root of the sample variance and the w_i are the 'a priori' weights. A variance factor of value unity was selected in the calculation of the weight matrix (See 13.3). The 'a posteriori' standard deviations will therefore be 21.4% smaller than the corresponding 'a priori' figures.

15.3.3 Standard Deviations of the Unknowns.

The standard deviations of the unknown quantities are given by

$$\sigma_x = S\sqrt{q_{xx}} \qquad \ldots\ldots \quad (15.7)$$

where the q_{xx} are the pertinent diagonal elements in the inverse normal equation matrix. Standard deviations calculated for the unknowns of the component net (Fig. 12.2) together with the corresponding corrections to be applied to their initial estimates are presented in Tables 15.1 and 15.2. The corrections to the assumed co-ordinates resulting from adjustment are always smaller than the standard deviations, suggesting that the assumed co-ordinates are already known to the precision attainable.

15.3.4 Error Ellipsoids.

Sample Error ellipsoids for typical points of the partial network are depicted in Figs. 15.11a and 15.11b, the symbols a, b and c referring to the semi-axes in decreasing order of magnitude. The direction cosines of the semi-axes are also given.

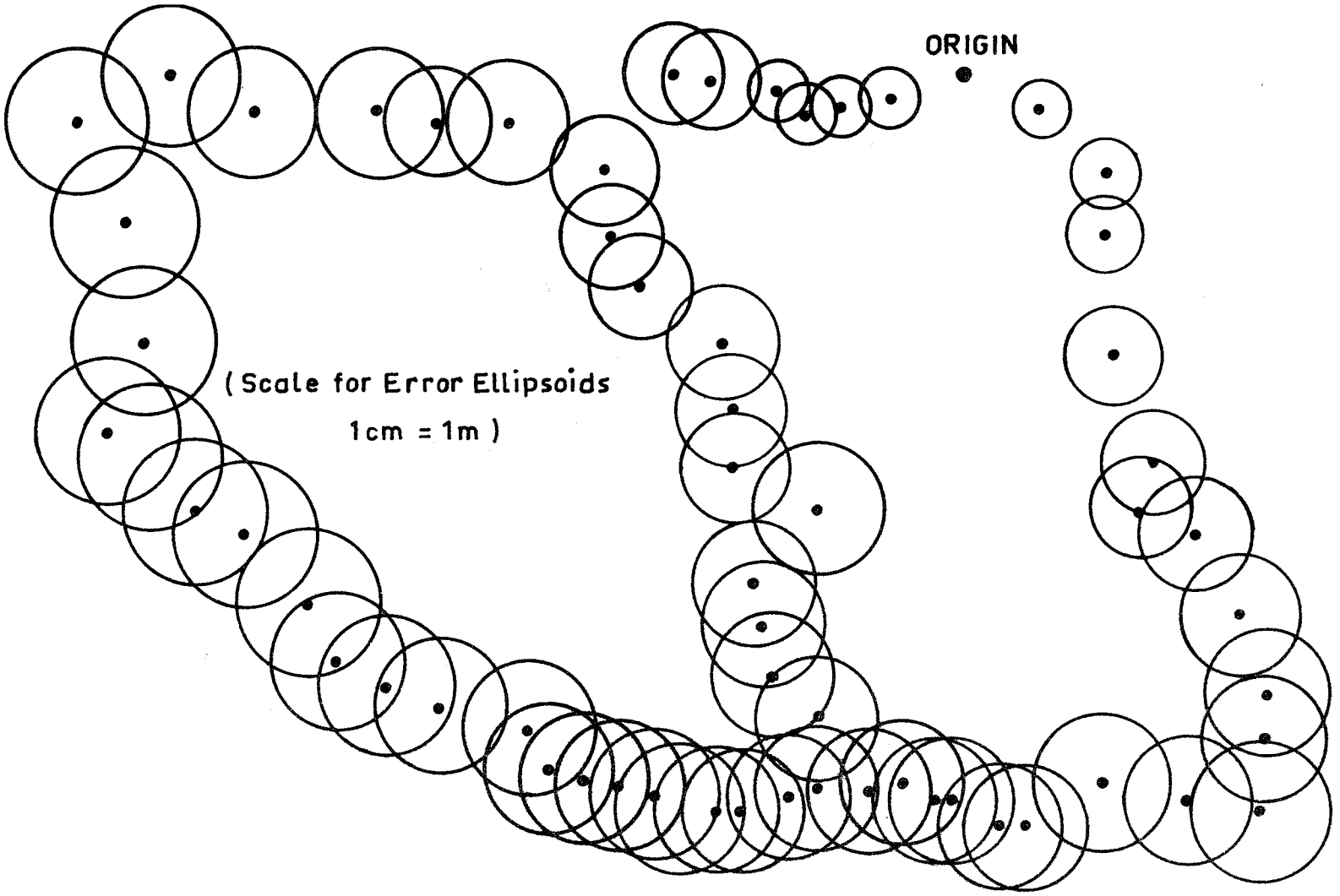


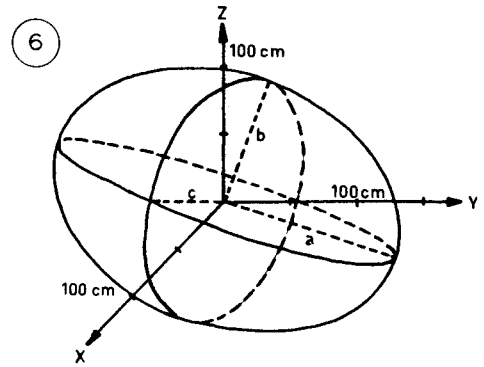
FIG. 15·10: ERROR ELLIPSOIDS (MEAN RADII)

STANDARD DEVIATIONS OF UNKNOWNS. CORRECTIONS RESULTING FROM ADJUSTMENT

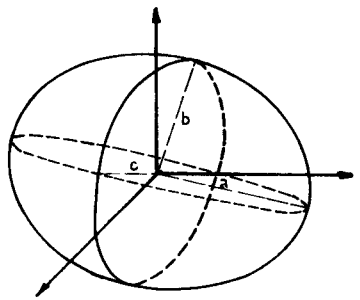
Serial	$\delta 0$	σ_0	$10^4\delta k$	$10^4\sigma_k$	Serial	$\delta 0$	σ_0	$10^4\delta k$	$10^4\sigma_k$	Serial	$\delta 0$	σ_0	$10^4\delta k$	$10^4\sigma_k$
1	3.08	0.70	- 4.7	0.47	22	- 3.19	0.97	4.3	0.37	43	1.31	0.87	- 3.4	0.40
2	2.14	0.83	9.2	0.44	23	0.47	0.74	2.7	0.41	44	3.40	0.91	1.7	0.51
3	3.16	0.88	16.3	0.53	24	- 1.80	0.90	3.1	0.44	45	4.30	0.75	3.7	0.52
4	1.02	0.71	- 3.1	0.41	25	- 3.09	1.01	- 3.2	0.38	46	3.52	0.96	1.5	0.51
5	2.94	0.87	- 7.0	0.59	26	- 0.20	0.86	7.6	0.43	47	4.07	0.78	4.9	0.53
6	-0.94	0.69	8.1	0.41	27	- 0.53	0.91	0.1	0.47	48	0.61	0.98	3.5	0.52
7	2.12	0.83	14.5	0.64	28	- 1.76	0.99	5.9	0.36	49	3.71	0.82	- 1.8	0.45
8	-2.70	0.75	- 8.1	0.41	29	0.11	0.94	- 3.3	0.49	50	0.46	0.97	- 0.2	0.52
9	1.99	0.80	3.7	0.67	30	- 0.66	0.92	3.4	0.48	51	2.66	0.85	4.4	0.46
10	-2.28	0.75	13.6	0.41	31	- 1.87	0.98	- 1.1	0.35	52	-1.86	1.00	8.8	0.51
11	3.41	0.92	- 7.9	0.66	32	0.90	0.94	19.1	0.53	53	2.46	0.85	0.5	0.54
12	-1.43	0.69	- 7.5	0.39	33	1.24	0.90	- 0.1	0.45	54	-0.06	0.99	- 5.2	0.51
13	4.27	0.95	12.0	0.64	34	1.90	1.02	3.0	0.39	55	1.43	0.83	3.8	0.62
14	1.23	0.70	6.7	0.38	35	- 2.45	0.96	1.1	0.55	56	-0.96	1.00	8.0	0.51
15	-2.95	0.96	4.5	0.38	36	- 1.20	0.87	1.4	0.45	57	-0.07	0.84	- 2.5	0.49
16	-0.28	0.85	1.1	0.57	37	- 1.60	0.98	- 0.6	0.36	58	-0.74	0.87	- 5.8	0.51
17	4.18	0.81	0.0	0.37	38	1.41	0.95	5.3	0.53	59	1.16	0.87	7.7	0.49
18	-3.15	1.01	1.1	0.37	39	- 1.14	0.83	3.5	0.41	60	-1.00	0.87	7.8	0.51
19	-2.95	0.86	1.3	0.40	40	3.18	0.97	7.3	0.39	61	3.72	0.90	- 7.7	0.50
20	-0.29	0.79	5.0	0.49	41	0.65	0.86	- 0.5	0.52	62	0.70	0.90	- 6.6	0.50
21	0.13	0.86	0.0	0.41	42	- 0.62	0.77	- 0.9	0.41	63	-2.07	0.87	11.6	0.50

TABLE 15.2

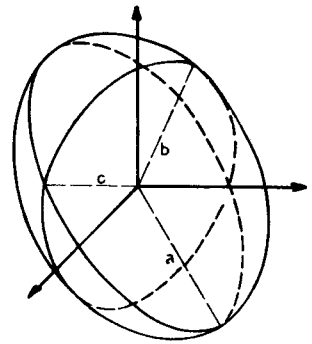
ERROR ELLIPSOIDS



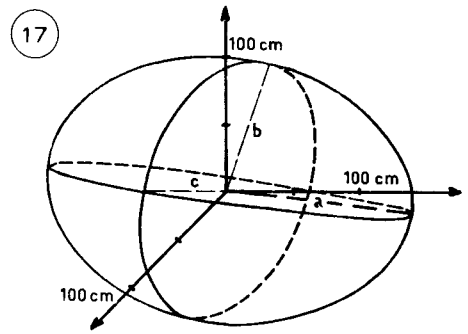
	0·574		−0·536		0·619
a = 1·45	0·764	b = 1·01	0·078	c = 0·57	0·640
	0·295		0·840		0·454



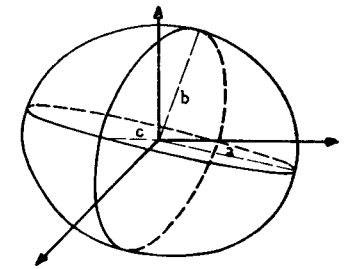
	0·551		−0·546		0·631
a = 1·21	0·778	b = 0·95	0·064	c = 0·47	−0·625
	0·301		0·835		0·461



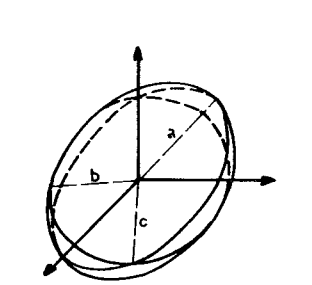
	0·677		−0·371		0·635
a = 1·30	0·733	b = 1·06	0·273	c = 0·70	0·622
	0·057		0·888		0·457



	0·542		−0·565		0·622
a = 1·50	0·765	b = 1·07	0·026	c = 0·66	−0·643
	0·347		0·825		0·446



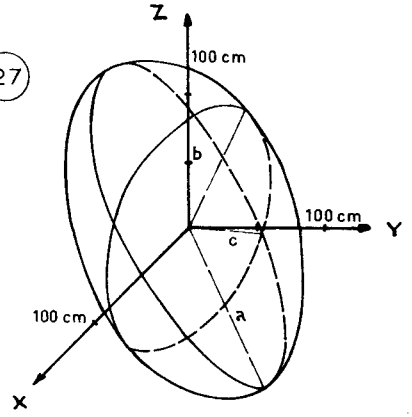
	0·546		−0·568		0·616
a = 1·12	0·776	b = 0·94	0·065	c = 0·47	0·628
	0·316		0·821		0·476



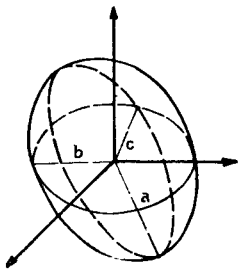
	0·213		0·678		0·703
a = 0·89	0·682	b = 0·71	−0·619	c = 0·66	0·390
	0·700		0·397		−0·594

FIG. 15·11a

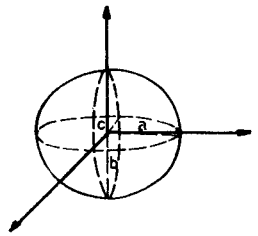
ERROR ELLIPSOIDS



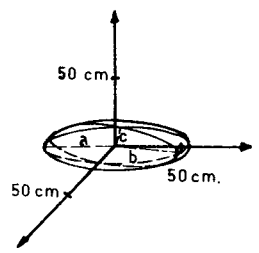
	0·703		−0·376		−0·604			
a = 1·43	0·709	b = 1·07	0·303	c = 0·59	0·637			
	0·056		0·875		0·480			



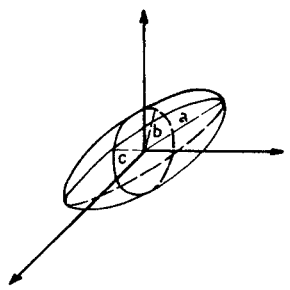
	0·698		0·646		−0·308			
a = 0·87	0·716	b = 0·66	−0·646	c = 0·46	0·265			
	0·028		0·405		0·914			



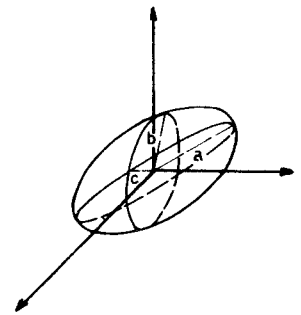
	−0·631		−0·772		−0·078			
a = 0·59	0·643	b = 0·53	0·576	c = 0·15	0·504			
	−0·434		0·268		0·860			



	0·633		0·531		−0·563			
a = 0·59	−0·634	b = 0·52	0·773	c = 0·16	0·014			
	0·443		0·348		0·826			



	0·405		−0·668		0·624			
a = 0·76	0·758	b = 0·35	−0·135	c = 0·25	−0·638			
	0·510		0·732		0·452			



	0·364		−0·680		0·636			
a = 0·77	0·750	b = 0·51	−0·192	c = 0·20	0·633			
	0·553		0·707		0·440			

FIG. 15·11 b.

The spatial orientation of the error ellipsoids, with the exception of near origin stations (Fig. 15.11b), appears to be similar, while their mean radius, as expected, increases gradually with distance from the datum point. The two stations adjacent to the datum point (Serials 55 and 59 - Fig. 12.2), each exhibit an error ellipsoid with a mean radius of 0.46 metres, whereas for the most distant station (Serial 24) a value of 1.17 metres is obtained. The previous magnitudes seem to be in agreement with the 'a priori' precision of the data and the law of propagation of errors.

16. Conclusions.

The principal problems in the coordination of terrestrial points which are connected by an extensive geodetic network were seen to be of a physical and a computational nature. That is to say, the adoption of a three-dimensional Euclidean mathematical model as a basis to computation necessitated the linearization of the measurements made through the atmosphere, whereas over-determination of the unknown parameters introduced large normal equation systems which had a tendency to be ill-conditioned.

The linearization of measurements burdened with refraction is a problem common to all existing methods of terrestrial point definition, whether two or three-dimensional. In practice, the torsion of the ray (lateral refraction) is generally ignored, while its curvature (vertical refraction) is deduced from meteorological data sampled only at the terminals of the line. Investigations into the form of the ray, although feasible, involve the higher differentials of curvature and torsion, which cannot be measured. The use of an atmospheric model, which makes as few assumptions as possible but accommodates the observations that can be taken - such as temperature, pressure and humidity at the two end points of the line - represents a suitable alternative. Thus, if the atmosphere were assumed to be in static equilibrium, which might approximately be so in settled weather during the afternoon, then the surfaces of equal density (isopycnics) which are virtually identical to surfaces of equal refractive index, are gravitational equipotentials. The present accuracy of measurement, however, hardly justifies the employment of an exact gravitational model. The use of a system of confocal ellipsoids *(Hotine 1969, 214-223)* would probably

suffice. Another approach, at least to the solution of the height problem, is to connect adjacent ground stations by precise levelling. Practical and economic considerations however, dictate that levelling ties to the trigonometric network be made at only a limited number of points. The inclusion of levelling in the computations in order to control the propagation of errors introduced by inadequate refraction data thus poses considerable computer storage problems, since widely separated connections imply a large band-width in the normal equation matrix. The use of iterative techniques avoids this complication because the coefficient matrix remains unaltered during the solution process and only the non-zero elements of the matrix need thus be stored.

Many normal equations arise in the least squares adjustment of extensive configurations, using the variation of coordinates method. The number of equations to be solved in the variation of coordinates approach equals the number of necessary observations, while by "conditions", it is equal to the number of redundant observations. In a configuration of the test network type (a series of closed loops) the number of redundancies will be small when compared to the number of necessary observations. Accordingly, since it requires less computer time and, in view of the rounding errors which occur, is more accurate to solve the adjustment problem using the least number of normal equations, it would be of immense advantage to solve the adjustment by means of correlates. However, before the "conditions" can be formulated as implicit relations of the observations, refraction must be entirely removed from the zenith distances, necessitating as a prerequisite the definition of a model atmosphere. The programming of a general configuration by the method of correlates, moreover, cannot be considered as trivial. Further, since only the corrections to the observations are obtained from an adjustment by correlates, and the coordinates of the network stations are desired, coordinates must still be calculated. Whilst the calculation of coordinates over and above adjustments might be considered inconvenient and unnecessary, particularly since the variation of coordinates approach yields the end product directly, for large problems, it is always faster - note that the condition method is a one cycle solution whereas in the variation of coordinates technique several iterative refinements are commonly required - and more accurate to choose the method involving the least number of normal equations.

At network stations where the latitude has not been observed astronomically the diagonal coefficient of $\delta\phi$, the error in the observed latitude, as well as off-diagonal terms in the row and column pertaining to $\delta\phi$ in the normal equation matrix will be significantly zero. A near-zero row or column in a square matrix however implies a near-zero determinant and the coefficient matrix is thus almost singular. An attempt at solving such an equation system by Gaussian elimination, using a particular computer word length, exhibits all the symptoms of an ill-conditioning problem - that is, the emergence of 'small' pivotal elements, a 'large' computed solution and a 'large' residual vector - while in the Cholesky approach the diagonal elements corresponding to the relevant coefficient of $\delta\phi$ tend to become negative during the triangularization algorithm. The previous predicament may be avoided by interpolating the latitude gravimetrically, that is, when sufficient gravity coverage is available. Otherwise, the use of partitioning and higher precision arithmetic is recommended. Unfortunately, it does not follow that the calculation of the least squares solution with a given accuracy is always straight forward. The spectral condition number for the normal equations is $(\kappa(W^{\frac{1}{2}}A))^2$, so that difficulty can be expected with any method if $\kappa(W^{\frac{1}{2}}A)$ is large. The existence of satisfactory computational procedures which avoid the formation of the normal equations is thus of considerable importance. Let S be a square and non-singular matrix of order n and let $L = W^{\frac{1}{2}}A$. Multiplying the normal equations by S^T yields

$$S^T L^T L x = S^T L^T b \qquad \ldots\ldots \quad (16.1)$$

which on replacing LS by B become

$$B^T L x = B^T b \qquad \ldots\ldots \quad (16.2)$$

and the solution of the normal equations and (16.2) will be identical since the non-singularity of S ensures their equivalence. The aim is to select B in such a manner that

$$\kappa(B^T L) = \kappa(L) \qquad \ldots\ldots \quad (16.3)$$

which is possible if the columns of A are linearly independent as L can then be factorized into $L = QR$, where Q is an orthogonal $m \times n$ matrix and R is upper triangular *(Householder 1964, 7)*. Moreover, $B = LR^{-1} = Q$.

Accordingly, since $B^TL = Q^TQR = R$ is an upper triangular matrix, (16.2) may be solved using back-substitution. Further, it will be noted that (16.3) is satisfied, since

$$\kappa(R) = [\lambda_n(R^TR)/\lambda_1(R^TR)]^{\frac{1}{2}}$$

$$= [\lambda_n(R^TQ^TQR)/\lambda_1(R^TQ^TQR)]^{\frac{1}{2}}$$

$$= \kappa(QR) = \kappa(L)$$

and the formation of the normal equations can be avoided if the factorization B^TL is generated directly. Two techniques may be used to obtain this factorization, namely:-

(1) the Björck method and

(2) the Gram-Schmidt orthogonalization method (*Björck 1967*).

However, since the accepted approach to error analysis in the least squares method is via the inverse normal equation matrix, an alternative error analysis must then be found.

Application of the model to a major portion of the network stations which were included in the 1966 Australian National Adjustment revealed large discrepancies between the two systems of computation, particularly at the south-eastern extremety of the test network. Although the resulting coordinate differences were in part attributable to the use of the data not being identical and the test network comprising only a small segment (10%) of the National configuration, significant contributions were expected as a result of approximations within the modified Bowie technique of computation - that is, error free Laplace azimuths, over-simplified weighting scheme of derived quantities, and convenient exclusion of overlaps during the forced adjustment. The intention is to readjust the National Net in 1972 with the improved height data which is now available. A specific study into the effects of the previous approximations on an extensive configuration, such as that selected in Chapter 10, would be of considerable benefit in the interpretation of the results.

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EXPLANATORY NOTES.

p.45 - p.48 THE PARAMETERS $\delta\phi$ AND $\delta\omega$ IN THE OBSERVATION EQUATIONS.

The angular and linear data, once corrected for curvature, pertains to the straight line segment connecting the observed and the occupied ground points. The angular measurements are made solely with reference to the plumb-line at the occupied station and the lengths are independent of the direction of the plumb-line at either of the points in question. Hence, in the observation equations for (α), (α^*) and (β) and for (s), $\delta\bar{\phi} = \delta\bar{\omega} = 0$ (Equations 6.13, 6.17 and 6.24) and $\delta\phi = \delta\bar{\phi} = \delta\omega = \delta\bar{\omega} = 0$ (Equation 6.21), respectively.

p.97 TABLE 10.9 AND TABLE 10.10

The figures in brackets located below the station names refer to the elevation of the ground point above Mean Sea Level.

p.108 - p.112 REJECTION OF THE OBSERVED ZENITH DISTANCES AS DATA.

Further proof that all was not well with the observed zenith distances arose when they were included in the solution. The results exhibited: (1) 'enormous' increments to the approximate co-ordinates; (2) corrections to observed longitude exceeding 10 arc seconds; and (3) refraction coefficients which were pure fantasy. It was believed that these events could be attributed to an over-estimation of the precision of the zenith distances. The computations were therefore repeated using a variance of $\sigma_\beta^2 = 100$ arc seconds^2. Decomposition of the normal equation matrix by the Cholesky method however, led to negative diagonal elements, signifying an ill-conditioning problem, that is, the observed zenith distances and the other measured data were no longer compatible.

p.114 CONSTRAINTS AT THE SURVEY ORIGIN.

The orientation of the survey scheme with respect to the global Cartesian system was thus not strictly effected in accordance with the principles of a 'free net' adjustment which were described in Section 5.2.3.

p.129 TABLE 13.3.

FIG. 10.2 should be consulted for the approximate locations of the stations quoted.

p.135 ITERATIVE REFINEMENT OF THE NORMAL EQUATIONS.

The first application of the iterative refinement process to the solution of the normal equations already yielded increments δx to x which were smaller than the 'a posteriori' standard deviations depicted in TABLE 15.1 (p.160), indicating that the data did not really warrant the use of the technique. The results quoted in TABLE 15.1 were however derived from the component net shown in FIG. 12.2 (p.107). For the complete solution it remained to determine whether successive residuals r converged. Two iterations provided the necessary information.

p.150 ADOPTION OF THE MEAN VALUE OF TWO SIMULTANEOUS RECIPROCAL AZIMUTHS.

One 'logical' reason for adopting the mean value of two simultaneous reciprocal azimuths is that experience in the United States and Australia has indicated that the mean value gives loop closures which reflect a precision in excess of that implicit in individual azimuths.

p.150 ADDITIONAL OBSERVATIONS AND THE SOLUTION OF THE NORMAL EQUATIONS.

There is no doubt that additional observations produce larger diagonal coefficients in the normal equation matrix. Non-diagonal terms and the vector of constants however, may either be enhanced or reduced in magnitude. Additional observations thus do not necessarily imply a smaller solution.

ERRATA.

Page	Line	
1	10	For 'System' read 'system'
15	11	For (ibid, 12 & 47) read (Hotine 1969, 145)
16	25	For $t_2\cos\phi/K$ read $t_1\cos\phi/K$
19	2	For 'provice' read 'provide'
27	)	For μ^r read μ_r
	) FIG.5.1	
27	)	For λ^r read λ_r
29	FIG.5.2	For $\bar{\mu}$ read μ
39	3	For $\bar{\rho}_r - \rho_r$ read $\lvert\bar{\rho}_r\rvert - \lvert\rho_r\rvert$
46	21	For $(\alpha' - \alpha + 0')$ read $(\alpha' - \alpha - 0')$
48	15	For 'to' read 'of'
48	17	For (Rainsford 1968) read (Rainsford 1969)
51	19	For 'possibly' read 'possible'
54	25	For ', c' read 'c,'
63	15	For Wilkinson (1964, 93) read Wilkinson (1963, 93)
143	26	For 'systemmatic' read 'systematic'
145	1	For 'Ellipsoidal' read 'ellipsoidal'
145	19	For $\overline{\nu(1 - e^2)} + h\sin\phi$ read $\overline{\nu(1 - e^2) + h}\sin\phi$
148	35	For 'systemmatic' read 'systematic'
173	18	For 'extremety' read 'extremity'

BIOGRAPHICAL NOTES.

ARTUR STOLZ joined the staff of the University of New South Wales in 1969. He is employed as a Lecturer in the School of Surveying. Mr. Stolz graduated from the University of New South Wales in 1963 with a B. Surv. degree and worked with the N.S.W. Department of Lands till 1967, becoming registered as a Surveyor in 1964.

He then joined the South Australian Institute of Technology as a Lecturer, where he remained till returning to the University of New South Wales.