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Proc.Symposium on Earth's Gravitational Field & Secular Variations in Position (1973),51-75.

THE EARTH'S GRAVITATIONAL FIELD FROM THE COMBINATION OF SATELLITE AND TERRESTRIAL DATA *

ABSTRACT

This paper reviews techniques and results in the combination of gravimetric and satellite data. The first section of the report deals with theoretical procedures, while the second section deals with current solutions. In the first section, the estimation of mean anomalies for use in combination studies is discussed with the location of current gravity material being described. Specific techniques for combination solutions are discussed for various models. These models include those where the gravitational field is represented by a set of potential coefficients, or by a set of discrete blocks distributed on the Earth. The potential coefficient solutions compared are those of the SAO Standard Earth II and III, the Goddard Earth Model 6 (GEM 6) and a solution by the author. These solutions are compared in terms of root mean square coefficients,undulations and anomaly differences, and implied anomaly degree variances. In addition, comparisons were made through terrestrial anomaly comparisons, astro-geodetic undulation comparisons and orbit fitting tests. Some solutions compared reveal certain solutions to be better for some purposes than others.

1. Introduction

The purpose of this paper is to discuss some of the procedures used for the combination of gravimetric and satellite data and to compare some of the current solutions that attempt to describe the Earth's gravitational field.

We should first define what we mean by the Earth's gravitational field. We note first that in some cases it is convenient to talk about the gravity field of the Earth instead of the gravitational field. Gravity and gravitation are related in that gravity, for a point rotating with the Earth, is the vector sum of the gravitational attraction and the centrifugal force. For points located on the surface of the Earth we are primarily concerned with gravity while for points located at satellite altitude we are interested in the gravitational field.

On the surface of the Earth we can determine gravity by making measurements of its normal component (using gravimeters) and its other components through the determination of the deflection of the vertical. Primarily, however, we can consider that we can determine the normal component of gravity at various discrete points on the surface of the Earth. These points may be located on the land, on the oceans, on ocean bottoms, or (in a few cases) at points of aircraft altitude. (Here we should note that measured gravity is a function not only of the Earth's attracting masses and centrifugal force, but also is effected by lunar and solar tides, by Earth tides, and to a very minor extent, the atmosphere. Formally, these latter effects are removed from measured gravity to yield a result independent of these quantities.) If we could obtain a dense global coverage of point gravity measurements we could say that the gravity field on the Earth has been determined and through the application of suitable equations, the gravitational vector at points in space could be determined (HEISKANEN & MORITZ 1967,chapter 5). Unfortunately the area gravity coverage for the Earth is not complete with

* Revised January 1974

wide areas (especially in some oceanic areas) having little gravity data, or data not generally available to the scientific community. A discussion of gravity measuring techniques may be found in HEISKANEN & VENING MEINESZ (1958) while a discussion on the utilization of gravity data for various geodetic purposes may be found in HEISKANEN & MORITZ (1967).

This lack of data had prevented an accurate analysis of the broad variations of the Earth's gravity field until artificial satellites were launched. A satellite moves within the gravitational field of the Earth (plus other forces, of course). Observing the precise motion of the satellite allows us to infer information with respect to the Earth's gravitational field. Since a satellite's position will be perturbed only in a minor way, due to local gravity irregularities, we generally think of the analysis of satellite information being used to determine the broader variations of the Earth's gravitational field. (These statements do not consider new satellite observation techniques such as satellite-to-satellite tracking, or satellite gradiometry that provide a measurement at satellite altitudes, of quantities that depend on the gravity field of the Earth.) A discussion on how a satellite's motion depends on the Earth's gravitational field may be found in (KAULA 1966a), or (MUELLER 1964).

If we say that the local variations of the Earth's gravity field are best determined by direct gravity measurements, and that the broad variations are determined through the analysis of satellite motion, the most complete description of the Earth's gravitational field must come from some combination of the two techniques.

This paper specifically concerns itself with some of the techniques used for this combination. I will not attempt to analyze all techniques used for combination solutions. Rather, I will discuss certain techniques that are currently being used, with appropriate references to other pertinent papers when necessary.

2. Gravity Material

Discrete measurements of gravity (g) are generally converted to gravity anomalies, Δg, by subtracting from the observed gravity some reference gravity due (generally) to an equipotential rotational ellipsoid. We have

$$\Delta g = g' - \gamma \tag{1}$$

where g' is the observed gravity reduced to some reference surface. In most applications carried out to date, this reference surface has been taken to be the geoid with no masses external to it. The specific way in which the reduction is done in obtaining g' from g will yield different types of anomalies. The starting anomaly for use in combination studies is the free-air anomaly. For this and other types of anomalies, see (HEISKANEN & MORITZ 1967,chapter 3). (We will discuss later modifications to the free-air anomaly for more precise statements of the combination procedures.)

The discrete anomalies available on the surface of the Earth are not directly usable for most combination solutions. (However, BJERHAMMAR (1963) has, for example, proposed to use discrete values for some purposes.) Because of this, the anomalies are formed into values representative of various size areas on the Earth's surface. We write:

$$\overline{\Delta g} = \frac{1}{A}\iint_A \Delta g \; dA \qquad (2),$$

where $\overline{\Delta g}$ is the mean anomaly in a block whose area is A. Usually A is defined to be an area bordered by meridians and parallels. These areas (or blocks) may be equiangular (such as 1° × 1°) or equal area (RAPP 1971a) in size. For a given basic block area there will be fewer equal area blocks than equiangular blocks on the surface of the Earth. Thus, for computational efficiency and statistical effectiveness, it is most efficient to use equal area blocks in combination studies. However, the gravity data used in recent combination studies has been supplied in 1° × 1° blocks (ACIC 1971). Such blocks need to be used in the estimation and prediction of mean anomalies in five degree equal area blocks, The larger size blocks are currently used in some types of combination studies. Other types of combination work use mean anomalies in ten degree and fifteen degree blocks (HAJELA 1973). The estimation of the five degree equal area blocks from the 1° × 1° data requires several decisions on the appropriate procedure. KAULA (1966b) used a linear regression technique to determine five degree equal area anomalies based on 1° × 1° data. RAPP (1972a) used a modification of Kaula's work that carried out anomaly prediction, and the prediction accuracy considering the location of the 1° × 1° anomalies within the 5° equal area blocks, as well as the accuracy of the 1° × 1° data. The specific procedure used is as follows. The 1° × 1° means were first formed into mean anomalies for areas 60 nautical miles (nm) (in latitude) and 60±30 nm (in longitude). These anomalies were then used to estimate the mean anomalies in 300 nm (in latitude) and 300±30 nm (in longitude) blocks by predicting, in the 300 nm block any missing 60 nm blocks by linear regression. The 300 nm mean anomaly was then formed as a straight average of the 25, 60 nm blocks in the 300 nm blocks. No estimations were made for a 300 nm block unless it contained one or more observed 60 nm blocks.

The specific equation used for predicting a 60 nm anomaly (g*) was given by MORITZ (1969):

$$g^* = \underline{C}_p (\underline{C} + \underline{D})^{-1} \underline{g} \qquad (3),$$

where $\underline{C}_p$ is a column vector whose elements are the covariance between the block (p) to be predicted and the observed anomalies; $\underline{C}$ is a matrix whose elements are the covariances between the observed anomalies; $\underline{D}$ is an error covariance matrix for the observed anomalies; and $\underline{g}$ is a column vector of the observed anomalies within the 300 nm block in which g* was situated. For these computations $\underline{D}$ is taken as a diagonal matrix with each diagonal element being equal to m_j^2, where m_j is the standard deviation of the observed anomaly g_j. The standard deviation (m) of the 300 nm anomaly was computed from (IBID,p.11)

$$m^2 = \overline{\overline{C}} - \underline{\overline{C}}_i (\underline{C} + \underline{D})^{-1} \underline{\overline{C}}_i \qquad (4),$$

where $\overline{\overline{C}}$ is the mean square value (or variance) of the 300 nm mean anomalies, and $\underline{\overline{C}}_i$ is a column vector representing the covariance between the i-th observed anomaly and the 300 nm block in which it lies.

The above equations were applied to a set of 23,355 1° × 1° anomalies which were based on a set of 1° × 1° anomalies of ACIC (1971) supplemented by additional material not present in the ACIC set. For example, using data not made available by ACIC, 323 ACIC anomalies in the Canadian area were replaced by updated values, and 1989 new, 1° × 1° values in the Canadian area were added. This data enables the prediction of 1283, 5° equal area anomalies and their accuracy. The location of these blocks is shown in figure 1. Since these predictions have taken place, new gravity material has

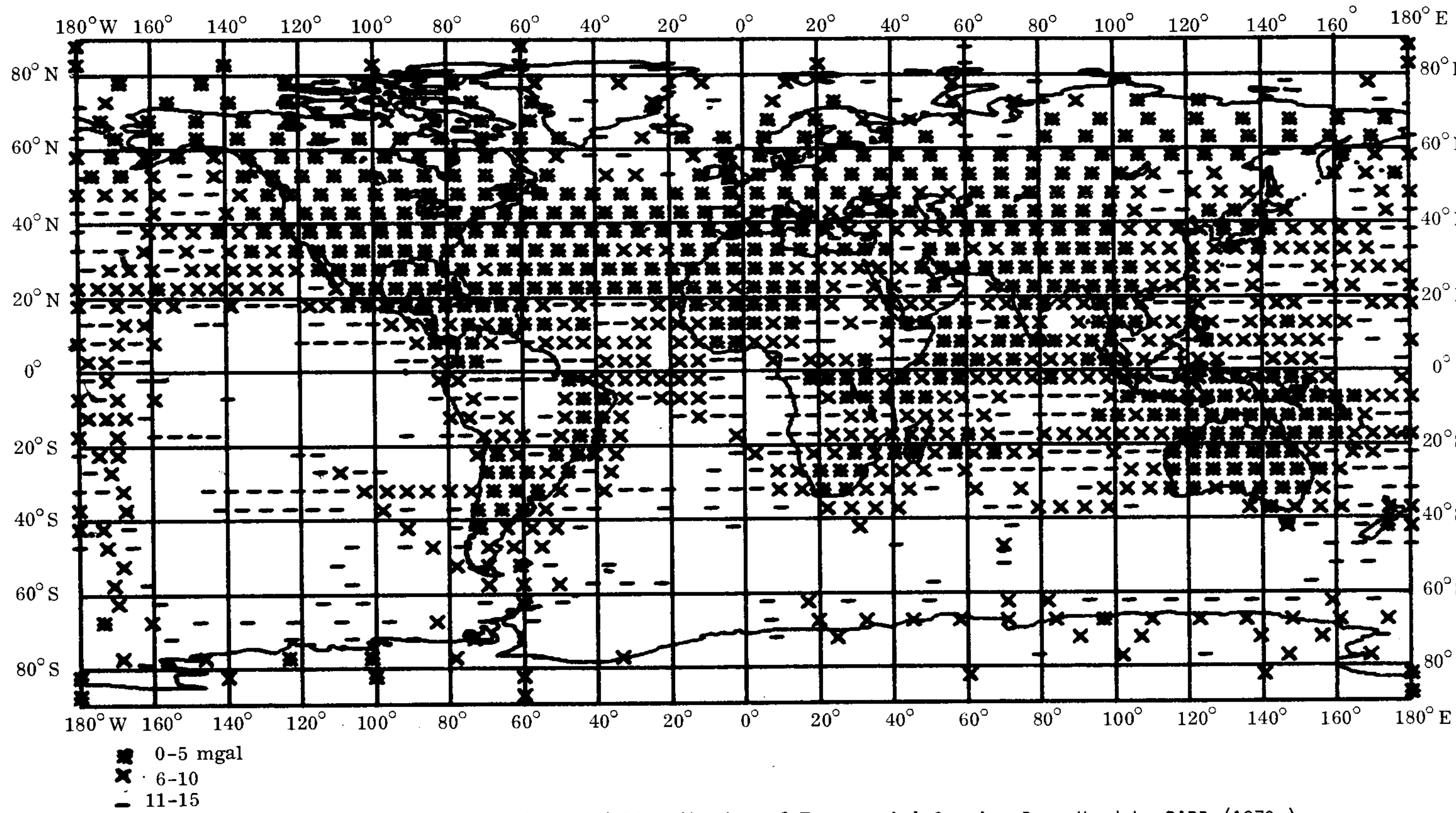

Figure 1. Quality and Distribution of Terrestrial Gravity Data Used by RAPP (1972a)
(Symbols are Located at the Centre of the 5° Equal Area Blocks)

become available in several areas. One large block of data are the results from Project SEAMAP of the National Geodetic Survey.

The procedures described above are only one of several that have been used by various investigators. More sophisticated techniques are possible. For example, the use of a localized covariance as opposed to global functions in developing the C matrices of equations 3 and 4 may lead to more precise estimates of some quantities. However, RAPP (1964) found that localized covariances did not yield significantly different anomaly predictions but did yield different anomaly standard deviation predictions. The inclusion of more 60 nm anomalies in the prediction of a 300 nm mean may be warranted but the resultant error correlation between the 300 nm blocks may cause more problems than is worth. The studies of GROTEN (1966) indicate the addition of known anomalies somewhat distant from anomalies to be predicted have little influence on the predicted anomaly.

Since we do not have global gravity coverage, a decision must be made for combination solutions as to what anomalies should be assigned (if any) for these empty areas. (Some combination methods do not require global estimates for the terrestrial gravity field. However, to assure that distortion - in terms of geoid heights or anomalies - is not given to the empty areas, it is usual practice to utilize global anomaly estimates in any combination solution.) The estimation of anomalies in unsurveyed areas is a field of study in itself (ORLIN 1966) which will not be gone into here. For recent combination solutions the following procedures may be found:

1) estimation of empty areas by linear regression from known 5° equal area anomalies (KAULA 1966a);
2) incorporation of model anomalies computed on the basis of topography and an isostatic hypothesis (RAPP 1968);
3) computing anomalies from a set of potential coefficients derived from satellite orbital analysis (KÖHLNEIN 1967); and
4) setting the anomalies in the unsurveyed areas to zero with large standard deviations (GAPOSCHKIN & LAMBECK 1971).

The linear regression is valuable because of its well defined applicability. However, its accuracy is subject to how well is the linear regression model formed (e.g., is the correlation of free air anomalies with topography considered) and how strong are the correlations between the known anomalies and the anomalies to be predicted. The model anomalies are useful as they incorporate independently estimated data (such as topographic heights). They may not reflect actual mass anomalies. The use of satellite anomalies in the empty areas is fine when we can ignore the havoc that such a procedure raises in trying to carry out a rigorous least squares solution. The use of a zero anomaly admits our ignorance and keeps the combination solution from "blowing up" in the empty areas if no other information were used.

3. Satellite Data

The fundamental satellite observations are the direct observation of satellites. These observations include those of right ascension and declination, range, and range-rate (or some function of range-rate). This data is generally processed to extract gravitational field information (as well as station co-ordinates, tidal parameters, orbital parameters, etc.). This gravitational field information is then combined with terrestrial gravity information in a variety of ways which are discussed in the following section.

4. Combination Methods

There are many methods to combine satellite and terrestrial gravity data. A summary of some of these may be found in (HOPKINS 1972). Space does not permit a discussion of all methods nor complete details of methods to be described. Basically, there are two types of combination methods that differ in the manner in which the gravitational field is represented. The first method represents the Earth's gravitational field using potential coefficients while the second method uses discrete blocks on the surface of the Earth.

4.1 Combination Methods Using Potential Coefficients

We represent the Earth's gravitational potential (V) by a set of fully normalized potential coefficients $(\overline{C}_{\ell m},\overline{S}_{\ell m})$ as follows:

$$V = \frac{kM}{r}\left[1 + \sum_{\ell=2}^{\infty}\left(\frac{a}{r}\right)^{\ell}\sum_{m=0}^{\ell}\left(\overline{C}_{\ell m}\cos m\lambda + \overline{S}_{\ell m}\sin m\lambda\right)\overline{P}_{\ell m}(\sin\overline{\Phi})\right] \qquad (5),$$

where: kM is the geocentric gravitation constant;

r is the geocentric latitude
$\overline{\Phi}$ is the geocentric latitude } of the point at which V is being computed;
λ is the longitude

a is a nominal equatorial radius; and

$\overline{P}_{\ell m}$ are the fully normalized associate Legendre functions.

Potential coefficients can be related to gravity anomalies in two ways. The first is through the following equation:

$$\begin{Bmatrix}\overline{C}_{\ell m}\\ \overline{S}_{\ell m}\end{Bmatrix} = \begin{Bmatrix}\overline{C}_{\ell m}\\ \overline{S}_{\ell m}\end{Bmatrix}_{ref} + \frac{1}{4\pi\gamma(\ell-1)}\iint_{\sigma}\overline{\Delta g}\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\overline{P}_{\ell m}(\sin\overline{\Phi})\,d\sigma \qquad (6)$$

where: $(\overline{C}_{\ell m},\overline{S}_{\ell m})_{ref}$ are potential coefficients implied by the reference figure (or gravity formula) to which the mean anomalies $\overline{\Delta g}$ (given as blocks $d\sigma$) are referred to. When the anomalies are referred to a rotational equipotential ellipsoid gravity formula, all $(\overline{C}_{\ell m},\overline{S}_{\ell m})_{ref}$ are zero except for $\overline{C}_{\ell 0}$ when ℓ is even. In that case the reference values of $\overline{C}_{\ell 0}$ with $\ell > 4$ are generally considered negligible. The integration in equation 6 is taken over the Earth which is here approximated by a spherical surface. Equation 6 follows from the discussion in (HEISKANEN & MORITZ 1967, section 2-20).

The combination adjustment is carried out by estimating $\overline{\Delta g}$ values and their standard deviations from terrestrial gravity material, and by taking values of the potential coefficients and their standard deviations from satellite analysis. Then equation 6 is used as a mathematical model to formulate a weighted least squares adjustment where the following quantity is minimized:

$$V_g' D_g^{-1} V_g + V_c' D_c^{-1} V_c \qquad (7),$$

where V_g and V_c are the residual vectors for the gravity anomalies and potential coefficients, and

D_g and D_c are the variance-covariance matrices of the observed anomalies and the a priori known potential coefficients.

The details of the adjustment procedure using this method are found in (RAPP 1969a). The first computation using this approach was done by KAULA (1966b) where an adjusted set of potential coefficients complete to $\ell = 6$ plus some additional terms were sought. The results of this method will be a set of adjusted potential coefficients and a set of mean gravity anomalies consistent with the adjusted potential coefficient set, yet retaining the greater detail that would exist in anomaly blocks smaller than that defined by a set of potential coefficients solved for up to a degree less than $180°/\theta°$, where θ is the block size in which the $\overline{\Delta g}$ values are given.

Anomalies and potential coefficients may also be related by the following equation (RAPP 1967)

$$\Delta g = \frac{kM}{r^2} \sum_{\ell=2}^{\ell_{max}} \left(\frac{a}{r}\right)^{\ell} (\ell - 1) \sum_{m=0}^{\ell} \left(\overline{C}^{*}_{\ell m} \cos m\lambda + \overline{S}_{\ell m} \sin m\lambda \right) \overline{P}_{\ell m}(\sin \overline{\phi}) \qquad (8),$$

where ℓ_{max} is the maximum degree for which potential coefficients are to be found or are given. $\overline{C}^{*}_{\ell m}$, $\overline{S}_{\ell m}$ are the potential coefficients referred to an ellipsoid whose flattening is specified. Actually, equation 8 does not give the standard anomaly, but rather the anomaly component in the radial direction. The difference is insignificant (RAPP 1972b). In some combination studies spherical approximations to equation 8 have been used by letting r be a mean radius (R) of the Earth. Considering the accuracy of current gravity material such approximations do not appear critical.

Knowing terrestrial estimates of $\overline{\Delta g}$ and a priori values (from satellite analysis) of the potential coefficients, equation 8 may be used as a mathematical model to obtain weighted least squares estimates of the potential coefficients under the same condition as specific in equation 7. The details of the adjustment procedure are given in (RAPP 1969a). This method was used by RAPP (1968) to obtain adjusted potential coefficients to $\ell = 14$ plus some additional terms.

It should be noted that both equations 6 and 8 can be used to determine potential coefficients given only terrestrial gravity material. Whether these equations are used in this manner, or as the foundation for a combination solution, the results for the two methods (assuming equivalent spherical approximations) will be different unless all anomalies (given in equal area blocks) have the same standard deviations (RAPP 1969a).

PELLINEN (1965; 1969) also discusses combination solutions using equations similar to 6 or 8.

The exclusive use of equations 6 or 8 ignores the fact that satellite estimated potential coefficients are not estimates independent of other unknowns such as station co-ordinates and parameters dependent on the satellite orbit (such as initial condition, drag effects, etc). To consider this information we consider two sets of normal equations, one arising from the satellite adjustment and the other arising from the potential coefficient solution from terrestrial data using equation 8. We have:

$$q\, N_{S,X}\, X + (q\, N_{S,P} + N_G)\, P = q\, U_S + U_G \qquad (9),$$

where X are station co-ordinate parameters, P are the potential coefficient parameters, $N_{S,X}$ are the normal equations for the station co-ordinates from the satellite data; $N_{S,P}$ and N_G are the normal equations for the potential coefficients from the satellite and terrestrial gravity data respectively. U_S and U_G are the constant vectors of the normal equation, q is a scaling parameter used to provide the most effective weighting between the satellite and terrestrial information. Equation 9 is solved to obtain the X and P vectors. The arc (or orbit) dependent parameters do not appear in equation 9 as they are usually eliminated by the technique described by KAULA (1966a,p.105). The combination technique represented by equation 9 with some version of 8, has been used by GAPOSCHKIN & LAMBECK (1971), LERCH ET AL (1972a;1972b) and GAPOSCHKIN (1973). BJERHAMMAR (1969) gives a more general view of combining data from different satellite solutions where he shows how observation and/or normal equations may be combined to determine station co-ordinates and potential coefficients on a basis of normal equations supplied by different investigators.

The preceding method has only used gravity information to infer information with respect to potential coefficients. However, gravity data may also be used to determine deflections of the vertical and geoid undulations which may be used to determine geocentric station positions. This additional information can be incorporated into a combined solution with satellite data to determine not only station positions and potential coefficients but datum shifts and an equatorial radius. This general method is described by RAPP (1971b) with computer programs for the implementation of the method given by GOPALAPILLAI ET AL (1971). To date no results from this procedure have been published.

FISCHER (1968a;1968b) and GROTEN (1970) have described techniques where astro-geodetic undulation solutions can be incorporated in combination solutions. Although no numerical results were given by Groten, Fischer computes a set of potential coefficients to degree 13. Because of the non-global extent of astro-geodetic undulation, the development of global information from them is more a theoretical procedure than a practical one.

MORITZ (1970) derived a procedure that can be used for a combination solution using the concept of least squares collocation. In this type of solution, the quantity minimized is

$$s' C^{-1} s + V_g' D_g^{-1} V_g + V_c' D_c^{-1} V_c \qquad (10),$$

where s is the signal array and C is the covariance matrix of the signal. When a combination solution using equation 6 is considered, the signal array corresponds to the gravity anomalies. When equation 8 is used, the signal is considered to be the potential coefficients. In this latter case the relationship between a set of potential coefficients, p_o, estimated under the usual least squares principle (equation 7) can be related to a set, p, determined from the least squares collocation principle through :

$$p = (I + N^{-1} C^{-1})^{-1} \; p_o \qquad (11),$$

where N are the coefficients of the unknowns (NX = U) in the usual least squares solution (where N contains a priori information on the estimated potential coefficients) and C is the covariance matrix between the potential coefficients that are being estimated.

4.2 Combination Methods Using Discrete Representation of the Earth's Gravitational Field

The use of a spherical harmonic expansion to represent the Earth's gravitational field has played the major role in combination solutions because of the advantageous manner in which the potential coefficients can be used for satellite orbital analysis (KAULA 1970). However, for future data acquisition schemes and as an independent alternative to potential coefficients, it is reasonable to consider the surface of the Earth (or some approximation to it) divided into discrete areas for which a function related to the Earth's gravitational field is to be determined.

In 1965, Arnold proposed a method where gravity anomalies in discrete blocks could be determined throught the analysis of the orbital variations of various satellites. In this procedure the perturbations on the orbital elements were related to the gravity anomalies through the generalized Stokes' equation and its derivatives. In the Arnold procedure, known anomalies were held fixed in solving for a limited number of anomalies considered as unknowns. Additional details may be found in (ARNOLD 1966; ARNOLD 1972).

OBENSON (1970; 1972) extended the idea of Arnold to the extent that a global anomaly field could simultaneously be found from a generalized least squares solution with satellite data alone, or in combination with terrestrial gravity material. Obenson's procedure considered the influence of the gravity anomalies on the satellite position expressed through orbital element variations.

KOCH (1968) proposed that the Earth's gravitational field be represented by a low degree set of potential coefficients plus a disturbing potential represented by the potential of a single layer distributed in discrete blocks on the surface of the Earth. The density of the surface layer for each of the blocks can be determined from orbital analysis and in combination with surface gravity material if necessary.

RAPP (1971c) outlined a procedure where the Earth's gravitational field could be represented simply by a set of discrete gravity anomalies. Such anomalies can be determined from orbital analysis or can be solved in a combination solution with observed terrestrial gravity material. As with the Koch solution, a global set of discrete parameters is determined in a generalized least squares adjustment where the terrestrial material is incorporated as a priori information.

For either the Koch or Rapp procedure we may represent a set of satellite observations as follows:

$$O(r_o, \dot{r}_o, t_o, t, R, A, G, P) = 0 \qquad (12),$$

where r_o and $\dot{r}_o$ are the initial position and velocity vectors at time t_o. t is the time from t_o, R represents a reference set of potential coefficients, A are parameters depending on the satellite arc; G are the parameters of the gravitational field, and P are the co-ordinates of the observed station. The unknowns of greatest interest are G and P.

To further consider G, we represent the gravitational potential, W, as a sum of a normal potential due to a low degree reference field and a disturbing potential T.

$$W = U + T \qquad (13).$$

In Koch's case, T is:

$$T = \iint_E \frac{\chi'(\phi,\lambda)}{\ell} dE \qquad (14),$$

where χ' is an auxiliary density of the surface layer (referred to the U field) in a surface element dE and ℓ is the distance between the block and the point at which the evaluation of T is done. In

Rapp's case, T is given by the generalized Stokes' equation (HEISKANEN & MORITZ 1967,p.93):

$$T = \frac{R}{4\pi} \iint_E S(r,\psi^*) \, \Delta g' \, dE \qquad (15),$$

where $S(r,\psi^*)$ is the generalized Stokes' function dependent on the distance from the centre of the Earth to the point at which T is being computed, and ψ^* is the spehrical arc between the element dE on the spherical approximation to the Earth, and the sub-computation point. The anomalies $\Delta g'$ are referenced to the U field.

In order to develop the observation equations for this type of procedure is is necessary to differentiate equations 14 or 15 in three directions to determine the accelerations acting on the satellite due to the density layer or the anomalies. These derivatives can then be used in the variational equations which are numerically integrated (along with the equations of motion of the satellite) to obtain the derivatives of the satellite observation with respect to the gravitational parameters.

For a combination solution with the surface gravity material and the surface density solution, a global terrestrial field must be transformed into surface density values using the following equation (KOCH & MORRISON 1970):

$$\chi = \frac{\Delta g - G}{2\pi} + \frac{3}{(4\pi)^2} \iint (\Delta g - G) \, S(\psi) \, dE \qquad (16),$$

where G is the mean global anomaly. The standard deviations of the χ values can then be determined from an error propagation with equation 16 using the standard deviation of the terrestrial anomalies. This terrestrial derived information may be used as a priori information in a generalized least squares adjustment.

A combination solution with surface gravity material, and the analysis with anomalies incorportaed as unknowns is made simply by adding the a priori anomaly data into the adjustment scheme. No transformation is required. Details of this adjustment are in (RAPP 1971c). For comparisons with the usual potential coefficient determinations, the density values or the anomalies resulting from these discrete block solutions can be converted into potential coefficient sets. In the case of anomalies, equation 6 can be used. In the case of surface density values, the following equation can be used (KOCH & WITTE 1971):

$$\begin{Bmatrix} \bar{C}_{\ell m} \\ \bar{S}_{\ell m} \end{Bmatrix} = \begin{Bmatrix} \bar{C}_{\ell m} \\ \bar{S}_{\ell m} \end{Bmatrix}_{ref} + \frac{1}{(2\ell + 1) k \, M \, a^{\ell}} \iint_E \chi' \, r \begin{Bmatrix} \cos m\lambda \\ \sin m\lambda \end{Bmatrix} dE \qquad (17)$$

where the integration is carried out numerically over the blocks used in the solution.

The use of surface density values as unknowns has the advantage of computational simplicity over the use of anomalies in the evaluation of the derivatives needed in the observation equations. In addition, the use of surface density values may have a strong theoretical basis when considered with respect to the assumptions on the use of the generalized Stokes' equation. On the other hand, the use of anomalies as unknowns permits a very simple combination solution to be carried out without the need for any transformation of data. In addition, any theoretical problems can apparently be resolved by proper interpretation of what anomaly is being found, or what precisely is the anomaly to be used in a combination solution.

4.3 Theoretical Concerns

Further discussion is warranted with respect to some problems connected with the combination procedures discussed in the preceding sections.

Equation 7 has two approximations connected with it. First, the reference surface on which the anomalies are given is assumed spherical. LELGEMANN (1972) gives equations that show how corrections for this assumption can be made and that in fact such corrections are negligible at least up to degree 30. An analogous discussion may be found in OSTACH & PELLINEN (1966) who do not give numerical results. We also have assumed in equation 7 and also in equation 15 that our terrestrial anomalies (or the anomalies being solved for), refer to a spherical surface. In fact, the terrestrial anomalies we use refer to the surface of the Earth. Formally these anomalies may be reduced to a common spherical surface (e.g., the Bjerhammar sphere) using procedures discussed by BJERHAMMAR (1969). PELLINEN (1969) and OSTACH & PELLINEN (1970) indicate that the anomalies, used in formulas such as equation 7, should be $\Delta g + G_1$, where G_1 is the Molodensky correction term, which is near the mean terrain correction for the $d\sigma$ area. Thus, the anomaly to use in the combination studies is not the free air anomaly but rather the Faye anomaly which contains the terrain effect. The large size of the anomaly blocks used to date and the inaccuracy of the terrestrial data, has not warranted such corrections to data (RAPP 1970).

A related concern is often expressed with respect to the convergence at the surface of the Earth, of the spherical harmonic series expressed by equation 8. If the series diverges (which formally it does) some concern is then stated with respect to combination solutions made with terrestrial gravity data given at the Earth's surface. The concern is bypassed by the knowledge that we are not solving for an infinite set of potential coefficients, but rather for a finite set which are being determined from a finite data set.

PELLINEN (1965) also suggests that the G_1 correction be made to the anomalies used in equation 8.

At the current data accuracy level, and in the size of the anomaly blocks being used, and the highest degree to which potential coefficients are being determined, the theoretical refinements considered above are probably not important at this time. On the other hand, PELLINEN (1962) showed that the effect of neglecting the G_1 terms in the coefficient determinations from anomalies may lead to errors on the order of 15-20% for low degree coefficients. Since we purport to know these low degree coefficients to better than this relative accuracy, we need to look in more detail at the case of the G_1 terms in our terrestrial anomaly data. As smaller block sizes are used, with different types of satellite observation data, the precise definition of the boundary value problem as related to satellite and gravimetric data will be needed both in the theoretical and practical sense.One start in this area is the work of BAUSSUS VON LUETZOW (1972).

OSTACH & PELLINEN (1970) continue the discussion on the proper anomalies to be used in determining, in essence, potential coefficients. In this paper, a spherical harmonic expansion of the G_1 correction term is given. Such an expression could be used to determine a global, but smoothed representation of G_1, which in turn could be used to find the corresponding effect on the potential coefficients.

5. Results

In this section we will discuss some of the current (Nov 1973) solutions that are postulated to describe the Earth's gravitational field. Although results exist for both potential coefficient determinations and discrete block determinations, more discussion will be devoted to the former as this is the area in which extensive computational effort has been expended.

5.1 Potential Coefficient Results

We will consider four potential coefficient determinations computed from the combination of gravimetric and satellite data. These are:

A: 1969 Smithsonian Standard Earth II (GAPOSCHKIN & LAMBECK 1970; GAPOSCHKIN & LAMBECK 1971) (called SE II). This solution consisted of 316 (20 of which were fixed zonal coefficients) potential coefficients which are complete to degree 16 with additional zonal and resonance terms to degree 22. The combination procedure was carried out basically using equation 9 with equation 8, solving simultaneously for the geocentric co-ordinates of 46 stations. The terrestrial gravity data used consisted of 935, 5° equal area anomalies used by KAULA (1966b), with the empty areas estimated from a linear regression procedure applied to the 935 anomalies.

B: Smithsonian Institution Standard Earth III (GAPOSCHKIN 1973). This solution consisted of 386 potential coefficients which are complete to degree 18 with additional zonal coefficients to degree 36 and resonance terms to degree 23. The combination procedure was carried out as with SE II but with an updated anomaly set, setting anomalies in the empty areas to zero. In addition, no partial derivatives of the anomalies with respect to the zonal harmonics or tesseral harmonics less than the 9th degree were computed. The justification for this was that such derivatives are negligibly small. The results of RAPP (1969b) whereby potential coefficients were found solely from terrestrial data do not support this justification.

C: GEM6. The GEM6 solution is the combination solution made with the pure satellite solution GEM5. These new models are updated models of the GEM1 and GEM2 solutions (LERCH ET AL 1972b), and the GEM3 and GEM4 solutions (LERCH ET AL 1972a). The GEM6 solution consists of 337 potential coefficients and the geocentric co-ordinates of 134 tracking stations. The coefficients are complete to degree 16 with additional terms to degree 22. The combination procedure was carried out using essentially equations 8 and 9. The anomalies used for the combination solution were those used by RAPP (1972; 1973a).

D: RAPP (1973a). This solution is a combination solution made with equation 11 using the least squares collocation principle. This model, which started from the GEM3 potential coefficient set, contains 449 coefficients being complete to degree 20, with additional terms to degree 22. The gravity data used in this solution was based on 23,355 $1° \times 1°$ anomalies processed using equations 3 and 4 to determine 1283, 5° equal area anomalies and their standard deviations (RAPP 1972). The remaining blocks were filled up using model anomalies based on topographic isostatic information (UOTILA 1964). In addition to the potential coefficients, this solution provided a set of 1654, 5° equal area anomalies adjusted, through equation 6 to be consistent with the adjusted potential coefficient set.

5.2 Potential Coefficient Comparisons

5.21 Potential Coefficient Differences

In this section we will compare the four potential coefficient sets in several ways. In table 1, we give the root mean square difference (Δ), the average percentage difference ($\overline{\%}$), the root mean square undulation (ΔN) and the anomaly difference (δg), when the coefficient sets are compared in terms of common coefficients. In table 2, we give the root mean square undulation and anomaly difference between the solutions when all the coefficients of a solution are considered. To estimate the maximum undulation or anomaly difference between a solution, multiply the RMS difference by four.

Table 1

Root Mean Square Potential Coefficient, Undulation and Anomaly Differences, for Various Potential Coefficient Sets

	SE II				SE III				GEM 6			
	$\Delta\times10^6$	$\overline{\%}$	RMS ΔN (m)	RMS δg (mgal)	$\Delta\times10^6$	$\overline{\%}$	RMS ΔN (m)	RMS δg (mgal)	$\Delta\times10^6$	$\overline{\%}$	RMS ΔN (m)	RMS δg (mgal)
SEII	--	--	--	--	0.066	81	7.5	10.6	0.049	90	5.5	9.3
SEIII	0.066	81	7.5	10.6	--	--	--	--	0.056	80	6.4	8.6
GEM6	0.049	90	5.5	9.3	0.056	80	6.4	8.6	--	--	--	--
Rapp	0.049	62	5.5	9.2	0.058	72	7.2	10.0	0.026	49	3.0	5.3

Table 2

Root Mean Square Undulation and Anomaly Differences Between Complete Potential Coefficient Sets

	SE II		SE III		GEM 6	
	RMS ΔN (m)	RMS δg (mgal)	RMS ΔN (m)	RMS δg (mgal)	RMS ΔN (m)	RMS δg (mgal)
SEII	--	--	7.7	11.4	5.5	9.4
SEIII*	7.7	11.4	--	--	6.6	9.6
GEM6	5.5	9.4	6.6	9.6	--	--
Rapp	5.9	10.8	7.4	11.1	3.7	7.7

* SEIII to ℓ = 23 only

5.22 Anomaly Degree Variance Comparisons

The potential coefficients may be converted to anomaly degree variances using the following:

$$\sigma_\ell^2(\Delta g) = \gamma^2(\ell - 1)^2 \sum(\overline{C}_{\ell m}^2 + \overline{S}_{\ell m}^2) \quad (18),$$

where the $\overline{C}_{20}$ and $\overline{C}_{40}$ are referred to an ellipsoid of a specified flattening. Values of the anomaly degree variances as computed from the potential coefficient sets described in section 5.1 are given in table 3 along with the anomaly degree variances computed from gravity data alone (RAPP 1972) using the procedures described by KAULA (1966b).

Table 3
Anomaly Degree Variances†
(mgal)^2

ℓ	SEII	SEIII	GEM6	Rapp	Gravity*
2	7.4	7.2	7.5	7.5	12.7
3	33.0	33.5	33.7	33.9	31.3
4	20.0	20.4	19.3	19.2	13.6
5	17.8	25.9	21.7	21.6	15.1
6	15.7	18.2	20.1	18.9	19.9
7	15.5	19.3	17.5	18.8	15.5
8	6.7	19.6	8.4	10.4	7.5
9	12.7	11.7	8.8	11.1	16.1
10	12.9	11.0	11.4	11.4	9.6
11	12.2	10.0	7.7	8.4	10.8
12	5.1	9.1	4.2	4.8	3.9
13	11.1	9.3	10.7	11.7	8.2
14	8.4	8.1	5.9	5.5	8.3
15	13.2	8.3	8.6	7.3	8.6
16	13.8	10.1	6.8	6.5	8.7
17		9.9		5.7	8.6
18		8.7		10.7	9.6
19				11.0	7.4
20				8.9	6.8

* RAPP (1972) † Reference Flattening 1/298.256

The biggest difference between the SEII and the SEIII occur at ℓ = 5 and 8 where the SEIII values are 8.1 and 12.9 mgal^2 higher than the SEII values. For these two cases the SEII values agree better with the gravimetrically derived anomaly degree variances than the values obtained from the SEIII. The lower value at ℓ = 8 also occurs in the GEM6 and Rapp solutions.

5.23 *Astro-Geodetic Undulation Comparisons*

A set of potential coefficients can be used to derive a set of geoid undulations which may be compared to astro-geodetic undulations after an appropriate transformation. The agreement of the transformed undulations with the astro-geodetic undulations (as judged by the root mean square undulations differ difference after the transformation) can be used to infer the value of the potential coefficients in describing geoid undulations.

Results for such comparisons have been described by RAPP (1973a) for all solutions given here except for the SEIII which is reported here along with the other values. These comparisons have been made on the North American Datum with 3112 points and on the Australian Datum with 1084 points. The root mean square differences are given in table 4. Considering the information from both datums, the best agreement is found with the GEM6 coefficients with the poorest agreement found from the SEIII set.

Table 4

RMS Difference (After Adjustment) Between Astro-geodetic Undulations and Undulations Computed from Potential Coefficients (metres)

	North American Datum	Australian Datum
SEII	± 5.2	± 2.9
SEIII	± 6.1	± 2.6
GEM6	± 3.9	± 2.2
Rapp	± 4.4	± 2.0

5.24 Anomaly Comparisons

Anomalies may be computed from potential coefficients as seen from equation 8. These anomalies may then be compared to the actual terrestrial anomalies to consider their agreement. KAULA (1966a) gave procedures for such comparisons when the potential coefficients were determined solely from satellite data. These comparisons have also been made with potential coefficients derived from combination solutions. The analysis of these comparisons must be balanced against the fact that the anomalies against which comparisons are being made have usually been used in the combination solution being judged. Thus, the comparisons may not reveal absolute truth, but rather relative truth. In this section we carry out these comparisons with the four potential coefficient sets of section 5.1, recognizing that 2 out of 4 sets used different gravimetric data in the combination solution. The terrestrial anomalies used for the comparisons are the 927 values given by RAPP (1972) (with the anomalies in the Canadian area corrected by -2 mgal), having standard deviations less than or equal to ±10 mgal. Other subsets of the terrestrial field were tested, but the results to be reported here are representative.

The comparison terms are as follows:

$E\left((g_T - g_S)^2\right)$: the mean square difference between terrestrial anomalies (g_T) and those derived from potential coefficients (g_S);

$E\left(\varepsilon_S^2\right)$: mean square value of the error associated with the potential coefficients;

$E\left(\delta g^2\right)$: mean square value of neglected higher order terms in the computation of g_S; and

$E\left(\delta g_H^2\right)$: mean square effect of the true contribution to g_S from the potential coefficients

The above values have been computed for two subsets (to ℓ = 12, and 16) of each potential coefficient set as well as the complete set. The results are given in table 5. The results of the comparison made to ℓ = 12 indicate that the SEIII coefficients are less accurate than the other three sets. The agreement with the terrestrial anomalies is about the same in the SEII and SEIII, and the GEM6 and Rapp solutions.

The results of the comparisons made to ℓ = 16 indicate that the SEII coefficient set is less accurate than the other three sets all of which are about the same accuracy. The comparison of the complete coefficient sets indicates the SEII is again poorer than the other sets. The Rapp set shows a slightly better agreement with the terrestrial field but this is due to the inclusion of more potential coefficients in its solution than are found in the other solutions. In all cases, the

Table 5

Comparison of Anomalies Derived from Potential Coefficients with Terrestrial Anomalies $(\mathrm{mgal})^2$

	To ℓ = 12				To ℓ = 16				Complete			
	$E((g_T-g_S)^2)$	$E(\varepsilon_S^2)$	$E(\delta g^2)$	$E(g_H^2)$	$E((g_T-g_S)^2)$	$E(\varepsilon_S^2)$	$E(\delta g^2)$	$E(g_H^2)$	$E((g_T-g_S)^2)$	$E(\varepsilon_S^2)$	$E(\delta g^2)$	$E(g_H^2)$
SEII	174	28	110	160	177	57	84	186	182	66	80	187
SEIII*	175	45	94	176	148	42	70	200	141	48	56	213
GEM6	154	25	93	177	138	31	71	199	143	38	69	201
Rapp	164	34	95	175	146	39	71	199	133	47	50	220

* Complete to ℓ = 23

GEM6 coefficients appear to be the most accurate as judged by $E(\varepsilon_S^2)$.

5.25 Orbit Fitting

A set of potential coefficients can be used in the representation of the Earth's gravitational field for satellite orbit computations. In the fitting of an orbit to the observations, the fit (as judged by the root mean square residual after adjustment) is a measure of how well the gravitational field is represented by the set of coefficients being tested. For complete testing of a set of coefficients, orbits not used in the original estimation of the potential coefficient sets should be used. In addition, the testing should be over as wide of range (in terms of inclinations, heights, etc) of orbits as possible. For this paper, however, I have the results (table 6) of orbit fitting for a single Geos 1 arc using laser data in a seven day arc.

Table 6

Root Mean Square Orbit Fit (metres)

Solution	Fit
SEII	± 4.8
SEIII	7.9
GEM6	7.2
Rapp	5.0

The poorest fit is found with the SEIII coefficients with little difference between the SEII and the Rapp sets. More extensive testing in this area needs to be carried out similar to what was done by MARSH & DOUGLAS (1970) and WAGNER (1972).

5.26 Undulation and Anomaly Maps

Potential coefficients can be converted to geoid undulations through the following equations which are spherical approximations:

$$N = R \sum_{\ell=2}^{\ell_{max}} \sum_{m=0}^{\ell} \left(\overline{C}^*_{\ell m} \cos m\lambda + \overline{S}_{\ell m} \sin m\lambda\right) \overline{P}_{\ell m}(\sin \overline{\phi}) \qquad (19),$$

and

$$\Delta g = \gamma \sum_{\ell=2}^{\ell_{max}} (\ell - 1) \sum_{m=0}^{\ell} \left(\overline{C}^{*}_{\ell m} \cos m\lambda + \overline{S}_{\ell m} \sin m\lambda\right) \overline{P}_{\ell m}(\sin \overline{\phi}) \quad (20),$$

where $\overline{C}^{*}_{\ell m}$ are the differences between the observed values and those values implied by an equipotential ellipsoid of a defined flattening. As mentioned in section 4.1, the differences involving $\overline{C}_{20}$ and $\overline{C}_{40}$ are the only ones usually considered. The ellipsoid to which the N and Δg values refer is a mean Earth ellipsoid (see HEISKANEN & MORITZ 1967, section 2-20 & 2-21). R is a mean Earth radius, and γ is an average value of gravity over the Earth. Values of N and Δg computed through equations 19 and 20 with the coefficients of RAPP (1973b) referred to a flattening of 1/298.256 are given in figures 2 and 3. To demonstrate some of the differences between the various solutions, figure 4 gives the anomaly differences between the RAPP (1973a) solution and the SEII solution. The maximum anomaly difference is 40 mgal occurring in the south Pacific area. This area is represented in more detail in figure 5 where anomalies in the block -10° to -40° latitude, and 190° to 235° east longitude are given. The anomalies have been computed for the SEII, SEIII, GEM5, GEM6 and the Rapp potential coefficient solutions from an integration over 5° equal area blocks strating from equation 8. In addition, we give the terrestrial anomalies (and their standard deviations) where they existed in the area. The GEM5 solution, which is based on satellite data alone, shows a smooth anomaly field in this area where the maximum anomaly range was 16 mgal. The SEII solution and the SEIII solution have a high and a low in this area with the maximum anomaly variation reduced by 19 mgal in the SEIII solution relative to the SEII. No observed gravity data was used in this area in the SEII. What gravity material was used in this area for the SEIII is not known. The Rapp solution also shows several maxima and minima, but not to the extent found in the other solutions. Additional gravity material is needed to verify the actual existence of the highs and lows in this area, or to see if, in fact, they are simply the result of a distortion introduced into the area by the adjustment procedure.

5.26 Summary

Current potential coefficients found from a combination of gravimetric and satellite data differ because of the satellite data used, the gravity data used, the combination procedure, and the degree at which the coefficients have been truncated. It is difficult to define what is the best coefficient set, although some of the comparisons made in the preceding sections indicate that one or more of the sets tested may be better suited for a specific purpose than one of the others.

5.3 Discrete Block Solutions

Gravitational field solutions using discrete blocks have been reported by ARNOLD (1972), KOCH & MORRISON (1970), KOCH (1970), KOCH & WITTE (1971), and RAPP (1973b). Arnold estimated 52, 20° × 20° mean anomalies out of 101 values on the Earth, assuming 49 anomalies as perfectly known, using 1182 observation equations based on the observation of six satellites. Koch & Morrison solved for 48, 30° × 30° surface density values based on 8692 optical observations of five satellites and a set of surface density values determined from terrestrial gravity material. They give potential coefficients to degree 8, based on a satellite solution and on the combined solution. KOCH (1970) used the 48 block satellite solution in conjunction with more detailed gravity material to estimate a 192 block solution which must be dominated by the gravity material since only the 30° × 30° blocks from the satellite solution were used.

KOCH & WITTE (1971) reported a satellite alone solution for 104, 20° surface elements computed using Doppler data from five satellites. The potential coefficients derived from their solution implied

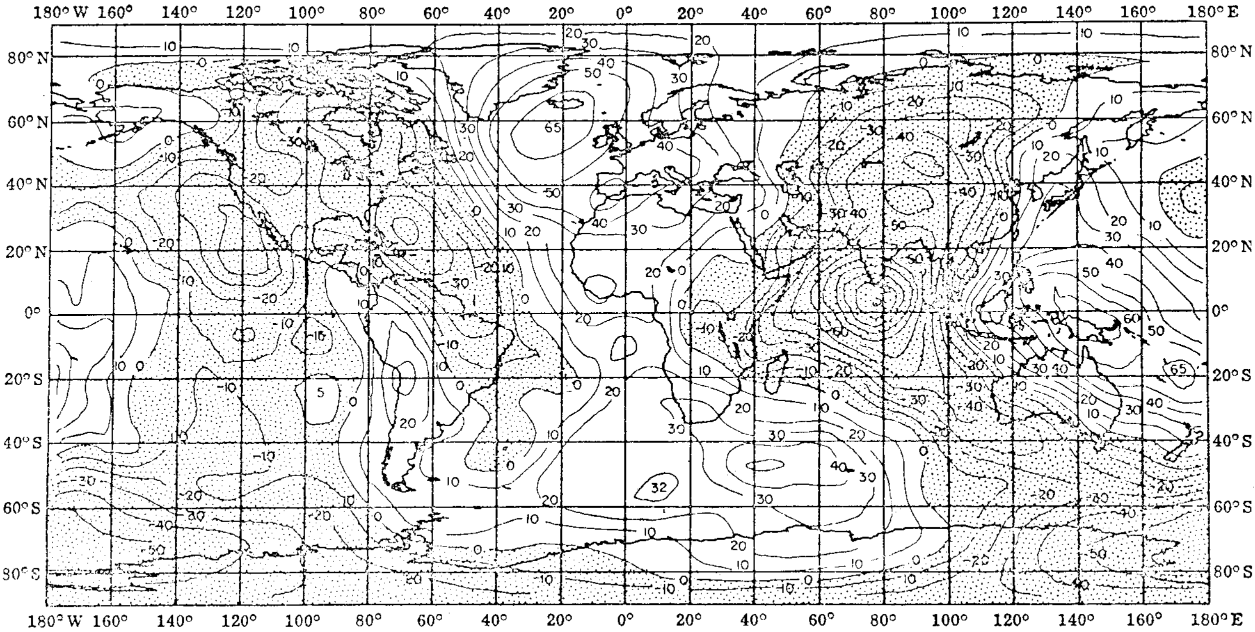

Figure 2. Geoid Undulations from Rapp Coefficients
Contour Interval: 10 m ; Reference Flattening = 1/298.256

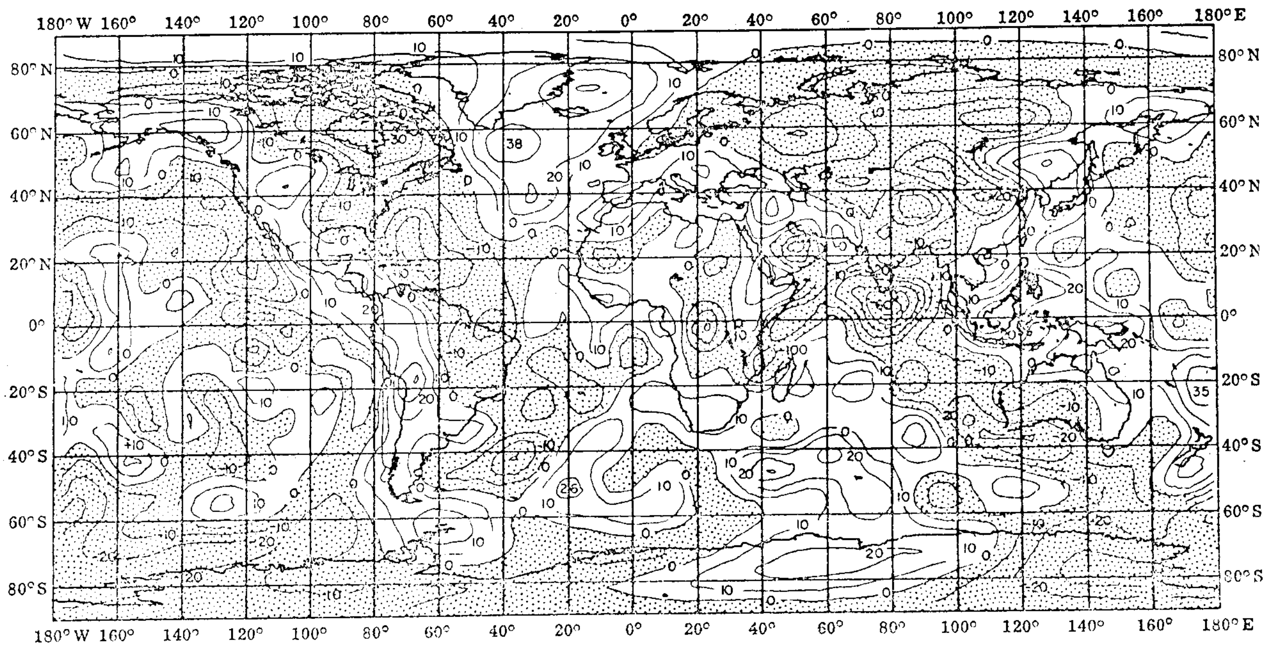

Figure 3. Anomalies from Rapp Coefficients
Contour Interval: 10 mgal; Reference Flattening = 1/298.256

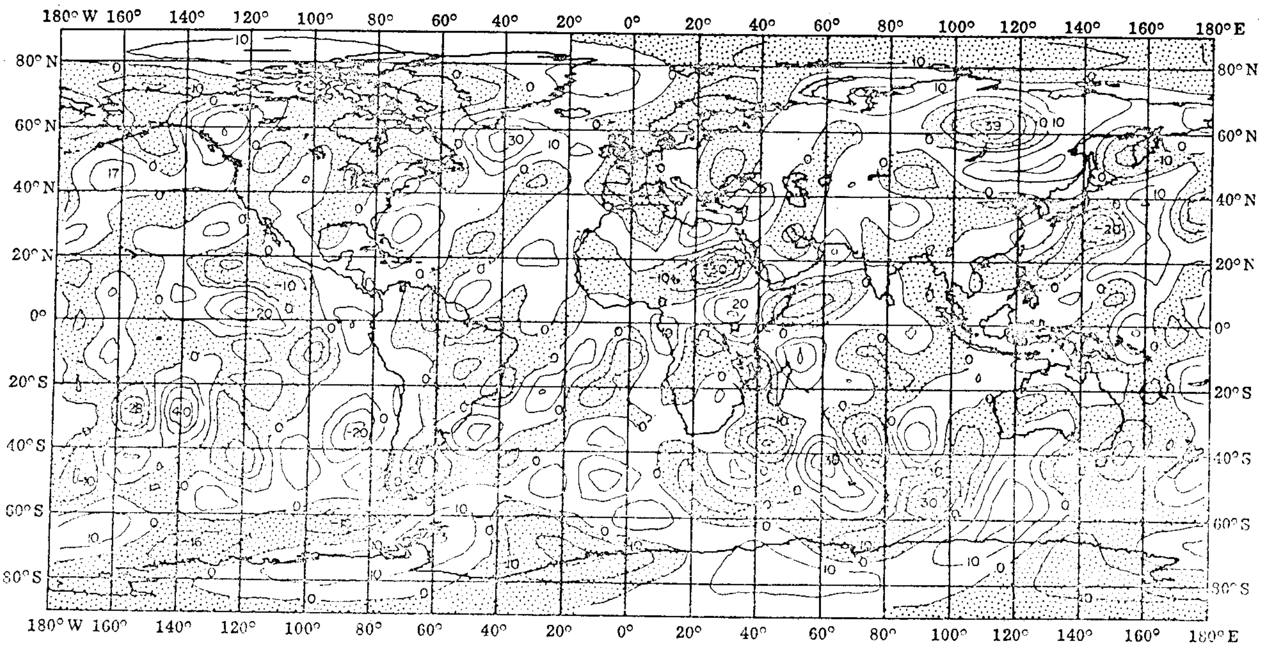

Figure 4. Anomaly Differences: Rapp Solution minus SEII solution
Contour Interval: 10 mgal; Maximum Difference = 40 mgal;
RMS Difference = 10.8 mgal

anomaly degree variances much larger than found by other investigations (at least up to degree 11) indicating that their solution has distorted potential coefficients.

RAPP (1973b) using the generalized Stokes equation (e.g., see equation 15), solved for 184, 15° equal area anomalies, and selected station co-ordinates. A satellite alone, and a combination solution were made. The satellite solution processed 17,651 optical observations from ten satellites in 29, 5 to 7 day arcs. The anomalies and their standard deviations for the 29 arc combination solution are shown in figure 6.

As with the potential coefficient combination solution, the combination solutions of a least squares nature, require some decision on a relative weighting between the two data types. This is needed as the satellite alone solution yields standard deviations for the unknown anomalies of unrealistically small value. For example, the standard deviation of the anomalies found from the 29 arc satellite solution average ±2 mgal where comparisons of the solution anomalies to well known terrestrial anomalies indicated a realistic standard deviation to be on the order of ±6 mgal. Thus, a scaling factor was applied to the satellite normal equations, before the combination solution was made, to yield a more realistic weighting scheme between the satellite and terrestrial information.

The potential coefficients implied by the anomalies found in the RAPP (1973b) solution agreed fairly well with potential coefficients derived from the more standard type of analysis. In addition, the anomaly degree variances computed from the potential coefficients agreed well with those computed from other sources.

The results obtained from current discrete block solutions represent sets of methods rather than

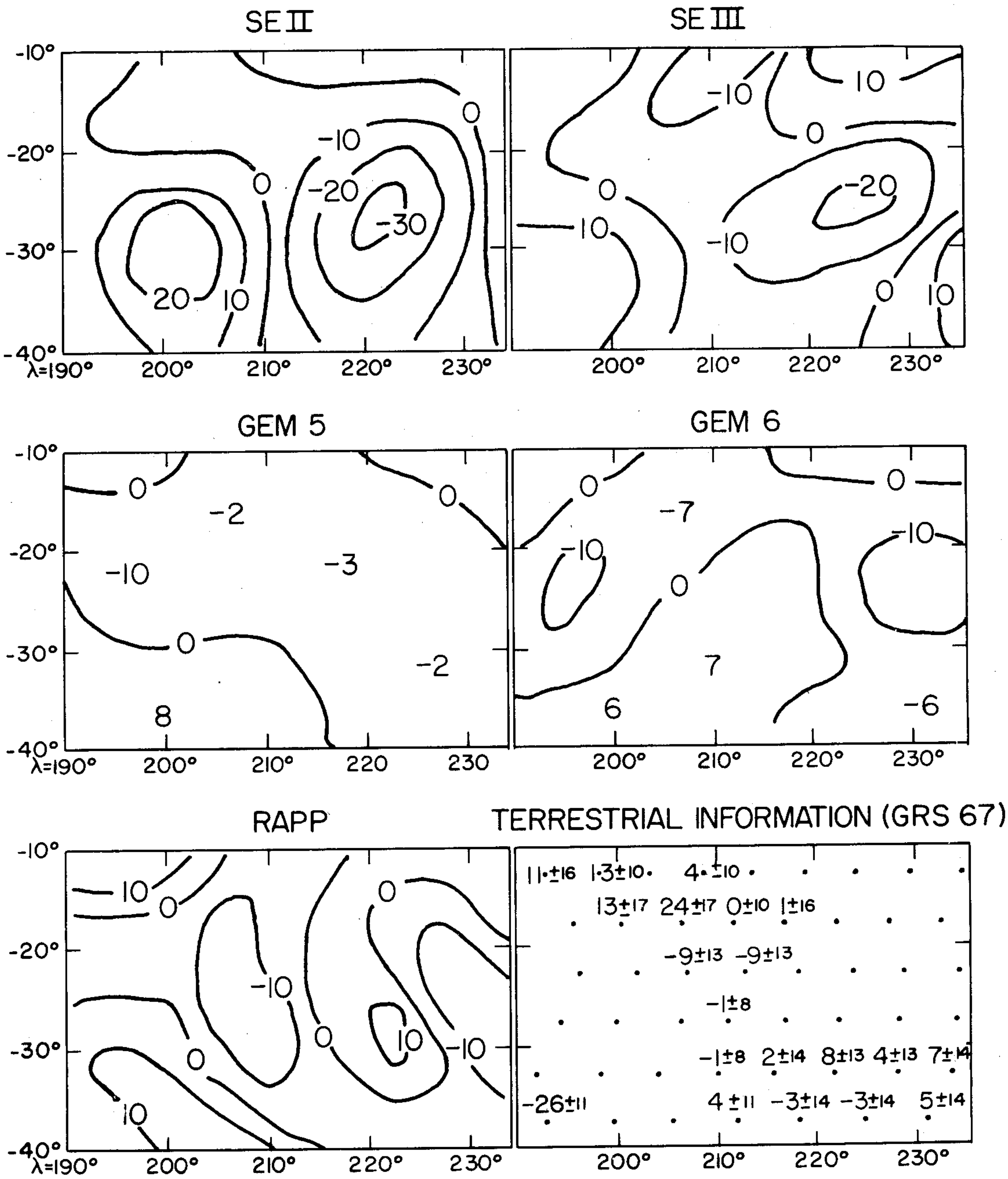

Figure 5. Anomalies in a South Pacific Area

results that are to be regarded as most current representations of the gravitational field. This is because the amount of satellite data analysed in the discrete block solutions is considerably less than that used in such solutions as the SEII, SEIII, or GEM6.

6. The Representation of the Earth's Gravitational Field

The combination of satellite and gravimetric data in the future will face the use of new observation

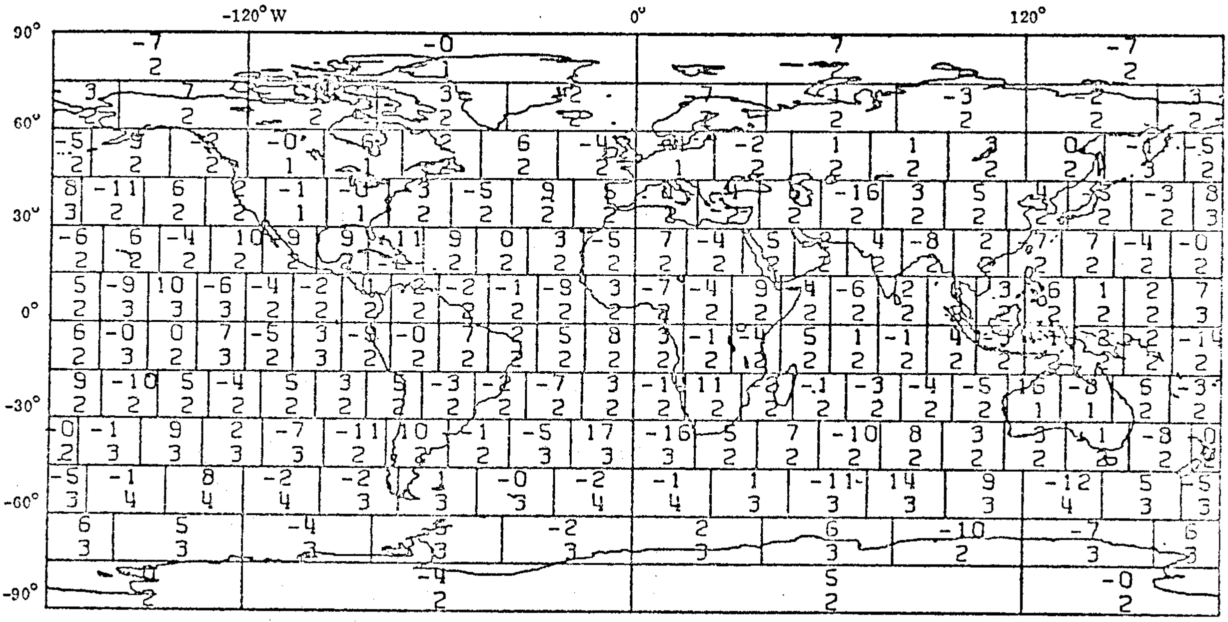

Figure 6. Anomalies (Upper Figure) and their Standard Deviations From 29 Arc Combination Solution (RAPP 1973b) - (mgal)

data obtained from satellites. The two new data types that will be available are satellite altimetry and satellite-to-satellite tracking. In using this data, and of course, existing satellite data, the proper way in which to represent the Earth's gravitational field must be considered. In addition, it may be that a global gravity field will not be sought in one given solution. Rather, solutions for the gravity field in local or regional areas could be found with the global field constructed from regional sets.

The representation problem is a rather important one at this time with a good deal of effort being placed on trying to define a system that can define the gravity field in an appropriate manner. In addition, to the potential coefficient and discrete block (surface densities or gravity anomalies) representative investigations have been carried out with point masses (BALMINO 1972; NEEDHAM 1970); sampling functions (LUNDQUIST & GIACAGLIA 1972; LUNDQUIST, GIACAGLIA & GAY 1973); or some analytic description of the anomaly field through some power series (STRANGE 1972), or through "regional functions" (DUFOUR & KOVALESKY 1970), or through the disturbing potential being represented in an analytic way, by a surface density layer distributed over a sphere bounding the Earth (VINTI 1971).

For future work several of the above systems will undoubtedly be used to describe the local part of the Earth's gravitational field. What procedure is best is not clear at this time. If we wish to easily incorporate existing terrestrial gravity material into a combination solution with a new type of representation; if we wish a function or quantity that can theoretically be related to the gravimetric boundary value problem, so that the external gravitational field as well as the field on the Earth's surface can be defined, and if we wish to be able to geophysically interpret the parameters of the local representation, it would appear that discrete blocks of gravity anomalies referred to some high degree (e.g., ℓ = 12 or 20) potential coefficient field may be the appropriate form of representation desired.

7. Summary

This paper has attempted to outline some of the procedures, problems and results that are found in the combination of satellite and terrestrial gravity material. Details have been relegated to the references where possible. We have now just about passed the era of the broad wavelength determination of the Earth's gravitational field. Although current solutions do not agree in detail, the broad character is well defined.

If detailed gravity material were available on a global basis, we could consider the gravity field determined. Such is not the case because of unsurveyed ocean areas and land areas for which data is not available. Gravity material in the ocean areas is increasing but complete coverage cannot be expected for many years, and in some areas perhaps not in the foreseeable future. Consequently, the need for satellite systems that can refine our knowledge of the Earth's gravity field. The use of this new data could, in the limit, provide a global gravity coverage in 2° × 2° blocks on the surface of the Earth. The future accuracy of this field is not clear although the studies of SCHWARZ (1972) and REED (1973) indicate the possibility of obtaining anomalies to a standard deviation on the order of 1 to 3 mgal. Such results on a global basis can be used to satisfy almost all but the most local requirements for our knowledge of the Earth's gravitational field.

8. Acknowledgment

The work presented here was sponsored, in part, by the Air Force Cambridge Research Laboratories under Contract No. F 19628-72-C-0120, and by the National Aeronautics and Space Administration (Goddard Space Flight Center) under research grant no. NGR 36-008-161. Jim Marsh of GSFC provided the information for table 6. Some computer support was received through the Instruction and Research Computer Center of the Ohio State University.

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10. Discussion

TAPLEY: What technique of weighting did you use to combine your satellite data with surface data?

RAPP: Are you referring to the block data or the geopotential data?

TAPLEY: The original geopotential data.

RAPP: There was not a unique weighting factor applied to the whole system. A systematic analysis of every single potential coefficient was gone through to estimate uniquely for each potential coefficient, a standard deviation. When those standard deviations were used, we found that it was not possible to get good orbit fits. So we tightened up the standard deviations on the a priori potential coefficients from satellites. On doing so, we got good orbit fits. A whole series of solutions were obtained. We finally picked the one which would give good orbit fits but did not distort terrestrial gravity data significantly.

TAPLEY: The second question is on your single orbit fits for laser tracking data. Which tracking station locations were used?

RAPP: The tracking station locations used were those obtained by Marsh, Douglas & Klosko using the GEODYN program.

DUNN: For a forty degree inclination satellite, what gravity model would you go for?

RAPP: I don't know. If I were to say "My own solution", I would be said to be biased. GEM 6 looks quite good.

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Proc.Symposium on Earth's Gravitational Field & Secular Variations in Position (1973),76-92.

EFFICIENCY OF METHODS FOR IMPROVING THE PRESENTLY AVAILABLE INFORMATION ON THE EARTH'S GRAVITY FIELD

ABSTRACT

Test computations and statistical considerations are used in estimating the possible improvement of geoid and gravity field computation based on satellite orbit analysis. Satellite altimetry and gradient measurements are mainly dealt with.

New methods and techniques for improving the present knowledge of the Earth's figure and gravity field have been discussed more or less in detail, e.g.WILLIAMSTOWN REPORT (1969). Altimetry from artificial satellites (GREENE 1972; STANLEY ET AL 1972) and, to some extent, gravity gradient observations (FORWARD 1972) might be of primary importance in this connection besides satellite-to-satellite techniques which will not be dealt with in the present paper. The full exploitation of such new methods will, of course, lead to improved computational methods and theoretical procedures. Until now, satellite altimetry was mainly considered in connection with other methods, leading to new types of "combination solutions"; see, for instance RAPP (1971). A somewhat different approach in altimetry has been proposed by ARNOLD (1972). The latter paper leads to the question to what extent can geoid undulations be directly predicted.

A similar question could be raised in considering the present knowledge of the gravity field. Ten years ago, the prediction of gravity in un-surveyed areas was a major topic in physical geodesy; meanwhile large scale information is obtained, e.g., from satellite orbit analysis. In many areas where detailed geoid heights are to be computed, detailed knowledge in the neighbourhood zones is available. Because of

$$-\Delta g = \frac{\partial T}{\partial n} - \frac{1}{\gamma}\frac{\partial \gamma}{\partial n} T \qquad (1),$$

where T = disturbing potential;
γ = normal gravity;
n = normal direction; and
Δg = gravity anomaly,
the prediction of T and the geoid undulation

$$N = T/\gamma \qquad (2)$$

is, in general, simpler than the Δg-prediction. For the prediction of any quantity is in most cases easier than the prediction of its derivatives. The discrepancy, in general,is not so big at sea where altimetry is relevant as on land, but even when dealing with mean values or smoothed data, T-prediction is remarkably advantageous.

In order to get information on the statistical parameters of N, we have analyzed the global gravimetric geoid by VINCENT & MARSH (1973); as this geoid has been basically evaluated from $1^\circ \times 1^\circ$ mean free air anomalies, it might be used, within its reliability, to study the spectrum of the gravity field up to degree $n \doteq 180$. However, as the corresponding gravity material is certainly affected by bias, spectral methods might lead to perturbed results, with severe perturbations in areas where biased gravity material has been used. On the other hand, the direct evaluation of statistical parameters of T and N from those of Δg by simple integral transform is not basically superior.

Accuracy investigations for the zone of geographic longitudes λ and latitudes ϕ

$$20^\circ \leq \phi \leq 80^\circ \quad \text{(north)} \qquad ; \qquad 200^\circ \geq \lambda \geq 50^\circ \quad \text{(east)},$$

and specifically for continental areas within this zone, indicate sufficient reliability for determining statistical parameters. A comparison has, for instance, been done with a very local geoid section (GROTEN & RUMMEL 1973) in central Europe. Figures 1 to 3 show the latter geoid section where regional gravity data have been combined with the latest coefficients model by RAPP (1973a), the SAO Standard Earth III and the Goddard Earth Model (GEM) 5. When N_o = -17.0 m is added to the numbers given in these figures, they are then related to the 1967-reference ellipsoid adopted at Lucerne where, however, Rapp's latest set of constants (GM, a, ω,...) was used in evaluating N_o. For details, see (GROTEN & RUMMEL 1973).

Under the assumption of isotropy and stationarity, the autocovariance functions within the above given area were evaluated for the global geoid, and for the local geoid within the zone shown in figures 1 to 3. In eliminating the trend, simply the effect of the outer zones corresponding to the above mentioned (global) coefficient sets has been omitted. Even though the local shape of the geoid section changes remarkably with changes in the outer zones, corresponding variations in the statistical parameters are small.

In addition, assuming stationarity, autocorrelation functions along meridians (λ = const) and latitude circles (ϕ = const) have been studied. Even when the samples are relatively small, they indicate systematic changes in cov(N) for profiles of constant latitude which are not fully explained by the heterogeneity of Δg-material. Figure 5 shows a few examples for ϕ = const; more details will be given in a forthcoming paper (GROTEN 1974). Figures 6 to 9 show a few examples for the autocorrelation along meridians λ = const. As these data are basically obtained from $1^\circ \times 1^\circ$ mean gravity values, the corresponding N-values are, of course, smoothed data. But on comparing the results for the local geoid with those for the global section it is seen that the tendency in very high harmonics seems to corroborate earlier assumptions by Kaula and specifically those by RAPP (1972).

On applying the well known linear autoregression prediction formulas to the local geoid section, it was found that within $d \doteq 20$ km or so, N can be predicted along profiles with a relative error of about 5 to 10 %. For the global geoid, the same accuracy is obtained within $d \doteq 50$ km. These accuracy estimates were evaluated by directly comparing "observed" with predicted values.

Unless collocation procedures are applied from the very beginning in satellite altimetry, it might be useful to extend the geoid obtained along altimeter profiles to "unsurveyed" areas.

In order to get a few hints on the very high harmonics we compared the local geoid section with a small geoid part obtained from astro-geodetic* and torsion balance measurements within the same area. It came out that even though the gravimetric geoid section was basically obtained from mean Δg-values for $6' \times 10'$ blocks, the corresponding loss of information is quite small. Such a comparison is, of course, always affected by local bias. Part of it is removed by topographic and terrain correction;

* Heitz-geoid as transformed by GROTEN (1970)

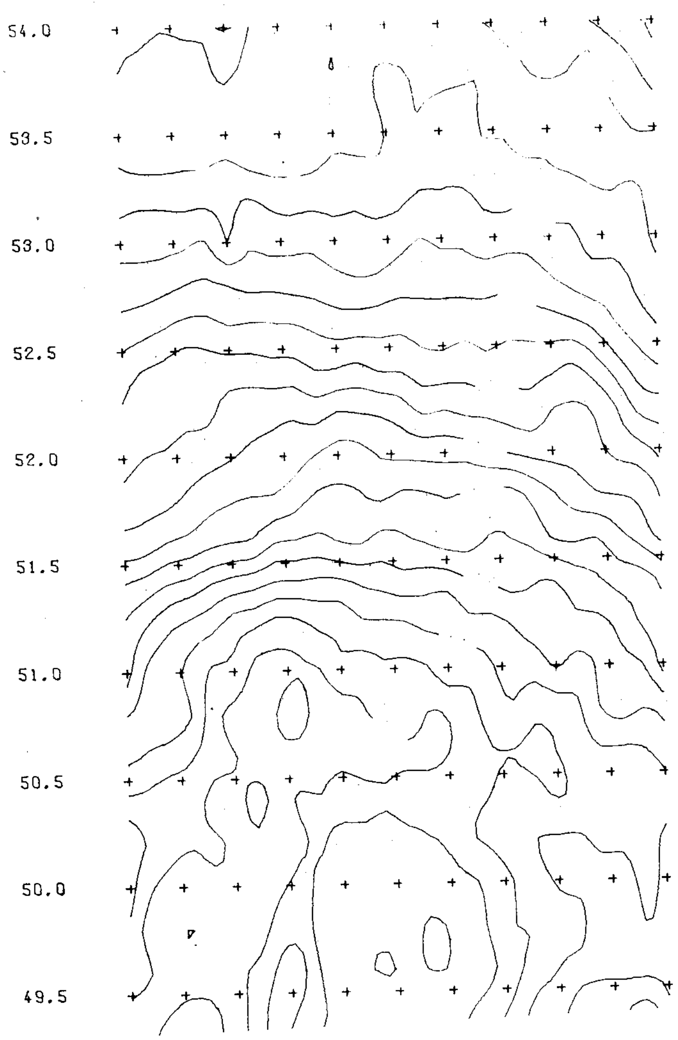

Figure 1.
Geoid Section at
Central Europe
from
Regional Gravity
and
Rapp's Coefficients

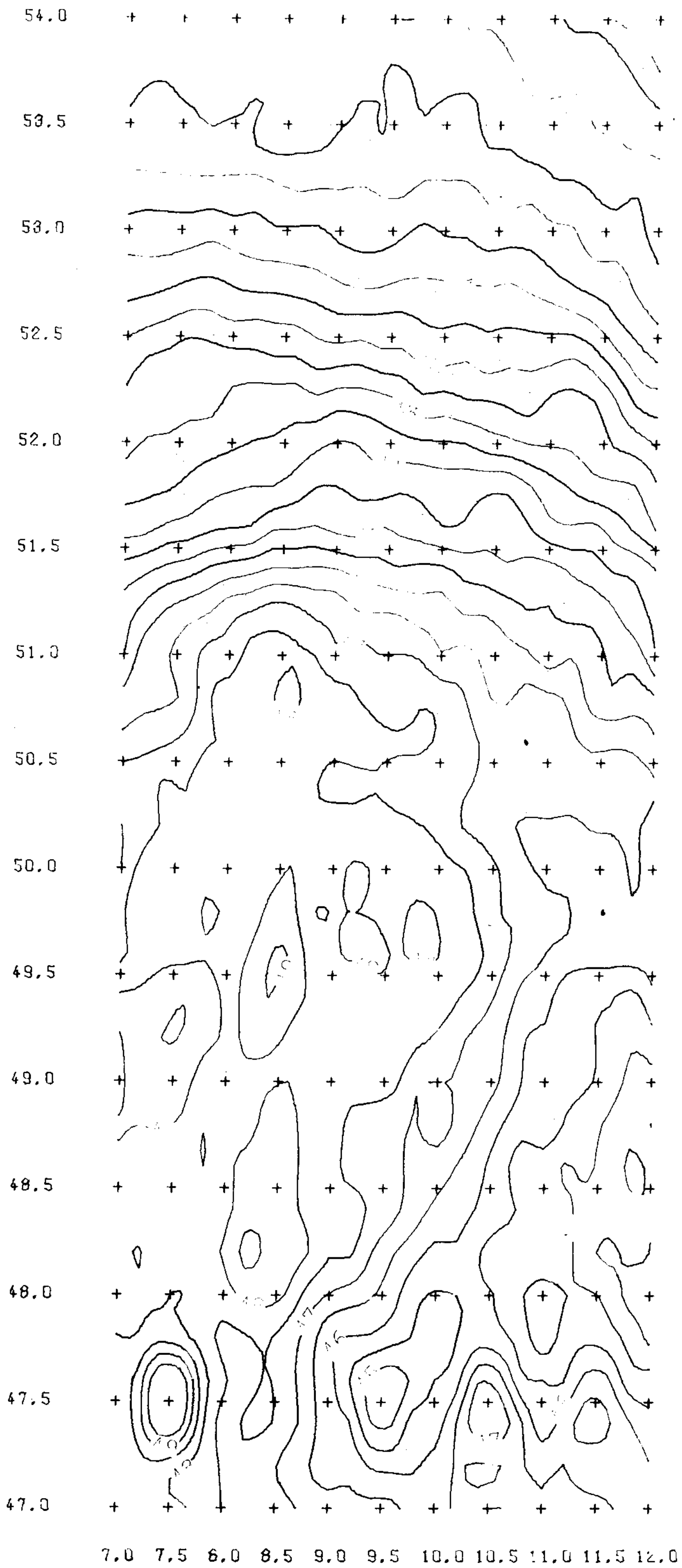

Figure 2.
Geoid Section at
Central Europe
from
Regional Gravity
and
SAO SEIII

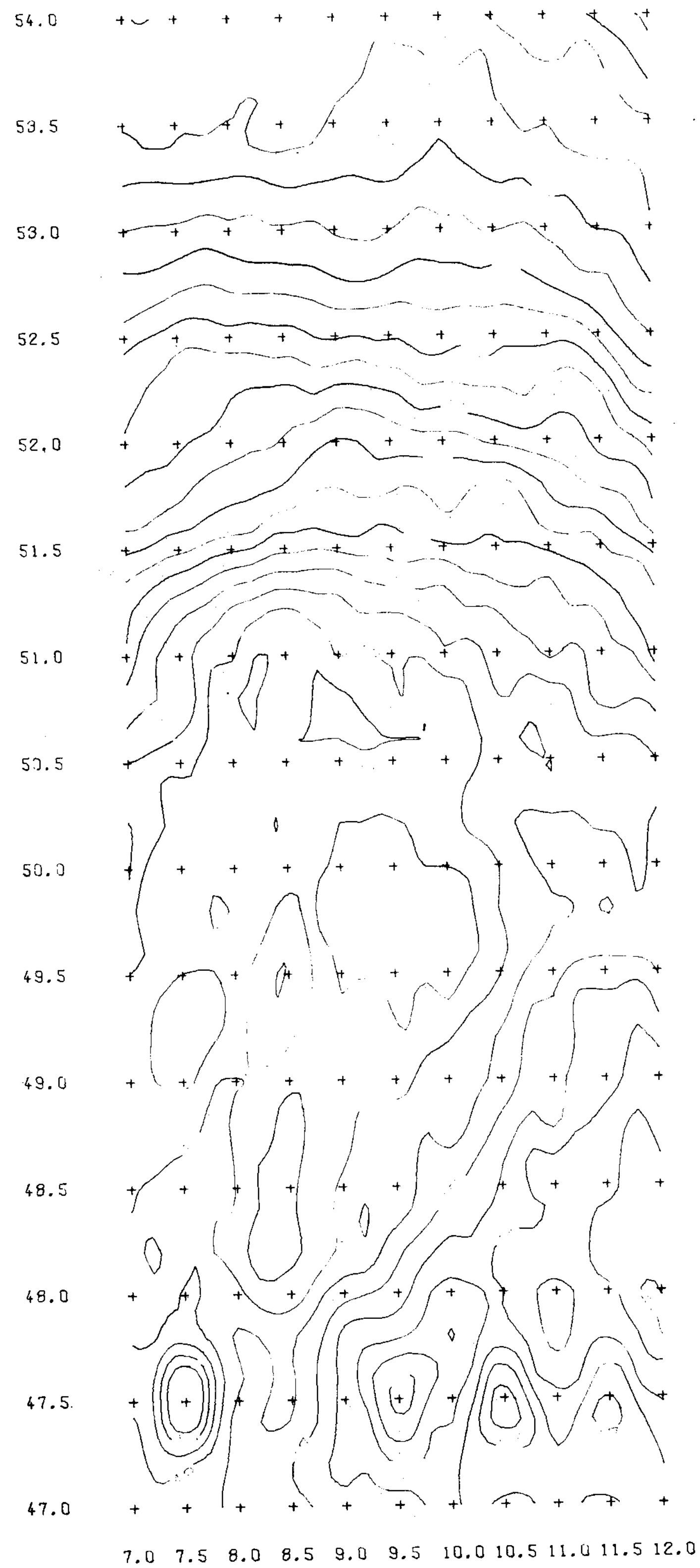

Figure 3.
Geoid Section at
Central Europe
from
Regional Gravity
and
GEM 5

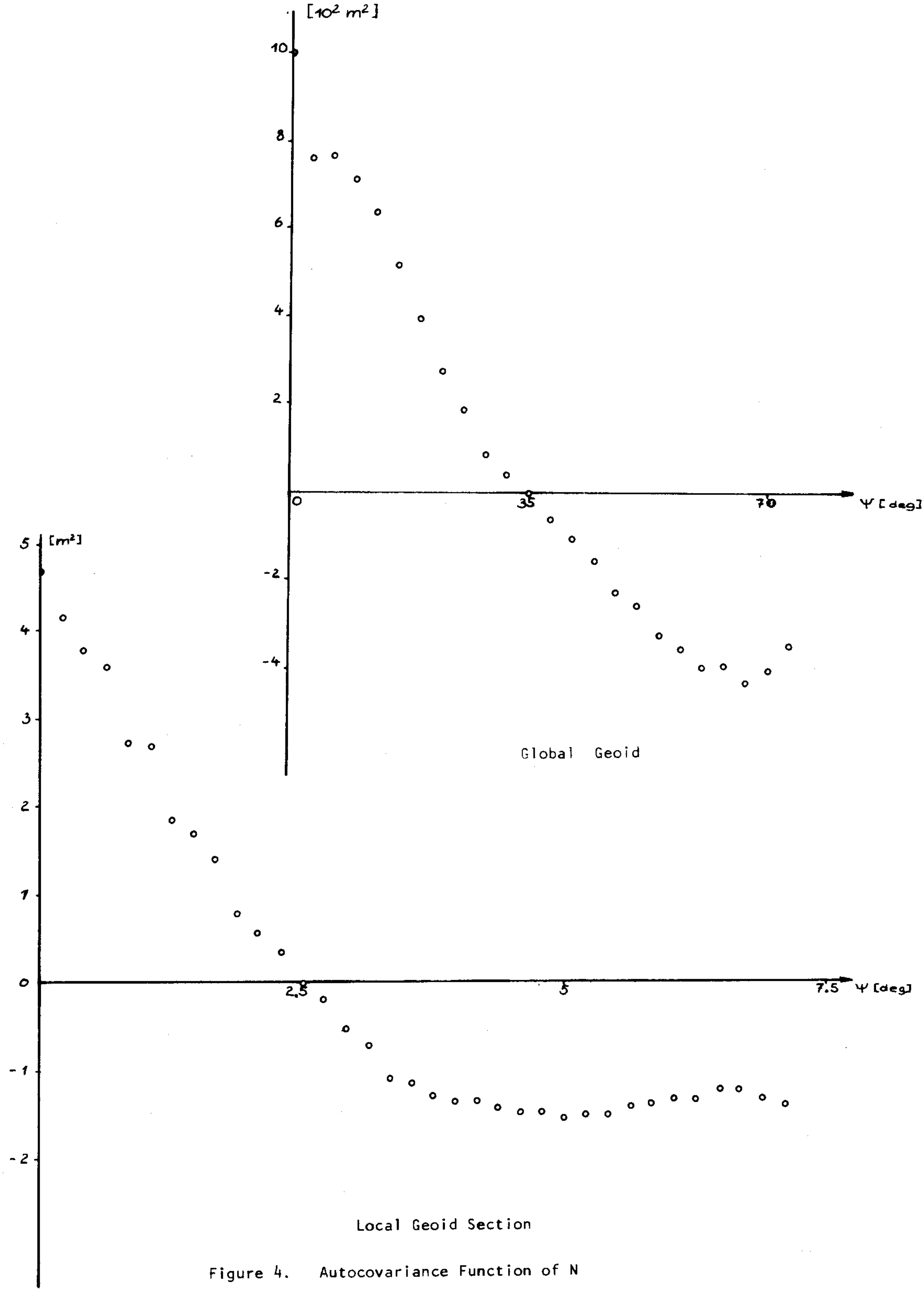

Figure 4. Autocovariance Function of N

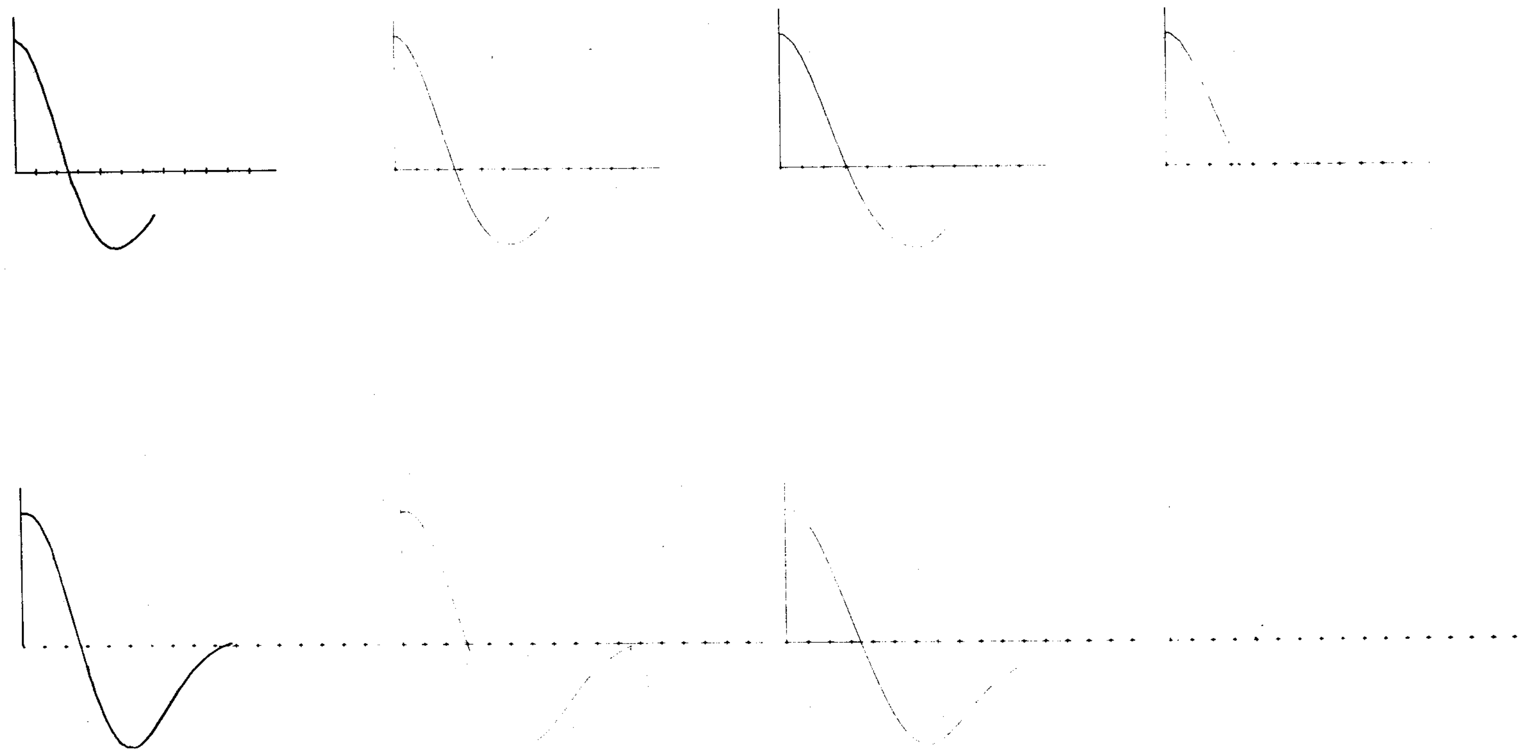

Figure 5. Autocorrelation* of **N** along ϕ = Constant around $\phi = 50°$**

* where in the upper part $\mathrm{cov}(t) = \frac{1}{T-t} \sum_i N_i N_{i+t}$,

and in the lower row $\mathrm{cov}(t) = \frac{1}{T} \sum_i N_i N_{i+t}$

was used with T = length of profile, t = separation and N_k = centred geoid heights

** one part on abscissa corresponds to 4000 km

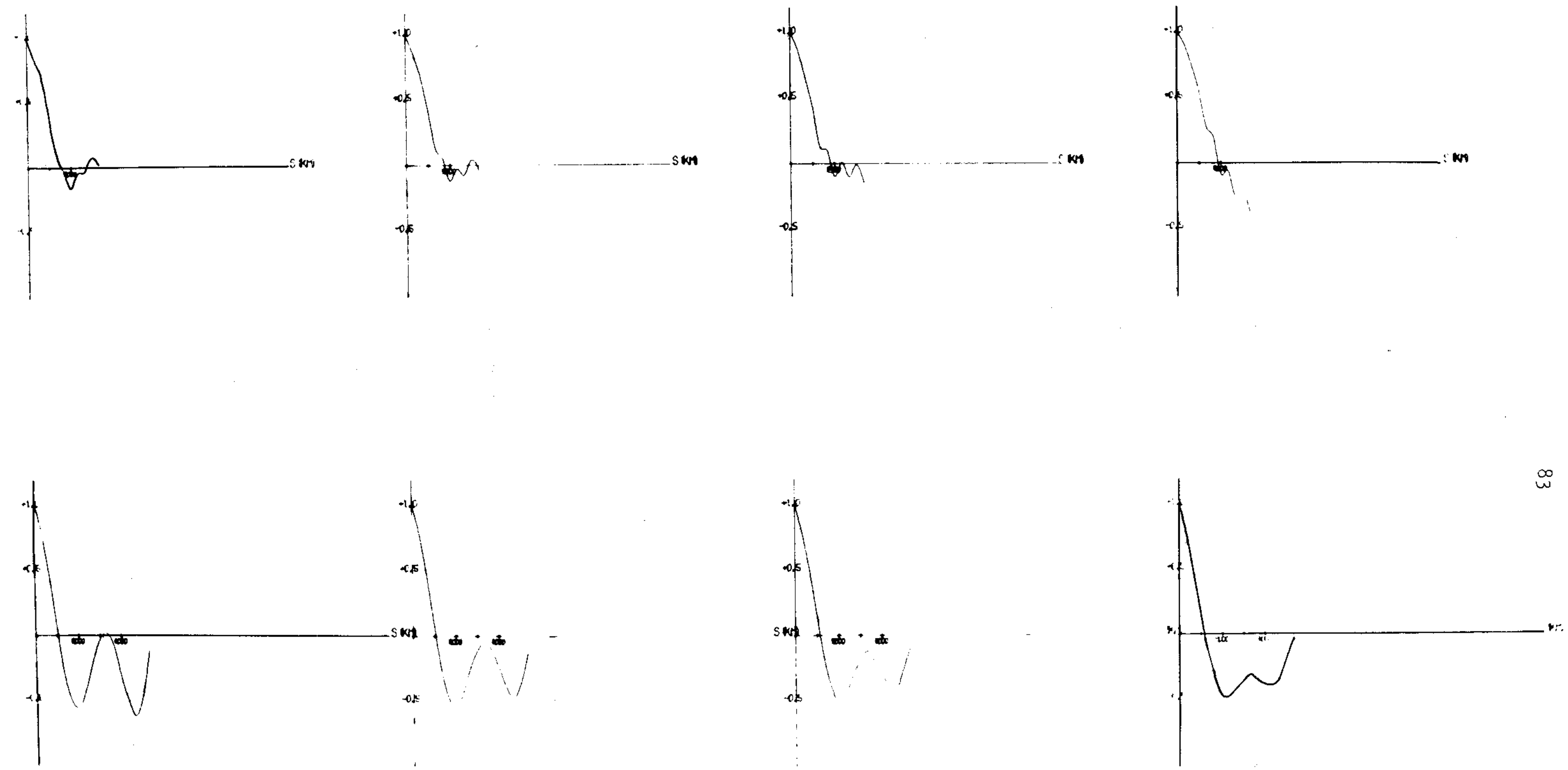

Figure 6. Autocorrelation* of **N** along λ = Constant for the Pacific Area ($\lambda \doteq 200°$)

*Refer note to figure 5

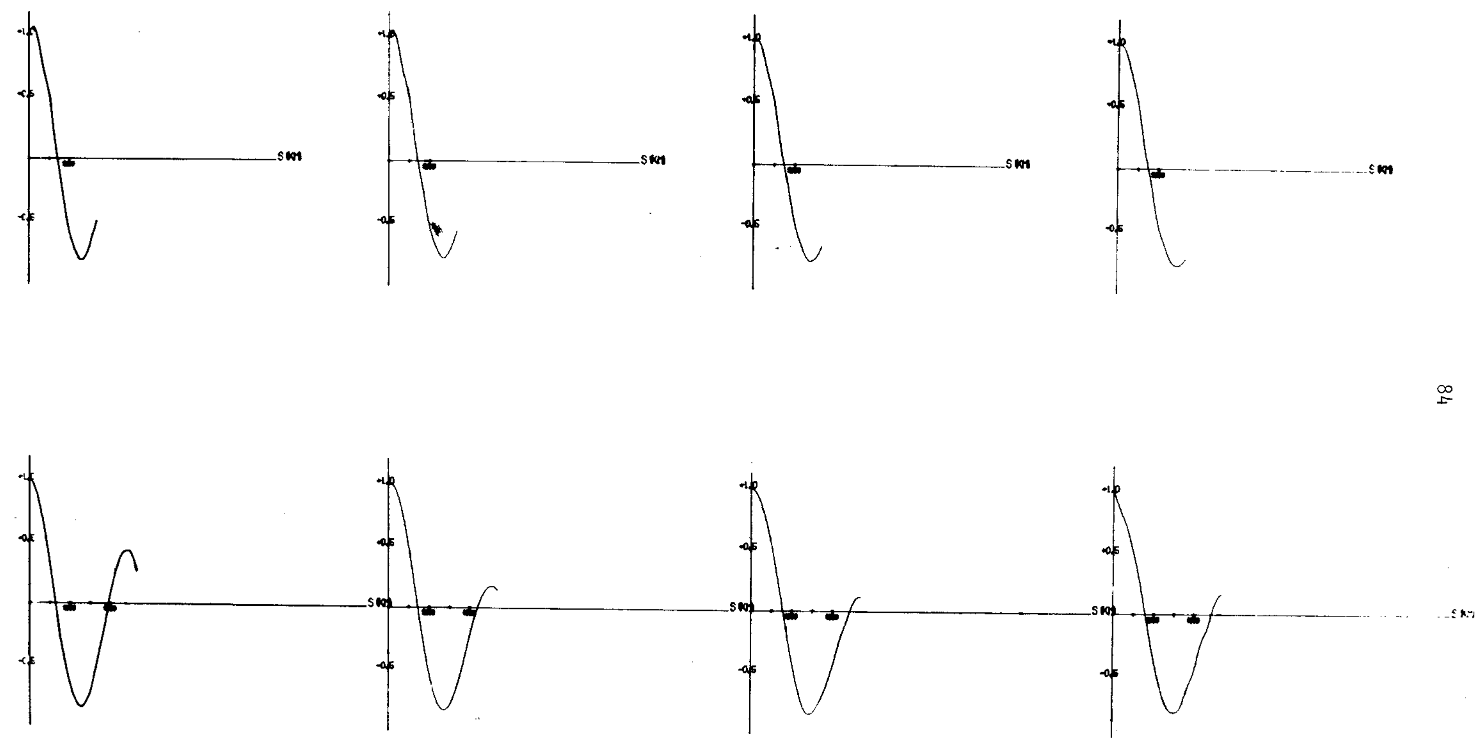

Figure 7. Autocorrelation* of **N** along λ = Constant for the North American Continent

* Refer note to figure 5

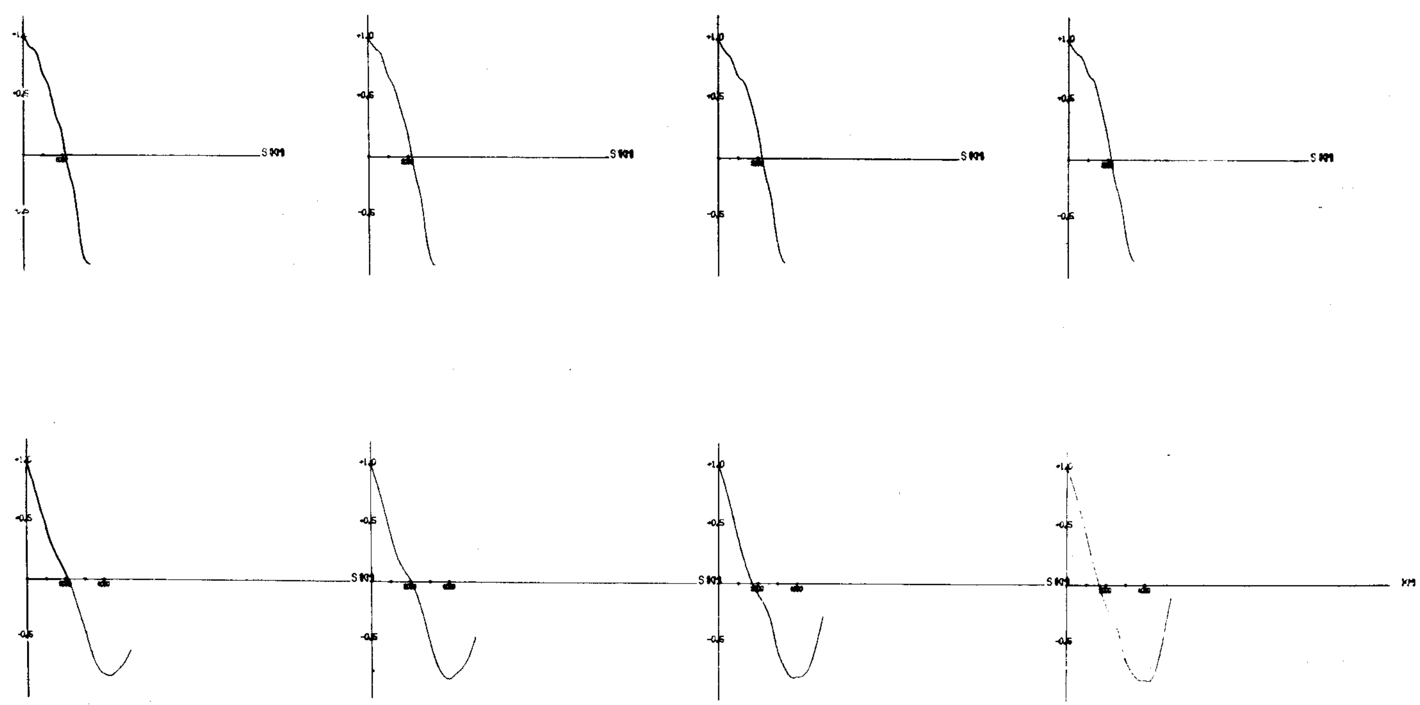

Figure 8. Autocorrelation* of **N** along λ = Constant for the Atlantic Ocean

*Refer note to figure 5

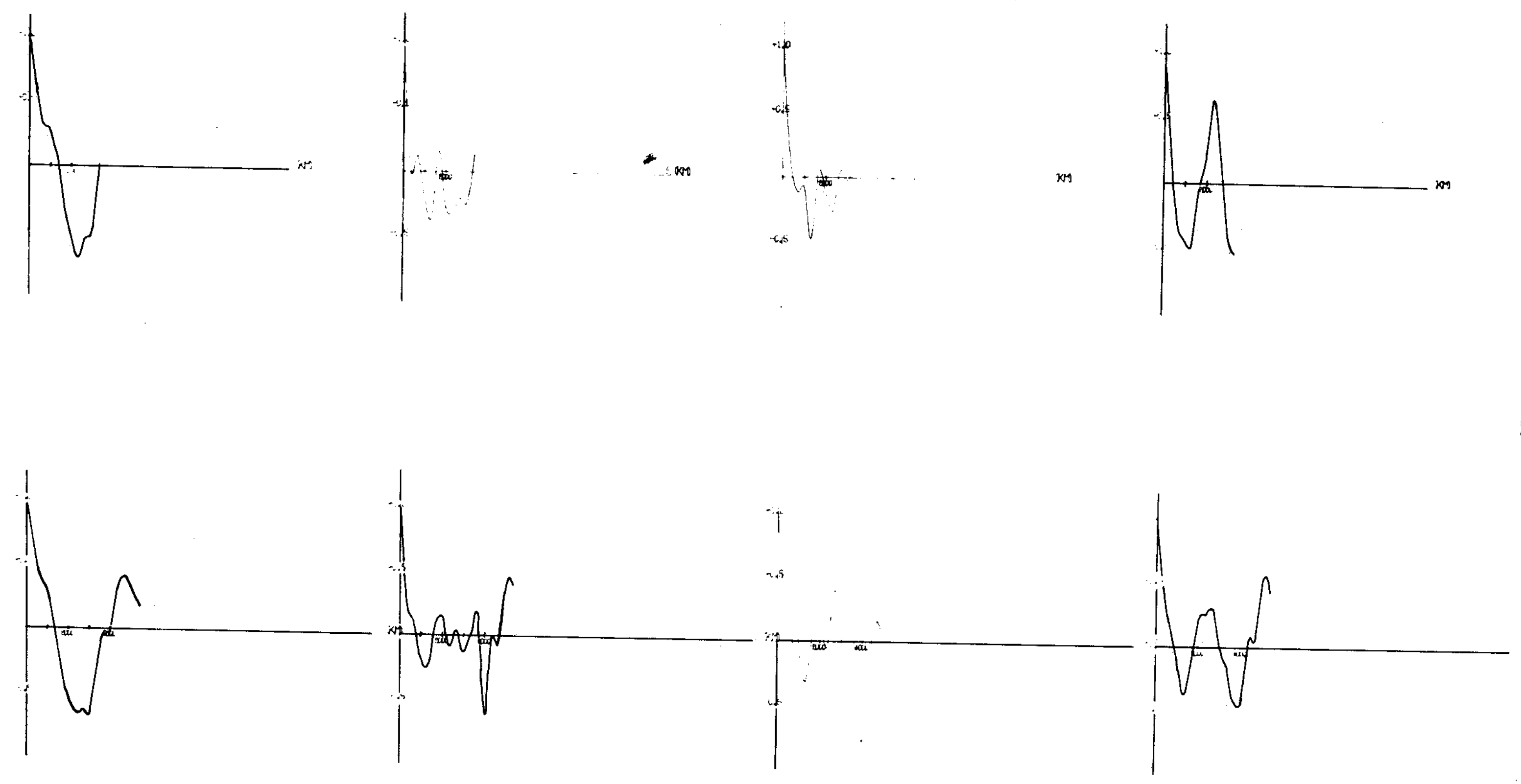

Figure 9. Autocorrelation* of **N** along λ = Constant for the European-African Area

*Refer note to figure 5

but local geology cannot be fully taken into account in most cases.

By comparing the autocorrelation functions given by KAULA (1963) with those few examples given here, the potentialities in predicting N or T directly become evident. In contrast to the above discussion of N-values alone, the detailed results of combination solutions with altimetry, etc. as given in (GROTEN 1970a) will be dealt with in (GROTEN 1974).

The application of recent gradiometry techniques in space and in the air as presented for example during the Symposium on Dynamical Gravimetry at Fort Worth in 1970, give rise to methods which are not so affected by very local topographic influence which bother gradiometer observers on the Earth's surface for several decades. One of the basic advantages of such methods lies in the fact that the influence of Δg at P (at the Earth's surface) on $\partial\Delta g/\partial h$ at a point P' (at elevation h) decreases as ℓ^{-5}, where $\ell = \overline{PP'}$. This rule holds for the lower altitudes above the Earth's surface. The basic *principles* are in these cases best represented by using Fourier series where for Δg_h at elevation h we get the well known relation

$$\Delta g_h = \sum F_{nm} \exp\left(i(u_n x + v_m y) - h(u_n^2 + v_n^2)^{\frac{1}{2}}\right) \qquad (3).$$

By transforming to a series of trigonometrical functions using Euler's formula we usually get for $\partial\Delta g/\partial h$ the coefficients

$$- a A_{nm} e^{-ah} \qquad (4)$$

and so on, whenever the coefficients of the corresponding Δg-series are A_{nm}, B_{nm} ...., where

$$a = \sqrt{(n^2 + m^2)}.$$

Analogously, the coefficients for the horizontal gradients are found to be

$$nm A_{nm} e^{-ah} \qquad (5)$$

and so on. For "two dimensional models" as often used in exploration geophysics, we have

$$n \gg m \qquad \text{or} \qquad n \ll m,$$

one of them being small. In these cases both types of gradients do not differ significantly; otherwise the latter formula behaves approximately like

$$a^2 A_{nm} e^{-ah} \qquad (6).$$

For $A_{nm} = 1$, the above expressions are discussed in figures 10 and 11. From gradients observed at elevations h, we arrive of course, at Δg at the Earth's surface by factor multiplications using

$$a\, e^{+ah} \qquad (7)$$

for vertical gradients, and by using

$$(nm)^{-1} e^{ah} \qquad (8).$$

The advantage, at least in theory, becomes evident by comparing the corresponding factor in downward

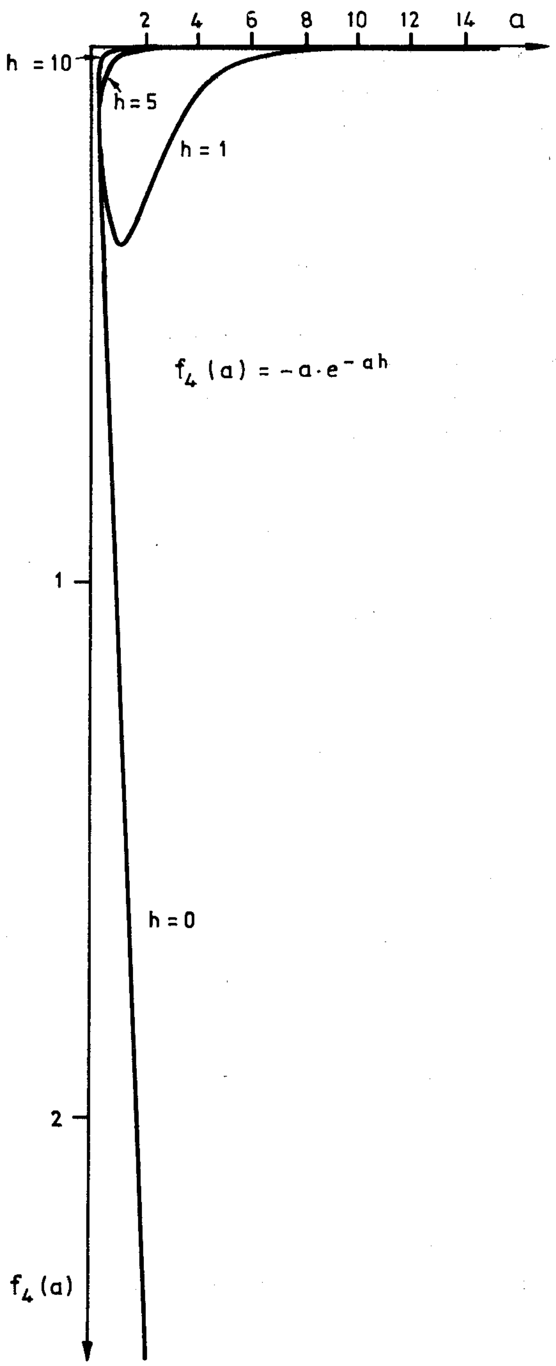

FIGURE 10: BEHAVIOUR OF TRIGONOMETRIC COEFFICIENTS DEPENDING ON a

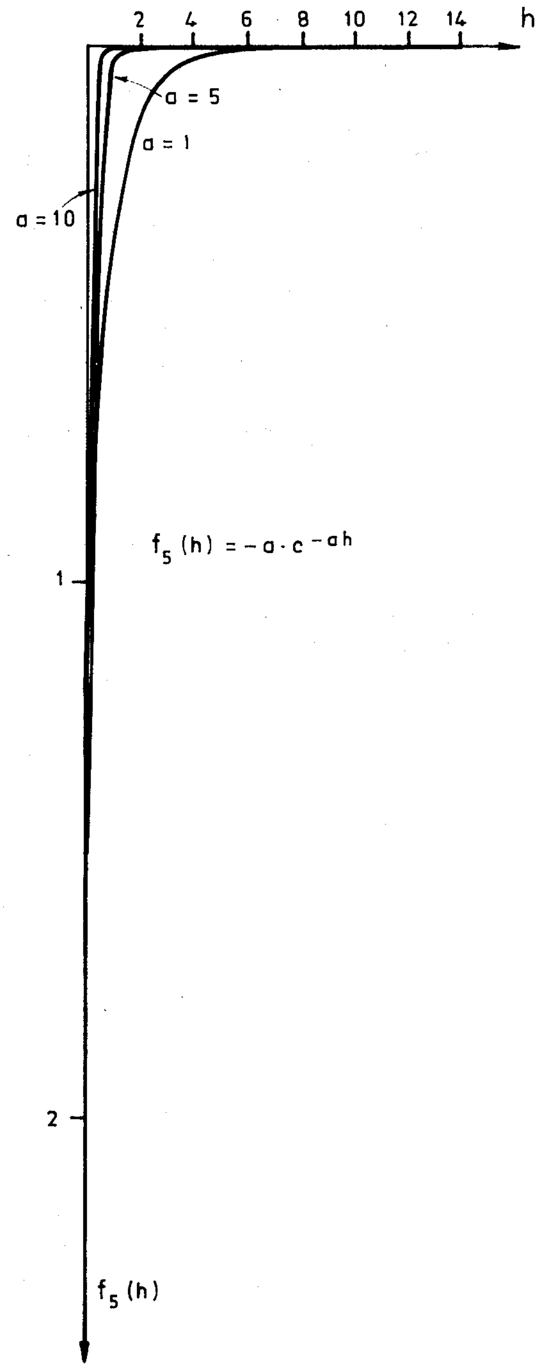

FIGURE 11: BEHAVIOUR OF TRIGONOMETRIC COEFFICIENTS DEPENDING ON h

continuation of Δg.

At higher elevations, the decrease of the effect of local phenomena is advatageous. FORWARD (1972) has given results of detailed studies using Kaula's rule $10^{-5}/n^2$. The difference in information obtained at high altitudes h > 250 km, at one side, and at altitudes $h \doteqdot 10$ km is best explained by figures 12 in the case of the "vertical gradient". It is realized that the superiority of gradient observations at satellite altitudes is relevant for harmonics of degree n > 40.

Figures 12 are self explaining as far as the general features are concerned. In dealing with spherical harmonic series, the slow decrease of the high harmonics of $\partial g/\partial h$ with increasing degree can be a remarkable disadvantage whenever real orbits are considered instead of the simple model orbits. For in this case averaging out of harmonics beyond a certain degree is no longer feasible so truncation errors arise which are higher than in the case of conventional orbital analysis. Even if the harmonic series is replaced by alternative representations of the gravity field, as done for example by Koch and others, any improvement cannot be anticipated. A term-by-term attribution of $\partial g/\partial h$ to the potential does not seem to be feasible in space, in any case. But in the case of "low" altitudes as for example in the case of aircraft profiles (as earlier mentioned), the application of linear integral equations is superior to spherical harmonic expansions as flat approximations are sufficient. Then, of course, truncation errors are avoided by simply smoothing the data.

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3. Acknowledgments

E. Tengstrom kindly made available recent data material of SSG 5:16, International Association of Geodesy; J. Brennecke and R. Rummel did most of the computer work.

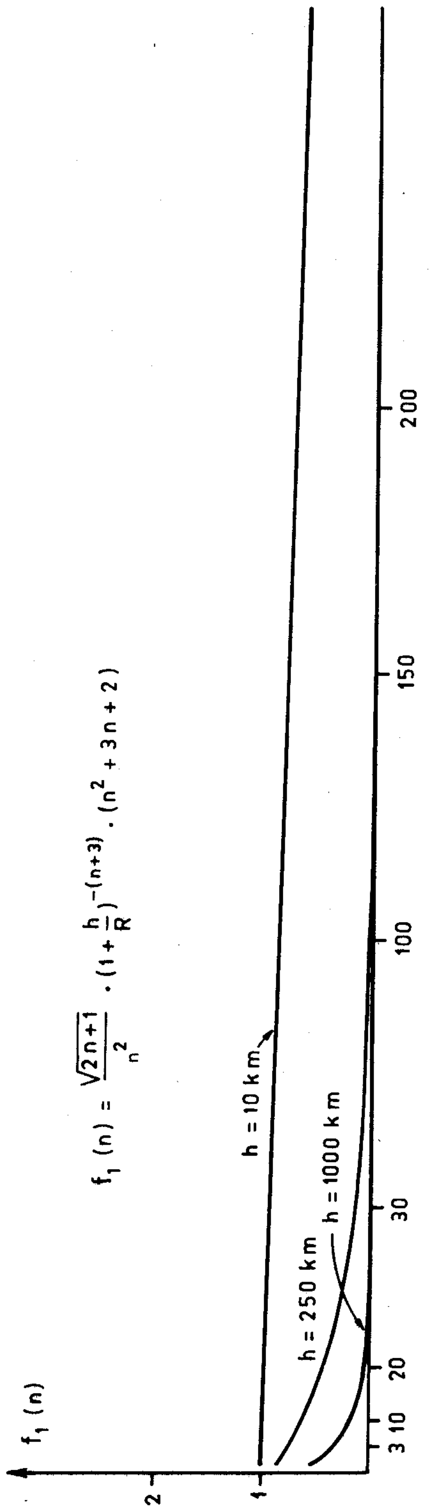

FIGURE 12 a: VERTICAL GRAVITY GRADIENT AT DIFFERENT ELEVATIONS.

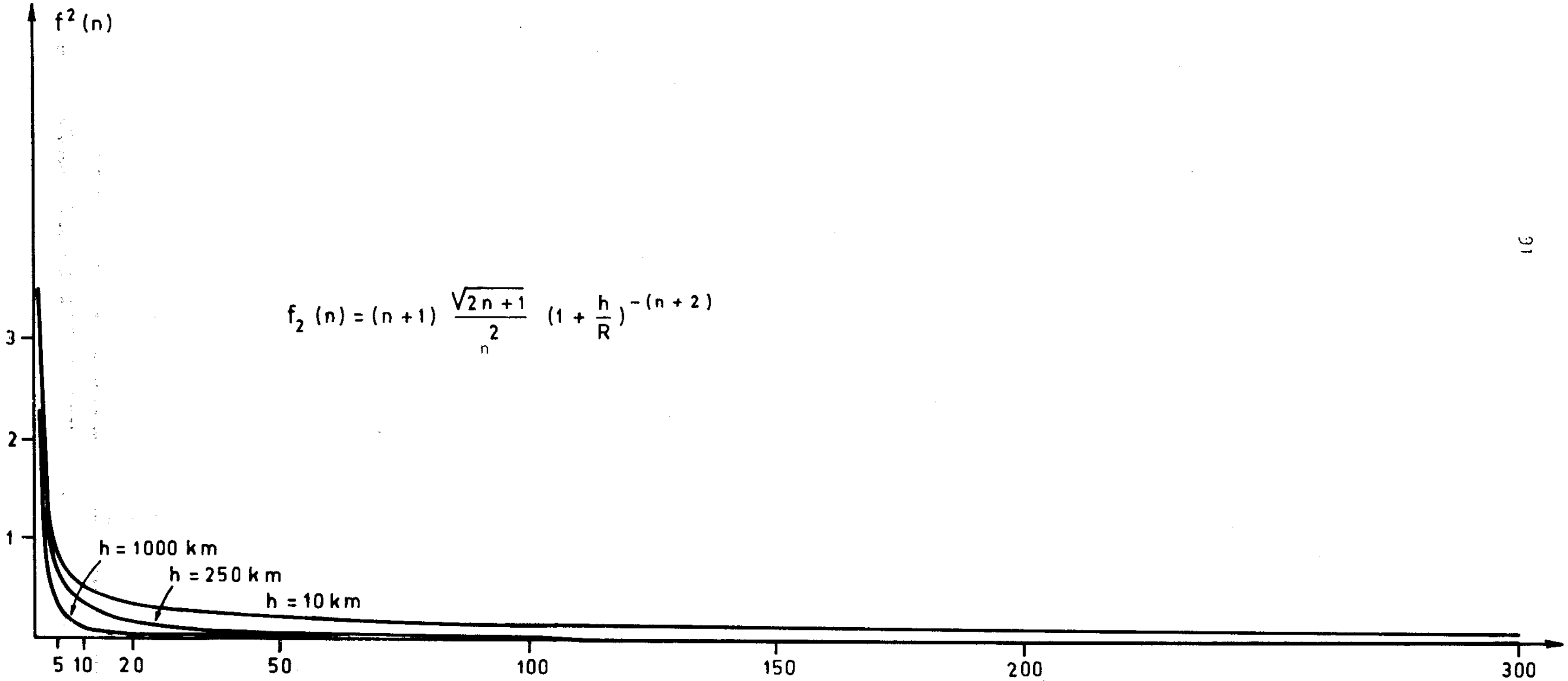

FIGURE 12 b: VERTICAL GRAVITY GRADIENT AT DIFFERENT ELEVATIONS

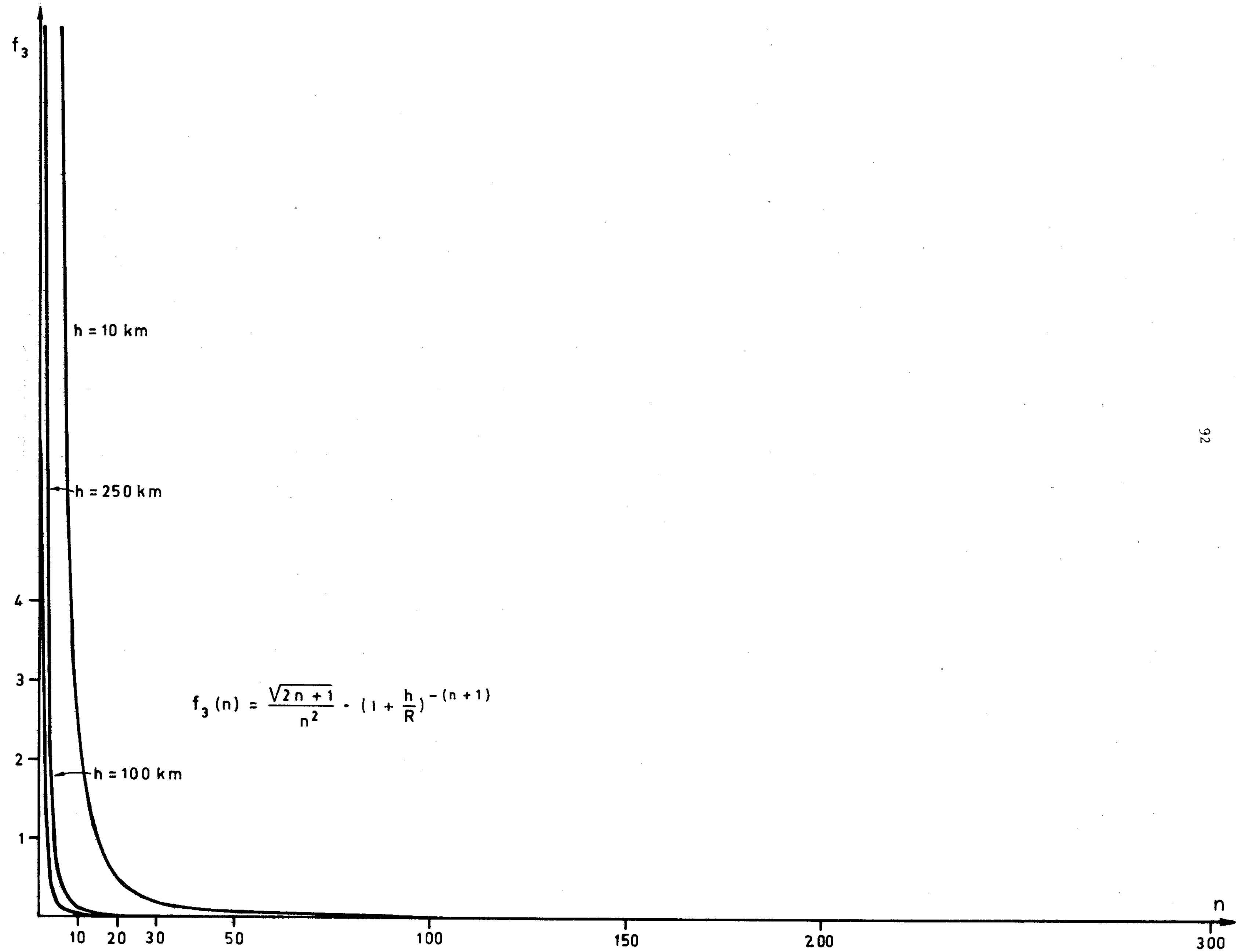

FIGURE 12 c: VERTICAL GRAVITY GRADIENT AT DIFFERENT ELEVATIONS.

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MATHEMATICAL MODELS OF GEOPOTENTIAL GRADIENTS

ABSTRACT

The gravitational potential of the Earth and its gradients are most frequently expressed in terms of spherical harmonics. This representation can be wasteful of computer time and/or core storage and can present numerical difficulties when higher degree coefficients are involved. A few viable alternatives to spherical harmonics such as point mass sets and surface density layers are examined. This evaluation is conducted in terms of goodness of fit and second order gradients of the geopotential and in terms of facility of computation.

1. Introduction

The most frequently used expression for the gravitational potential of the Earth at an exterior point is a summation of spherical harmonics. This expression can be easily differentiated with respect to the spherical co-ordinates of the external point to derive expressions for the first, second and higher order gradients (spatial derivatives) of the geopotential. The number of coefficients required for the spherical harmonic representation increases as the square of the highest degree used. In the electronic computer, then, this representation might utilize an undesirable amount of processing time and core storage. Further, numerical difficulties are encountered in many computers due to exponent limits on floating point numbers.

What we seek, then, is an alternate model which meets three criteria

(1) readily differentiable to any desired order;

(2) minimal number of coefficients; and

(3) avoidance of very large or very small exponents in the computations.

Representations considered in this paper are

(a) point masses (EALUM 1971);

(b) density layers (KOCH & WITTE 1971); and

(c) Taylor's series (JUNCOSA & JOHNS 1965).

Although the partial derivatives required for these representations are not as succint as those in the spherical harmonic representation, a little care is all that is required.

We might select a "best" model from these alternatives by considering the goodness of fit to test data in various applications. Possibly, we may select as "best" that model which is most conservative of computer time and/or core storage. Again, we may wish to consider the trade-offs between goodness of fit and computer resource conservation. The purpose of this paper is to present the derivation of the models rather than propose a definitive selection of a "best" model.

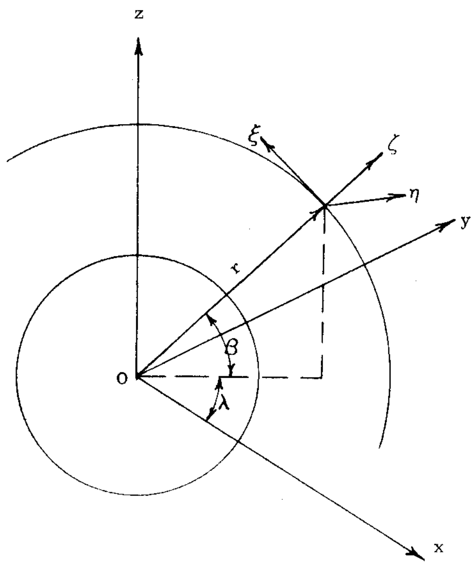

Figure 1. Global and Local Co-ordinate Systems

2. Fundamentals

The geopotential and its gradients are herein represented in terms of global spherical co-ordinates, global Cartesian co-ordinates, and local Cartesian co-ordinates (see figure 1). We define the global spherical co-ordinates (β,λ,r) in the usual sense, i.e., β is the geocentric latitude for the exterior point, λ is its geocentric longitude (from Greenwich), and r its geocentric radius. We define the global Cartesian co-ordinates (x,y,z) as being Earth centred - Earth fixed with the z axis coincident with the Earth's rotational axis and the xy plane coincident with the Earth's equatorial plane. The x axis is directed through the Greenwich meridian and the y axis completes the right-handed orthogonal triad. The local Cartesian co-ordinates (ξ,η,ζ) are defined as being centred at the exterior point and oriented so that the ζ axis coincides with the outward directed radius vector. The $\xi\eta$ plane is normal to this axis with the ξ and η axes directed north and east, respectively.

The first order derivatives of the geopotential express the components of the gravity vector at the exterior point. In terms of partial derivatives with respect to spherical co-ordinates, they are (MORITZ 1971)

$$V_\zeta = V_r \quad ; \quad V_\xi = -\frac{1}{r} V_\beta \quad ; \quad V_\eta = \frac{1}{r \cos \beta} V_\lambda \qquad (1),$$

where $V_r = \partial V/\partial r$, etc. The second order derivatives express the components of the gravity gradient tensor at the exterior point. Again, in terms of partial derivatives with respect to the spherical co-ordinates (IBID)

$$V_{\xi\xi} = \frac{1}{r^2} V_{\beta\beta} + \frac{1}{r} V_r \qquad (2a),$$

$$V_{\xi\eta} = -\frac{1}{r^2 \cos \beta} V_{\beta\lambda} + \frac{1}{r^2 \sec \beta \tan \beta} V_\lambda = V_{\eta\xi} \qquad (2b),$$

$$V_{\xi\zeta} = -\frac{1}{r} V_{r\beta} + \frac{1}{r^2} V_\beta = V_{\zeta\xi} \quad (2c),$$

$$V_{\eta\eta} = \frac{1}{r^2\cos^2\beta} V_{\lambda\lambda} + \frac{1}{r} V_r + \frac{1}{r^2}\tan\beta \, V_\beta \quad (2d),$$

$$V_{\eta\zeta} = \frac{1}{r\cos\beta} V_{r\lambda} - \frac{1}{r^2\cos\beta} V_\lambda = V_{\zeta\eta} \quad (2e),$$

and

$$V_{\zeta\zeta} = V_{rr} \quad (2f).$$

Gradients can be transformed between local and global Cartesian co-ordinates by a rotation matrix (IBID)

$$\begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} = A \begin{pmatrix} V_\xi \\ V_\eta \\ V_\zeta \end{pmatrix} \quad (3),$$

and

$$\begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{pmatrix} = A \begin{pmatrix} V_{\xi\xi} & V_{\xi\eta} & V_{\xi\zeta} \\ V_{\eta\xi} & V_{\eta\eta} & V_{\eta\zeta} \\ V_{\zeta\xi} & V_{\zeta\eta} & V_{\zeta\zeta} \end{pmatrix} A^T \quad (4)$$

where

$$A = \begin{pmatrix} -\sin\beta\cos\lambda & -\sin\lambda & \cos\beta\cos\lambda \\ -\cos\beta\sin\lambda & \cos\lambda & \cos\beta\sin\lambda \\ \cos\beta & 0 & \sin\beta \end{pmatrix} \quad (5),$$

or by its inverse, which equals its transpose due to orthogonality

$$\begin{pmatrix} V_\xi \\ V_\eta \\ V_\zeta \end{pmatrix} = A^T \begin{pmatrix} V_x \\ V_y \\ V_z \end{pmatrix} \quad (6)$$

and

$$\begin{pmatrix} V_{\xi\xi} & V_{\xi\eta} & V_{\xi\zeta} \\ V_{\eta\xi} & V_{\eta\eta} & V_{\eta\zeta} \\ V_{\zeta\xi} & V_{\zeta\eta} & V_{\zeta\zeta} \end{pmatrix} = A^T \begin{pmatrix} V_{xx} & V_{xy} & V_{xz} \\ V_{yx} & V_{yy} & V_{yz} \\ V_{zx} & V_{zy} & V_{zz} \end{pmatrix} A \quad (7).$$

We note here, also, that the gradient tensors, both local and global are symmetric. Also, per Laplace's equation

$$V_{xx} + V_{yy} + V_{zz} = V_{\xi\xi} + V_{\eta\eta} + V_{\zeta\zeta} = 0.$$

3. Mathematical Models

3.1 Spherical Harmonics

The usual expression for the gravitational potential of the Earth on an external point in terms of spherical harmonics is (e.g., HOTINE 1969)

$$V(\beta,\lambda,r) = \frac{GM}{r}\left\{1 + \sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\beta)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right)\right\} \qquad (8),$$

where (β,λ,r) = global spherical co-ordinates of exterior point,
GM = Earth's gravitational constant,
a = Earth's equatorial radius,
$P_{nm}(\sin\beta)$ = associated Legendre functions, and
C_{nm}, S_{nm} = conventional geopotential coefficients.

This expression can be easily differentiated to yield the required partial derivatives.

We see that frequent references to sine and cosine routines are required in the computation process. as the coefficient set grows larger, the number of associated Legendre functions required increases rapidly. Recursive relations (HOPKINS 1973; BURINGTON 1949) are available to relieve the situation somewhat. Still, the process is quite formidable. If in the above equation, we choose to eliminate the effects of the constant term and of the coefficients C_{20} and C_{40}, we operate upon the anomalous potential T, i.e.,

$$T = V - U \qquad (9)$$

where

$$U = \frac{GM}{r}\left\{1 + \left(\frac{a}{r}\right)^2 P_{20}(\sin\beta)\, C_{20} + \left(\frac{a}{r}\right)^4 P_4$$

a readily differentiable form. The anomalous potential le
(e.g., HEISKANEN & MORITZ 1967,p.86)

$$\Delta g = -\frac{\partial T}{\partial r} - \frac{2}{a}T$$

In terms of spherical harmonics (e.g., RAPP 1972)

$$\Delta g = \frac{GM}{r^2}\left\{\sum_{n=2}^{\infty}\left(\frac{a}{r}\right)^n (n-1)\sum_{m=0}^{n} P\right.$$

where the C_{20} and C_{40} are omitted.

3.2 Point Masses

The basic formulation for the gravitational potential at a p
terms of point masses is (EALUM 1971)

$$V_i(\beta,\lambda,r) = GM\sum_{j=1}^{N}\left(\frac{m_j}{M}\right)\frac{1}{\rho_{ij}}$$

where

(m_j/M) = ratio of the j-th point mass to the Earth's mass;

ρ_{ij} = distance from the point mass j to the exterior point i; and

N = number of point masses in the set.

In the global Cartesian co-ordinate system (x,y,z),

$$\rho_{ij}^2 = (x_i - x_j)^2 + (y_i - y_j)^2 + (z_i - z_j)^2 \qquad (14),$$

while in the global spherical co-ordinate system (β,λ,r)

$$\rho_{ij}^2 = r_i^2 + r_j^2 - 2r_i r_j \cos\psi \qquad (15).$$

The subscript $_i$ refers to the exterior point and the subscript $_j$ refers to the j-th point mass. The spherical distance ψ is given by

$$\cos\psi = \sin\beta_i \sin\beta_j + \cos\beta_i \cos\beta_j \cos(\lambda_i - \lambda_j) \qquad (16).$$

The partial derivatives which are form-invariant within and between the global co-ordinate systems, by the chain rule, are

$$V_s = GM \sum_{j=1}^{N} - \left(\frac{m_j}{M}\right) \frac{1}{\rho_{ij}^2} \frac{\partial \rho_{ij}}{\partial s} \qquad (17a)$$

and

$$V_{st} = GM \sum_{j=1}^{N} \left(\frac{m_j}{M}\right) \left(\frac{2}{\rho_{ij}^3} \frac{\partial \rho_{ij}}{\partial s} \frac{\partial \rho_{ij}}{\partial t} - \frac{1}{\rho_{ij}} \frac{\partial^2 \rho_{ij}}{\partial s \partial t}\right) \qquad (17b),$$

where $s,t = x_i, y_i$ or z_i or $s,t = \beta_i, \lambda_i$ or r_i.

Computations of gradients in the global Cartesian co-ordinates (x,y,z) is accomplished directly by taking the partial derivatives in the above equation with respect to x, y and z. Should gradients be required in the local Cartesian co-ordinate system (ξ,η,ζ), transformations reflected by equations 6 and 7 are applied.

On the other hand, if the partial derivatives are taken with respect to β, λ and r, the computation of gradients in the local Cartesian co-ordinates (ξ,η,ζ) proceeds through equations 1 and 2. Should gradients be required in global Cartesian co-ordinates (x,y,z), transformations reflected by equations 3 and 4 are applied.

The gravity anomaly, a function of the anomalous potential, can readily be expressed in global Cartesian co-ordinates (NEEDHAM 1970) as

$$\Delta g_i = GM \sum_{j=1}^{N} \left(\frac{m_j}{M}\right)\left(\frac{r_i^2 - F_{ij}}{\rho_{ij}^3 r_i} - \frac{2}{\rho_{ij} r_i}\right) \qquad (18),$$

where

$$F_{ij} = x_i x_j + y_i y_j + z_i z_j .$$

3.3 Density Layers

The basic formulation for the gravitational potential at a point exterior to the Earth expressed in terms of density layers is (KOCH & WITTE 1971)

$$V_i(\beta,\lambda,r) = \frac{GM}{r_i}\left[1 + \sum_{n=2}^{n_{max}} \left(\frac{a}{r_i}\right)^n \sum_{m=0}^{n} P_{nm}(\sin\beta_i)\left(C_{nm}\cos m\lambda_i + S_{nm}\sin m\lambda_i\right)\right] + \sum_{j=1}^{N} \chi_j \iint_{\Delta E_j} \frac{dE_j}{\rho_{ij}} \quad (19),$$

where
χ_i = density function of layer;
E_j = area of layer;
n_{max} = maximum degree of spherical harmonic portion; and
N = number of density layers.

If the potential be represented as the sum of the spherical harmonic portion and the density layer portion, i.e.,

$$V = U + T,$$

the density layers can be used to represent the residual potential above a certain harmonic degree n_{max}. Now let

$$T_i = \sum_{j=1}^{N} \chi_j \iint_{\Delta E_j} \frac{dE_j}{\rho_{ij}} \quad (20).$$

This equation expresses a critical point in the density layer model. Since the integral over the surface element E_j is solved numerically, it is subject to errors of quadrature. KOCH (1971) develops analytical expressions for the surface elements on ellipsoidal and spherical surfaces which can be used in lieu of the integral. Fineness of the grid about the computation point is analogous to the well known scheme for computation of geoid heights from surface mean gravity anomaly elements.

The integral in equation 20 is replaced by the area of the layer, i.e.,

$$E_j/\rho_{ij} = \iint_{\Delta E_j} dE_j/\rho_{ij} \quad (21)$$

and the anomalous potential can be expressed as

$$T_i = \sum_{j=1}^{N} \chi_j \frac{E_j}{\rho_{ij}} \quad (22).$$

The development of expressions for the gradients in the local Cartesian co-ordinate system (ξ,η,ζ) closely parallels the development in terms of point masses, i.e., equation 17. Quite simply,

$$T_p = \sum_{j=1}^{N} - \chi_j E_j \frac{1}{\rho_{ij}^2} \frac{\partial\rho_{ij}}{\partial p} \quad (21a)$$

and

$$T_{pq} = \sum_{j=1}^{N} \chi_j E_j \left[\frac{2}{\rho_{ij}^3} \frac{\partial\rho_{ij}}{\partial p} \frac{\partial\rho_{ij}}{\partial q} - \frac{1}{\rho_{ij}^2} \frac{\partial^2\rho_{ij}}{\partial p\partial q}\right] \quad (21b)$$

where

$$p,q = (\beta_i,\lambda_i,r_i).$$

The computation of the gradients then proceeds through equations 1 and 2.

3.4 Taylor Series

While the coefficients in the point mass and density layer approaches have some physical meaning, the Taylor series approach (JUNCOSA & JOHNS 1965) is unique in that the coefficients do not. For the basis of this method we return to fundamental potential theory. Let (β,λ,r) and (β',λ',r') be the spherical co-ordinates of the exterior point and some mass element dm of the attracting mass, as illustrated in figure 2. For the potential the the exterior point, we have

$$F(\beta,\lambda,r) = \frac{G}{r}\iiint \frac{dm(\beta',\lambda',r')\ d\beta'\ d\lambda'\ dr'}{\left(1 + (r'/r)^2 - 2(r'/r)\cos\theta\right)^{\frac{1}{2}}} \qquad (22).$$

The integration is taken over the Earth's total volume. Denoting the direction cosines of the two vectors by

$$(u,v,w) = (\cos\lambda\cos\beta, \sin\lambda\cos\beta, \sin\beta) \quad ; \quad \text{and}$$
$$(\xi,\eta,\zeta) = (\cos\lambda'\cos\beta', \sin\lambda'\cos\beta', \sin\beta'),$$

we have

$$\cos\theta = u\xi + v\eta + w\zeta\ .$$

Then equation 22 can be expressed in the form of a Taylor series

$$F(\beta,\lambda,r) = \frac{G}{r}\iiint \sum_i\sum_j\sum_k C_{ijk}u^i\xi^i v^j\eta^j w^k\zeta^k \left(\frac{r'}{r}\right)^{i+j+k} dm\ d\beta'\ d\lambda'\ dr' \qquad (23),$$

where C_{ijk} are coefficients that depend upon the various derivatives of the integrand with respect to $u\xi$, $v\eta$ and $w\zeta$ up to the (i+j+k)-th order.

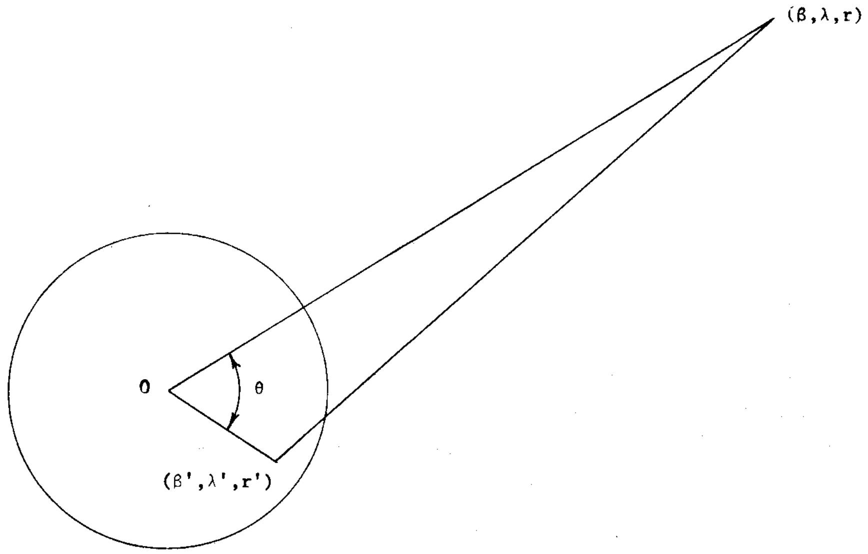

Figure 2. Geometry of Potential Theory

We now invert the order of integration and summation and perform the summations in the order

$i + j + k = 0 ; \quad i + j + k = 1 ; \quad i + j + k = 2 ; \quad$ etc.

Recognizing here that the possible derivatives are linearly dependent since $u^2 + v^2 + w^2 = 1$, we may finally write for the potential

$$
\begin{aligned}
F(\beta,\lambda,r) = {} & \frac{A}{r} + \frac{1}{r^2}\left[B_1 u + B_2 v + B_3 w\right] + \frac{1}{r^3}\left\{C_1(3u^2 - 1) + C_2(3v^2 - 1) + C_3 uv + C_4 uw + C_5 vw\right\} + \\
& \frac{1}{r^4}\left\{D_1(5u^2 - 3)u + D_2(5v^2 - 3)v + D_3(5u^2 - 1)v + D_4(5v^2 - 1)u + D_5(5u^2 - 1)w + \right. \\
& \left. D_6(5v^2 - 1)w + D_7 uvw\right\} + \frac{1}{r^5}\left\{E_1(35u^4 - 30u^2 + 3) + E_2(35v^4 - 30v^2 + 3) + \right. \\
& E_3(7u^2 - 3)uv + E_4(7v^2 - 3)uv + E_5(7u^2 - 3)uw + E_6(7v^2 - 3)vw + E_7(35u^2v^2 + 5w^2 - 4) + \\
& \left. E_8(7u^2 - 1)vw + E_9(7v^2 - 1)uw\right\}
\end{aligned}
\qquad (24),
$$

where A, B_1, B_2, ... E_9 are coefficients to be determined. While this expansion is given to the fifth power of r^{-1}, the pattern is readily seen, should it be desirable to extend the representation to higher powers of r^{-1}.

Obtaining the required partial derivatives of equation 24 with respect to the spherical co-ordinates of the exterior point is quite tedious, but the derivatives themselves are rather simple. The partial derivatives with respect to r, i.e., F_r and F_{rr} are direct, while the remaining partial derivatives involve the chain rule, i.e.,

$$F_x = \frac{\partial F}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial F}{\partial w}\frac{\partial w}{\partial x} \qquad (25),$$

and

$$F_{xy} = \frac{\partial^2 F}{\partial u^2}\frac{\partial u}{\partial x}\frac{\partial u}{\partial y} + \frac{\partial^2 F}{\partial v^2}\frac{\partial v}{\partial x}\frac{\partial v}{\partial y} + \frac{\partial^2 F}{\partial w^2}\frac{\partial w}{\partial x}\frac{\partial w}{\partial y} + \frac{\partial F}{\partial u}\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial F}{\partial v}\frac{\partial^2 v}{\partial x \partial y} + \frac{\partial F}{\partial w}\frac{\partial^2 w}{\partial x \partial y} \qquad (26),$$

where

$$x, y = \beta, \lambda \text{ or } r.$$

We then proceed through equations 1 and 2 to obtain gradients in the local Cartesian co-ordinate system (ξ,η,ζ).

4. Characteristics of Observations

While the gravitational potential is not usually measured directly, satellite-to-satellite tracking provides a means of doing so (COMFORT 1971). Equations 8, 13, 19 and 24 express the potential in the various mathematical models under discussion. The first order gradient is the gravity vector at the exterior point. In practice, it might be measured as range and/or angular rates, but for the sake of this paper, it will be used directly.

The second order gradients or gravity gradients can be measured directly by gradiometers at or near the surface of the Earth or at satellite altitudes (HOPKINS 1972). In the surface case, we expect that a three sensor system will yield all nine components of the gradient tensor (of which five are independent). In the orbit case, the principal contribution to the signal is (GLASER & SHERRY 1971)

$$V_{\zeta\zeta} - V_{\xi\xi} = V_{rr} - \frac{1}{r^2} V_{\beta\beta} - \frac{1}{r} V_r \qquad (27).$$

Currently, the best data we have available for determining the geopotential is the set of mean gravity anomalies. Using potentials, first order gradients, second order gradients, or mean gravity data as observations, equations can readily be written for computation of spherical harmonic coefficients, point mass ratios, density values, or Taylor series coefficients by a least squares process. Generation of pseudo-observations for numerical tests is described in the following section.

5. Numerical Tests

Pseudo-observation data generated using the spherical harmonic model was used as a norm against which to compare the alternative models. Using a set of spherical harmonic coefficients complete to degree and order 36, potentials, gravity vector components and gravity gradients were computed for 1440 points defined in global spherical Earth fixed co-ordinates selected from polar, circular orbit at an altitude of about 300 km. A set of surface gravity gradient pseudo-observations was computed in a latitude band extending 45 degrees each side of the equator. The harmonic coefficients were themselves derived from recent unclassified mean free air gravity anomaly data.

Initially, a set of 186 point mass ratios, spaced over the Earth at a 900 nautical mile interval, was computed from the anomalous potential data, i.e., the input observations were adjusted to remove the contribution of the reference ellipsoid. The masses were placed at a distance of 5400 km from the Earth's centre. From these mass ratios, the first and second order gradients and mean gravity anomalies were computed and compared with the test data. The required gradients were computed both in local co-ordinates directly and in global co-ordinates rotated to local. To assess the effect of increasing the mass set size, a set of 410 point mass ratios, spaced over the Earth at 600 nautical mile intervals were computed from the V_{ζ} component of the orbit data. From this data the potential and the remaining gradients and gravity anomaly data were computed and compared with the test data. The density layer concept was applied by computing a set of density layer values located on the surface of the Earth coincident with the location of the point masses. The anomalous potential data was used to derive these values, without application of the quadrature techniques to represent the surface element. As in the point mass tests, the first and second order gradients were computed and compared with test values.

The Taylor series coefficients were computed from the full potential values. A total of 25 coefficients were computed, incorporating the fifth power of r^{-1}. From the coefficients, the first and second order gradients were computed and compared with the test data. Test results are given in Table 1.

We expect a good comparison between potentials as this is the data used to derive the coefficients for the representation. All comparisons are within an order of magnitude of the desired accuracy. The computation of the gravity vector components directly in local co-ordinates, more complex mathematically, yields no better results than the simpler global to local computation. The gravity gradients, especially those in orbit, are reasonably well predicted by either route. Generally the mean anomaly results are somewhat poor. However, a spherical harmonic representation for gravity anomalies can be just as poor for certain coefficient sets derived solely from orbit data (DECKER 1972). Another important consideration is that of computer time. One may wish to consider trade-offs in computer time versus accuracy. Typical computer times are given in table 2.

A substantial savings of computer time in the application of the point mass model can be achieved by the computation of the gradients in global co-ordinates and applying the appropriate rotation matrices. Each alternative model represents a savings over the spherical harmonic model in terms of computer time, the price being a loss of accuracy to some degree.

6. Conclusions

It is reasonable to conclude that all three alternatives to the spherical model are viable. The limited size of the point mass, density layer or Taylor series coefficient sets introduces a limit to the precision of representation. Each representation can be made more accurate by increasing the number of defining coefficients.

The Taylor series representation provides the most compact representation and is highly conservative of computer time. The point mass and density layer models are competetive with it and with each other. The trade off is an order of magnitude difference in computer time requirements versus a factor of two improvement in accuracy.

In such applications as orbit computations using a numerical integration scheme such as Runge-Kutta or Cowell's method, the necessity to re-compute the gravity vector component several times in one integration step makes the Taylor series model quite attractive. It is for this reason that heretofore ignored Taylor series model ought to be given further study. It is my intention to do so.

7. Acknowledgments

The assistance rendered to me by my son, John L. Hopkins, student at Florissant Valley Community College, St. Louis, Missouri, in checking my mathematics is sincerely appreciated. His satisfaction in detecting errors committed by his father has enhanced my confidence in the algebra involved.

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Table 1

Data Comparisons (RMS - Computed vs Test Data)

Model	Potential cm^2/sec^2	V_ζ	V_η cm/sec^2	V_ξ	$V_{\zeta\zeta}$ (Surface) EU**	$V_{\zeta\zeta} - V_{\xi\xi}$ EU**	Δg cm/sec^2
Point Masses (186)	8.9×10^4						
Local		3.0×10^{-3}	2.1×10^{-3}	2.2×10^{-3}	4.2×10^{-1}	1.6×10^{-1}	1.1×10^{-2}
Global to Local		1.1×10^{-2}	7.2×10^{-3}	6.9×10^{-3}	*	1.6×10^{-1}	1.1×10^{-2}
Point Masses (410)	1.5×10^6						
Local		1.6×10^{-3}	1.7×10^{-2}	3.4×10^{-3}	1.3	8.5×10^{-1}	4.6×10^{-2}
Global to Local		1.6×10^{-3}	1.7×10^{-2}	3.2×10^{-3}	*	3.3×10^{-1}	4.6×10^{-2}
Density Layers	2.7×10^5	4.1×10^{-2}	9.0×10^{-3}	7.6×10^{-3}	*	7.7×10^{-1}	*
Taylor Series	6.4×10^5	7.6×10^{-3}	5.4×10^{-3}	4.7×10^{-3}	4.2×10^{-1}	2.1×10^{-1}	1.3×10^{-2}
Representative Values	6.0×10^{11}	8.8×10^2	7.6×10^{-3}	1.2	3.1×10^3	4.5×10^3	1.4×10^{-2}
Desired Accuracy	1.0×10^5	3.0×10^{-3}	3.0×10^{-3}	3.0×10^{-3}	1.0	1.0×10^{-1}	2.0×10^{-3}

* Computation not made

** 1 EU = 10^{-9} cm $sec^{-2} cm^{-1}$ (Eötvös unit)

Table 2

Computation Times (UNIVAC 1108)

Model	Derive Coefficients (Total)	V_ζ, V_η, V_ξ sec/pt	$V_{\zeta\zeta}$ sec/pt	$V_{\zeta\zeta} - V_{\xi\xi}$ sec/pt	Δg sec/pt
Point Masses (186)	$8^m\ 15^s$				
Local		0.253	0.103	0.231	0.138
Global to Local		0.064	*	0.121	0.058
Point Masses (410)	$1^h\ 23^m\ 45^s$				
Local		0.491	0.206	0.815	0.292
Global to Local		0.114	*	0.212	0.094
Density Layers	$9^m\ 08^s$	0.220	*	0.226	*
Taylor Series	$0^m\ 28^s$	0.026	0.009	0.035	0.015
Spherical Harmonics	*	0.377	0.145	0.378	0.150

* Not computed

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9. Discussion

RAPP: Let me add a comment about gravity gradiometers because I think it is something that may be coming along. Many people think that gravity gradiometry is primarily going to help us determine the details of the gravity field; if you observe a gradiometer over an area it reacts very locally.

HOPKINS: I should like to add that gravity gradiometry can assist in getting global information of long wavelength.

RAPP: The question is whether the gravity gradiometer will be accurate enough to obtain the resolution obtained presently from potential coefficients using the orbital technique.

HOPKINS: The accuracy of the gradiometer is one part in a billion plus or minus another twenty (bias).

RAPP: Second; the representations you tested. There is another one that would have been nice to test - the gravity anomaly, which is something you can measure. You don't measure point masses or terms in a Taylor series. So if you are looking for a representation which can easily be combined, you use the anomaly. A report was completed nine months ago by a graduate student (G. REED of the US Army) at the Ohio State University on satellite gradiometry and its implications in determining the gravity field of the Earth. He studied the problem of recovering gravity anomalies or point masses from rotating gradiometer data or a hard mounted system. He concluded that the most reasonable representation was obtained using anomalies. The use of an anomaly representation allows straightforward combination solutions to be made. So I add that as a comment; there are additional views to those expressed by you.

ECKHARDT: I have a question on Taylor series. You have a spherical solid harmonic converted to a Cartesian type solid harmonic. Is it the identical thing whether you do it in spherical or rectangular harmonics. The Taylor series should give the same answer as the spherical harmonics. You are comparing your models with much higher degree models. In a way it is not fair.

HOPKINS: That's what I am doing with the point masses. I am comparing smaller models with the rather larger spherical harmonic models.

ECKHARDT: You should compare Taylor series models with the same degree spherical harmonic models - with the same amount of information - and see what you get in the shortest time. That will give the answer.

GRAFAREND: You mention the transformation between the local system and the geocentric system done by Moritz. This only holds for a spherical Earth. It does not hold for an ellipsoidal Earth or the real Earth. Your formulae are therefore restricted. The exact transformation is given in (HOTINE, M. *Mathematical Geodesy*. ESSA Monograph 2, Washington DC 1969).

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Proc. Symposium on Earth's Gravitational Field & Secular Variations in Position (1973),106-116.

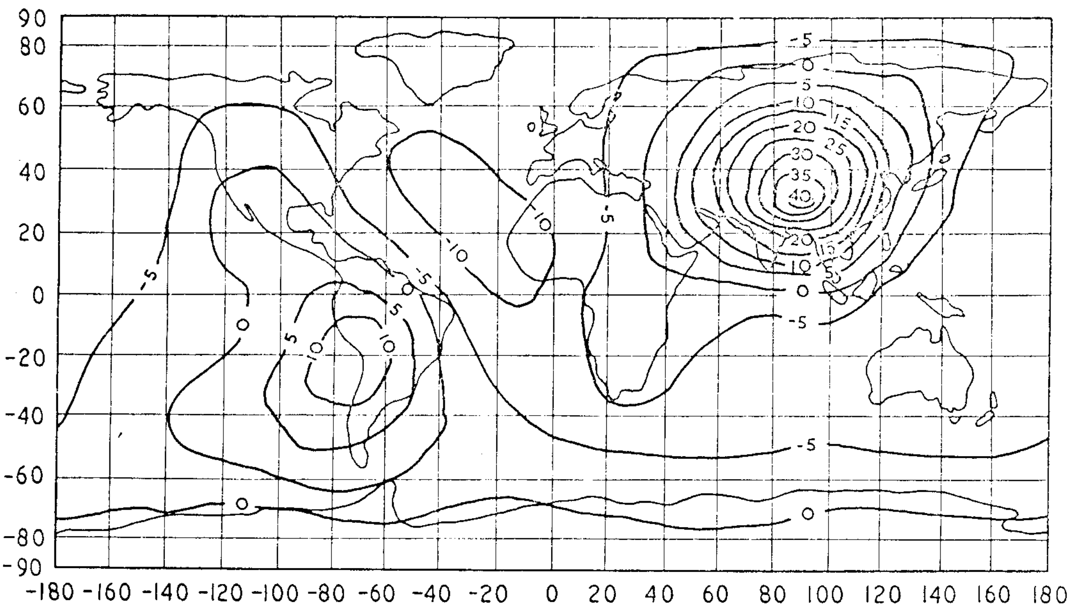
EVALUATION OF THE GRAVITATIONAL POTENTIAL AND ATTRACTION OF A MODEL OF THE EARTH'S TOPOGRAPHY AND COMPENSATION

ABSTRACT

Development of a method of directly computing the gravitational potential and attraction of the earth's topography and isostatic compensation, at geoidal, surface, and orbital elevations, is reported. Some preliminary results, from a programme of world-wide numeric evaluation, illustrate the basic features of these fields and demonstrate some practical limitations to the attainable accuracy.

1. Introduction

In a recent study by FRYER (1970) estimates of the global indirect effect for the free-air geoid were computed. It has been shown (e.g. HEISKANEN & MORITZ 1967, p. 145; MATHER 1968 A, p. 45) that the free-air "co-geoid" closely approximates the geoid, and the associated indirect effect is, therefore, expected to be comparatively small. Fryer's estimates (summarized in figure 1) displayed appreciably greater magnitude than anticipated and remarkably slow attenuation of the effect with increasing distance from the major topographic masses.



(After FRYER 1970,p.155)

Figure 1. Total Indirect Effect for Global Solution

Consequently, the study reported here has been directed towards a general assessment of the effects of the earth's topography and isostatic compensation on its gravitational field and thus ultimately, the evaluation of the geoid. The approach has been to evaluate directly the Newtonian gravitational potential and attraction of a digital model of the topography and compensation. This procedure has assumed the compensated topography to be in complete isolation from the remainder of the earth, and all other energetic influences. The direct nature of this approach and its investigative bent, may be contrasted with the solution oriented, and thus more involved, derivations of Fryer, wherein the properties of the effect were not of primary concern.

As the major computational work for this study is continuing, only results of preliminary investigations are presented, and any conclusions must be tentative.

2. The Indirect Effect

Stokes' formula for the determination of the geoid requires gravity anomalies representing boundary values at the geoid. Consequently, matter exterior to the bounding surface (that is, topography above the geoid) is mathematically removed or displaced inside the boundary. The resulting disturbance of the gravitational potential, and hence the geoid, is referred to as the indirect effect. A formulation of the indirect effect for the free air geoid has been derived by MATHER (1968 A, 1968 B) and shown to comprise, like the geoid solution itself, a zero-order term, a potential dependent term, and a term with Stokesian characteristics. The free air geoid closely approximates a non-regularized geoid, in that the "direct" effect, due to the attraction of the topography and compensation, is dismissed from the reduction procedure.

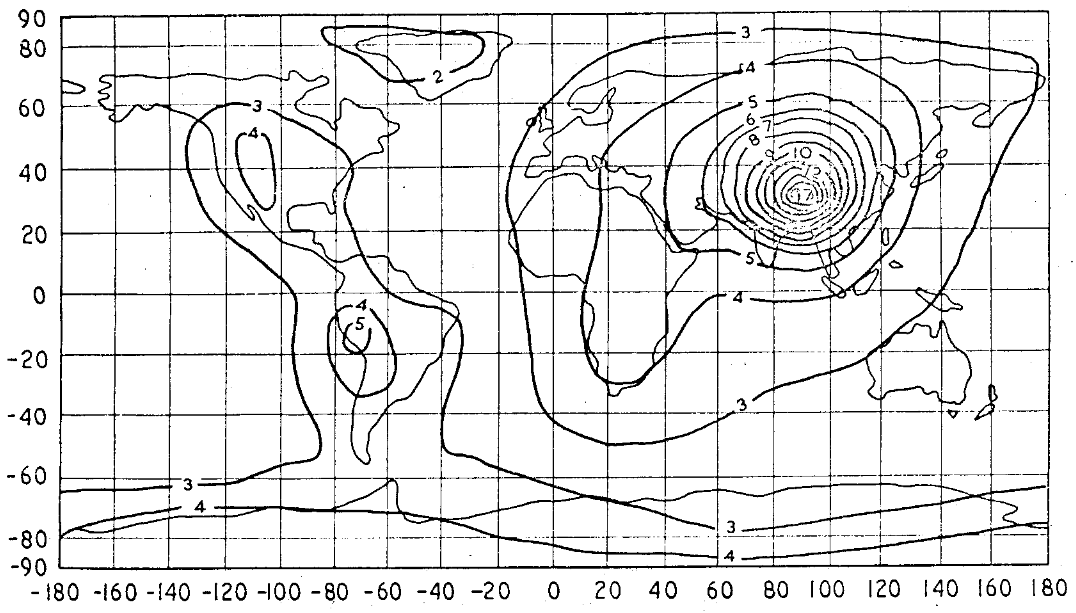
Fryer's detailed derivations of the potential and Stokesian terms for the indirect effect (FRYER 1970, p. 45 et seq) show the former to be predominantly a function of a residual potential, representing the difference between the potential due to the topography exterior to the geoid and that of its mathematical condensation as a surface layer on the geoid. The major contributing component of the Stokesian term arises from a Stokes integration of the differential terrain correction, being the difference in attraction of the topography between the surface and the geoid. Fryer's global estimates of these terms are illustrated by figures 2 and 3, respectively.

3. Computational Methods

Evaluation of the potential and attraction of the topography and compensation at a point, requires the solution of the following fundamental integrals, in 3 dimensions, derived from Newton's law of gravitation, (see figure 4).

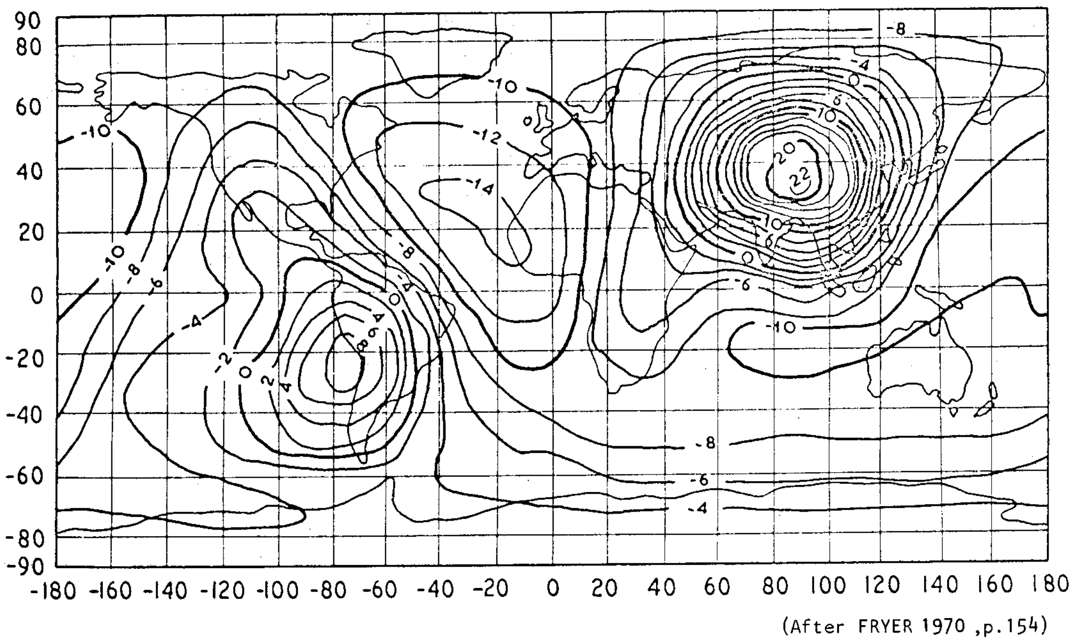
POTENTIAL: $$V = k \iiint \frac{\rho\, dv}{r}$$

ATTRACTION COMPONENTS: $$x_i = -k \iiint \frac{(x_i - \xi_i)\rho\, dv}{r^3} \qquad (i = 1, 3)$$



(After FRYER 1970,p.153)

Figure 2. Non-Stokesian Term in Global Indirect Effect Solution Derived from Numerical Integration Method. Contour Interval 1 m



(After FRYER 1970 ,p.154)

Figure 3. Stokesian Term to Order (6,6) in Global Indirect Effect Solution Contour Interval 2 m

where: k is the gravitational constant (6.67×10^{-11} $m^3kg^{-1}s^{-2}$)

dv is an elemental volume of matter,

ρ is the density of matter

r is the distance of the elemental volume from the point of computation,

x_i are the rectangular coordinates of the point of computation, and

ξ_i are the rectangular coordinates of the elemental volume.

A total solution for a particular point requires the limits of integration to be such as to include the whole of the topography and compensation. Then, to determine the spacial qualities of the gravitational field, evaluation may be repeated for a number of points distributed globally. Bearing in mind the contributing terms of the indirect effect, these points may be chosen at the geoid and at the surface of the earth. Points at an elevation representative of satellite orbits (500 km) may be similarly considered.

Whatever method of evaluating the integrals is employed, the limits of integration must be defined by adopting mathematical models to represent the extent of both the topography and compensation. The most commonly available digital data representing the topography is in the form of mean elevations, based on various "square" subdivisions of the geographical coordinate graticule. Isostatic compensation limits are defined in terms of the topographic elevations, after adopting a particular compensation model. Topographic elevations are usually referred to the geoid, however it is necessary to assume that this coincides everywhere with some basic figure of thé earth, so as to enable definition of the spatial distribution of the topographic masses with respect to a local coordinate system at each computation point. Polar coordinates in the local system are easily derived by a transformation from the general, geocentric reference frame.

A number of methods are available for solution of the integrals, including rigorous evaluation, numerical integration (cubature), and series expansion. If each topographic block, defined by a geographical coordinate system, is approximated by a homogeneous, rectangular parallelopiped, a rigorous solution of the potential and attraction integrals is available in general form (MACMILLAN 1930, p. 72). Alternatively, without need of the rectangular approximation, conventional numerical integration techniques can be applied, by dividing the block into smaller elements. A further solution is provided by series expansion of the reciprocal distance (1/r) in terms of Legendre polynomials (Ibid, p. 81), however the convergence of this series as r approaches zero requires careful consideration. The applicability of each of these methods is dependent upon the accuracy sought, and is therefore governed mainly by the distance of the topographic block from the computation point. This leads naturally to the introduction of a set of zones, surrounding the computation point, within each of which, a particular method is applied, consistent with the required accuracy. Accuracy is also affected by the degree of subdivision in each zone, because of the redistribution of matter which results from adopting a mean elevation over the extent of each block.

4. Assumptions and Their Effects

In view of the nature of the indirect effect as essentially a small correction term in the overall determination of the geoid, it has been assumed that accuracy corresponding to the order of the flattening should be sought throughout the derivations and calculations. This criterion may not always be fulfilled in practice, largely because of external limitations imposed by the available data. The major assumptions and models involved in the evaluations are summarized in table 1.

TABLE 1

ASSUMED MATHEMATICAL MODELS

1. Figure of the Earth:
   International Ellipsoid, 1967
   Semi-Major Axis $a = 6\ 378\ 160$ m
   Flattening $f = 1/298.25$

2. Isostatic Compensation:
   Airy-Heiskanen Model
   Crustal Thickness $T = 30\ 000$ m
   Sub-crustal Density $\rho = 3270$ kg m^{-3}

3. Density of Topography:
   de Graaff-Hunter Model
   Density of Column (kg m^{-3}) $= 2770 - h/21$ ($h \leq 2100$ m)
   $= 2670$ ($h > 2100$ m)
   where h is the mean elevation in metres.

4. Topographic Data:
   A Set of Digital Models, Comprising Mean Elevations
   Based on Geographical "Square" Subdivisions of 5' and 1°.

A figure of the earth is required to determine the distribution of the topography with respect to the computation point, but the influence of its departures from reality on the results is negligible. Indeed a plane model would be adequate in the inner zone and the difference between a spherical and a spheroidal model in the outer zones is quite insignificant. A spheroidal model was adopted, since only a slight saving in computation time would accrue from the use of any less realistic model.

The Airy-Heiskanen model for compensation was adopted as one which reasonably fits the available evidence, without introducing the unmanageable complexity of a regional model into the process of storing and accessing the digital data. It has the further advantage of facilitating comparisons with results of other studies.

Rather more concern must arise from the choice of a density model. Any departure from reality of the densities used is propagated directly into the solution, since they act much like a common scaling factor in the calculations. It is most unlikely that any available model is correct to the order of the flattening and the effect of mass stratification on the gravity field is ignored. Certainly, a better model could be compiled, from the available geological mapping at least, though the task would be a major undertaking.

Topographic mean elevations introduce two important sources of error. The first depends on the primary accuracy of the data, which is determined by its source and method of compilation. Clearly, the user of such data has little or no control over these factors. Mean elevation data used in the present study has been gathered from four sources:

(1) One degree mean values were compiled from the available five minute data, as listed in (2) and (3) below and, where this coverage is incomplete, from one degree values prepared by W. H. K. Lee of the University of California.

(2) Five minute data for North America and Europe was supplied by the U.S.A. Defense Mapping Agency (CZARNECKI 1970).

(3) Five minute values for Australia were derived by linear interpolation from the tenth degree data compiled by MATHER (1968 A).

(4) Five minute values for the remainder of the world are being simulated. A linear correlation of regional geomorphology, based on the one degree data, is used to provide a linear model for conversion of known five minute values to areas where such data is lacking.

There are approximately 3.2 million positive five minute mean values in the global model.

The second prominent source of error relates to the use of a digital model composed of mean values. Representation of the topographic elevations within some finite and relatively large area, imposes an apparent shifting of mass and an overall smoothing of gradients. Both factors contribute to a distortion of the gravity field being evaluated, to the extent that the digital model misrepresents reality. It should be noted that the five minute data, including the simulated values, enter into the calculations pertaining to the inner zones only. Outer zone calculations are therefore based entirely on "realistic" data.

5. Preliminary Investigations

A number of preliminary comparisons of formulae and methods and numerical checks have been completed. In particular, the behaviour of formulae, and the effect of approximations in the contact zone (within 5' of the computation point), where the value of r approaches zero, have been closely investigated. A comparison of evaluations of the potential of the topography in this zone, using a cylindrical assumption (Ibid, pp. 92 - 104), by numerical integration based on Simpson's formula, and by a rigorous formula for a rectangular prism, is summarized in table 2.

TABLE 2

COMPARISON OF METHODS OF EVALUATION OF THE POTENTIAL OF THE CONTACT ZONE

Dimensions of Block: 11 120 m ($0^{\circ}.1$) sq. x 5000 m high
Density: 2670 kg m^{-3}

METHOD	POTENTIAL (Jkg^{-1})
Cylindrical Assumption	58.42
Rad. = 12 547 m	
Numerical Integration	
No. of Intervals = 2 x 2 x 2	79.20
4 x 4 x 4	55.63
6 x 6 x 6	61.20
8 x 8 x 8	57.40
10 x 10 x 10	59.21
12 x 12 x 12	57.78
Rigorous Formula	57.68

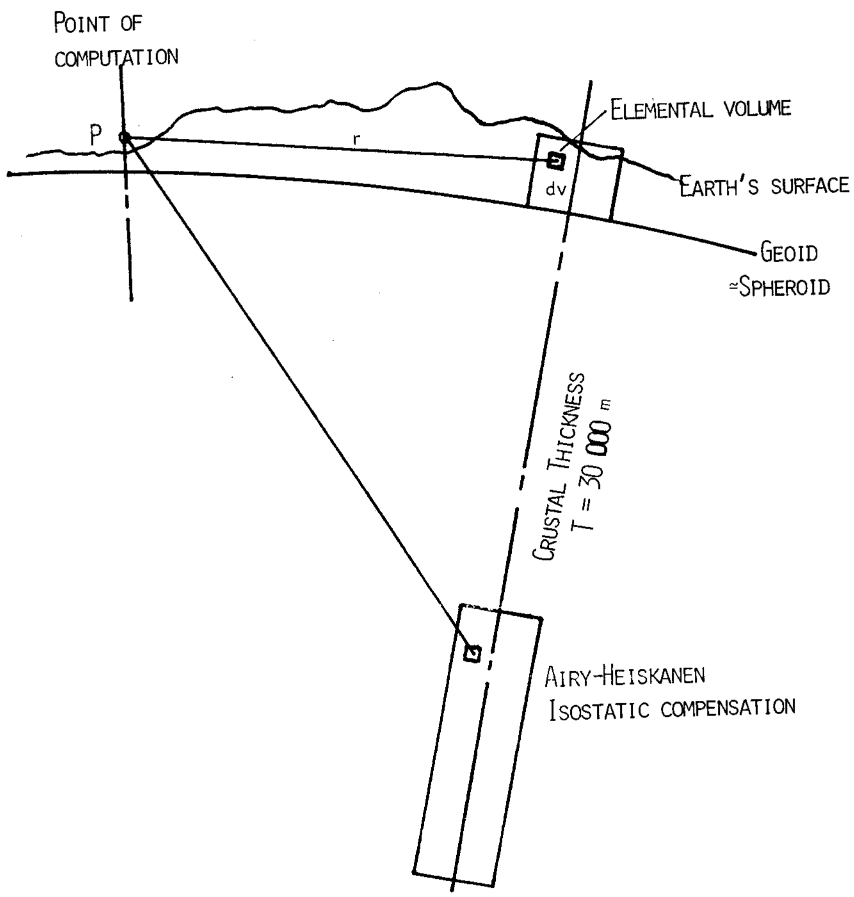


Figure 4. Integration of Topography and Compensation

As might be anticipated, the cylindrical assumption tends to slightly over-estimate the solution, though it provides a valid result in this unstable region, near the discontinuity in the reciprocal distance. Although numerical integration does not entirely fail under these circumstances, it is seen to present an oscillating solution as the number of intervals is increased. Further investigation has revealed that this effect is not present beyond the contact zone. However, any saving in computation time for this method, relative to a rigorous solution, is marginal.

Another method, relevant to all but the contact and inner zones, is provided by expansion of the reciprocal distance in a series, dependent on Legendre polynomials. The first term of such a series is no more than the commonly employed sphere, or point mass, approximation. Experimental calculations, with up to five terms, have shown that the first term alone provides a satisfactory result, when the distance r is greater than about ten times the maximum dimension of the topographic block. Consequently, this approximation may be safely applied beyond about 30' from the computation point.

Table 3 contains the definition of zones used, the size of data subdivisions within each zone, and the method of evaluation employed. A degree of flexibility of zone boundaries and data subdivisions has been programmed into the computer routines.

TABLE 3

METHODS OF EVALUATION AND DEFINITION OF ZONES

ZONE	LIMITS	SIZE OF DATA SUBDIVISIONS	METHOD
CONTACT	0 to 5'	5' x 5'	RIGOROUS
INNER	5' to <2°*	5' x 5'	RIGOROUS
NEAR	30' to 2°*	5' x 5'	SERIES EXPANSION
OUTER	>2°	30' x 30' † 1° x 1° 5° x 5°	SERIES EXPANSION

* Boundary of inner and near zones varied, depending on mean elevations

† Data subdivisions in outer zone varied, depending on "ruggedness" of topography

Hence the change from rigorous formula to series evaluation is varied depending on the mean elevations. A similar refinement is included in the outer zone evaluation, where the presence of steep topographic gradients within a data subdivision can introduce a significant error, if they are not depicted more realistically by smaller data subdivisions. For instance, if a 1° block of topography, 5° from the computation point, contains elevation differences of about 3000 metres (such as may occur in the Himalayas), an error approaching 3% is introduced by adopting a mean elevation for the whole area. Under these circumstances, the computer routines are designed to resort to a smaller data subdivision.

Results of some early evaluations for the contact zone are illustrated in figure 5. Gravitational potential at the surface, due to the topography and compensation within five minutes of the computation point is plotted for a profile along the 19^{o} north parallel of latitude, where it crosses the Eastern Sierra Madre in Mexico. This area depicts well the effects of steep topographic gradients, relatively large elevations, and the sea coast. A high degree of correlation clearly exists between the potential and the elevations, and there is no evidence of slow attenuation of the field. The vertical component of the gravitational force behaves similarly. Simplified test evaluations have also provided a provisional indication of the general trends to be expected for the outer zones. Apparently, the influence of the topography and compensation on the gravitational field extends further than anticipated, resulting in a noticeable smoothing out of the high correlation with local topographic variations, seen in the contact zone. This is substantially in agreement with Fryer's results, where the outer zone contribution to the potential term was found to be fairly constant at 6.3 metres, or about 37% of the total (FRYER 1970, p. 127).

6. Summary

1. Fryer's global estimates of the indirect effect for the free air geoid are larger than expected and display slow attenuation away from the sources of topographic influence.

2. This study attempts to investigate these results, using a different approach, based on a direct global evaluation of the gravitational potential and attraction components due to the topography and isostatic compensation.

3. Mathematical models have been adopted to define the figure of the earth, the isostatic compensation, and the density. A digital model, including some simulated data, is used to represent the topography.

4. Preliminary investigations indicate that the influence of nearby topography (the contact zone) does not contribute to the effect described by Fryer. These investigations further suggest that the topography in the outer zones (at distances of the order of the separation of the continents) may exert a significant influence on the magnitude and gradient of the gravitational field.

7. References

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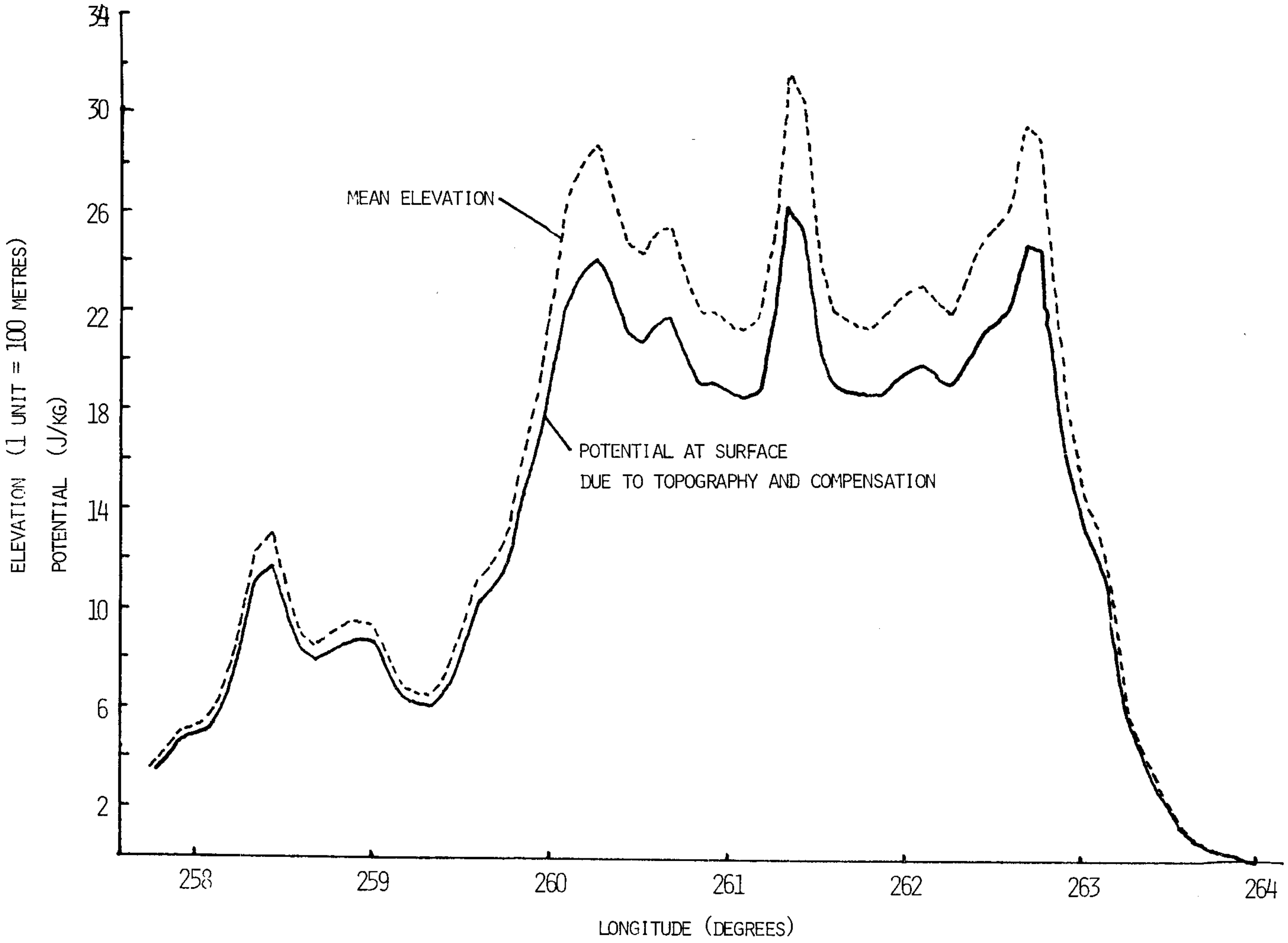

Figure 5. Gravitational Potential Due to Contact Zone (4 × 5' Squares) and Mean Elevations at Latitude 19° north

8. Discussion

FRYER: About the matter of slow attenuation; my calculations agreed with ANDERSON's that the attenuation from a topographic block in isolation is quite rapid and can be modelled against topographic shape - however, when considered from a global calculation, the actual distribution of data has a cumulative effect, making the attenuation around a large mass appear slow.

ANDERSON: I can't comment at this stage as I don't have global solutions but only preliminary calculations over selected areas.

QURESHI: Is the calculation being carried to an angular distance of 5 degrees from the computation point?

ANDERSON: At each point I evaluate; so all topography is taken into account at each computation point.

REILLY: In the topographic-isostatic model you are adopting, are the masses of the topography and compensation of a given column equal and opposite? I note that you use a variable density for the topography.

ANDERSON: The variable density is constant for a given block. It varies from block to block. As the density of compensation is based on the value assumed for the density of the topography, the net effect is equal and opposite.