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POSITION FROM GRAVITY

ABSTRACT

Procedures for obtaining position from surface gravity observations and reviewed and their relevance assessed in the context of the application of modern geodetic techniques to programs of Earth and ocean physics. Solutions based on the use of surface layer techniques, the discrete value approach, and the development from Green's third identity are stated in summary, the latter being extended to order e^3 in the height anomaly.

The representation of the surface gravity field which is required in order that this accuracy may be achieved is discussed. Interim techniques which could be used in the absence of such a representation are also outlined.

The role which can be played by the determination of changes in observed gravity to a few microgal, in the definition of geodetic reference systems for long period studies in Earth physics, is discussed and the consequences of changes of zero degree summarized. The possible use of these techniques in future geodetic practice is also assessed.

1. Introduction

1.1 Preamble

At first glance, it would appear that geodesists of today should be grateful for the activities of exploration geophysicists which have made significant contributions to all the well known data banks available at present. After all, it is only in the past two decades that the course of events has looked favourably on the collection of gravity information with solely geodetic objectives in view. All the available surface gravity information has still not been able to provide meaningful definitions of position on its own at the time of writing. The techniques of satellite geodesy revolutionized practical determinations related to position from a consideration of the Earth's gravity field, and there is little argument that these methods give relevant position-related parameters with a precision of about 2 - 4 %, the higher precision estimate being obtained on the use of combination techniques incorporating surface gravity information with satellite data.

Many factors have changed since the advent of the satellite era less than two decades ago. A technical advance of importance is the development of the surface ship gravimeter which provides a means of defining the surface gravity field in ocean areas with a resolution which if used advantageously, can be shown to be adequate for all present day requirements in Earth and ocean physics. A second development of great significance is the sensational improvement achieved in the precision with which absolute determinations of gravity based on interferometric techniques can be made (e.g., COOK 1965; FALLER 1965). The permanent installation maintained by the Bureau International des Poids et Mesures (BIPM) at Sevres, France, has been achieving a measuring precision of ±3 μgal for some years now (SAKUMA 1971), improving the resolution of g from 2 parts in 10^6 to about 3 parts in 10^9 in less than a decade. To this should be added the capability which has been available for some time, and enables the measurement of *differences* in gravity with an accuracy better than 1 part in 10^5 on land

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without any appreciable measurement time.

These significant improvements in metrology pose a series of interesting problems which must be dealt with before the maximum geodetic information can be obtained from surface gravity measurements. In the first instance, it becomes necessary to review the implications of adopting a rigid body model for the Earth as the basis for computations of position from surface gravity data. Further effect to be considered is the change in the Earth space location of the rotation vector and their influence on the determination of position from gravity. Short period mass changes smaller in magnitude than Earth tide effects, and possibly more difficult to model, may also have to be considered. Into this category fall changes in atmospheric circulation patterns from some model, variations in the water table and similar phenomena. Over a long time scale, it is necessary to consider the implications of a possible secular variation in the gravitational constant G.

As most of these effects are 7 - 8 orders of magnitude smaller than that of g, it has been accepted practice to convert observed gravity g to the gravity anomaly Δg by differencing g from the value γ of normal gravity for a model of the Earth, afforded by the value of GM, the rate ω of rotation of the Earth, assumed to be constant, together with the equatorial radius a and flattening f of an ellispoid of revolution which "best fits" the geoid. No allowance is made for the possibility of variations with time, in any of the parameters defining this system of reference. This is not inconsistent with the concept of determinations relevant to a certain epoch, provided

(a) the observations used are all made during the epoch considered; and
(b) the accuracy sought is less than 1 part in 10^7 in g.

The magnitude of gravitational deviations from a solid Earth model are smaller than $o\{10^{-6}g\}$. The largest effect is the diurnal Earth tide variation with magnitude $o\{10^{-7}g\}$. It has not been considered necessary at the present time, to recommend the adoption of a systematic procedure for modeling and removing the effect of the Earth tides from observed gravity except when establishing gravity standardization networks, in view of the limited accuracy of elevation data used in computing the gravity anomaly. This would call for the adoption of a universally acceptable model for Earth tides, which would then be used as a matter of routine to correct observed gravity prior to use in geodetic computations. Such corrections are only necessary at fundamental gravity stations where determinations are made with the highest possible resolution for either the definition of global gravity standardization networks (MATHER 1973,p.68) or when attempting to locate changes in the position of the Earth's centre of mass (geocentre) with time (MATHER 1972,p.13). The need for applying such corrections at other stations will depend on the extent of gravity coverage available locally and whether elevation datums have been unified at the 50 cm level.

Current practice accepts the validity of each individual nation's elevation datum as well as its gravity datum. The continuance of such a practice is unwise if systematic errors at the 50 cm level are not to occur in the final results. The most taxing goal is the definition of the geoid in ocean areas to the highest possible accuracy, in order that such results could be used with satellite altimeter data to study ocean circulation by defining the sea surface topography. On present trends, it would appear that 5-10 cm accuracy is desirable in geoid determinations for this purpose.

It is in this context that the use of gravimetric techniques in the determination of geodetic position should be reviewed. The present development covers

(a) the basic developments underlying the determination of position from gravity;
(b) a review of some of the methods suggested to the present time, for solving the boundary value problem in physical geodesy;
(c) techniques for the preparation of data sets for this task; and
(d) the geodetic interpretation of such solutions in Earth space.

In all sections, an assessment is made of the requirements which will have to be met in order that the independent evaluation of selected geodetic characteristics available from surface gravity determinations, can be used in the resolution of some possible ambiguities from other methods when applied to studies of high precision for Earth and ocean physics.

1.2 A guide to notation

1.2.1 Recurring Symbols

a = equatorial radius of the ellipsoid of reference

$\vec{d}$ = separation vector between equivalent points P on the Earth's surface and Q on telluroid

dz = increment in orthometric elevation

$d\sigma$ = element of surface area on unit sphere

e = eccentricity of the meridian ellipse; $e^2 = 2f - f^2$

$F(\psi) = f(\psi) \sin\psi$

f = flattening of meridian ellipse

$f(\psi)$ = Stokes' function = $\operatorname{cosec} \tfrac{1}{2}\psi + 1 - 5\cos\psi - 6\sin\tfrac{1}{2}\psi - 3\cos\psi\left(\log\{\sin\tfrac{1}{2}\psi(1 + \sin\tfrac{1}{2}\psi)\}\right)$

g = gravity as observed at the surface of the Earth

h = ellipsoidal elevation

h_d = height anomaly

h_n = normal elevation

M = mass of the Earth, including the atmosphere

$M\{X\}$ = global mean value of X

$m = a\omega^2/\gamma_e$

$m' = a^3\omega^2/GM = m + o\{f^2\}$

N = elevation of geoid above ellipsoid

$\vec{N}$ = unit vector normal to the surface of the Earth

N_f = free air geoid; the Stokesian contribution to h_d

N_c = indirect effect to free air geoid; non-Stokesian contribution to h_d

R = geocentric distance

$\overline{R}$ = radius of sphere containing all topography - the Brillouin sphere

R_b = radius of sphere which is internal to the Earth's surface - the Bjerhammar sphere

R_m = mean radius of the Earth

r = distance between the point of computation P and the element of surface area dS

$\overline{r} = 2\overline{R}\sin\tfrac{1}{2}\psi$

$r_o = 2R_m \sin\tfrac{1}{2}\psi$

S = surface of the Earth

U = spheropotential due to the system of reference

U_o = spheropotential on the surface of the reference ellipsoid

V_d = disturbing potential

W = geopotential

W_o = potential of the geoid

X_i = geocentric rectangular Cartesian co-ordinate system $X_1X_2X_3$

x_i = local rectangular Cartesian co-ordinate system $x_1x_2x_3$ with the x_3 axis along the local normal, the x_1x_2 plane defining the local horizon, with axes oriented north, east.

α = azimuth

β = ground slope; subscripts $_1$ and $_2$ refer to components north and east

γ = normal gravity due to reference system; subscripts $_o$ and $_e$ refer to values on reference ellipsoid and equatorial gravity respectively.

Δg = gravity anomaly at the surface of the Earth, defined by equation 10

ΔW = geopotential difference with respect to the geoid

δg = gravity disturbance

λ = longitude, positive east

ξ = components of deflection of the vertical; subscripts $_1$ and $_2$ refer to valuesin the north and east directions respectively

ζ = deflection of the vertical, positive if outward vertical lies north, east of normal

Φ = density of surface layer, except in section 3.4

ϕ = latitude, positive north; subscripts $_c$, $_a$, $_g$, refer to geocentric, astronomically determined and geodetic latitudes respectively.

ψ = angular distance at geocentre between the point of computation P and the element of surface area dS

ω = angular velocity of rotation of the Earth

$$\vec{\nabla} = \sum_{i=1}^{3} \frac{\partial}{\partial x_i}\vec{i}$$

1.2.2 Conventions

$a = b + o\{b^2\} \equiv$ terms whose order of magnitude is equal to or less than b^2 are neglected $(b < 1)$

$x_\alpha y_\alpha = x_1y_1 + x_2y_2$

$x_i = a_{ij}b_j \equiv x_i = a_{i1}b_1 + a_{i2}b_2 + \ldots$, there being as many equations as possible values of i

$a \doteq c \equiv$ a is approximately equal to c

2. Basic principles

2.1 The system of reference

The principle behind the determination of Earth space position from gravity observations made at the surface of the Earth is implied from deviations of observed gravity from values at an "equivalent" point on some Earth model, whose parameters are completely defined. Current geodetic practice (IAG 1970,p.12) specifies a rigid body model by the following parameters.

(a) The value $\mu(= GM)$ where G is the gravitational constant & M the mass of the Earth.

(b) The constant rate ω of rotation of the rigid Earth model.

(c) The equatorial radius a of an ellipsoid of revolution which presumably is one of best fit to the geoid.

(d) The dynamic form factor J_2 which is equivalent to a value for the flattening f for the reference ellipsoid.

It is conventional to choose the value for a such that the ellipsoid has the same volume as the geoid. This is not a necessary condition if zero degree effects are taken into account when formulating a solution for the boundary value problem. What is more important in solutions which aspire to accuracies greater than the order of the flattening (i.e., ±30 cm in the height anomaly), is that the ellipsoid lies everywhere within the physical surface of the Earth. This permits the use of Laplace's equation in the representation of the appropriate disturbing potential without approximation. The adoption of such a procedure without an equivalent adjustment in μ could cause larger numerical values of the gravity anomaly, which would in turn, call for greater caution in developing computer algorithms for evaluation purposes.

It can be stated without being contentious that the value adopted for a has to be based on some determination of the scale of Earth space. This would be provided by either the measurement of long arcs at the surface of the Earth by classical techniques (e.g., the PAGEOS baselines) or else by laser ranges to either satellites or the moon. All determinations of scale are therefore based on the velocity of light and the definition adopted for the basic interval of time. The value of the flattening f of the reference ellipsoid is best deduced from the second degree zonal harmonic obtained from the secular variations in the right ascension Ω of the node and the argument ω of perigee of near Earth satellites. The precision claimed at the present time (e.g.,LERCH ET AL 1972, p.27) for this harmonic is 1 part in 10^5, the required relation being (e.g., MATHER 1971a,p.85)

$$C_{20} = \frac{1}{3}m' - \frac{2}{3}f - \frac{3}{7}m'f + \frac{1}{3}f^2 + o\{f^3\} \qquad (1),$$

where

$$m' = \frac{a^3\omega^2}{GM},$$

ω being the angular velocity of rotation of the Earth. The exact relation between the observed secular variations $\dot{\Omega}_{20}$ and C_{20} is (e.g., IBID,p.151)

$$\dot{\Omega}_{20} = \frac{3(GM)^{\frac{1}{2}}}{2(1-e_s^2)^2}\frac{a^2}{a_s^{7/2}}\cos i\; C_{20} \qquad (2),$$

a_s being the equatorial radius of the satellite orbit, e_s its eccentricity and i the inclination.

The change df in f due to changes da, $d\omega$ and d(GM) in a, ω and GM are given by

$$df = \frac{3}{2}m'\left(3\frac{da}{a} + 2\frac{d\omega}{\omega}\right) - \frac{1}{2}\frac{d(GM)}{GM}(3m' + C_{20}) + o\{f^2 df\} \qquad (3).$$

The ratio df/f is therefore of the same order of magnitude as d(GM)/GM for a specified value of a as the ratio $d\omega/\omega$ is at least an order of magnitude smaller, if these ratios reflect the precision with which each of these quantities are determined.

It is all-important that the rotational characteristics assigned to the reference model are exactly equivalent to those influencing gravity as measured at the Earth's surface. This is implicit in deriving equation 60 from equations 58 and 59. The rotation vector in Earth space is not fixed and hence deviations from the rigid body model occur in practice. The rate ω of rotation is subject to secular variations due to tidal friction. Certain shorter period effects have to be accounted for, at least in theory, when reducing gravity to a rigid body equivalent of the Earth; the practical consequences are negligible as their magnitude is less than 1 part in 10^9 in g. A second factor is the change in position of the instantaneous axis of rotation with respect to the Earth's crust. The total contribution of the rotation to observed gravity is the appropriate resolute of

$$g_r = p\omega^2 \qquad (4)$$

directed away from the axis of rotation and perpendicular to it. The changes dg_r in g_r due to changes dp in p, which is the distance of the point at which gravity is measured, from the axis of rotation, and $d\omega$ in ω is given by

$$dg_r = g_r\left(2\frac{d\omega}{\omega} + \frac{dp}{p} + o\{10^{-12}\}\right) \qquad (5).$$

The effect of the ratio $d\omega/\omega$ will be less than 1 μgal on observed gravity for a 10 msec variation in the length of day, and hence this term is not of significance in the reduction of observed gravity if the latter were restricted to some epoch of observation. The effect of polar motion is reflected in the second term in equation 5 and is o{1 μgal} (BURSA 1972).

Short period changes in observed gravity with larger magnitudes have been reported by SAKUMA (1971) after modeling the effects of Earth tides. It should be noted that a 1% change in the local atmospheric density is of order 10 microgal, while quasi-stationary changes in the local geological formations, e.g., in the local water table, could cause gravitational effects of this same magnitude.

It is therefore important that both the atmosphere and the local geology be modeled in the vicinity of those gravity stations at which g is to be re-measured at regular intervals with the highest possible precision.

The dominant gravitational variation with time is that due to Earth tides, with magnitude o{10^2 μgal} and Earth models for this effect are well known in the literature (e.g., MELCHIOR 1966). It is important that an unambiguous Earth tide model at the 10 μgal level be uniformly adopted when specifying gravity values at stations comprising the global gravity standardization network described in section 4.

The final parameter defining the reference system is μ(=GM). The commonly accepted values of GM are presently based on the analysis of interplanetary space probes. The technique used can be briefly summarized as follows (ESPOSITO & WONG 1972). Doppler data from inter-planetary space probes is analysed using numerical integration procedures for the determination of the motion of the probe with reference to a geocentric inertial co-ordinate system. Perturbations due to the Earth's departure from a sphere, solar radiation pressure, planetary and lunar gravitational effects and spacecraft attitude control forces are modeled when effecting this solution, which also provides revised estimates of tracking station co-ordinates as a by-product of the solution. The main conclusions of relevance to the present review are the following. Firstly the value of GM is based on the velocity of light. Secondly, the potential U_o on the surface of the reference ellipsoid, assumed to be an equipotential, is related to the values adopted for ω, a, f and μ by the relation (e.g., MATHER 1971a,p.83)

$$U_o = \frac{GM}{a} \frac{\sin^{-1} e}{e} + \frac{1}{3} a^2\omega^3 ,$$

where

$$e^2 = 2f - f^2.$$

As GM has an uncertainty of 1 part in 10^6 at the present time, it follows that U_o will differ from the true potential of the geoid consistent with Newtonian gravitation, as scaled by the velocity of light by at least 1 part in 10^6. If U_o were assumed to be equal to the potential W_o of the geoid, it would be tantamount to imposing a second scale constraint when using gravitational techniques in geodesy. The only way out of this impasse is to use external geometrical information which when combined with the gravitational solution, will give an improved estimate of W_o as discussed in section 5. Such an estimate would be consistent with the scale provided by the velocity of light. In the interim, it should be borne in mind that all position determinations based on gravity alone may have a constant scale error of upto 1 part per million on this account.

In summary it may be stated that

(a) Only Earth tide effects need be allowed for in all work except those determinations required for the monitoring of co-ordinate systems;

(b) Position determinations based on gravity alone are liable to have a constant scale error of upto 1 part per million due to the uncertainty in the assumption $U_o = W_o$.

The following conclusions may therefore be drawn about the adoption of a rigid body model of the Earth as an intermediary in the definition of position from gravity, with the highest possible precision as the ultimate goal.

(i) The only departures from rigidity which need be considered for solutions of the boundary value problem are the effect of Earth tides on gravity observations especially when establishing gravity standardization networks.

(ii) An error in GM will give rise to ambiguities in scale if W_o is forced to be equal to U_o. For further discussion, see section 5.1

2.2 Data requirements

Observed gravity will be the result of two kinds of determinations. The first type will be absolute determinations with the highest precision possible, while the second will be point values established by differential techniques based on the former, and with a precision which is about one order of magnitude inferior. The practice adopted in gravimetric determinations is the use of gravity observations and a knowledge of the surface topography of the Earth to determine the separation vector $\vec{d}$ between equivalent points on the reference model, whose Earth space position is known, and the physical surface of the Earth, as illustrated in figure 1. The separation vector can be completely defined by the height anomaly h_d and the angles ξ_α which are more completely defined in the next sub-section.

The most exacting requirements are called for in the definition of position from gravity when determining the geoid for ocean physics applications, where present estimates of requirements call for resolution at the ±10 cm level. The equivalent order of magnitude is e^3(i.e., 5 parts in 10^4) which can be assessed as ±50 μgal in the gravity anomaly Δg. On the basis of the discussion in the previous sub-section, it would be adequate to maintain a rotating rigid body model as the system of reference and apply the appropriate reductions to observed gravity to make the measurements compatible with the model. The position so defined will be unaffected by short period time variations in the Earth's gravitational field.

The nature of the reductions necessary will depend on the purpose for which the gravity data is required, the accuracy with which it has been established and the nature of the elevation data available for its reduction. Earth tide corrections necessary at stations monitoring changes in g for global reference system definition (MATHER 1972,section 3.3) should be capable of resolution to 1 gal. Some difficulty may be experienced in removing ocean lodaing effects, especially in coastal areas, as these could influence the tidal correction in the second significant figure (HENDERSHOTT 1972). Corrections for short period variations of the atmosphere and the stratigraphy in the vicinity of gravity stations monitoring the reference system, are also necessary. This presupposes the existence of "accepted" models for both the atmosphere and the local stratigraphy, and the effect meteorological changes may have on them. While Earth tide effects should be allowed for when making any gravity determination, the summary in section 4 indicates that the effect of omitting the correction on determinations of position from gravity is likely to be negligible as the tidal effect has the characteristics of a random measurement error. All further discussion will therefore assume observed gravity g as having been observed on a rigid Earth, rotating with uniform angular velocity, the gravitational effects of polar motion being corrected for when using gravity data for the definition of geodetic reference systems.

The formulation of relations at the surface of the Earth is based on the following principles.

(a) An estimate is available of the geocentric co-ordinates of the point P at the surface of the Earth. In classical terms, these surface co-ordinates (ϕ_a,λ_a) are related to the vertical at P by astronomical determinations, and can be evaluated at best to a factor of one or two better than 1 part in 10^6 (i.e., ±6 m in each co-ordinate). This estimate differs from the true geocentric co-ordinate by amounts upto 10^{-4} radians depending on the magnitude of the local deflection of the vertical.

(b) The displacement of P above the equivalent point P_o on the ellipsoid is defined by the normal elevation h_n, which is related to the difference in geopotential ΔW between the equipotential datum for elevations (the geoid) and P as obtained from levelling, by the relation

$$\Delta W = - \int_{\text{geoid}}^{P} g \, dz,$$

g being the observed gravity for the section of the line of levelling where the orthometric height difference is dz. The equation defining h_n in terms of ΔW is the relation (e.g., MATHER 1971a,p.100)

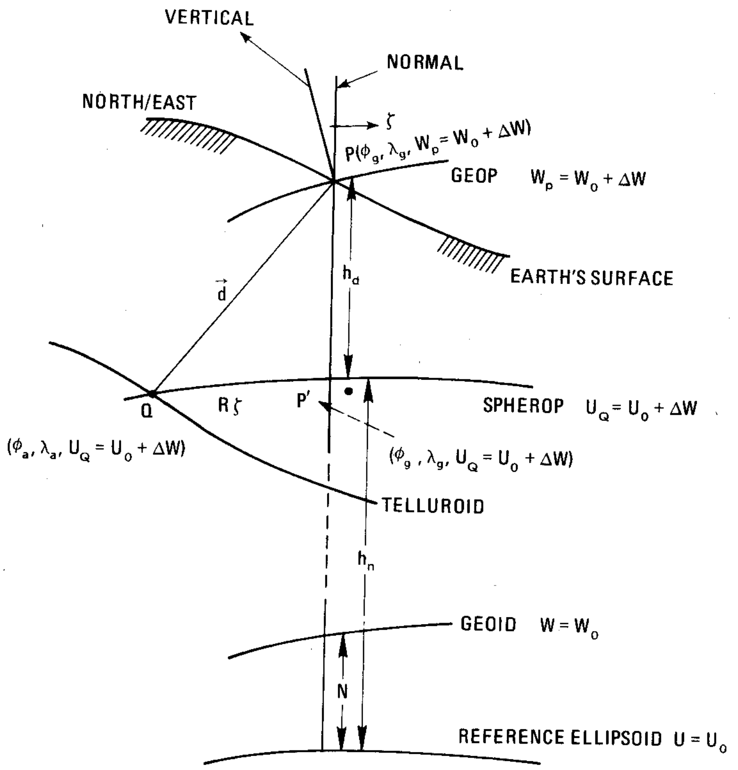

FIGURE 1 : The Separation Vector $\vec{d}$ and the Reference System

$$h_n = \frac{\Delta W}{\gamma_o}\left\{1 + \frac{\Delta W}{a\gamma_o}\left(1 + m + f - 2f \sin^2\phi\right) + \left(\frac{\Delta W}{a\gamma_o}\right)^2 + o\{f^3\}\right\},$$

where γ_o is the value of normal gravity on the reference ellipsoid, and

$$m = \frac{a\omega^2}{\gamma_e} \qquad (6),$$

γ_e being the value of normal gravity at the equator, and ΔW is treated without regard to sign for points exterior to the geoid.

It has been shown (MATHER 1973,section 4.3) that the data requirements for the determination of the height anomaly h_d with a precision equivalent to that possible in establishing h_n are well within the capabilities of measuring and data sampling techniques available at the present time. Thus position determination from gravity in any absolute sense

(i) requires a knowledge of astronomical co-ordinates; and

(ii) calls for a global representation of the gravity anomaly field.

The resolution of the information from positional astronomy at the present time will have to improve by a factor of 50 before the horizontal determinations are of adequate accuracy for the complete determination of position by this method alone. Such a determination will also require the determinations of the deflections of the vertical ξ_α to $o\{10^{-4}\xi_\alpha\}$.

It can therefore be concluded that the determination of geocentric position from positional astronomy and surface gravity to accuracies much in excess of 1 part in 10^6 may not be a practical possibility in the foreseeable future. The determination of the height anomaly on the other hand, will remain a problem of fundamental interest as it forms an integral part in the definition of sea surface topography from space. The ensuing development will continue to deal with the complete development necessary for the determination of position from gravity, but only in outline. More detailed review will be confined to the techniques for determining the height anomaly.

2.3 Basic Relations

The formulation of the Molodenskii problem (HEISKANEN & MORITZ 1967,p.291) can be treated as one which seeks the determination of the separation vector $\vec{d}$ between "equivalent" points $P(\phi_g,\lambda_g,W_p=W_o+\Delta W)$ on the Earth's surface and $Q(\phi_a,\lambda_a,U_Q=U_o+\Delta W)$ on the associated spherop $U=U_Q$ of the reference system, as illustrated in figure 1. If the subscripts $_a$ refer to values determined astronomically at P, the separation vector is given by

$$\vec{d} \;=\; R_\alpha \xi_\alpha \vec{\alpha} \;+\; h_d \,\vec{3} \qquad (7),$$

where R_α are the meridian and prime vertical radii of curvature of the associated spherop, $\vec{\alpha}$ are unit vectors oriented along the tangent plane to the spherop at Q in the meridian and prime vertical respectively, while $\vec{3}$ is the unit vector along the outward normal at Q. The subscripts $_g$ refer to the surface co-ordinates of the point P' in figure 1 on the associated spherop $U = U_Q$ whose normal passes through P.

The locus of the point Q is called the telluroid and mirrors the physical surface of the solid Earth and oceans to order f^2. The system of reference is based on the family of spherops $(U=U_o+\Delta W)$ exterior to the reference ellipsoid defined by the geometrical parameters a and f, and the gravitational characteristic μ(=GM), with the constraint that the surface of the reference ellipsoid is the equipotential surface $U = U_o$. As pointed out in section 2.1, there is no necessity for the ellipsoid to be forced to have the same volume as the geoid $(W=W_o)$ provided terms of zero degree were retained in the solution. In such circumstances it is no difficult to show that (e.g., MATHER 1973, p.14) the height anomaly (h_d in figure 1) is given by

$$h_d \;=\; \frac{1}{\gamma}\left(V_d \;-\; (W_o - U_o)\right) \;+\; o\{10^{-3}m\} \qquad (8),$$

where the disturbing potential at P is given by

$$V_{dp} \;=\; W_p \;-\; U_p \qquad (9).$$

The quantity W_p is defined by the value W_o of the geopotential on the equipotential surface used as the datum for geodetic levelling, and the observed difference of geopotential ΔW between this surface at P, by the relation

$$W_p \;=\; W_o \;+\; \Delta W.$$

The datum in use at the present time is that afforded by mean sea level derived from tide gauge readings over periods in excess of one year. As solutions of the geodetic boundary value problem require definitions which are applicable globally, it is essential that all regional definitions of mean sea level are correlated on a world wide basis to a common epoch in the first instance before the differences in geopotential ΔW can be considered to be referred to an equipotential surface of the Earth's gravitational field. It has been estimated that systematic errors of $o\{\pm 10$ cm$\}$ could result in solutions of the boundary value problem if errors on this account were of $o\{\pm 30$ cm$\}$ and each datum covered $o\{10^6$ km$^2\}$ (IBID,p.68).

A second problem of consequence and of which little is known at the present time, is the possibility of quasi-stationary departures of the sea surface from the equipotential surface defined by the results of geodetic levelling. The phenomenon, known as stationary sea surface topography, has been reported along coastlines in many parts of the world. A summary of some results is given by HAMON & GREIG (1972), indicating magnitudes of 30-50 cm being commonplace, with one reported as large as 1.7m over 3000 km. This effect is discussed further in section 4.

The gravity anomaly Δg at the surface of the Earth is defined by

$$\Delta g = g_p - \gamma_{p'} \tag{10}$$

where g_p is the value of gravity observed at P, corrected for departures of the Earth from a rigid body model, as described in section 2.1, and $\gamma_{p'}$ is obtained from the equivalent value γ_o of normal gravity on the reference ellipsoid, given by the commonly used relations of the type (e.g., HEISKANEN & MORITZ 1967,p.79)

$$\gamma_o = \gamma_e\left(1 + \beta \sin^2\phi_g + \beta_2 \sin^2 2\phi_g + o\{f^3\}\right) \tag{11}$$

where γ_e is equatorial gravity defined by the values adopted for a, f, GM and ω(e.g., IAG 1970,p. 48;MATHER 1971a,p.87), $\beta = o\{f\}$ and $\beta_2 = o\{f^2\}$, using the relationship (e.g., IBID,p.101)

$$\gamma_{p'} = \gamma_o - 2\frac{\Delta W}{a}\left(1 + f + m - 2f \sin^2\phi - \frac{1}{2}\frac{\Delta W}{a\gamma} + o\{f^2\}\right) \tag{12}$$

ΔW having the same significance as in equation 6.

Possible sources of systematic error in the computed value of $\gamma_{p'}$ and hence Δg arise in the definition of ϕ_g and ΔW. While the effect of errors in the latter have already been described, ϕ_g should be defined to ±0.4 arcsec if $\gamma_{p'}$ is not to have an error of approximately ± 10 μgal. It is therefore important to use any of the global solutions available at the present time for the definition of geocentric position, to evaluate geocentric orientation parameters for each of the regional geodetic datums (MATHER 1973,p.16) before computing gravity anomalies for high precision determinations, rather than use values referred to regional geodetic datums.

The equation described as the fundamental equation in physical geodesy (HEISKANEN & MORITZ 1967,p.86) defines the relationship between the disturbing potential V_d and the gravity anomaly Δg as (MATHER 1973,p.18)

$$\frac{\partial V_d}{\partial h} = - \Delta g + \frac{\partial \gamma}{\partial h} h_d \left(+ \frac{1}{2} g \zeta^2 + o\{1 \text{ μgal}\}\right) \tag{13}$$

where the terms within the bracket take into account effects smaller than $o\{f \Delta g\}$, ζ being the deflection of the vertical at the point considered.

The philosophy underlying equation 13 is the contention that the geocentric position of P is not known, though estimates adequate for the linearization of the quantities involved are available. Circumstances may well arise in the future where accurate horizontal and vertical surveys may be available and the principal practical role of techniques in physical geodesy is the determination of the geoid in ocean areas for study of ocean circulation. In such a situation, it is envisaged that all the land masses are linked to a geocentric system of reference using laser ranging methods and/or VLBI, giving at least one fundamental station on each geodetic datum. Horizontal survey methods together with geodetic and astro-geodetic levelling, will provide data for completely defining geocentric position of points on any regional network which includes at least one fundamental station, with an accuracy of 1 part in 10^6. As surface ship locations can be routinely determined to within one order of magnitude greater, it is of relevance to examine the gravity disturbance δg(e.g., HOTINE 1969,p.312), given by

$$\delta g \;=\; g_p \;-\; \gamma_p \;=\; -\frac{\partial V_d}{\partial h} + \frac{1}{2}\, g\zeta^2 \;+\; o\{1\ \mu gal\} \qquad (14),$$

where the uncertainties in defining the position of P can be estimated as ±0.2 arcsec in horizontal position and ±2 m in normal displacement, if the astro-geodetic levelling is based on an adequate distribution of stations. The effect of errors due to the first source on δg are $o\{1\ \mu gal\}$ while that of those due to the second are $o\{5\times10^2\ \mu gal\}$. Thus the gravity disturbance, whose order of magnitude is not significantly different to that of the gravity anomaly, is likely to have errors of $o\{5\times10^2\ \mu gal\}$ which are probably correlated with wavelengths in excess of 1000 km (e.g., MATHER, BARLOW & FRYER 1971,figure 4.2) unless radically new techniques are available for determining

either geocentric position at each gravity station such that the radial component is resolved with systematic biases of wavelengths longer than 1000 km held to below the 20-30 cm level;

or the contribution of astro-geodetic levelling with the same resolution as geodetic levelling.

The projection of present day techniques does not lead to the conclusion that there would be significant advantages in using the gravity disturbance δg in preference to the gravity anomaly Δg in formulating solutions of the boundary value problem.

The separation vector $\vec{d}$, illustrated in figure 1, can be represented by components along the axes of a local Cartesian co-ordinate system x_i at Q, with the x_3 axis oriented along the spherop normal at Q, as illustrated in figure 2, in accordance with equation 7. $\vec{d}$ is of importance in defining the geocentric orientation vector $\vec{O}$ for regional geodetic datums using surface gravity data (MATHER 1971b, p.62). A description of how such information could be used to assemble a world geodetic system linking the major land masses by comparing the separation vectors as obtained from gravimetry and astro-geodesy is given by MATHER (1971c).

The mainstream of practical endeavours at the present time is in the determination of the height anomaly h_d. Over 90% of the power in such determinations comes from the "free air geoid" N_f, obtained by the use of free air anomalies (i.e., surface gravity anomalies to the order of the flattening) in Stokes' integral which is set out out in equation 15. The latter is a solution of the boundary value problem for a spherical Earth which is exterior to all matter and whose bounding surface is an equipotential (STOKES 1849).

The deflections of the vertical ξ_α are usually obtained using the principles generally attributed to VENING MEINESZ (1928). Working on a spherical reference system, he showed that if the separation N_f between the physical and reference surfaces were given by Stokes' integral

$$N_{fp} \;=\; \frac{R}{4\pi\gamma} \iint f(\psi)\ \Delta g\ d\sigma \qquad (15),$$

where Δg is the value of the gravity anomaly at the element of surface area $d\sigma$ on unit sphere which is at an angular distance ψ from the point of computation P, $f(\psi)$ being Stokes' function (e.g., HEISKANEN & MORITZ 1967,p.94), then

$$\xi_\alpha \;=\; -\,\frac{\partial N}{\partial x_\alpha} \qquad (16),$$

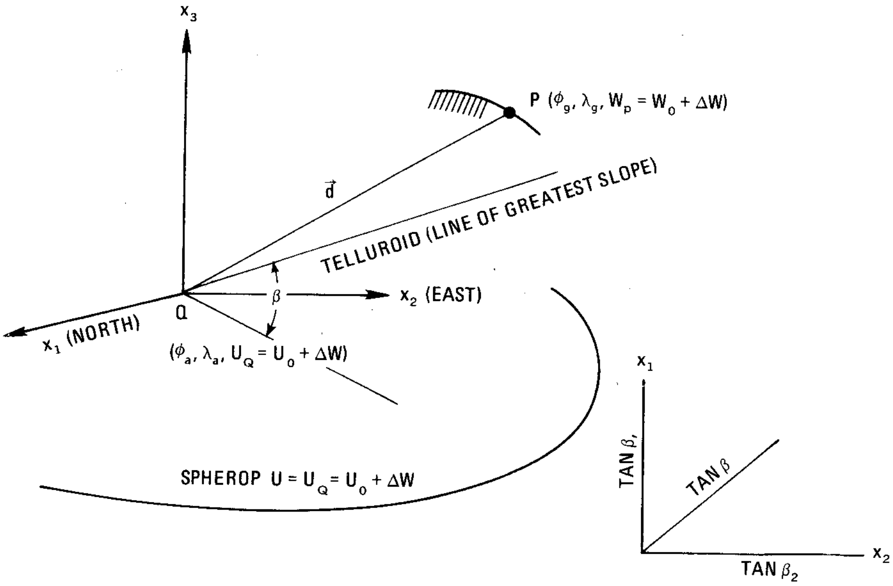

Figure 2. The Separation Vector and the Local Cartesian Co-ordinate System

as illustrated in figure 3, where x_α is a two dimensional Cartesian system in the horizontal plane at the point of computation P, with the x_1 axis oriented north and x_2 axis east. It is not difficult to show in the case of Stokes' problem that

$$\xi_\alpha = \frac{1}{4\pi\gamma} \iint \frac{\partial\{f(\psi)\}}{\partial\psi} \cos A_\alpha \; \Delta g \; d\sigma \qquad (17),$$

as

$$\frac{\partial}{\partial x_\alpha} = - \frac{1}{R} \frac{\partial}{\partial\psi} \cos A_\alpha \qquad (18),$$

where

$$A_1 = \alpha \qquad \text{and} \qquad A_2 = \tfrac{1}{2}\pi - \alpha \qquad (19),$$

α being the azimuth of $d\sigma$ from P. This follows as only ψ in the kernel of the integral at 15 varies as the point of computation changes from P to some adjacent point Q in the case of the Stokes problem.

The required expression for the Molodenskii problem is not the same as the elevation h_p of P appears in the kernel of the integral. As the deflection of the vertical at the surface of the Earth is obtained from the height anomaly h_d (IBID, p.312), which is given by

$$h_d = h_d(\phi,\lambda,h_p) = h_d(\psi,\alpha_\sigma,h_p) \qquad (20),$$

where α_σ is the azimuth of P from the element of surface area $d\sigma$, it can be shown that (MATHER 1971c, p.88)

$$\xi_\alpha = \frac{1}{h_\alpha} \left(\frac{\partial h_d}{\partial\psi} \frac{\partial\psi}{\partial u_\alpha} + \frac{\partial h_d}{\partial\alpha_\sigma} \frac{\partial\alpha_\sigma}{\partial u_\alpha} \right) \qquad (21),$$

u_α being the set of curvilinear surface co-ordinates on the reference surface, and h_α the associated

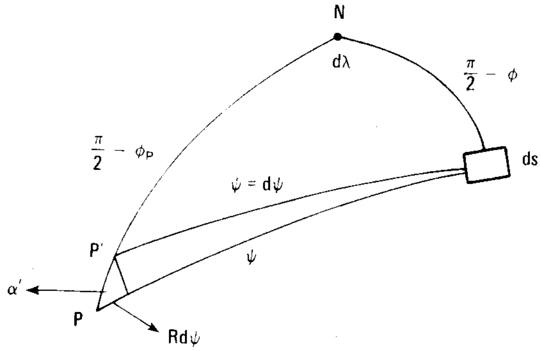

Figure 3. The Vening Meinesz Problem

linearization parameters. For the latitude-longitude system

$$u_1 = \phi \quad ; \quad u_2 = \lambda \tag{22}$$

while

$$h_1 = R \quad ; \quad h_2 = R \cos \phi \tag{23}$$

for a spherical approximation of the Earth. In the case of solutions to order e^2,

$$h_1 = \rho + h; \quad h_2 = (\nu + h) \cos \phi \tag{24}$$

ρ and ν being equivalent to the R_α defined in equation 7. The detailed development of solutions of the boundary value problem in this case use the geocentric latitude ϕ_c instead of the geodetic latitude ϕ, all parameters referring to quantities relating angular displacements between the geocentric radii to the pole P and $d\sigma$.

As explained above, the principal task in determining position from gravity is the definition of the height anomaly h_d, which is equal to the geoid height N in ocean areas, where $\Delta W = 0$. The next section deals in summary with some of the methods which have been proposed for defining the height anomaly.

3. Techniques for the Solution of the Boundary Value Problem

3.1 Introduction

Attention will be confined to three techniques whose use to obtain solutions to the boundary value problem have been extensively reported in the literature. The methods considered deal with formulations of solutions to what is known as Molodenskii's problem, at the physical surface of the Earth. It is not intended to formulate solutions for surfaces other than that of the Earth, e.g., the geoid, obtained by defining N instead of h_d. Neither is any attempt made to discuss the merits of regularization (e.g. MOLODENSKII ET AL 1962,p.45), where the conditions applicable to Stokes' problem are artificially created by the transfer of mass to within the geoid. The main advantage claimed for such techniques is a utility which is desirable when the surface gravity coverage is poor, as the adoption of certain types of mass transfers enables a more reliable prediction of gravity anomalies *for the chosen model*. The validity of such claims is open to question if the end-product of the calculations is to be a meaningful determination of positional parameters, *which can only be as good as the available data.*

The three techniques which will be covered, ostensibly do not require a knowledge of the stratification of matter within the Earth, defining solutions in terms of an "adeqate sampling" of the gravity field at the surface of the Earth, in conjunction with a complete definition of the associated topography. They can be classified as

(1) Surface layer solutions;

(2) Solutions from data sampled at discrete points on the Earth's surface; and

(3) Solutions from Green's third identity.

It is of interest to summarize the basis of each of these methods.

3.2 The Surface Layer Technique

This method, initially developed by Molodenskii (MOLODENSKII ET AL 1962,p.118 et seq.) was first published in 1949. Considerable material is available on the problems associated with the practical use of this technique by MORITZ (1966; 1970; 1972) and members of the Soviet school (e.g., BROVAR 1964; MARYCH 1969; YEREMEEV 1969; PELLINEN 1972). The derivation calls for the representation of the disturbing potential V_d at the surface of the Earth by a surface layer of density Φ such that the former can be represented at any point P either on the surface of the Earth or exterior to it by the relation

$$V_{dp} = \iint_S \frac{\Phi}{r}\, dS \qquad (25),$$

where there is no restriction on the shape of the surface S. It can be shown that

$$\left(\frac{\partial V_d}{\partial h}\right)_p = -2\pi\Phi_p \cos\beta_p + \frac{\partial}{\partial h_p}\left\{\iint_S \frac{\Phi}{r}\, dS\right\} \qquad (26),$$

where the subscript $_p$ refers to evaluation at P, β_p being the ground slope at P. The first term on the right appears because of the indeterminance at P itself. The inner zone in this region is treated as a disk (e.g., HEISKANEN & MORITZ 1967,p.129), the negative sign being introduced as the outward derivative is required, while the attraction of the disk is toward the geocentre. The cos β term allows for the slope of the surface of the disk with respect to the vertical. No approximations are involved in the derivation of equation 26.

The ensuing development which is well documented (IBID,p.300) can be summarized as follows, on retaining those terms whose contributions are greater than $o\{fh_d\}$. On using equations 8,13,25 and 26,

$$\Delta g = 2\pi\Phi_p \cos\beta_p - \frac{W_o - U_o}{\gamma_p}\left(\frac{\partial\gamma}{\partial h}\right)_p - \iint \left\{\frac{\partial}{\partial h_p}\left(\frac{1}{r}\right) - \frac{1}{\gamma_p}\left(\frac{\partial\gamma}{\partial h}\right)_p \frac{1}{r}\right\}\Phi\, dS + o\{f\,\Delta g\} \qquad (27).$$

As

$$\frac{1}{\gamma_p}\left(\frac{\partial\gamma}{\partial h}\right)_p = -\frac{2}{R_p} + o\{f\,\frac{1}{\gamma}\frac{\partial\gamma}{\partial h}\} \qquad (28),$$

and

$$r = (R_p^2 + R^2 - 2R\,R_p\cos\psi)^{\frac{1}{2}} \tag{29},$$

it follows from figure 4 that

$$\frac{\partial}{\partial h_p}\left(\frac{1}{r}\right) = \frac{\partial}{\partial R_p}\left(\frac{1}{r}\right) + o\{f^2\frac{\partial}{\partial h_p}\left(\frac{1}{r}\right)\} = -\frac{1}{r^3}\left(R_p - R\cos\psi\right) = -\frac{R_p}{2r^3} + \frac{R^2}{2R_p r^3} - \frac{1}{2R_p r}.$$

Thus

$$\frac{\partial}{\partial h_p}\left(\frac{1}{r}\right) - \frac{1}{\gamma_p}\left(\frac{\partial\gamma}{\partial h}\right)_p\frac{1}{r} = \frac{3}{2R_p r} + \frac{R^2 - R_p^2}{2R_p r^3} + o\{f\frac{1}{R_p r}\} \tag{30}.$$

Equation 27 can therefore be written as

$$\Delta g = 2\pi\Phi_p\cos\beta_p - \frac{W_o - U_o}{\gamma_p}\left(\frac{\partial\gamma}{\partial h}\right)_p - \iint\left(\frac{3}{2R_p r} + \frac{R^2 - R_p^2}{2R_p r^3}\right)\Phi\,dS + o\{f\,\Delta g\} \tag{31}.$$

The solution suggested by Molodenskii for equation 31 is based on the method of successive approximations where the surface of the Earth (S) is transformed into the surface $\overline{S}$ using a parameter k which specifies the relationship between the geocentric radii R and $\overline{R}$ to equivalent points on S and $\overline{S}$ by the relation (MOLODENSKII ET AL 1962,p.120)

$$\overline{R} = R_m + k(R - R_m) \tag{32},$$

where R_m is the mean radius of the Earth and $0 \leq k \leq 1$. Thus S and $\overline{S}$ coincide when $k = 1$, while the classical Stokesian case in which no topography exists exterior to the geoid, is obtained when $k = 0$. This is equivalent to scaling all elevations and grades by k from h and tan β to $\overline{h}$ and k tan β where

$$\overline{h} = kh,$$

and the related angle $\overline{\beta}$ is given by

$$\overline{\beta} = \cos^{-1}\left((1 + k^2\tan^2\beta)^{-\frac{1}{2}}\right) \tag{33}.$$

Other relevant conversions are

$$\overline{r} = \left(r_o^2 + k^2(h - h_p)^2\right)^{\frac{1}{2}} + o\{f\overline{r}\} \tag{34},$$

where r_o is the expression for the spherical case, given by

$$r_o = 2R_m\sin\tfrac{1}{2}\psi \tag{35}.$$

Molodenskii simplifies the solution by introducing the parameter χ defined by the equation

$$\chi = \frac{R^2}{R_m^2}\,\Phi\,\sec\beta \tag{36},$$

which, when taken in conjunction with the relation

$$dS = R^2 d\sigma \sec \beta \tag{37}$$

where $d\sigma$ is the element of surface area on unit sphere, enables equation 31 to be written as

$$\Delta g = \frac{R_m^2}{R_p^2} 2\pi\chi_p \cos^2\beta_p - \frac{W_o - U_o}{\gamma_p}\left(\frac{\partial\gamma}{\partial h}\right)_p - \frac{3R_m^2}{2R_p}\iint \frac{\chi}{r}\, d\sigma - \frac{R_m^2}{2R_p}\iint \frac{R^2 - R_p^2}{r^3}\chi\, d\sigma + o\{f\,\Delta g\} \tag{38}$$

It can be shown that (IBID,p.121) that if χ were expanded in a power series of the form

$$\chi = \sum_{i=0}^{\infty} \chi_i k^i \tag{39}$$

and on introducing a set of functions G_i, the use of equations 33, 34 and 39 in equation 38 gives the system of integral equations

$$G_i = 2\pi\chi_i + 2\frac{W_o - U_o}{R_m} - \frac{3}{2}R_m \iint \frac{\chi_i}{r_o}\, d\sigma, \qquad i=1,\infty \tag{40}$$

on equating the coefficients of k^i, and as $R^2 - R_p^2 = 2R\,(h - h_p) + o\{fR^2\}$. The quantities G_i are obtained in this manipulation as

$$G_o = \Delta g \tag{41}$$

$$G_1 = R_m^2 \iint \frac{h - h_p}{r_o^3}\chi_o d\sigma \tag{42}$$

with more complex expressions for higher values of i (IBID,p.122), being of the form

$$G_i = G_i(h, h_p, \chi_0, \chi_1, \ldots\ldots\ldots \chi_{(i-1)}) \tag{43}$$

Equation 38 reduces to equation 40 when $h = h_p = 0$ and $\beta = 0$. The Molodenskii problem is equivalent to Stokes' problem in such a case, the solution of which is equation 15; which, on taking zero degree effects into account (MATHER 1971c,p.85), can be written as

$$N_f = \frac{1}{\gamma}(V_d + W_o - U_o) = \frac{W_o - U_o}{\gamma} - R_m \frac{M\{\Delta g\}}{\gamma} + \frac{R_m}{4\pi\gamma}\iint f(\psi)\,\Delta g\, d\sigma \tag{44}$$

where $M\{\Delta g\}$ is the global mean value of Δg. The substitution of equation 25 in equation 40, in an appropriately modified form gives

$$\chi_i = \frac{1}{2\pi}\left(G_i + \frac{3V_{di}}{2R_m}\right) \tag{45}$$

on adoption of the representation

$$V_d = \sum_{i=0}^{\infty} k^i V_{di} = \sum_{i=0}^{\infty} \iint k^i \frac{X_i}{r_o} d\sigma + o\{fV_d\} \qquad (46).$$

The second equality in equation 46 would be consistent with equation 25 only if there were no topography. If this inconsistency were removed (MOLODENSKII ET AL 1962,p.123), the final expression for the height anomaly would be aseries in G_i embedded in the form set out at 44 with some topographic correction terms whose effects are purely local in character and need only be considered in areas of rugged topography, being functions of r_o^{-3}, the series being obtained when $k = 1$. In this case,

$$h_d = \frac{W_o - U_o}{\gamma} - R_m \frac{M\{G\}}{\gamma} + \frac{R_m}{4\pi\gamma} \iint f(\psi)\ G\ d\sigma \quad + T \qquad (47),$$

where

$$G = \sum_{i=0}^{\infty} G_i \qquad (48)$$

and T are the series of topographic correction terms whose form is given by MOLODENSKII (IBID). MORITZ (1966,p.91) has given alternate forms for G_1, and shows that if the gravity anomalies are linearly correlated with elevation, G_1 reduces to the terrain correction. Thus the combination of equations 42 and 45 gives

$$G_1 = \frac{R_m}{2\pi} \iint \frac{h - h_p}{r_o^3} \Delta g\ d\sigma = \frac{1}{2}\pi\ \rho\ R_m^2 \iint \frac{(h - h_p)^2}{r_o^3}\ d\sigma \qquad (49),$$

the second equality being based on the assumption of linear height correlation of gravity anomalies (IBID,p.88).

Notes

(1) This technique will be practically effective only if the contributions of the higher G_i are significantly smaller than those obtained for $i = 0$ and 1. The evaluation of any particular G_i presupposes a knowledge of all $X_j (j < i)$, which in turn are defined through equation 45. The solution is therefore iterative, and as the series in X_i is theoretically infinite, it is desirable that

$$X_i = o\{10^{-1}\ X_{i-1}\} \qquad (50)$$

for efficient practical evaluation. As G_o is the gravity anomaly, the first iteration gives the free air geoid, which contains over 90% of the power in the solution. It should therefore require only three iterations to obtain a solution to order e^3 in h_d if equation 50 were satisfied.

(2) There would be little difficulty in meeting this criterion if the ratio $(h - h_p)/r_o = o\{10^{-1}\}$. As the oceans comprising 70% of the Earth's surface and non-mountainous regions make little or no significant contribution to topographical effects, the magnitude of the corrections to the free air geoid would be small if the above criterion were satisfied. All topography with grades in excess of 5° pose problems in this respect when they occur within a few km of the point of computation, distant zone effects being rapidly submerged by the r_o^{-3} term. Also see section 3.5

(3) Serious embarrassment is caused when slopes exceed $\frac{1}{4}\pi$. Divergent series are obtained, making an iterative approach unstable. Discussions on the problem of convergence are available in the literature (MORITZ 1970; MORITZ 1972; KRARUP 1972). For a further discussion see section 3.5

(4) The quantity G can have no first degree harmonic, as the solution of Stokes' problem forbids the existence of such harmonics. Consequently the reference ellipsoid used for computing normal gravity is situated at the centre of mass of the mass distribution needed to produce values of gravity at the surface of the Earth which would give rise to a gravity anomaly distribution equivalent to that of G. The writer is not aware of any detailed investigation of this problem but it is unlikely that the nett effect would exceed o{5×10 cm}.

(5) The extension of this theory to orders of accuracy greater than that of the flattening is possible in theory. Such a solution could be obtained on including all effects of relevant magnitude in equations 27 and 28, and on allowing for the existence of the atmosphere, noting that Stokes' integral is strictly valid only if there is no mass exterior to the phsyical surface. In addition, it is necessary to take into account the Earth's ellipticity, when utilizing the orthogonal properties of surface harmonics.

3.3 Solution from Discrete Values

This technique was originally proposed by Bjerhammar who summarizes the problem as follows (BJERHAMMAR 1964,p.14).

> "A finite number of gravity data (gravity anomalies) is given for a non-spherical surface, at it is required to find such a solution that the boundary values for the gravity data (gravity anomalies) are satisfied in all given points."

The Bjerhammar problem is differently posed to that of Molodenskii and a different approach is used for the representation of the gravity anomalies at the surface of the Earth. Working on the basis that the representation of the surface gravity field can only be in terms of samples taken at discrete points at the Earth's surface, Bjerhammar proposes the interpretation of such data in terms of a set of model anomalies Δg^* on the surface of a sphere of radius R_b which is less than or equal to the polar radius of the best fitting ellipsoid. The appropriate requirement in Earth space is that any point on the Earth's surface lies exterior to the sphere of radius R_b (the Bjerhammar sphere), whose centre is collocated with the geocentre.

The technique of solution can be summarized as follows. The surface of the Bjerhammar sphere is partitioned into a grid, each element of which has a surface area $R_b^2 d\sigma$, and is represented by the model gravity anomaly Δg^*, assumed constant over the area. The disturbing potential V_{dp} at any exterior point P whose geocentric distance, as illustrated in figure 4, is R_p, is given by

$$V_{dp} = \frac{R_b^2}{4\pi R_p} \iint \Delta g^* \sum_{n=2}^{\infty} \frac{2n+1}{n-1} \left(\frac{R_b}{R_p}\right)^n P_{no}(\cos\psi)\, d\sigma \tag{51}$$

under conditions applicable to Stokes' problem. Δg^* obviously cannot have a first degree harmonic and the possibility of satisfying this condition in conjunction with the geocentric collocation of the Bjerhammar sphere is subject to the same arguments as outlined in note 4 to section 3.2.

The observational data is in the form of gravity anomalies Δg as determined at discrete points at the surface of the Earth. Using the fact that Poisson's integral

$$H_p = \frac{R_b(R_p^2 - R_b^2)}{4\pi} \iint \frac{H}{r^3}\, d\sigma \tag{52}$$

applies without approximation to any function H which is harmonic exterior to the Bjerhammar sphere, is is possible to define gravity anomalies Δg at all exterior points, if a surface distribution of the data set Δg^* were available on the sphere. Alternately, if Δg_i are the gravity anomalies measured at the surface of the Earth, the equation defining Δg_i in terms of the Δg^* is obtained from equation 52 as

$$\Delta g_i = \frac{R_b(R_{pi}^2 - R_b^2)}{4\pi R_p} \sum_j \frac{\Delta g_j^*}{r_{ij}^3} d\sigma_j \qquad (53),$$

where r_{ij} is the distance between P_i on the Earth's surface, with geocentric distance R_{pi}, and the surface element $d\sigma_j$ on the Bjerhammar sphere.

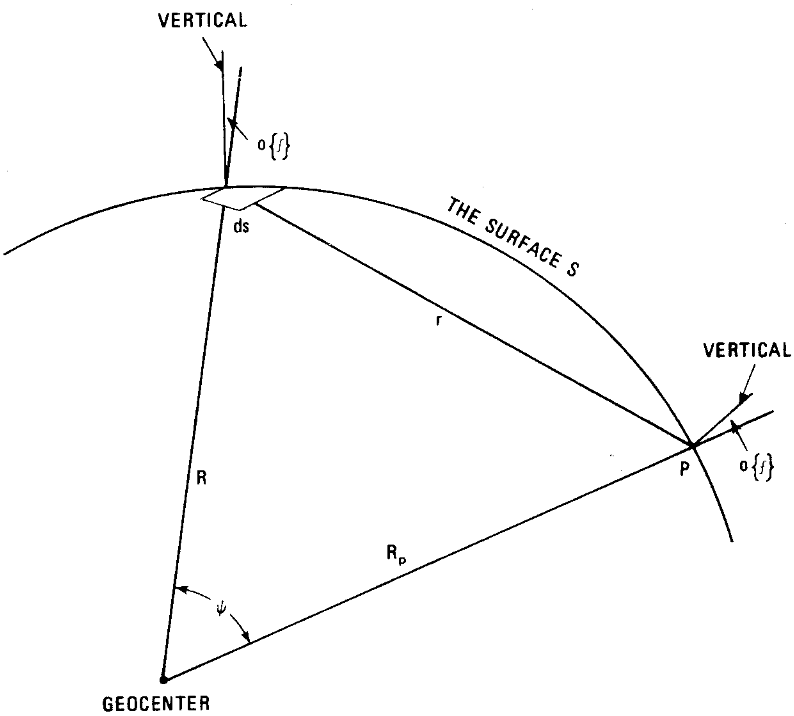

Figure 4. Relations for r

Equation 53, called the *discrete integral equation* by BJERHAMMAR (1968,p.6), can be treated as a set of observation equations which can be solved by standard techniques for the elements Δg^*. The technique is subject to certain practical difficulties when tested on models with heavy point masses between the sphere and the Earth's surface (IBID,p.67). In such cases, Bjerhammar advocates the use of the disturbing potential rather than the gravity anomaly in equation 53, presumably by recourse to an iterative procedure. The validity of the technique hinges on whether the disturbing potential computed at all points P_i at the Earth's surface due to the Bjerhammar system is identical with that due to the Earth. For a summary of the proof of this condition, see (BJERHAMMAR 1969, pp.452-6).

The instability of the inversion procedure due to the nature of gravitational attraction and its susceptibility to large masses locally (e.g., mountainous regions) led Bjerhammar to suggest that the more stable disturbing potential V_d^* on the Bjerhammar sphere be used as an intermediary in the solution on the following lines (IBID,p.498 et seq). The disturbing potential V_{di} at P_i on the Earth's surface is given by equation 52 as

$$V_{di} = \frac{R_b(R_{pi}^2 - R_b^2)}{4\pi} \iint \frac{V_d^*}{r_i^3} d\sigma \qquad (54).$$

As

$$\left(\frac{\partial V_d}{\partial h}\right) = \frac{R_b}{8\pi R_{pi}} \iint \frac{4R_{pi}^2 r_i^3 - 3r_i(R^2 - R_b^2 + r^2)(R_p^2 - R_b^2)}{r_i^6} V_d^* \, d\sigma + o\{f^2 \frac{\partial V_d}{\partial h}\} \qquad (55),$$

Δg is obtained from equation 13 on considering terms larger than $o\{f\ \Delta g\}$, as

$$\Delta g_i = - \frac{R_b}{8\ R_{pi}} \iint \left(\frac{4R_{pi}^2}{r_i^3} - \frac{3(R_{pi}^2 - R_b^2)^2}{r_i^5} + \frac{R_{pi}^2 - R_b^2}{r_i^3} \right) V_d^* \, d\sigma + o\{f\ \Delta g\} \qquad (56).$$

Equation 58 is simplified by differencing V_d^* from the value V_{do}^* which is the value of V_d^* at the point on the Bjerhammar sphere corresponding to P. It can be shown (IBID,p.499) that equation 56 can be transformed to

$$\Delta g_i = - \frac{R_b V_{do}^*}{R_p^2} - \frac{R_b}{8\pi R_{pi}} \iint \left(\frac{5R_{pi}^2 - R_b^2}{r_i^3} - \frac{3(R_{pi}^2 - R_b^2)^2}{r_i^5} \right) (V_d^* - V_{do}^*) d\sigma + o\{f\ \Delta g\} \qquad (57),$$

which is a generalized version of the Molodenskii inverse of Stokes' integral (MOLODENSKII ET AL 1962, p.50).

Notes

(1) The use of this system would, at first glance appear to be a prohibitive task. This is not the case as the terms being integrated are scaled by r^{-3} and hence only limited regions need be considered around each primary point at which evaluations are made. Details of tests in the West Alps using a $5^o \times 5^o$ area with a $15^o \times 15^o$ buffer zone, with basic sub-divisions of 5'x 5', are given by Bjerhammar (IBID,p.508), an iterative procedure being used to recover Δg^*.

(2) The intellectual elegance of the method is enhanced by its ability to combine all manifestations of the Earth's gravitational field into a single solution entity. It must be added that this same end can be achieved by using the methods proposed by KRARUP (1969), though the problems associated with practical implementation have yet to be tackled in the case of high precision determinations.

(3) The factors which have to be taken into account to extend the solution to orders smaller than that of the flattening are similar to those outlined in section 3.2(5).

(4) The solution, like that from Krarup's method, is unique for a given distribution of data. This of course, doen not mean that the answer obtained is correct to the order of accuracy with which the problem is formulated. Data needed for solutions of the boundary value problem are dealt with in section 4.

(5) For completeness, the solution should incorporate terms of zero degree as in section 3.2.

3.4 Solutions from Green's Third Identity

Considerable work has been done in this field (e.g., ARNOLD 1959; KOCH 1965; MORITZ 1965; MATHER 1971c). The basic integral used is Green's third identity which is obtained by the application of Green's theorem to two scalars r^{-1} and W which is harmonic in the volume V_e exterior to a surface S. On combining the gravitational and rotational effects (e.g., HEISKANEN & MORITZ 1967,p.15), the final expression obtained for the gravitational potential (W_p) at a point P on the surface S, on the assumption that all matter is contained within S and rotates with constant angular velocity ω, is

$$W_p = \frac{1}{2\pi} \iint \left(\frac{1}{r} \vec{\nabla}\cdot\vec{N}\, W - W\, \vec{\nabla}\cdot\vec{N}\, \frac{1}{r}\right) dS \; - 2\omega^2 \iiint \frac{1}{r}\, dV_i \tag{58}$$

where

$$\vec{\nabla} = \frac{\partial}{\partial x_i}\vec{i} \; ,$$

$\vec{i}$ being unit vectors along the axes x_i of a Cartesian co-ordinate system and r the distance of the elements of surface area dS and volume dV_i interior to S, from P.

A similar expression is obtained for the potential U_p at P due to a gravitating ellipsoid of reference which has the same gravitational characteristics as the Earth, including rotation. On considering the same surface S which is that of the Earth,

$$U_p = \frac{1}{2\pi} \iint \left(\frac{1}{r} \vec{\nabla}\cdot\vec{N}\, U - U\, \vec{\nabla}\cdot\vec{N}\, \frac{1}{r}\right) dS - 2\omega^2 \iiint \frac{1}{r}\, dV_i \tag{59}$$

$\vec{N}$ in both equations 58 and 59 being the unit vector normal to S at dS. Both equations hold exactly if U and W are harmonic exterior to S. This condition requires that no matter exists on either system exterior to the Earth's surface. The practical consequences, which are of significance when resolution approaching the order of the flattening is sought as the end result of computations, are the following.

1. The reference ellipsoid must always lie within S. As S councides with the ocean surface over 70% of the Earth, the reference ellipsoid must be smaller than the ellipsoid which best fits the geoid by an amount greater than the largest negative geoid undulation if no condition is to be be imposed on the mass distribution within the equipotential ellipsoid.
2. There should be no atmosphere exterior to S if equation 60 is to hold to accuracies in excess of order $10^{-2}V_d$.
3. Both the reference ellipsoid and the Earth are assumed to rotate with the same constant angular velocity ω. Irregularities in the Earth's rotation have to be allowed for as corrections to observations in instances where such magnitudes are of significance. For details, see section 2.2 and 4.

The practice to date has been to treat the atmospheric effects as those which should be modeled and allowed for as corrections to observations prior to use in computations (e.g., IAG 1970,p.18). In a recent solution MATHER (1973,p.28 et seq.) formulated a solution of the boundary value problem to $o\{e^3 h_d\}$ by separating the gravitational effects of the atmosphere from those of the solid Earth and oceans.

In conventional solutions, the disturbing potential V_d is obtained on differencing equations 58 and 59, when

$$V_{dp} = W_p - U_p = \frac{1}{2\pi} \iint \left(V_d\, \vec{\nabla}\cdot\vec{N}\, \frac{1}{r} - \frac{1}{r}\, \vec{\nabla}\cdot\vec{N}\, V_d\right) dS \tag{60}$$

This equation is not valid to orders smaller than $o\{10^{-2}V_d\}$ as it assumes the geopotential to be harmonic outside S. A function which does satisfy Laplace's equation exterior to S is the potential W' due to the solid Earth and oceans, which is related to W by the relation

$$W' = W - V_a \tag{61}$$,

where V_a is the potential of the atmosphere, which is of order 10^{-6}W, and more significantly, $V_a = o\{10^{-2}V_d\}$. As such, it is desirable to construct a theory which allows for its existence in the course of the derivation.

The final solution using this technique can only be obtained by iteration, the number of iterations required, as in the surface layer method, being a function of the accuracy sought. Favourable conditions for the adoption of an iterativeprocedure are the following.

a. A significant amount (> 90%) of the power should be generated in the first iteration.
b. The iterative procedure should have the ability to converge to the correct result.
c. The number of iterations necessary for achieving the desiring degree of resolution should be as small as possible.

When surface gravity is the sole source of information, the only procedure available for obtaining an adequate first approximation to the height anomaly h_d is the use of Stokes' approach. Fundamental to this technique is the assumption that the disturbing potential is harmonic exterior to and on the surface S, and therefore can be expressed in the form

$$V_d = \sum_{n=0}^{\infty} \frac{A_n}{R^{n+1}} \quad , \; n \neq 1 \tag{62}$$,

where

$$A_n = \sum_{m=0}^{n} A_{nm} \tag{63}$$,

and

$$A_{nm} = P_{nm}(\sin \phi_c)\left(C_{nm}\cos m\lambda + S_{nm}\sin m\lambda\right) \tag{64}$$,

the last equation being the standard expression for a surface harmonic.

The adoption of this model enables the combination of the effects of the disturbing potential V_d and its vertical gradient $\partial V_d/\partial h$ on using equation 13, thereby transforming the formulation to a representation of the observed quantity, the gravity anomaly Δg. Details of the problems involved in obtaining a solution of the boundary value problem to order $e^3 h_d$ are dealt with by MATHER (1973,p.31 et seq.). To preserve flexibility in the formulation of results to any required order of accuracy, it is desirable to retain physical relevance in the derivation by constructing the integral at 60 such that the disturbing potential V_d is replaced by the quantity

$$V'_d = V_d - V_a \tag{65}$$,

where V_a is the potential of the atmosphere.

A generalized solution which did not consider either the existence of the atmosphere or the fact that the potential of the geoid was not known, the latter being disregarded after due consideration as a quantity which correctly cannot be determined from gravimetric methods alone (MOLODENSKII ET AL

1962,p.104), was formulated by Molodenskii in 1945 (IBID,p.93). A specific solution was given by ARNOLD (1959), which could be written as

$$h_{dp} = \frac{R_m}{4\pi\gamma}\iint (\Delta g - \gamma\ \xi_\alpha \tan\ \beta_\alpha) f(\psi)\ d\sigma + \frac{R_m^2}{2\pi\gamma}\iint \frac{1}{r_o^3}\left\{(h_p - h) + R\ \sin\ \psi\ \frac{dh}{dr}\right\} V_d\ d\sigma \qquad (66),$$

where ξ_α are the components of the deflections of the vertical, $f(\psi)$ is Stokes' function, tan β_α being the components of the gradient of the ground slope in the north and east directions,

$$\frac{dh}{dr} = \cos\ A'_\alpha\ \tan\ \beta_\alpha \qquad (67),$$

where

$$A'_1 = \alpha_\sigma \quad ; \quad A'_2 = \tfrac{1}{2}\pi - \alpha_\sigma \qquad (68),$$

and r_o is given by equation 35. A revision of the derivation to the order of the flattening showed that (MATHER 1971c,p.85)

$$h_{dp} = \frac{W_o - U_o}{\gamma} - R_m\ \frac{M\{\Delta g\}}{\gamma} + \frac{R_m}{4\pi\gamma}\iint f(\psi)\ \ \Delta g\ d\sigma + \frac{R_m^2}{2\pi\gamma}\iint \frac{1}{r_o}\left\{\left((h_p - h) + R_m \sin\psi\frac{dh}{dr}\right)\frac{V_d}{r_o^2} - \gamma\ \xi_\alpha\ \tan\ \beta_\alpha\right\} d\sigma$$

$$+\ o\{f\ h_d\} \qquad \text{if} \quad \left(\frac{(h_p - h)}{r_o}\right)^2 = o\{f\} \qquad (69),$$

and Laplace's equation were satisfied to $o\{f\ \vec{\nabla}^2 V_d\}$ at all points exterior to and on S. Equations 66 and 69 would then be equivalent if

$$\frac{1}{2}\iint \gamma\ \xi_\alpha\ \tan\ \beta_\alpha\ f(\psi)\ d\sigma = R_m \iint \frac{1}{r_o}\gamma\ \xi_\alpha \tan\ \beta_\alpha\ d\sigma \qquad (70).$$

The effect of the terms common to the kernels of the integrals on either side of the equality in equation 70 can be expected to arise from only 30% of the surface area of the globe. Significant contributions to h_d will be restricted to only about 5% of the surface area, being about one order of magnitude smaller than Δg if $\xi_\alpha = o\{10^{-4}\}$ and $\tan\ \beta = o\{10^{-1}\}$. The high probability of correlation between the signs of ξ_α and β_α in regions where the latter is significant magnitude, indicates that this effect is likely to be always positive. As $h_d = o\{10^2 m\}$, it is realistic to estimate the effect of the above term as $o\{5\times10\ \text{cm}\}$, the effect being consequential if regions of mountainous topography occur near the point of computation.

The first term in the second integral at 69 converges much more quickly with increase of r_o and can be treated as a purely local effect. The advantage of the solution at 69 over that at 66 is the fact that all terms due to the interaction between the ground slope and the slope of the equipotential can be treated as purely local effects, giving solutions where the neglected effects do not have magnitudes much in excess of the order of the flattening. Another advantage of the solution at 69 is its unambiguous definition in Earth space as the Stokesian term defined a contribution with reference to an ellipsoid whose centre is at the geocentre of the Earth with no atmosphere.

The generalization to order e^3 in h_d (i.e., ±5 cm) deals not only with the effect of the topography, the interactions between the slopes of the topography and the equipotential surfaces of the Earth's gravitational field, as well as the atmosphere, but is also in keeping with the physical characteristics of the scalar potential (MATHER 1973). It also establishes the nature of the relationship between the Stokesian term and the indirect effect without limitations imposed by the simplistic approximations permitted by the adoption of a lower order of accuracy and identifies the anomaly to be used in Stokes' integral. The geometry of the solution is also specified in Earth space, as the centre of the reference ellipsoid is located at the centre of mass G' of the solid Earth and oceans, whose co-ordinates $\overline{X}_{ei}$ with respect to a geocentric Cartesian co-ordinate system are given by (IBID,p.26)

$$\overline{X}_{ei} = - \frac{M_a}{M_e} \overline{X}_{ai} \qquad (71),$$

where $\overline{X}_{ai}$ are the co-ordinates of the centre of mass of the model adopted for the Earth's atmosphere, M_a and M_e being the mass of the atmosphere and the solid Earth and oceans respectively. The final formulae obtained in the solution referred to are summarized below.

$$h_{dp} = N_{fp} + N_{cp} \qquad (72),$$

where the Stokesian term N_{fp} is given by

$$N_{fp} = \frac{W_o - U_o}{\gamma_p} - \overline{R}\,\frac{M\{\Delta g_c\}}{\gamma_p} + \frac{\overline{R}}{4\pi\gamma_p} \iint f(\psi)\;\; \Delta g_c \; d\sigma \qquad (73),$$

γ_p being the value of normal gravity at P' in figure 1, $\overline{R}$ being the radius of the Brillouin sphere whose centre is collocated with the centre of mass of the solid Earth and oceans G', and contains the solid Earth and oceans. The gravity anomaly Δg_c is defined by

$$\Delta g_c = \Delta g_1 + \Delta g_2 \qquad (74),$$

where

$$\Delta g_1 = \Delta g + \frac{\partial V_a}{\partial h} + 2\,\frac{V_a}{R_m} \qquad (75),$$

V_a being the potential of the atmosphere, and

$$\Delta g_2 = \frac{2V_d'}{R_m} c_\phi - \frac{1}{2} g\, \zeta^2 + dR\,\frac{\partial \Delta g}{\partial h} + o\{e^3 \Delta g\} \qquad (76),$$

if

$$\frac{1}{2}(dR)^2\,\frac{\partial \Delta g}{\partial h} = o\{e^3 \Delta g\},$$

where

$$dR = \overline{R} - a(1 - f\sin^2\phi_c) - h + o\{f\,dR\} \qquad (77),$$

$$\frac{\partial \Delta g}{\partial h} = -\gamma\left(\sum_{\alpha=1}^{2} \frac{\partial \xi_\alpha}{\partial x_\alpha} - \frac{\xi_1 \tan\phi_c}{R_m} - 2\,\frac{N_f}{R_m^2} + o\{f\,\frac{\partial \Delta g}{\partial h}\}\right) \qquad (78),$$

and

$$c_\phi = f + m - 3f\sin^2\phi_c + o\{f^2\} \qquad (79).$$

The angle ψ is computed in calculations to $o\{e^3 h_d\}$ from the geocentric latitude ϕ_c and longitude λ as

$$\psi = \cos^{-1}(\sin\phi_c \sin\phi_{cp} + \cos\phi_c \cos\phi_{cp} \cos d\lambda) \qquad (80),$$

where

$$d\lambda = \lambda - \lambda_p \qquad (81),$$

the value without subscript referring to $d\sigma$, and those with subscript $_p$ to the point of computation P. The indirect effect N_{cp} is given by

$$N_{cp} = \frac{V_{ap}}{\gamma_p} + \frac{1}{2\pi\gamma_p} \iint \frac{R^2}{r} \left\{ \frac{\partial V'_d}{\partial x_\alpha} \tan\beta_\alpha + V'_d \frac{x_\alpha \tan\beta_\alpha}{r^2} + \frac{1}{2R}\left\{3\left(c_\Delta + 3\frac{dR}{R}\right) - \Phi\right\} - dR\frac{\partial \Delta g}{\partial h} + \Delta g'\left\{c_\Delta + \frac{3}{2}\frac{dR}{R}\right\} + \right.$$

$$\left. o\{e^3 \Delta g\} \right\} d\sigma \qquad \text{if} \quad \frac{1}{2}(dR)^2 \frac{\partial^2 \Delta g}{\partial h^2} = o\{e^3 \Delta g\} \qquad (82),$$

r being the distance between $d\sigma$ and P, R being the geocentric distance, given by

$$R = a(1 - f\sin^2\phi_c) + h + o\{f^2 R\} \qquad (83),$$

and h the ellipsoidal elevation. The other expressions which need definition are

$$\frac{\partial V'_d}{\partial x_\alpha} \tan\beta_\alpha = -\gamma\,\xi_\alpha \tan\beta_\alpha + N_f \frac{\partial\gamma}{\partial x_1} \tan\beta_1 + o\{f^2 \Delta g\} \qquad (84),$$

the two dimensional co-ordinate systems x_α having the same significance as in figure 2,

$$\frac{x_\alpha}{r^2} \tan\beta_\alpha = \frac{R}{r^2}(1 + c_x)\sin\psi \frac{dh}{dr} \qquad (85),$$

where dh/dr is defined by equation 67,

$$c_x = \frac{\cos(\frac{1}{2}\psi - \theta)}{\cos(\frac{1}{2}\psi + \theta + \delta)} - 1 \qquad (86),$$

$$= -\tan^{-1}\left\{ \frac{2\sin\delta \sin\frac{1}{2}\psi - \frac{\Delta R}{R_m}\cos(\frac{1}{2}\psi + \delta)}{2\cos\delta \sin\frac{1}{2}\psi + \frac{\Delta R}{R_m}\sin(\frac{1}{2}\psi + \delta)} \right\} + o\{f^2 \tan\theta\}$$

$$= \frac{1}{2}\frac{\Delta R}{R_m}\cot\tfrac{1}{2}\psi - \delta + o\{f^2\} \qquad \text{if} \quad \psi > 5^\circ \qquad (87),$$

$$\Delta R = R_p - R \qquad (88),$$

and

$$\delta = f\sin 2\phi_c \cos\alpha_\sigma + o\{f^2\} \qquad (89).$$

The term Φ is given by

$$\Phi = \frac{2R}{r}\left\{R - R_p\cos(\psi + \delta)\right\} - 1 \qquad (90),$$

and

$$c_\Delta = \frac{1 + 2\frac{dR}{R}}{(1 + c_{\bar{r}})^{\frac{1}{2}}} - 1 \qquad (91),$$

where

$$c_{\bar{r}} = \left(\frac{\Delta R}{\bar{r}}\right)^2 - \frac{dR + dR_p}{R_m} + o\{f^2\} \qquad (92),$$

and

$$\bar{r} = 2\bar{R}\sin\tfrac{1}{2}\psi \qquad (93).$$

Notes

(1) The adoption of an iterative procedure to solve equations 72 to 93 cannot be avoided. See note 1 to section 3.2 for background. Mather (IBID,p.49 et seq.) has suggested an iterative procedure which should require three stages in the computation for determining N_c to 5 parts in 10^4 (i.e., $o\{e^3h_d\}$), while the Stokesian contribution need be evaluated only once, but in two stages.

(2) The first two terms in equation 73 are of zero degree and are meaningful only when the *surface gravity field* is sufficiently well defined to give an adequate value for the global mean surface gravity anomaly. Present day solutions which are heavily dependent on satellite determined low degree harmonics of the Earth's gravitational field, are insensitive to the effects of these terms. For a further discussion of zero degree effects which would be of conseqeunce in solutions based on adequate distributions of surface gravity data alone, see section 5.

(3) It could be construed that the conditions attached to equations 76 and 82 are a limitation on the development outlined above. The relevant terms which are omitted have a nett differential effect of

$$\frac{1}{4\pi\gamma}\iint\left\{\frac{1}{2}\bar{R}\,f(\psi) - \frac{R^2}{r}\right\}(dR)^2\,\frac{\partial^2\Delta g}{\partial h^2}\,d\sigma$$

on h_d which would be negligible if the quantity $\partial^2\Delta g/\partial h^2$ had random error characteristics over areas larger than 10^4 km^2 with magnitudes of order 10^{-8} mgal m^{-2}, which is only one order of magnitude smaller than that of $\partial^2\gamma/\partial h^2$.

(4) The components of the deflections of the vertical ξ_α are computed on the principles outlined in equations 16 to 24. A series of expressions which include effects with magnitudes of the order of the flattening are given by MATHER (1971c,p.86 et seq.). Such expressions should be adequate for the evaluation of h_d to $o\{e^3h_d\}$, but fail if accuracies of this type are required from the deflections themselves. Extension would principally require the use of an ellipsoidal co-ordinate system and a more careful evaluation of some of the higher derivatives of characteristics of the gravitational field which have been assessed as having insignificant effects on the height anomaly.

3.5 Conclusion

The formulation of a solution of the boundary value problem at the physical surface of the Earth, in contrast to Stokes' problem where there is no topography exterior to the geoid, calls for the evaluation of "topographical" terms which arise as a consequence of

(a) departures of the Earth's surface from a level surface; and

(b) elevation of the point of computation above or below the surrounding topography.

The first effect has contributions with long wavelength which, on present assessment of geoid determinations, should not have effects in excess of o{50 cm} unless rugged topography occurs in the vicinity of the point of computation. The second effect is a purely local one as it is scaled by the factor r^{-3}, as seen in equations 42, 57 and 69.

The limitation in theory of the surface layer method is the heavy reliance it places on the convergence of the series in G_i, defined at 43, in the mathematical sense. It may appear to be paradoxical in practical terms, that the *slope* of the topography of distant areas, which at least to a first order of approximation, are in isostatic compensation, can affect computations of the height anomaly. These terms occur because the gravity anomaly used in computations and defined by equations 10 and 11, reflect the mass distribution of the Earth as it exists. If, on the other hand, a suggestion similar to that made by DE GRAAFF HUNTER (1958) calling for the smoothing of the topography such that slopes in excess of 5^o did not exist, were adopted, there would be little to choose between the methods outlined in this section for the solution of the boundary value problem. In such a case, each of the iterative methods mentioned would not require more than three iterations to achieve a 5 cm resolution in h_d.

The conclusion that a model had to be adopted for the topography was also reached by MORITZ (1972, p.49) after a detailed study of the convergence of Molodenskii series. This approach has in the latter half of this century, become anathema to physical geodesists (e.g., MOLODENSKII ET AL. 1962, p.118) as it involves making assumptions about the density of material comprising the upper layers of the Earth's crust. In contrast, the quantities Δg, h, tan β and $\partial^i \Delta g/\partial h^i$ must be considered those which can be observed. In this sense, the solution described in section 3.4 exhibits favourable convergence characteristics, as the solutions involved are not open ended, but controlled in magnitude as terms in a rapidly convergent power series in the parameter f (= o{10^{-3}}). In practical terms however, the higher differential coefficients $\partial^i \Delta g/\partial h^i$ are unlikely to be determined with sufficient density to be of practical use, and the adequacy of equation 69 will depend largely on the magnitude and wavelength of the series $(i!)^{-1}h^i(\partial^i \Delta g/\partial h^i)$. Cumulative magnitudes of o{±0.5 mgal} with wavelengths of 100 km or less can be considered to be acceptable for solutions to order $e^3 h_d$, as discussed in section 4.

Reverting to the question of smoothening the topography in order that grades do not exceed 10^{-1}, the following problems will have to be attended to if such a procedure were adopted as everyday practice in physical geodesy

(a) The principles underlying the transfer of mass and their associated consequences should be clearly defined.

(b) The question of assigning a density for each element transferred will have to be dealt with, as the resulting corrections to observed gravity will depend on the model adopted for these masses.

If such procedures were deemed to be necessary, it would be madatory to adopt a model for the Earth with surface slopes less than 5^o. The transfer of matter to achieve this goal will change both observed gravity as well as the location of the center of mass of the physical system. The exact numerical values of the corrections made will depend on the principles adopted for the mass transfer. Physical geodesists advocating this type of approach will have to face up to the philosophical problem of which elements of topography to flatten out or fill. It is most important that a single model be adopted in order that the geodetic community is not subject to a confusing variety of results which are not in agreement, not because of significant factors, but merely as a consequence of the adopted smoothing procedure. The limited surface gravity data available at the present time continues to keep the above problem in the area of academic interest alone. It is one in which continued discussion is to be encouraged.

The solution of Molodenskii's problem by means of analytical continuation (MORITZ 1969; MARYCH 1969) has not been dealt with as it has been shown to be equivalent to the solution obtained using the surface layer approach (MORITZ 1970). Another method which may prove to have some benefits is the use of numerical integration techniques, on which published material is hard to come by.

4. Practical Considerations

4.1 Introduction

Practical considerations fall into two distinct categories. The first concerns the optimum sampling of data in order that the necessary precision can be achieved in numerical computations. The second is the extraction of the most probable results from whatever (inadequate) data is available. The second falls beyond the scope of this review and is covered elsewhere in this Symposium. One exception is the use of Molodenskii and Cook truncation functions to obtain the maximum information from satellite determined gravity anomalies and local gravity fields.

The problem can be summarized as follows. Over 90% of the power in h_d comes from Stokes' integral. Many regions exist where dense local gravity fields are available, but where beyond some limiting angular distance ψ_o, the available gravity data from the analysis of the orbital perturbations of near Earth satellites on combination with whatever surface gravity data exists, can be represented as a set of surface harmonics of the type given in equations 63 and 64. The free air geoid at equation 73 can be written as (MOLODENSKII ET AL 1962,p.147)

$$N_{fp} = \frac{W_o - U_o}{\gamma} - R\,\frac{M\{\Delta g\}}{\gamma} + \frac{R}{4\pi\gamma}\int_0^{\psi}\int_0^{2\pi} f(\psi)\ \Delta g\ \sin\psi\ d\psi\ d\alpha + \frac{R}{2\gamma}\sum_{n=2}^{\infty} Q_n\ \Delta g_n \qquad (94),$$

where the gravity anomaly Δg to be used in Stokes' integral is expressed by the set of surface harmonics

$$\Delta g = \sum_{n=0}^{\infty} \Delta g_n, \quad n \neq 1 \qquad (95).$$

Q_n is Molodenskii's truncation function, given by

$$Q_n = \int_{\psi_o}^{\pi} f(\psi)\ P_{no}(\cos\psi)\ \sin\psi\ d\psi \qquad (96),$$

$P_{no}(\cos\psi)$ being the Legendre zonal harmonic. Values of Q_n for various ψ_o are given by Molodenskii and his co-workers (IBID,p.150) to n=8, DE WITTE (1967) to n=25, and HAGIWARA (1972) to n=18. The computational efficiency of this method over the use of surface quadrature techniques for distant zone effects, in the present era where distant zone fields are heavily dependent for quality on satellite data, is a factor of 70 (OJENGBEDE 1973,p.32). In practice, only a limited number (at present, upto degree and order 20) of such harmonics are available and a rounding off error will occur in computations, due to the existence of a residual in the power spectrum of gravity anomalies on adopting the surface harmonic representation. Molodenskii uses an elegant technique to show that the use of harmonics to n=8 with surface gravity representations up to $\psi_o=23^\circ$, results in errors less than ±2 m, while extension of surface gravity coverage to $\psi_o=35^\circ$ reduces the truncation error to less than ±50 cm (MOLODENSKII ET AL 1962,p.164).

Similar considerations apply to the computation of the Vening Meinesz contribution to the deflections of the vertical using Cook's truncation function (COOK 1950,p.377), the equation equivalent to 94 in this case being (DE WITTE 1967,p.455)

$$\xi_\alpha = \frac{1}{4\pi\gamma}\int_0^{\psi_o}\int_0^{2\pi} \frac{\partial(f(\psi))}{\partial\psi}\ \Delta g\ \cos A_\alpha\ \sin\psi\ d\psi\ d\alpha\ + \frac{1}{2}\sum_{n=2}^{\infty}(n-1)c_{n1}q_n \qquad (97),$$

where

$$c_{\alpha n1} = \frac{1}{4\pi(n-1)\gamma}\iint \Delta g_n\ p_{n1}(\cos\psi)\ \cos A_\alpha\ d\sigma \qquad (98),$$

A_α being defined by equation 19 while Cook's truncation function q_n is given by

$$q_n = \int_{-1}^{\cos\frac{1}{2}\psi_o} \frac{\partial}{\partial\psi}(f(\psi))\ p_{n1}(\cos\psi)\ d(\cos\psi) \qquad (99).$$

The relationship between the functions Q_n and q_n has been established by HAGIWARA (1972,p.461) who gives a proof of the equivalence of the developments of Molodenskii and Cook. On using the same values of n and ψ_o described in the previous paragraph, Molodenskii shows that the truncation errors in ξ_α are less than 1.1 arcsec and 0.2 arcsec respectively in the two cases given. It can be concluded with confidence that the use of truncation functions is capable of giving a resolution equivalent to the best astro-geodetic results as is borne out by determinations in Australia (MATHER, BARLOW & FRYER 1971,p.19) and the tests carried out by OJENGBEDE (1973,p.41).

4.2 The Sampling of the Gravity Field at the Surface of the Earth

The overwhelming majority of surface gravity data available at the present time have been established for geophysical purposes motivated by regional considerations. Such information has to be carefully screened before being put to geodetic use. There are problems that arise in the establishment of the value of observed gravity itself. Until recently, most gravity determinations of quality were made by differential means using gravimeters. It is now possible to carry out an absolute determination of g with a resolution of ±50 μgal using a transportable apparatus (MORELLI ET AL 1971,p.17) while resolution at the ±3 μgal level has been reported by the apparatus at Sevres, France (SAKUMA 1971).

It is all-important in the first instance that all values of observed gravity are correctly referred to the unified gravity standardization network defined by the *International Gravity Standardization Network 1971* (IGSN71) or an equivalent global control network, in order that datum discrepancies may be minimized, if not eliminated. The solution of the geodetic boundary value problem requires an evaluation of the gravity anomaly. This calls for a knowledge of

(a) the geodetic latitude ϕ_g of the gravity station to 0.04 arcsec (± 1 m) for an accuracy of ±1 μgal; and

(b) the geopotential difference ΔW with respect to the geoid to ±0.003 kgal m for a resolution of ±1 μgal in the gravity anomaly Δg,

in addition to the requirements stated earlier for values of observed gravity. This also calls for the definition of a datum for the geopotential differences on a global basis, and to some desirable degree of resolution.

The status at the present time is as follows. While IGSN71 is available, it is most unlikely that any of the large gravity data banks are reliably connected to this network in toto at the present time. Most values of normal gravity are computed from regional geodetic co-ordinates of gravity stations which are unlikely to differ from geocentric values by more than ±10 arcsec. Thus all values of normal gravity computed in a given continental area covered by one of the regional datums (usually up to 5% of the Earth's total surface area), are subject to systematic errors not exceeding $\pm\frac{1}{4}$ mgal. The lack of a global datum for geopotential cannot cause errors much in excess of $\pm\frac{1}{2}$ mgal if the datum for elevations were based on at least one year's tide gauge readings, and there is no evidence available at present which indicates that the magnitude of stationary sea surface topography is much in excess of ±2 m. While these magnitudes appear to be small, their effect on the evaluation of Stokes' integral is significant, being systematic in character.

Present day geoid computations from surface gravity data are therefore limited in effectiveness as a consequence of irregularly distributed data which could be subject to systematic errors due to the effect of inadequately defined datums on the data set used in the computations. The existence of such effects cannot be tolerated when the data is required for the determination of the geoid with the highest possible precision in the study of sea surface topography, whose magnitude is unlikely to exceed 2-3 m. The term sea surface topography refers to departures of the ocean surface from an equipotential surface of the Earth's gravitational field and is partially due to salinity, meteorological and tidal effects. The magnitude of the residual departures on allowing for these factors, and termed stationary effects, can only be estimated from manifestations along coastlines which have been obtained by comparing the results of geodetic levelling with tide gauge readings. Departures which cannot as yet be explained, have been reported in Australia (e.g., HAMON & GREIG 1972), the United States (e.g., STURGES 1972) and elsewhere with slopes approaching or in excess of 0.1 arcsec. On balancing satellite altimeter technology presently available against oceanographic requirements, it would appear that a 10 cm resolution in the determination of the geoid is a desirable goal for this purpose (WILLIAMSTOWN REPORT 1969,3-2).

The criteria governing the factors which constitute a "desirable" representation of the gravity field for the solution of the geodetic boundary value problem is dependent on the requirements for the solution of Stokes' integral which, as discussed earlier, provides over 90% of the power. This would apply to any of the techniques of solution described in section 3. The following is a summary of a recent look at this problem (MATHER 1973,p.53 et seq.). A suitable form of Stokes' integral for quadratures evaluation is

$$N_f^{(cm)} = K \sum_i n_i \sum_j \mu_{ij} \, f(\psi_{ij}) \, \Delta g_{ij}^{(mgal)} \qquad (100),$$

where Δg_{ij} is the value of the gravity anomaly representing a $n_i^o \times n_i^o$ square,

$$K \doteqdot 1.58 \times 10^{-2} \qquad (101),$$

and $\mu_{ij} = \cos \phi_{cij}$ or $\sin \psi_{ij}$ depending on whether a latitude-longitude or azimuth-angular distance system of co-ordinates is used. Equation 100 would be adequate if the subdivision of the basic $n_i^o \times n_i^o$ into $N (= n_i^2/m^2)$ $m^o \times m^o$ squares $(m < n_i)$, where the k-th such square will be represented by the gravity anomaly Δg_k at an angular distance ψ_k from the point of computation P such that

$$\Delta g_k = \overline{\Delta g} + c_{gk} \quad ; \qquad F(\psi_k) = \overline{F(\psi)} + c_{\psi k} \qquad (102),$$

$\overline{\Delta g}$ and $\overline{F(\psi)}$ being given by

$$\overline{\Delta g} = \frac{1}{N}\sum_{k=1}^{N} \Delta g_k \quad ; \qquad \overline{F(\psi)} = \frac{1}{N}\sum_{k=1}^{N} f(\psi_k) \qquad (103),$$

and the use of these smaller sub-divisions in the quadratures evaluation in lieu of the $n_i^o \times n_i^o$ squares together with the appropriate area mean, did not reduce the quadratures error to below the desired order of accuracy ($o\{\varepsilon\}$). This would happen if

$$\sum_{k=1}^{N} c_{gk}\, c_{\psi k} = o\{\varepsilon\} \qquad (104),$$

implying no correlation whatever between variations in $f(\psi)$ and Δg over the $n_i^o \times n_i^o$ area. While the function $F(\psi)$, given by

$$F(\psi) = f(\psi)\sin\psi \qquad (105),$$

has predictable variations, Δg defies accurate prediction free from systematic bias except over very short distances and under carefully controlled conditions. As gravity has to be sampled at discrete points, the quadratures approach makes a representation procedure mandatory. Consequently, some finite element of surface area has to be represented by a single observation. It is useful to bear in mind that

(a) the global gravity standardization network available at present has a station accuracy of ±0.2 mgal (MORELLI ET AL 1971 ,p.6);

(b) errors in gravimeter ties seldom exceed ±0.2 mgal if performed with adequate instruments and any sort of minimal care; and

(c) geopotential errors of $o\{\pm 3$ kgal m$\}$ give rise to an error of $o\{\pm 1$ mgal$\}$ in the gravity anomaly.

A precision of ±1 mgal in the gravity anomaly is relatively easy to obtain in areas where the regional geodetic level network is reasonably dense. The gravity anomaly also undergoes changes with position within the basic square it is expected to represent. This penchant was characterized by a quantity introduced by de Graaff Hunter, called the error of representation $E\{\Delta g\}_{nm}$ for an $n^o \times m^o$ square, which in the case of a fully represented square, is given by (DE GRAAFF HUNTER 1935)

$$\left(E\{\Delta g\}_{nm}\right)^2 = \sum_{i=1}^{N} \frac{(\Delta g_i - \overline{\Delta g})^2}{N} \qquad (106).$$

A reliable value for $E\{\Delta g\}_{nm}$ is obtained from N evenly spaced values of Δg_i covering the $n^o \times m^o$ square, $\overline{\Delta g}$ being the mean value of the gravity anomaly, given by

$$\overline{\Delta g} = \frac{1}{N}\sum_{i=1}^{N} \Delta g_i \qquad (107).$$

Several estimates of this statistical characteristic of the gravity anomaly field at the surface of the Earth are available in the literature (e.g., IBID; HIRVONEN 1956; MOLODENSKII ET AL 1962,p.172; MATHER 1967,p.131). Samples which are available at the present time from different parts of the globe reflect the flatter continental areas. $E\{\Delta g\}_n$ in such areas is a function of square size and in general terms, can be expressed by the relations

$$E\{\Delta g\}_n = \begin{cases} \pm C_1 \sqrt{n} & \tfrac{1}{4}^o < n < 5^o \\ \pm C_2\, n & n < \tfrac{1}{4}^o \end{cases} \qquad (108)$$

for an $n^{o} \times n^{o}$ square, where n is in degrees and $E\{\Delta g\}_n$ in mgal, when $C_1 \doteqdot 12$ and $C_2 \doteqdot 3 \times 10$. It can also be shown that $E\{\Delta g\}_n$ is a function of unsigned ground slope $|\beta|$, with magnitudes which can be as much a five times as great in very rugged areas, especially when n is small. As such variations are functions of ground slope and not elevation, it is estimated that about 2-5 % of the Earth's surface will require values of C_1 and C_2 which are significantly greater than those given above, for an adequate representation of variations in the gravity anomaly.

The number of terms involved in the quadratures evaluation is a function of the accuracy desired in the computation. If the requirments of sea surface topography determinations (1 part in 10^4) were to be met, it would be necessary, for estimation purposes, to restrict square sizes to those over which the contribution of the terms containing the second differential coefficient of $F(\psi)$ were held to $o\{e^3 h_d\}$. The required number of summations is $o\{10^6\}$. The study of the propagation of systematic and random error characteristics through equation 100 under these circumstances shows that an adequate representation of the surface gravity field which would enable the achievement of an accuracy of ±10 cm in the final result would be one which had an $E\{\Delta g\}$ value of ±3 mgal, if the data were not subject to systematic error in excess of ± 50 μgal. Such a representation is afforded by a 10 km grid in non-mountainous areas. While the estimation characteristics of gravitationally disturbed regions are covered by the above figures, which assume that oceanic fields will have a similar tendency to vary as continental data, regions characterized by larger ground slopes have significantly greater values of $E\{\Delta g\}$. It would be necessary to reduce the size of the grid in such cases to retain $E\{\Delta g\}$ at ±3 mgal. The use of the smoothening techniques described in section 3.5 would of course reduce these values. It follows that present day techniques for establishing surface gravity anomalies are adequate for the determination of sea surface topography. It is interesting to note that the station spacing required on the above basis, is already available over large continental areas like the United States, Canada and Australia, at the present time.

The consequences of systematic errors in Δg which hold the same sign over considerable extents, on the values of h_d computed, are significant. A systematic error $e_{\Delta g}$ which holds its magnitude and sign over a $n^{o} \times n^{o}$ area, but has random characteristics over larger extents, is shown to have an effect e_{Ns} on the computed value of h_d given by (IBID,p.65)

$$e_{Ns} = \pm\, o\{K''n\, e_{\Delta g}\} \qquad (109),$$

where $K'' \doteqdot 10$, for e_{Ns} in cm, n in degrees and $e_{\Delta g}$ in mgal. If e_{Ns} were held at ±5 cm, the estimate of the magnitude of the tolerable systematic error e_{g}, which is inversely proportional to its wavelength, varies from o{±5 mgal} when $n = 0.1^{o}$ to o{±0.1 mgal} when $n = 5^{o}$.

Likely sources of systematic error have been listed at the commencement of this sub-section. The following conclusions can be drawn.

(1) IGSN71 would be an adequate gravity standardization network for sea surface topography studies only if the station density were 1 per 5×10^4 km^2 and the errors of adjacent stations were not correlated at the 200 μgal level. Neither of these conditions is likely to be satisfied. An adequate net would be afforded by stations at which absolute determinations had been carried out with ±50 μgal resolution, and a representation of 1 station per 10^6 km^2. In the interim, it would be advisable that all gravity data should be subject to randomization procedures at the level of the precision of the gravity standardization network, prior to use in solutions of the boundary value problem.

(2) Gravity anomaly information on each geodetic datum should be corrected for changes in normal gravity due to the datum not being geocentric (IBID,p.16).

(3) The term "geoid" which is synonymous with both the global datum for elevations

as well as the "undisturbed" free level of the sea, should be defined on the basis of models which afford resolution with an accuracy of ±10 cm.

A possible problem of some significance in the determination of sea surface topography and other high precision determinations of h_d, is the existence of the sea surface topography itself with not insignificant amplitudes (e.g., 3-4 m) and substantial wavelengths. The evidence for the existence of such phenomena is widespread but based on purely coastal phenomena, as obtained from levelling- tide gauge comparisons. Extended studies of the sea surface using short pulse high resolution altimeters should go a long way toward clarifying whether stationary sea surface topography is merely a coastal phenomenon, and if not, the dominant wavelengths with which it is prone to occur. The existence of stationary sea surface topography with 4000 km wavelengths and 2 m amplitudes would cause errors of o{±1 m} in h_d. While this estimate is based on the maximum magnitude of the phenomenon reported to date, the existence of such an effect will require an iteration in the determination of h_d. (For a later perspective on this problem, see the other paper by Mather in these Proceedings. The rapidity with which these iterations converge is more a function of the wavelength of the stationary sea surface topography than of its amplitude.

5. Gravity and Earth Space

5.1 Gravity and Scale

A problem which requires careful scrutiny is the possibility or otherwise of defining a scale for Earth space from gravity determinations at the surface of the Earth. It must be clearly emphasized that the ensuing development excludes the consideration of satellite data which constitutes the basis of low degree representations of the Earth's gravity field at the present time. The problem could be stated as follows. Given an adequate distribution of determinations of surface gravity, how are the effects of zero degree h_{do} in the global distribution of height anomalies to be interpreted. This effect can be written as

$$h_{do} = \frac{W_o - U_o}{\gamma} - R\,\frac{M\{\Delta g_c\}}{\gamma} + N_{co} + o\{fh_{do}\} \qquad (110),$$

on considering equations 73 and 74, N_{co} being the contribution of zero degree by the indirect effect N_c. *W_o is not known* and it is common practice to assume the first term to be zero. The second and third terms will have finite magnitudes. A change in the value of GM will provide nearly equal and opposite changes in the terms containing U_o and $M\{\Delta g_c\}$. Hence equation 110 cannot be evaluated unless the value of W_o were known, which is certainly not the case to order ±1 m at the present time.

Equation 110 could however be used to find out the potential of the geoid W_o if h_{do} were known from some independent determination (e.g, geometrical satellite geodesy). The numerical value of h_{do} can be established by analyzing the differences

$$v_i = h_{di} + h_{ni} - h_i \qquad (111),$$

where the subscript $_i$ refers to evaluation at the i-th station in a global satellite station network, h_d being determined gravimetrically with the first term in equation 110 supressed. The value so obtained for h_{do} on use in equation 110 will give a value for W_o which is consistent with the set of units defined.

Any "improvement" in the value of GM obtained from gravimetric determinations in the strictest sense will have to be based on the assumption that the potential of the geoid W_o is equal to that of the equipotential ellipsoid of revolution U_o. The geoid is a physical reality, being a manifestation of the mass distribution which gives the observed gravitational phenomena at the surface of the Earth, while U_o is defined by the chosen values for the parameters a, GM, ω and f which define the system of reference. It has been deduced that the term $(W_o - U_o)/\gamma$ is approximately 3 m if the ellipsoid were one of best fit to the geoid and the value adopted for GM was the best estimate available for the Earth (MATHER 1971c,p.98), provided the free air anomaly had no zero degree harmonic.

Thus any deductions which can be drawn about scale from gravimetric determinations alone are subject to ambiguity, if restricted to a single epoch. The effect of zero degree deduced from comparisons between geometrical satellite solutions and surface gravity determinations of h_d described by equation 111, should be used only for the purpose of determining the value of W_o. A second effect of importance is the term of zero degree obtained on studying changes in observed gravity determined by the use of absolute techniques with a resolution approaching ±1 μgal, as determined on specially designed observing platforms, well distributed about the Earth, between successive epochs in time. Such changes can be interpreted as either reflecting an expansion of the Earth, as measured within the framework of the velocity of light and the adopted standard for the measurement of time intervals, or else as a change in the value of GM. For a discussion see (MATHER 1972,p.15).

5.2 Gravity and Geodetic Reference Systems

The preceding development has assumed that the Earth has a fixed mass distribution subject to some periodic changes due to effects like Earth tides. Such a description would be adequate only if the observations were taken over a limited period of time, such as one or two decades. There is considerable evidence which seems to point to the large scale re-distribution of at least the masses constituting the Earth's crust, over very long periods of time, with the attendant possibility of mass variations at greater depth depending on the nature of the mechanism which could produce such crustal motions.

A possible consequence of such mass re-distributions could be the motion of the Earth's centre of mass (geocentre) with respect to the Earth's crust. The analysis of high precision determinations of absolute gravity at a well distributed net of observing platforms as described in the previous subsection, could provide a means of recovering the motion of the geocentre between epochs, on analyzing the first degree harmonic of changes in absolute g (IBID). It should be pointed out that a problem in filtering out short period effects due to meteorological causes has to be overcome before results of reliability are likely to be obtained. Fortunately an estimate of the same effect can be obtained on studying changes in geocentric position of a global network of laser tracking stations using dynamic techniques, to provide a verification of the effectiveness of the determination.

5.3 The Role of Gravimetric Methods in Earth and Ocean Physics

Until recently, it was generally held that gravimetric methods if used with adequate data, provided the only non-controversial technique for computing ellipsoidal elevations with the same resolution as that available from geodetic levelling, thus completing the definition of geocentric position of points on the Earth's surface in three dimensions. Position determination at the present time has not provided resolutions which can confidently claimed to be better than 1 part in 10^6.

It is now clear that the most precise determination of geocentric position is required primarily for studies in Earth and ocean physics, rather than for any direct engineering or technological purpose. It would not be exaggeration to state that resolution to 1 part in 10^8 would be the aim of geodetic techniques being currently developed for such schemes. While there is no clear indication that surface methods, subject to restrictions imposed by atmospheric uncertainties, can be improved to meet these goals, extra-terrestrial techniques like laser ranging to near Earth satellites and VLBI, promise that such goals may well be achieved in the near future. There is also no reason to doubt at this stage, that transportable versions of these systems could not achieve this same degree of resolution.

It would therefore appear that, with the passage of time, there would be less use of geodetic levelling and the systems of reference implicit in its concept, for use in Earth physics. The exception is of course the study of the instantaneous geocentric position of the ocean surface, and the interpretation of these results for the study of ocean circulation. The determination of the geoid with the highest possible precision is a necessary prerequisite for such studies. Gravity information will still have to be assembled and anomalies computed on the basis of elevations referred to an equipotential surface, the most convenient being the geoid.

Three matters of significance which should be closely studied before undertaking the task of assembling an adequate gravity anomaly field for computation of geoid heights to ±10 cm, are the following.

(a) The definition of the physical model to serve as a datum for elevations with an accuracy which is not more than a factor of three less than the highest precision sought in the geoid solution.

(b) Techniques to be used for minimizing the effect of gravity base station errors on geoid computations.

(c) The question of whether it is necessary to adopt a model for the "surface of measurement" and, if so, the nature of an acceptable model and the procedure to be adopted in converting measurements on the Earth's surface to equivalent quantities on the model.

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8. Discussion

QURESHI: What role do geological anomalies play in this type of work?

MATHER: In trying to determine geodetic information, it is preferable to treat the gravity effect in terms of the free air anomaly and to keep the other contributions as a set of topography dependent terms which are treated separately in terms of elevations or their geopotential equivalents.

QURESHI: But geological anomalies are not always related to the topography.

MATHER: Yes. For example if you take an isostatic type anomaly and use it in Stokes' integral, you obtain a co-geoid. Provided you compute the correct indirect effect by taking all factors into consideration, you will get the correct answer. But this is not the same problem. In this case you are trying to make a *geoid* determination; you are trying to create Stokes' conditions by manipulating the topography. But most geodesists want to avoid making assumptions about (the density distribution of) the topography.

MORITZ: This old question of density and its effects is more or less obsolete. We know that density does not have such a large effect on the problem. A much larger effect is introduced, as you mention, through the interpolation problem.

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GLOBAL DETAILED GEOID COMPUTATION AND MODEL ANALYSIS

ABSTRACT

A global detailed gravimetric geoid has been computed by combining the Goddard Space Flight Center (GSFC) GEM-6 gravity model derived from satellite and surface gravity data and surface 1°- by - 1° mean free air gravity anomaly data. The accuracy of the geoid has been assessed at ±2 m on the continents, and 5 to 7 m in areas where surface gravity data are sparse.

The GSFC GEM-4, -6 models, the SAO 2, 3 models and the Rapp '73 model were considered in order to arrive at the best base gravity model for use in detailed geoid computations. RMS differences between GEM-6 and the other models ranged from 3 m to 7 m. The maximum differences in all cases occurred in the southern hemisphere where surface data and satellite observations are sparse. These differences exhibited wavelengths of approximately 30° to 50° longitude. To study the source of these differences, detailed geoid heights were computed with models truncated to twelfth degree and order, as well as eighth degree and order. This truncation resulted in a reduction of the rms differences to a maximum of 5 m. Comparisons have been made with the astro-geodetic data of Rice (United States), Bomford (Europe) and Mather (Australia). Comparisons have also been carried out with geoid heights from satellite solutions for geocentric station co-ordinates in North America and the Caribbean.

1. Introduction

A global detailed gravimetric geoid has been computed by combining the Goddard Space Flight Center (GSFC) GEM-6 gravity model (LERCH ET AL 1973) derived from satellite perturbations and surface gravity data, with 1°- by - 1° mean surface free air gravity anomaly data. Previously, a local detailed geoid combining the above model and 1°- by - 1° surface gravity data was computed for the north east Pacific and the north Atlantic area to provide an independent base of comparison for the GEOS-C altimeter experiment (MARSH ET AL 1973).

In the process of computing the geoids presented in this reference, several satellite gravity models published in the past four years were tested in order to determine the best base gravity model for detailed geoid computations. The models used in the geoid computations, in addition to the GEM-6 model were GSFC GEM-4 (LERCH ET AL 1972), the SAO-2 (GAPOSCHKIN ET AL 1970), SAO-3 (GAPOSCHKIN 1973) models and the Rapp '73 model. The rms differences between geoid heights computed using the GEM-6 gravity model and those computed using other gravity models, ranged from 3 m for the Rapp 1973 model to 7 m for the SAO-3 gravity model when the computations utilized the complete set of spherical harmonic coefficients.

The largest geoid height differences occurring in the above comparisons were located in the southern hemisphere. These differences exhibited a wavelength of approximately 30° in longitude, indicating errors in the middle degree and order coefficients of the various models. This finding prompted recomputations of the geoid with the satellite models truncated to lower degree and order, starting with (12,12). As a result, the rms difference was reduced to 1 m for GEM-4, and to 5 m for SAO-3. Geoid profiles at 10° intervals in latitude were drawn for all models (complete and truncated).

Differences between the geoids along these profiles were generally 5 m in areas of relatively dense surface gravity data and as large as 25 m in areas of sparse or absent gravity data. However, when truncated models were used, the differences were reduced to a maximum of about 15 m.

The accuracy of the GEM-6 detailed geoid is assessed at ±2 m in areas of dense surface gravity coverage and 5 to 7 m in areas of less dense coverage based on comparisons with astro-geodetic geoids and dynamically derived station heights.

2. Method of Computations and Data Source

The method of computation is presented in detail in (VINCENT ET AL 1973). The detailed geoid heights were computed by combining the GEM-6 satellite gravity field and surface 1°- by - 1° gravity data. The component of the detailed geoid obtained using the GEM-6 gravity field is derived as a function of the spherical harmonic coefficients of the gravity model and the surface geoidal heights are derived by incorporating surface 1° - by - 1° gravity data into Stokes' equation for areas 20°- by - 20° centred at the computational points.

The surface gravity data used in the computations consisted of 23,947 records of 1°- by - 1° mean free air gravity anomalies obtained from the Defence Mapping Agency/Aerospace Center. This data set was complemented with collections from the National Ocean and Atmospheric Agency, and the Hawaii Institute of Geophysics. However, whenever possible, local data collected by local agencies were considered first in data preparation. When these data were not sufficient, the above mentioned sources were used to fill in the voids. The data file is discussed in detail in (IBID).

3. Analysis

The base gravity model used in detailed geoid computations provides information on the long wavelength (approximately $\geq$ 1000 km) undulations of the geoid. The short wavelength information is provided by the 1°- by - 1° surface gravity data. All models tested were complete to degree and order 16 with selected higher degree terms, and were therefore capable of providing the 1000 km information on the geoidal undulations. Since all models were combined with the same set of 1°- by - 1° surface gravity data, the resultant differences in the detailed geoid heights are due to variations in the gravity models. The analyses of the models and the final choice of the base model for use in detailed geoid computations were carried out by

1. inter-comparing the respective geoids of these models; and
2. comparison with external standards such as astro-geodetic geoids and dynamically derived tracking station co-ordinates.

3.1 Inter-Model Comparisons

Detailed geoids were computed using the full set of coefficients of the five gravity models and profiles were drawn along parallels of latitude around the entire globe at 10° intervals in latitude. Figures 1 through 5 present representative examples of these profiles. In the northern hemisphere, representative profiles were chosen at 20° and 40° north latitude (figures 1 and 2). These profiles show an average variability of about ±5 m; however, individual differences as large as 10 m do appear. For example, in figure 2, at longitude 180°E, the geoid computed using the SAO-2 model differs from the geoid computed using the GEM-6 solution by 10 m. The models show the largest scatter at 180°E mainly because of a lack of surface gravity data available. The dominant

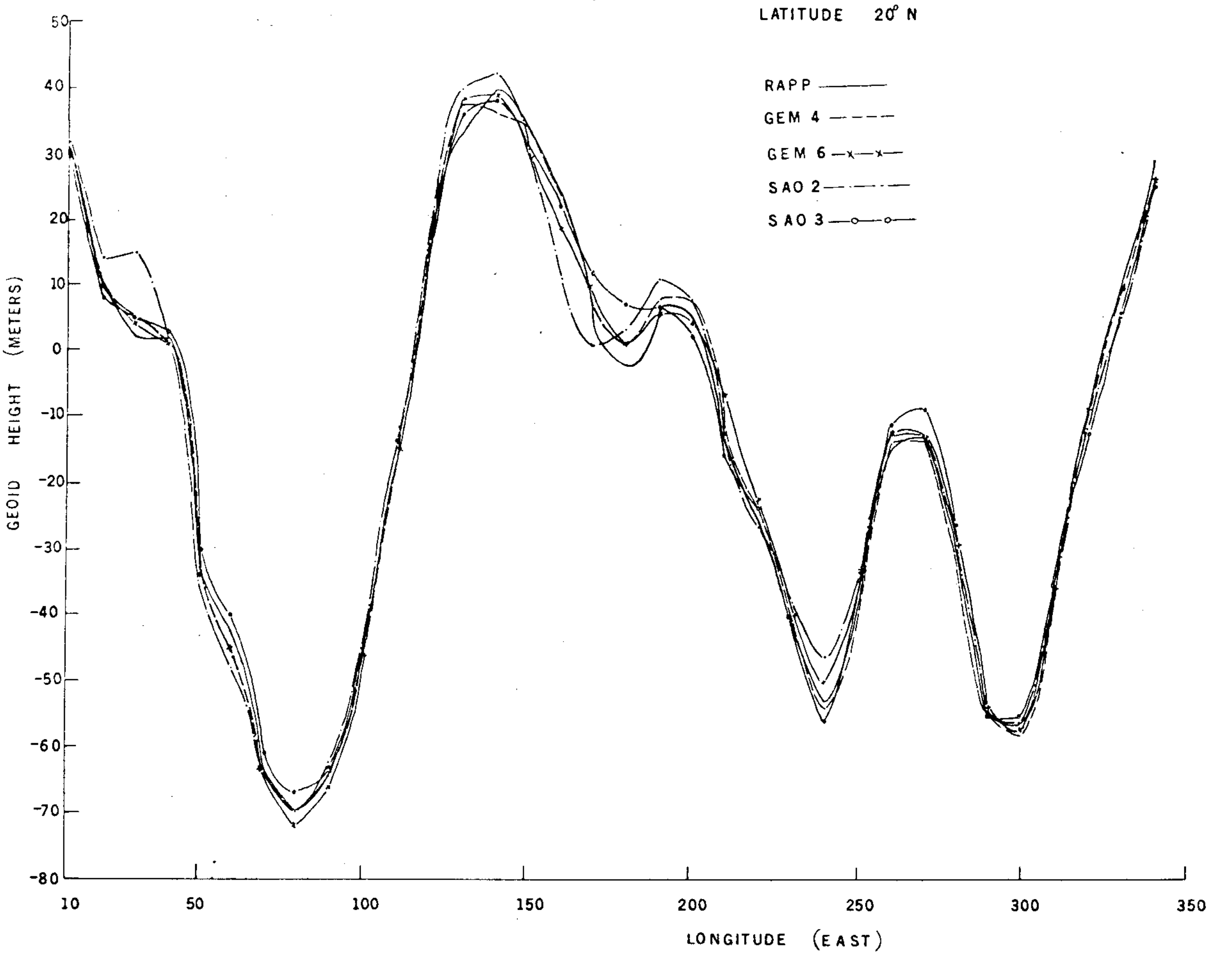

Figure 1. Detailed Geoid Profiles at Latitude 20°N

differences in these profiles are the amplitudes of the main features, rather than the slopes of the geoids. It is noted that there are many places along these profiles where the respective geoid profiles vary only by a few percent.

In figures 3 through 5 (southern hemisphere), a completely different picture emerges. The scatter is much more prevelant. This is largely attributed to the sparsity of surface gravity data, as well as a lack of satellite observational data. The scatter in the profiles increases gradually towards the Antarctic. For example, in figure 3, the scatter is evident only along longitudes 200°E to 350°E, but in figures 4 and 5, the divergence is noted along the entire length of the profile. In figure 5, the maximum difference reaches approximately 25 m at longitude 180°E. In contrast to the northern hemisphere, the geoid slopes in the southern hemisphere exhibit large variations.

MARSH ET AL (1973) showed differences between GEM-6 and other models to exhibit a wavelength of approximately 30° to 50° in longitude. This variation, when translated into spherical harmonic terms, corresponds to middle degree and order. The orbital perturbations arising from spherical harmonic coefficients of degree and order larger than (8,8) are generally on the order of a few metres, a fact which makes accurate recovery of the individual coefficient values difficult except in the case of resonance. This fact, plus the findings of BROWND & RICHARDSON (1973)

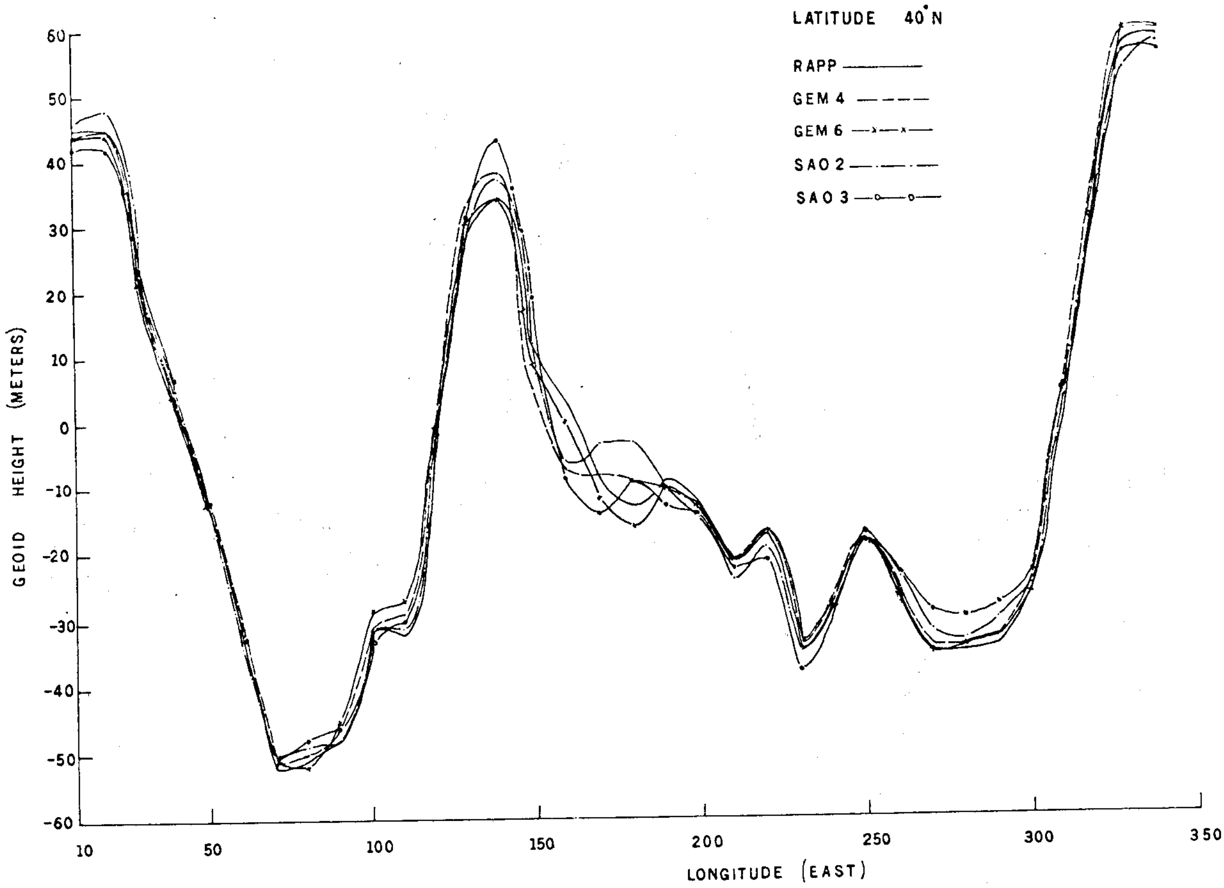

Figure 2. Detailed Geoid Profiles at Latitude 40°N

when conducting tests on the gravity anomalies derived from various gravity fields, coupled with the scarcity of surface gravity data in the southern hemisphere, led to the re-computation of the detailed geoids using truncated gravity models (12,12) and (8,8). Brownd & Richardson found that when satellite derived gravity anomalies were compared with surface gravity data using truncated models, the (12,12) field agreed best with the surface gravity data while the lower and higher degree models were divergent from the surface data.

Figures 6 through 10 present profiles of detailed geoids computed using models truncated to (12,12), plus SAO-2, and SAO-3 truncated to (8,8). One point noted throughout these profiles is that truncation to (12,12) for the GEM-4, GEM-6 and Rapp '73 models reduces the differences between then to an envelope of about 2 m. The SAO-2, SAO-3 (12,12) models on the other hand, show variations as large as 5 m with respect to each other. They are also in disagreement with the general trend of the other truncated models, as well as indicating features not portrayed by the general trend. However, further truncation of the SAO-2 and SAO-3 models to (8,8) generally reduces the differences along the main features and eliminates some of the extraneous features. As one truncates back to lower degrees, the results of the geoid computations tend to become identical because the satellite coefficients are nearly equal. But since the surface data within 10° of the computation point cannot completely represent the effect of the truncated wavelengths, this does not necessarily mean

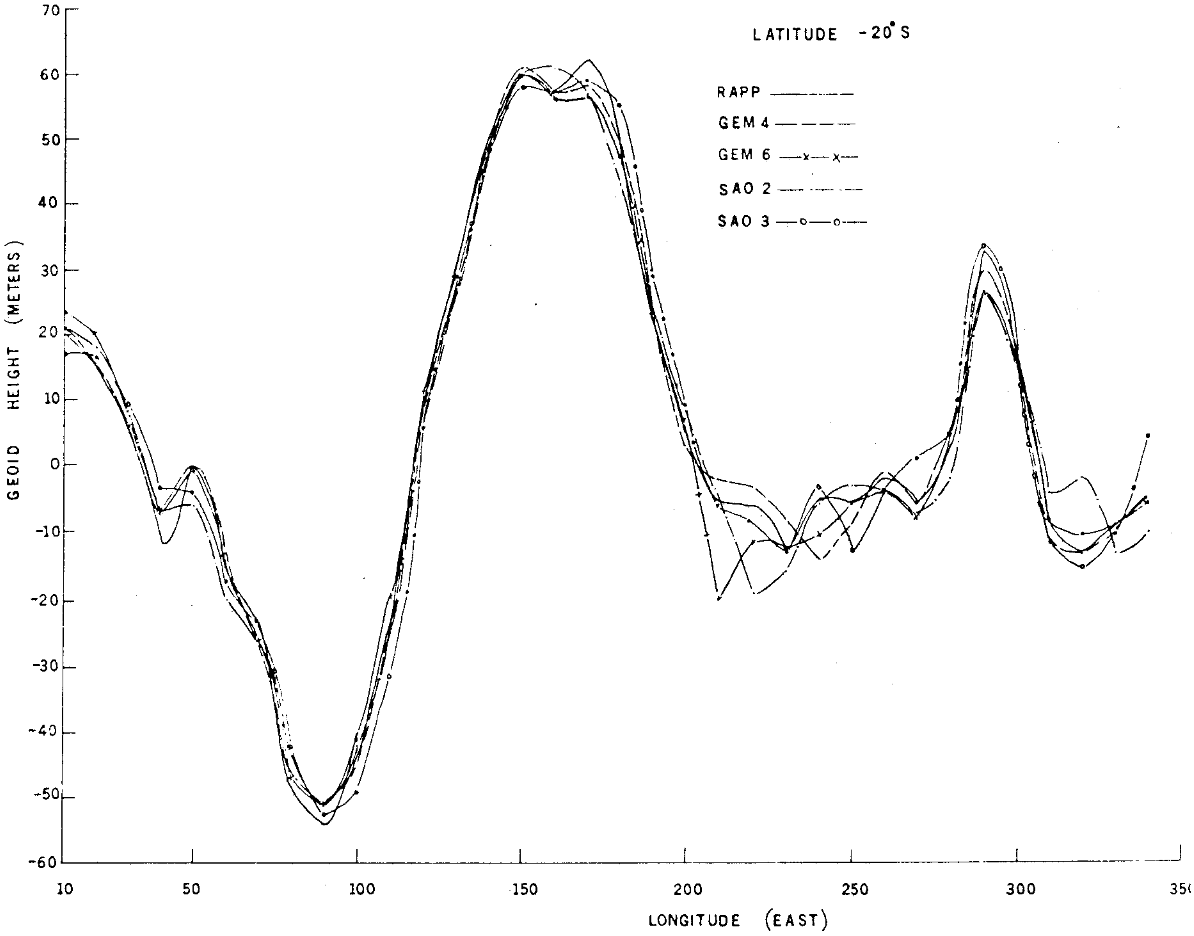

Figure 3. Detailed Geoid Profiles at Latitude 20°S

that the answers are getting better, but rather that the truncated models are identical. The GEM models and the Rapp '73 model represent the geoid better because of the higher accuracy of their higher harmonics as further upcoming tests show.

RMS differences have also been computed for the complete and truncated models versus GEM-6.

GEM-6 versus	Complete Model	(12,12)
Rapp	± 2.7 m	± 1.6 m
GEM-4	± 3.7 m	± 1.3 m
SAO-2	± 4.5 m	± 3.3 m
SAO-3	± 6.5 m	± 5.2 m

It is felt that global rms differences are probably not too meaningful, since the differences in the southern hemisphere are much larger than in the northern hemisphere. Furthermore, these relative differences could be interpreted as a lower bound for the absolute accuracy of the geoid. Figures 11 through 14 present geoid height differences in histogram form. As is noted in these histograms,

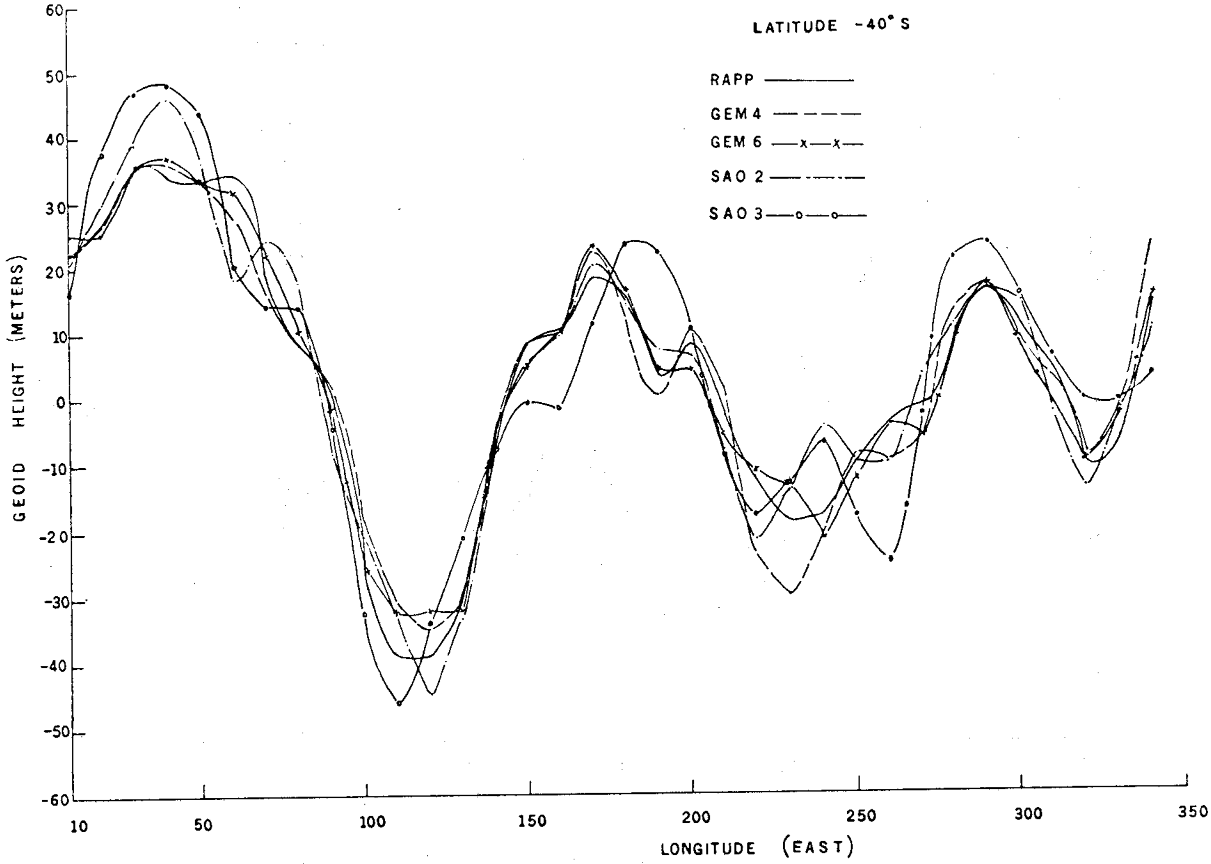

Figure 4. Detailed Geoid Profiles at Latitude 40^{o}S

the most frequent differences are in the range of -5 to +5 m.

3.2 Comparison with External Standards

3.2.1 Comparisons with Astro-geodetic Geoids

Detailed gravimetric geoids computed with the above mentioned models were compared with the astro-geodetic geoids of BOMFORD (1971) in Europe, RICE (1973) in the United States, and MATHER ET AL (1971) in Australia. In all cases, the astro-geodetic geoids were transformed to a centre of mass system using transformation sets of MARSH ET AL (1973) before comparisons were made.

In Europe, detailed gravimetric geoids were computed with the Stokes' functions integrated 10^{o} and 20^{o} around the point of computation. This was done

1. because of the availability of 1^{o}- by - 1^{o} data; and
2. to assess the long wavelength contribution of the gravity models.

A profile at latitude 48^{o}N recommended by Bomford as being the most representative, was used for the comparison. The SAO-2 and SAO-3 profiles were similar, but they were different from those for GEM-4, GEM-6 and the Rapp '73 models. In the case of 10^{o} integration, the detailed geoid using the SAO-3 model showed a tilt of 1.6 arcsec with respect to the astro-geodetic geoid.

However, when the GEM-6 model was considered, the differences became much less systematic and were

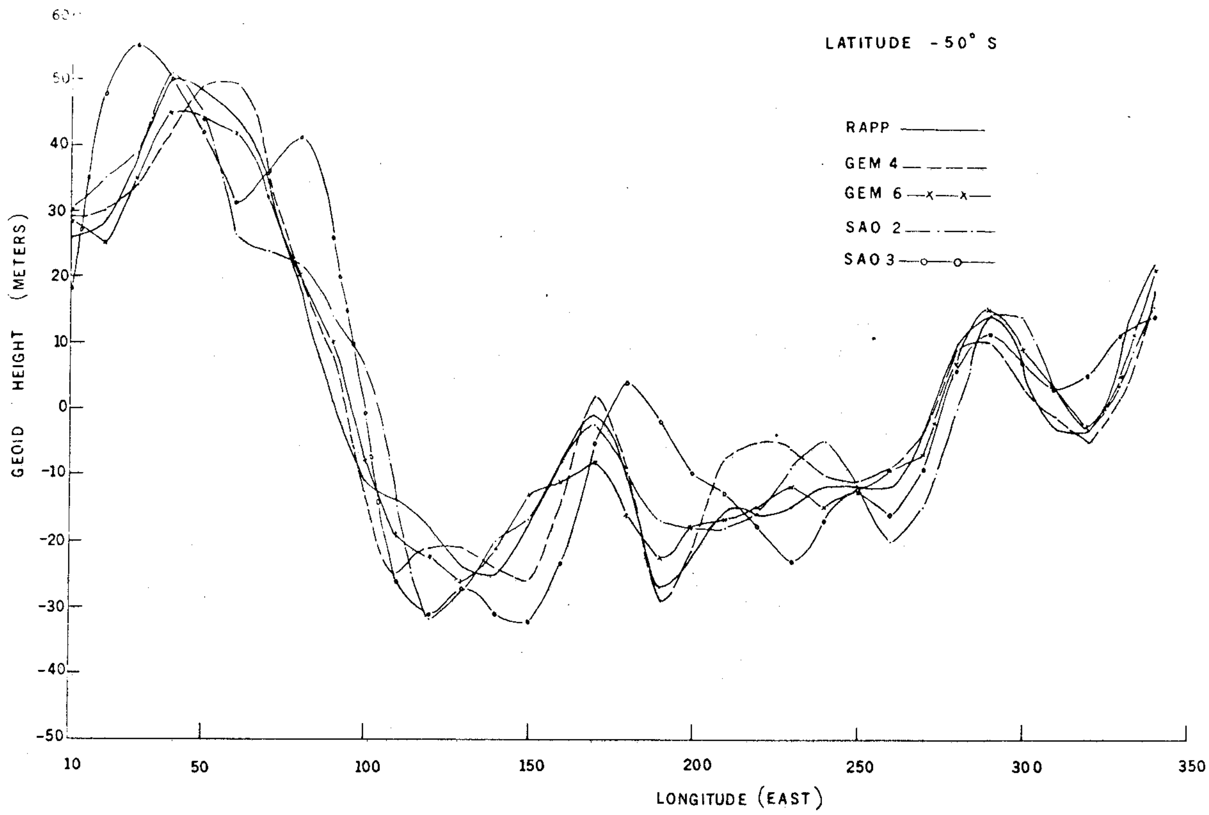

Figure 5. Detailed Geoid Profiles at Latitude 50°S

of the order of ±2 m (figure 15). When the 20° integration was performed, the SAO-3 model and the GEM-6 model both agreed well with the astro-geodetic geoid (figure 16). The GEM-6 detailed geoid did not change when the integration interval was increased from 10° to 20°, indicating a more accurate representation of the long wavelength features by the GEM and Rapp '73 models. In Australia, the comparisons with Mather's astro-geodetic geoid were conducted along a profile 26°S (figure 17). The detailed geoid, when based upon the SAO-3 model, exhibited a tilt of 1 arcsec with respect to Mather's geoid. However, the detailed geoid based upon the GEM-6 model showed only 0.5 arcsec tilt. GEM-6 matched the results Mather found in his studies on the Australian datum.

Another comparison was made with Rice's astro-geodetic geoid for a profile in the United States at latitude 35°N. Table 1 presents the differences between Rice's geoid and the detailed geoids computed using the various models. The differences in the geoid heights for all models were random except for SAO-3 and GEM-6 where an additional constant value of 2 m for SAO-3 and 1 m for GEM-6 had to be added. The agreement between Rice's geoid and all the models was on the order of ±2 m.

3.2.2 Comparison with Dynamic Station Heights

Goddard Space Flight Center Long-Arc Orbital Analyses have provided geocentric co-ordinates for tracking stations (MARSH ET AL 1973). Geoid heights of the tracking stations derived from this

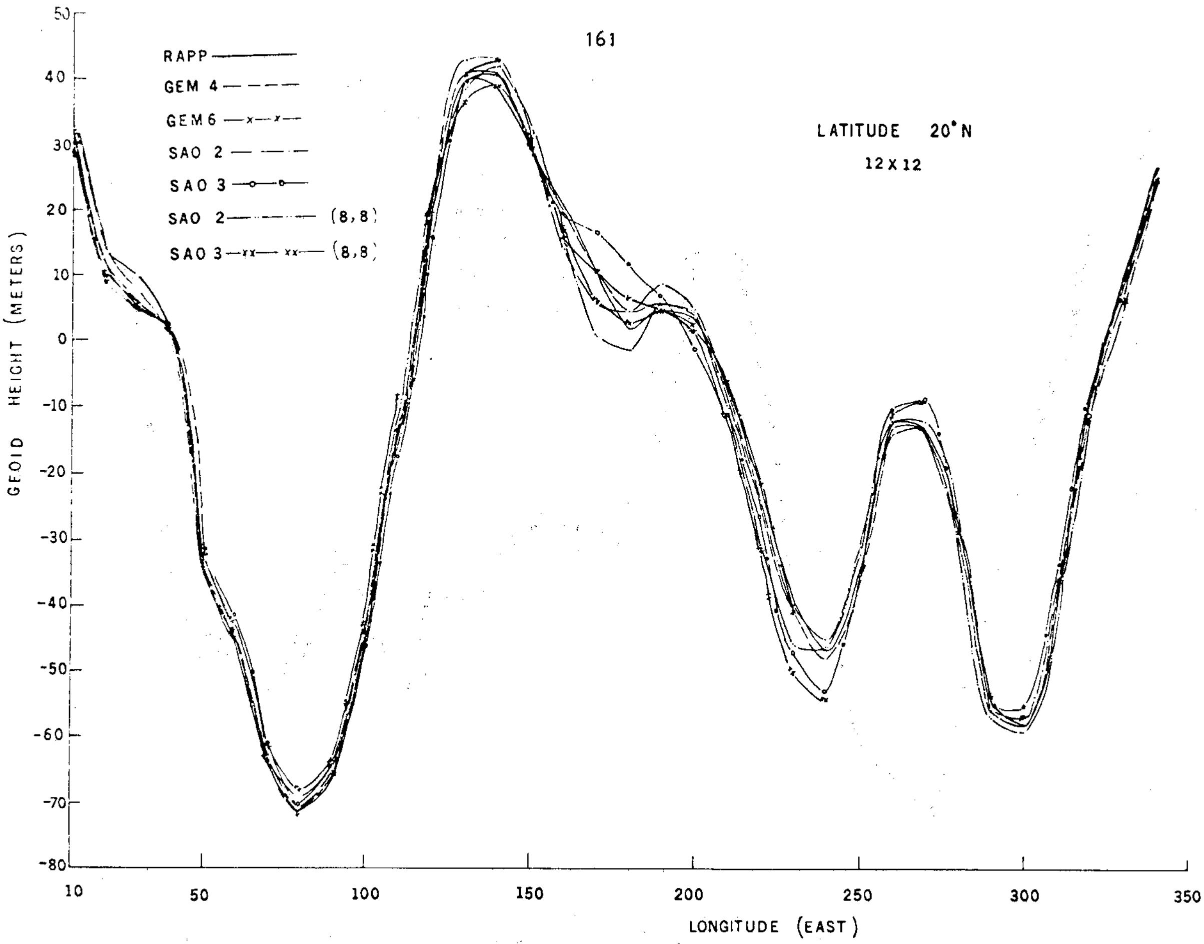

Figure 6. Detailed Geoid Profiles at Latitude 20°N (Truncated Models (12,12) and (8,8))

Table 1

Profile at Latitude 35°N

Difference Between Rice's Converted Astro-geoid and Detailed Geoid Computed using Various Models (units metres)

Longitude (deg)	Rice	GEM-4 *	GEM-6 *†	SAO-II *	SAO-III * ††	RAPP *
81	-34	-1	-2	-2	-2	-1
83	-30	1	0	0	-2	1
85	-32	-1	-1	-2	-2	-1
87	-29	0	0	-1	-2	0
91	-29	0	1	0	-2	0
92	-29	0	2	0	-1	0
93	-31	-1	1	-1	-1	-1
96	-29	2	3	1	-1	2
99	-28	1	1	0	0	1
101	-28	0	1	0	1	1
105	-20	1	2	2	3	3
107	-21	0	0	0	2	1
110	-23	0	-1	0	2	1
113	-27	0	-1	-1	2	0
115	-30	1	-1	0	1	0
117	-31	2	1	2	2	2

KEY:-

* Rice minus GEM-4 detailed geoid height, GEM-6, ...

† Systematic difference of 1 m has been added

†† Systematic difference of 2 m has been added

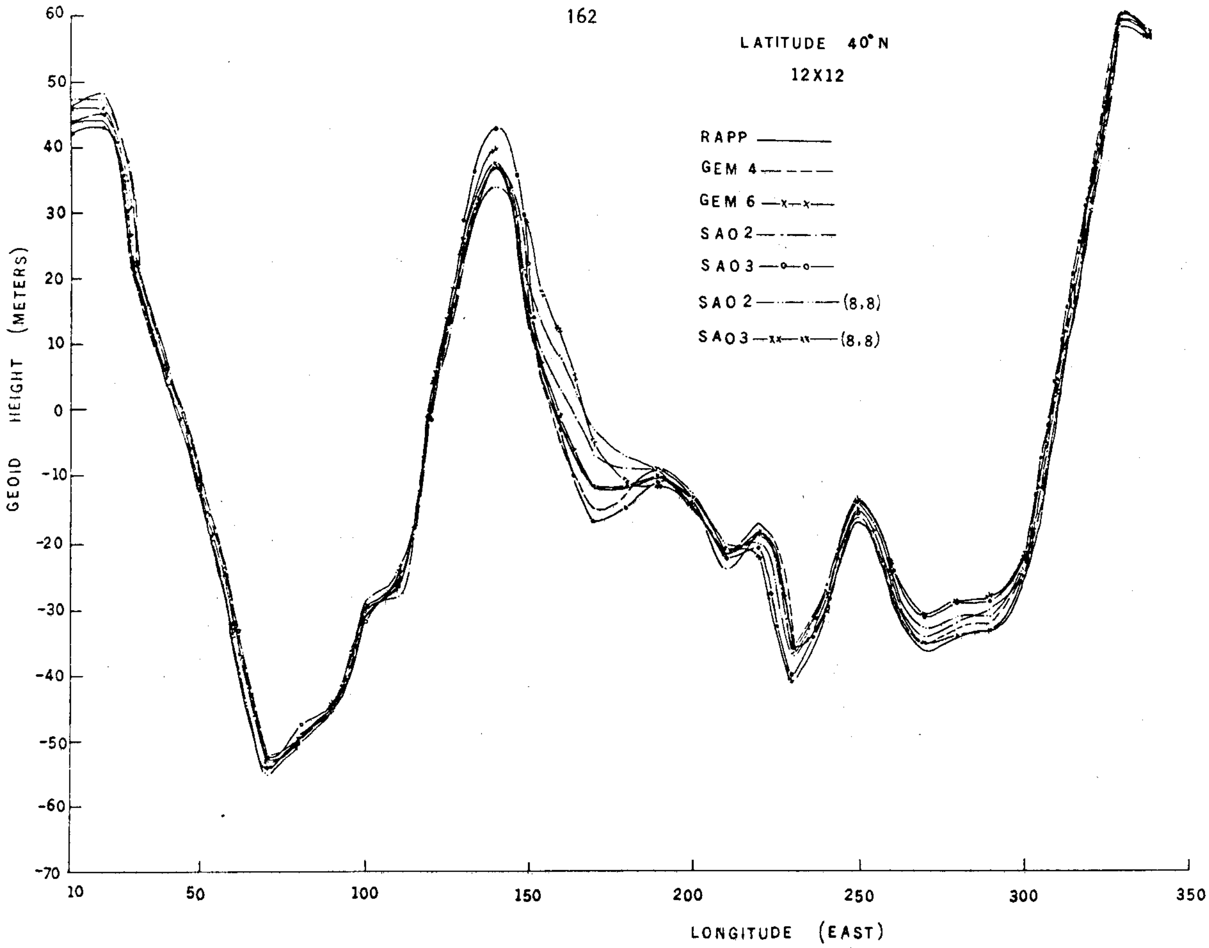

Figure 7. Detailed Geoid Profiles at Latitude 40°N (Truncated Models (12,12) and (8,8))

solution were compared with detailed geoid heights. Table 2 presents the results of these comparisons for stations in the United States and the Caribbean. The results obtained using all the satellite models are similar, except for those computed using the SAO-3 model where differences as large as 5 m versus the average are apparent. The rms agreement for all models is about ± 3 m. This agreement is considered excellent considering the various error sources inherent in this type of comparison. For example, errors can be attributed to

a. dynamically derived station heights;

b. mean sea level values; and

c. gravimetric geoid heights.

4. The Goddard Earth Model (GEM-6) Detailed Geoid

The GEM-6 gravity model was chosen to be the base model for detailed geoid computations. The GEM-6 model consists of a geopotential field in spherical harmonics and a centre of mass system of tracking stations. The GEM-6 solution was computed from a combination of the GEM-5 solution with surface gravimetric data and simultaneous satellite tracking data (LERCH ET AL 1973). The GEM-5 solution is based on satellite data only. The satellite data consisted of 350 weekly orbital arcs of optical, electronic and laser tracking data on 27 close Earth satellites. In addition, approximately 100 one- and two-day arcs of GEOS tracking data were employed for refinement of tracker co-ordinates. The surface gravimetric data consisted of a global collection of 300-by-300

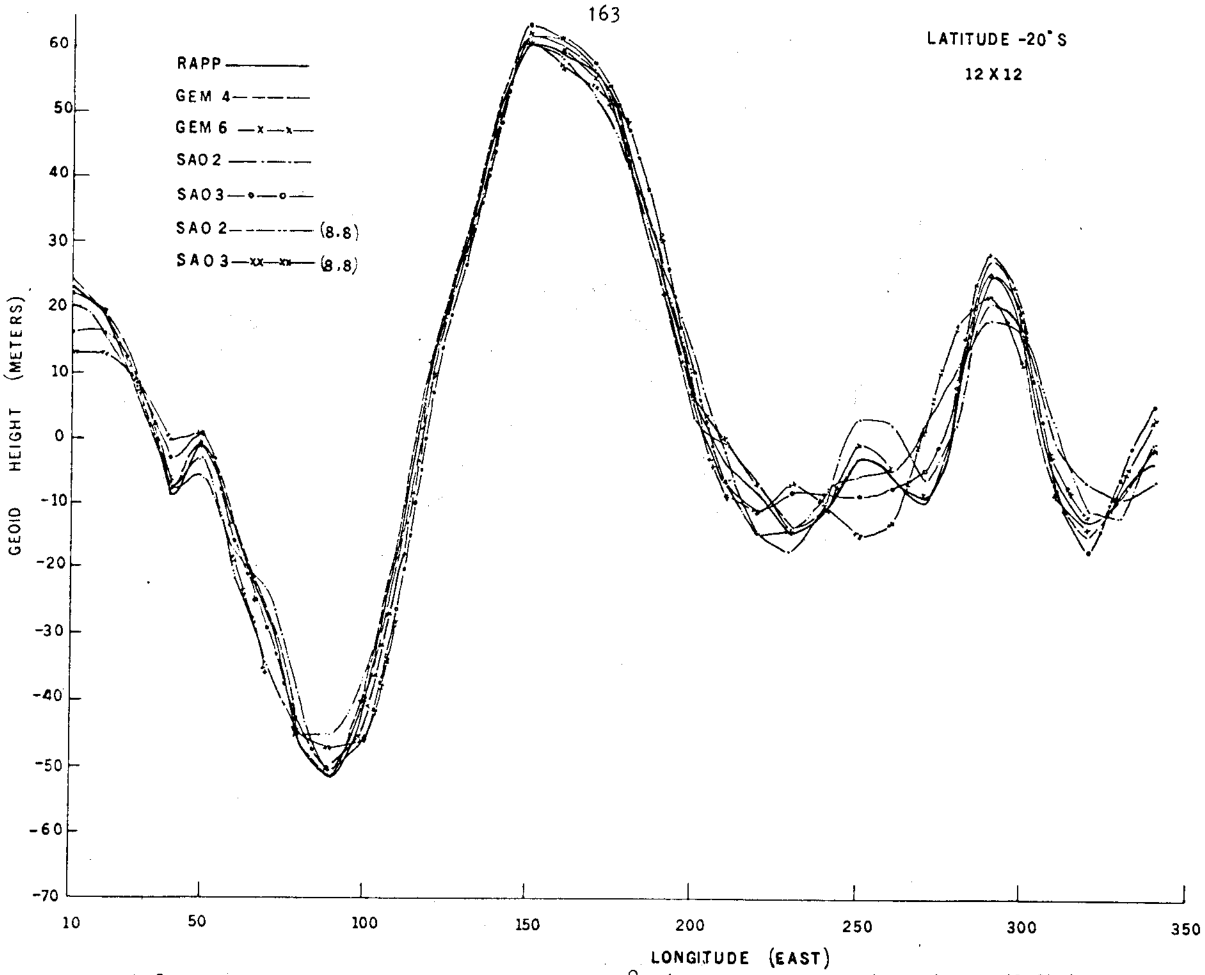

Figure 8. Detailed Geoid Profiles at Latitude 20°S (Truncated Models (12,12) and (8,8))

Table 2

Comparison Between Dynamic Station Heights and Gravimetric Geoid Using Various Models (Metres)

Station Number	Latitude (deg)	Longitude (deg)	GSFC 73* Long Arc	GEM-4†	GEM-6†	SAO-II†	SAO-III†	RAPP*	KEY
1032	48	307	12	-1	0	5	1	1	* Geoid Height = (Dynamic height above ellipsoid) minus (Mean Sea Level height)
1021	38	283	-43	-9	-10	-9	-14	-9	
1022	27	278	-29	2	1	-1	-3	1	
1030	35	243	-30	5	3	2	3	3	
1034	48	263	-27	1	-1	-3	-3	2	
1042	35	277	-34	-2	-3	-4	-7	-3	† GSFC '73 (MARSH, DOUGLAS & KLOSKO 1973) minus GEM-4 detailed geoid, GEM-6,....
7036	26	262	-27	-2	-3	-3	-5	-2	
7037	39	268	-35	-1	-1	-2	-6	-1	
7050	39	283	-40	-6	-8	-6	-12	-6	
7045	40	255	-18	0	0	1	-2	-1	
9001	32	253	-22	1	1	6	1	2	
9021	32	249	-30	-1	-2	-8	-2	-1	
7072	27	280	-32	4	0	-1	-2	1	
7075	46	279	-32	5	5	5	1	6	
7039	32	295	-35	4	4	6	2	6	
7040	18	294	-46	4	4	2	3	3	

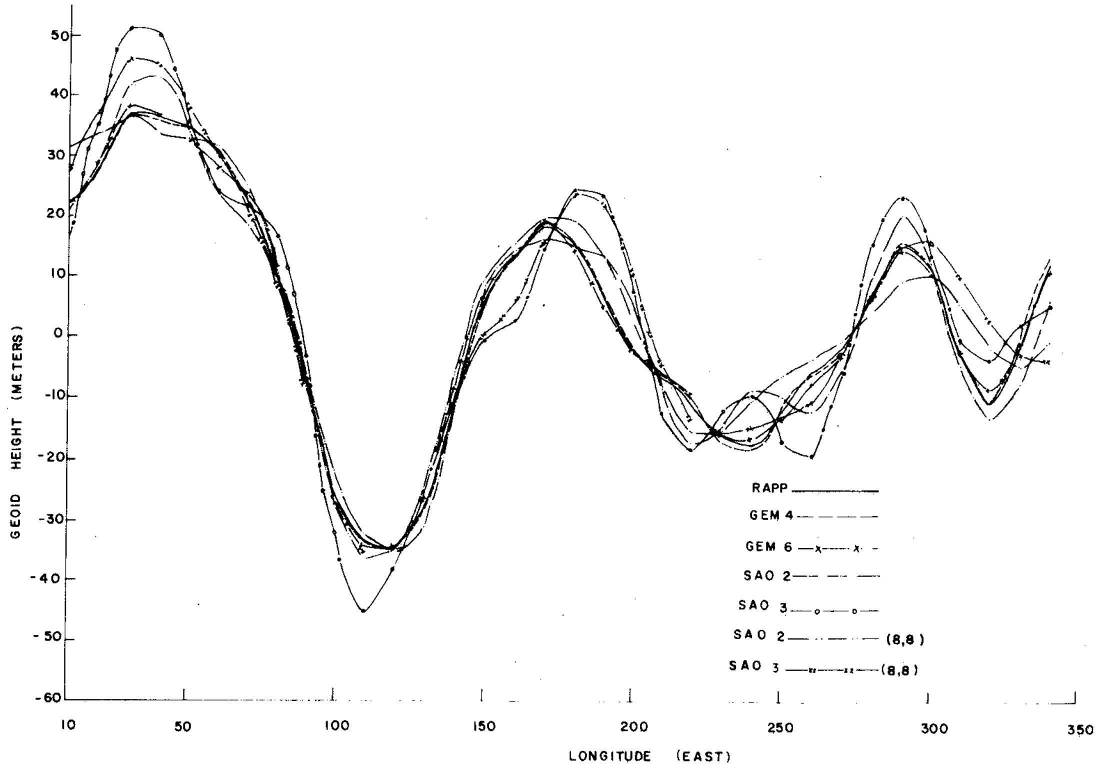

Figure 9. Detailed Geoid Profiles at Latitude 40°S (Truncated Models (12,12) and (8,8))

nautical mile equal area anomalies developed by RAPP (1972). The simultaneous observational data from the North American MOTS-Laser network and from the global BC-4 network were processed geometrically before inclusion in the the GEM-6 solution. The GEM-6 model is complete to degree and order 16, with higher degree zonal harmonics and selected satellite resonant terms extending to degree 22.

The global detailed geoid (GEM-6) is presented in figure 18. The parameters used in the computation of the detailed geoid are :

$$
\begin{aligned}
W_o &= 6263687.5 \text{ kgal m}, \\
\gamma_e &= 978032.2 \text{ mgal}, \\
a_e &= 6378.142 \text{ km}, \\
1/f &= 298.255, \\
GM &= 3.986009 \times 10^5 \text{ km}^3\text{sec}^{-2}, \text{ and} \\
\omega &= 0.72921151467 \times 10^{-4} \text{ rad sec}^{-1}.
\end{aligned}
$$

The general difference between the satellite geoid (GEM-6) and the detailed geoid are on the order of 10 m or less. However, large variations in geoid heights do exist in certain areas. For example, in Australia, prominent differences of 10-12 m occur in the eastern part of the country due to the dominance of mountain ranges that adjoin relatively flat plains and shallow continental slopes. A difference of 15 m over the Puerto Rico trench occurs, which is a function of a large gravity gradient over a small region. These differences, coupled with numerous others, are the representation of the surface gravity short wavelength contributions to the geoid that are

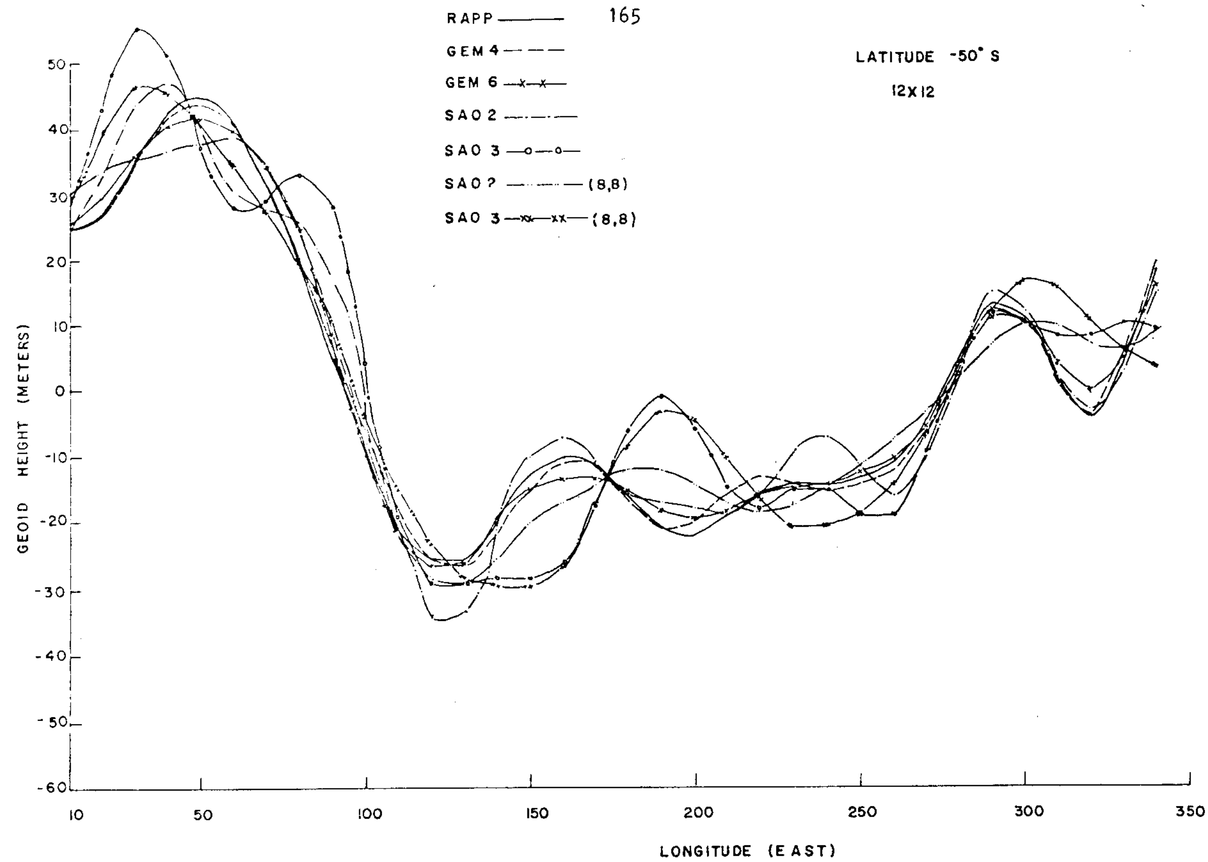

Figure 10. Detailed Geoid Profiles at Latitude 50°S (Truncated Models (12,12) and (8,8))

not provided by the above satellite model.

5. Conclusions

The accuracy of the GEM-6 detailed geoid is assessed as ±2 m in areas of dense surface gravity coverage.

The greatest divergence in these models appears in areas of sparse surface data coverage, notably in the southern hemisphere. The magnitude of these differences was as large as 25 m with a wavelength of approximately 30° to 50°,

Analysis in the southern hemisphere with models truncated to (12,12) indicated that caution should be exercised in interpreting geoid details provided by higher degree and order harmonic coefficients when surface data is lacking.

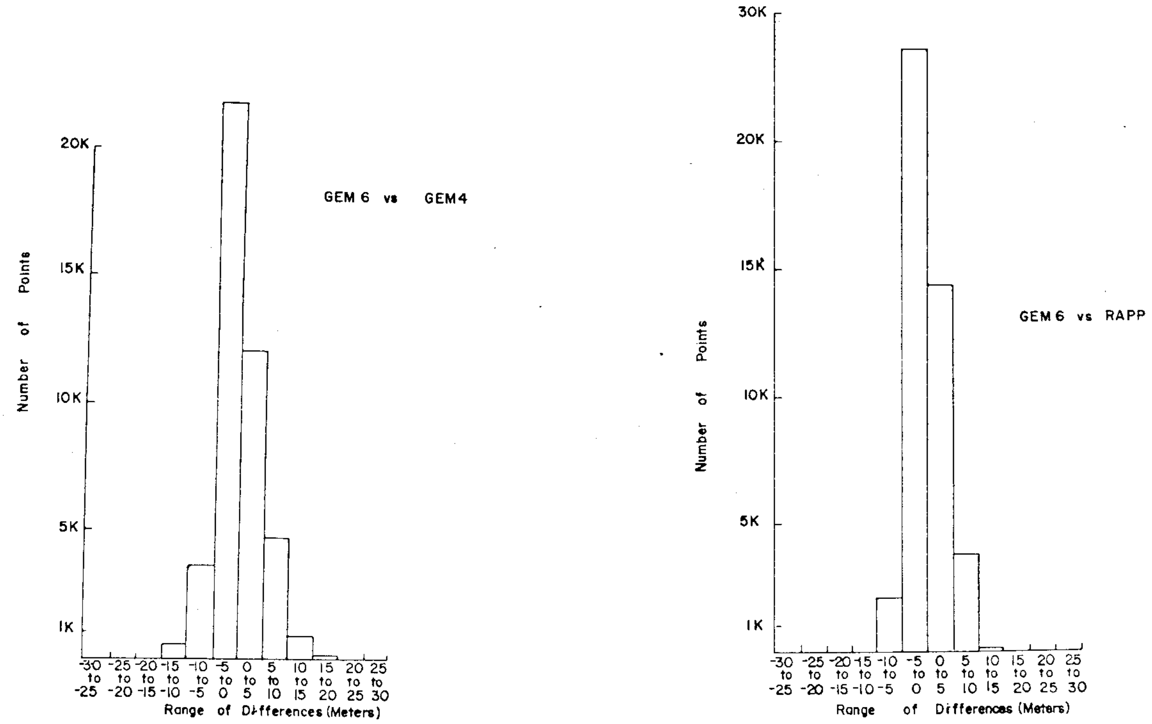

Figure 11. Histogram Showing Differences Between GEM-6 Detailed Geoid versus Detailed Geoids of GEM-4 and Rapp '73 (Complete Model)

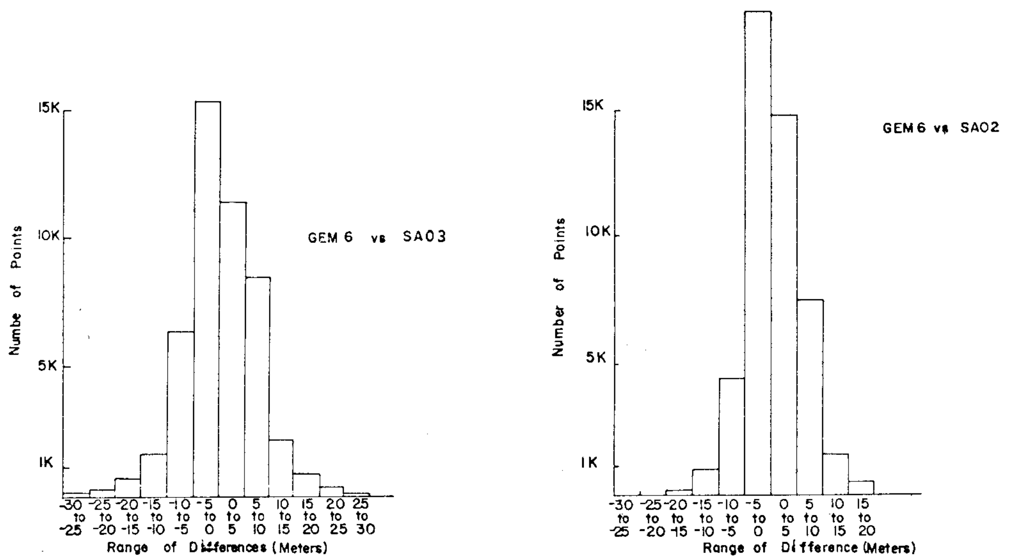

Figure 12. Histograms Showing Differences Between GEM-6 Detailed Geoid versus Detailed Geoids of SAO-2 and SAO-3 (Complete Models)

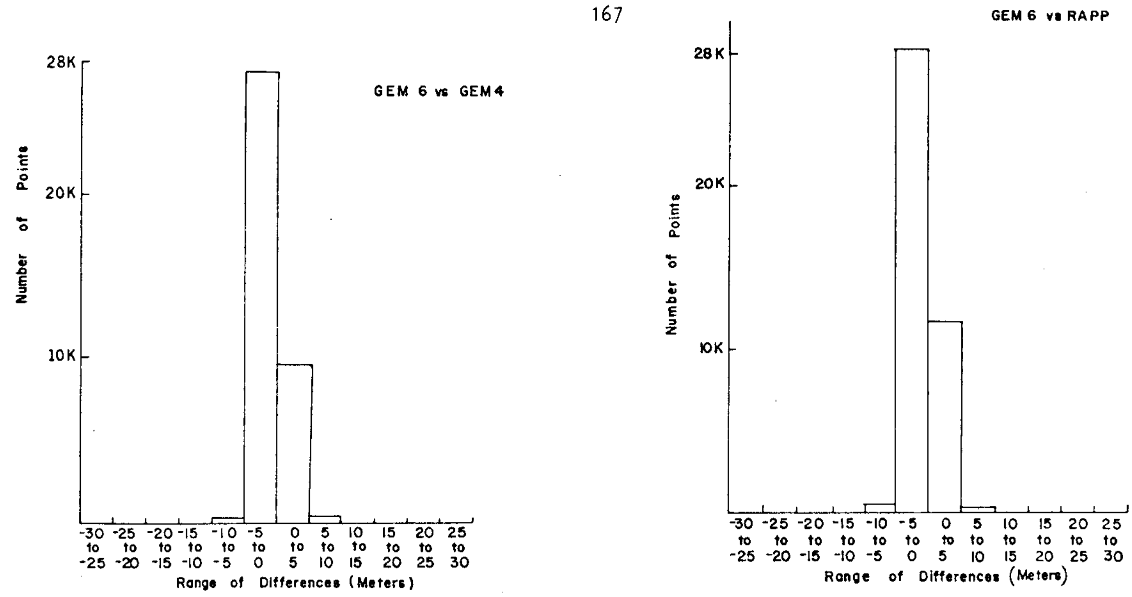

Figure 13. Histograms Showing Differences Between GEM-6 Detailed Geoid Versus Detailed Geoids of GEM-4 and Rapp [Truncated Models (12,12)]

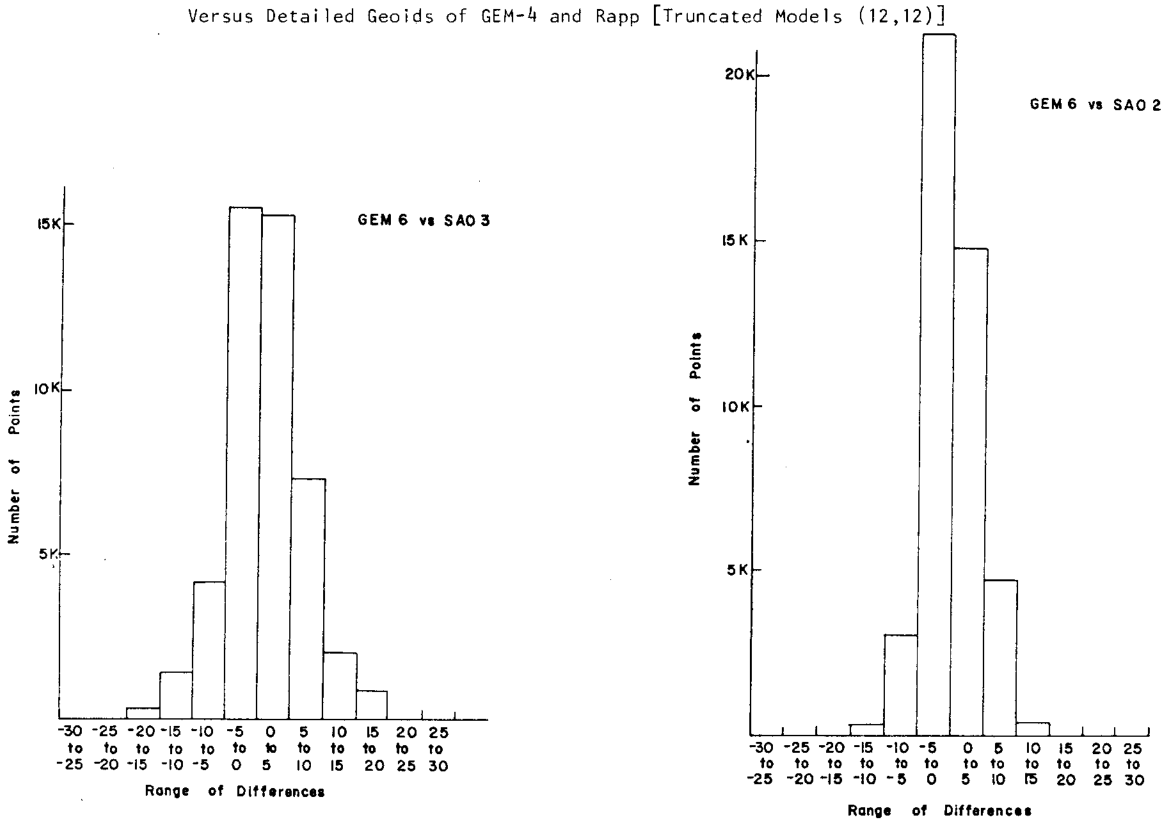

Figure 14. Histograms Showing Differences Between GEM-6 Detailed Geoid versus Detailed Geoids of SAO-2 and SAO-3 [Truncated Models (12,12)]

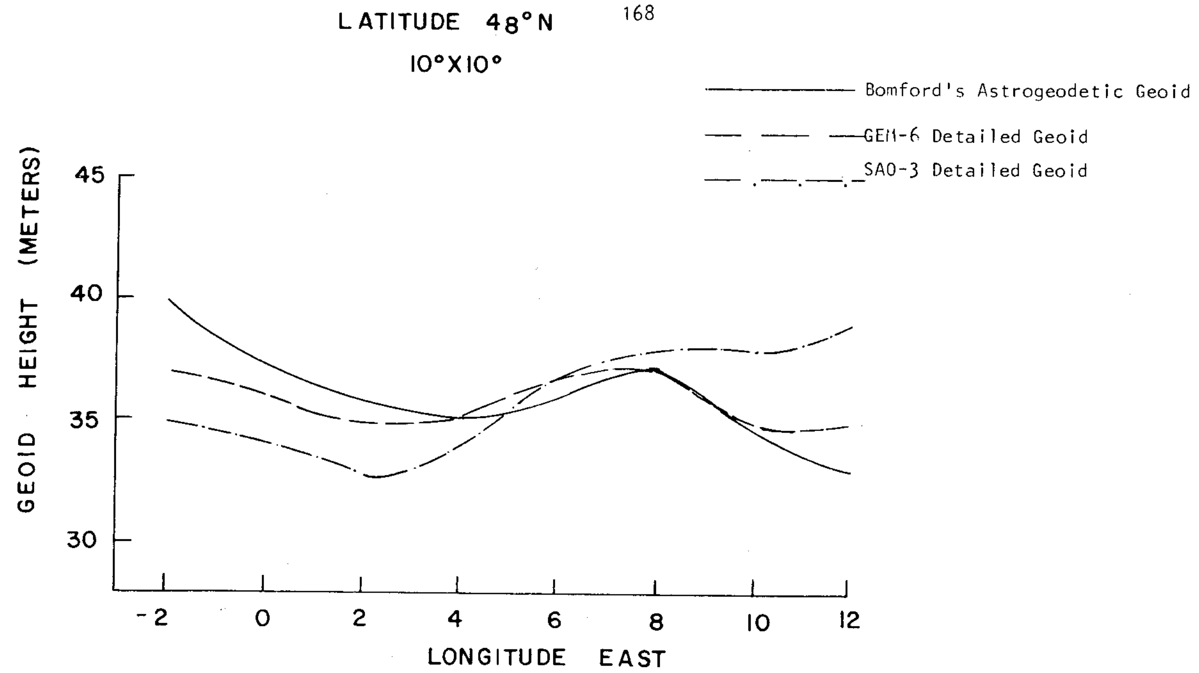

Figure 15. Comparison Between Bomford's Transformed Astrogeodetic Geoid and Detailed Gravimetric Geoid (GEM-6 and SAO-3) Integrated 10° Around Computation Point in Europe

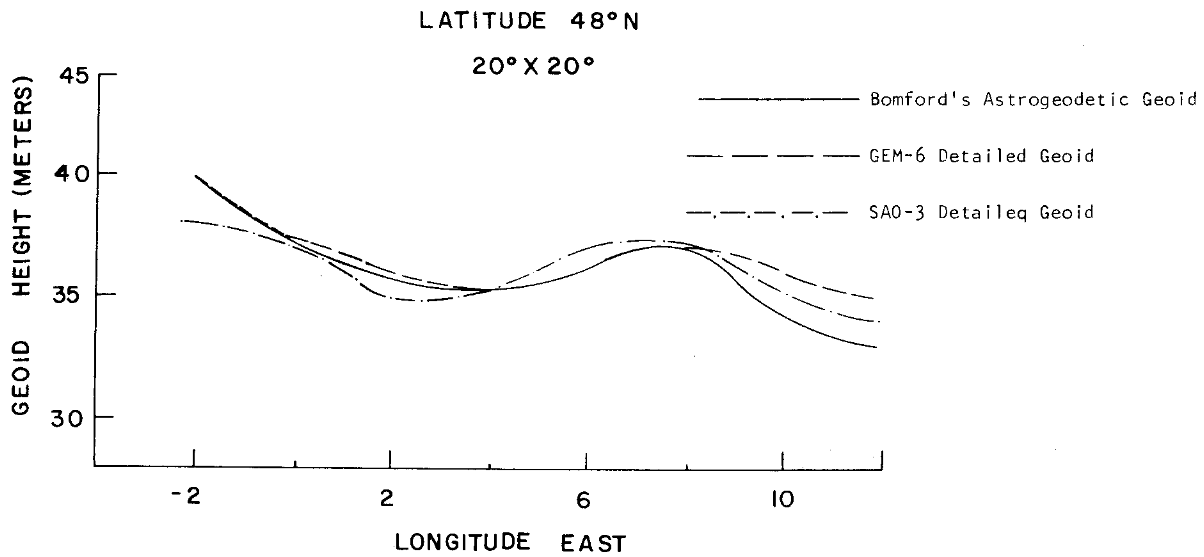

Figure 16. Comparison Between Bomford's Transformed Astrogeodetic Geoid and Detailed Gravimetric Geoid (GEM-6 and SAO-3) Integrated 20° Around Computation Point for Europe

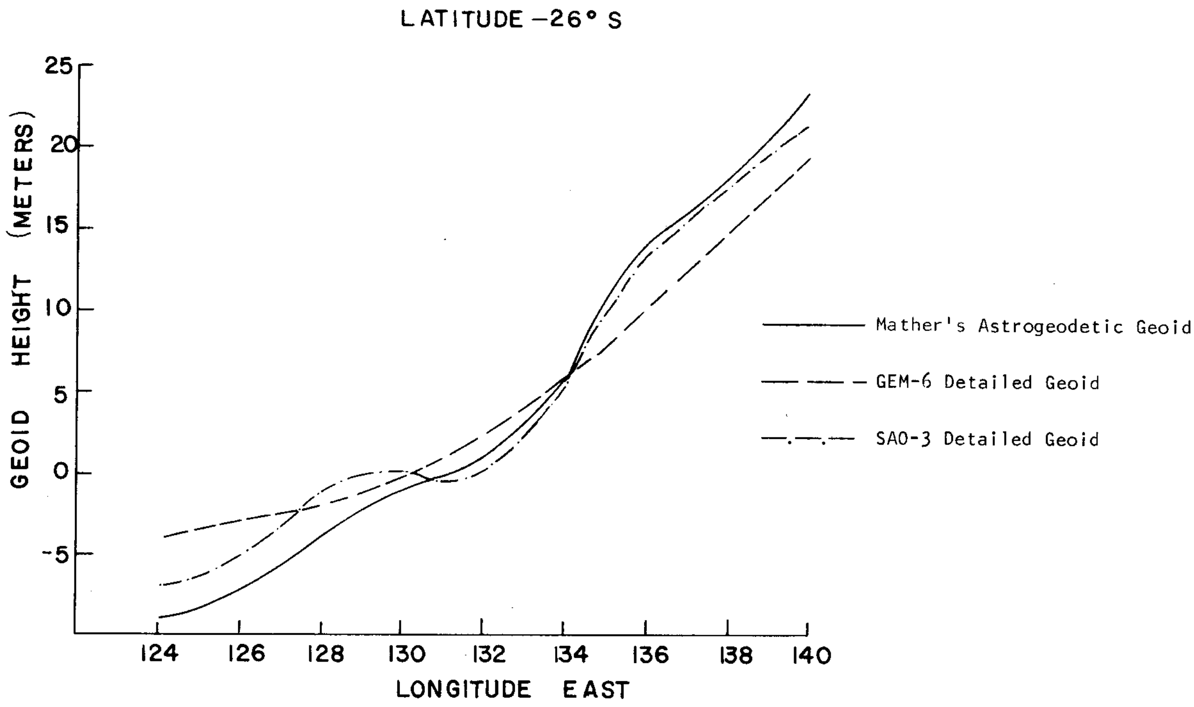

Figure 17. Comparison Between Mather's Transformed Astrogeodetic Geoid and Detailed Gravimetric Geoid (GEM-6 and SAO-3) in Australia

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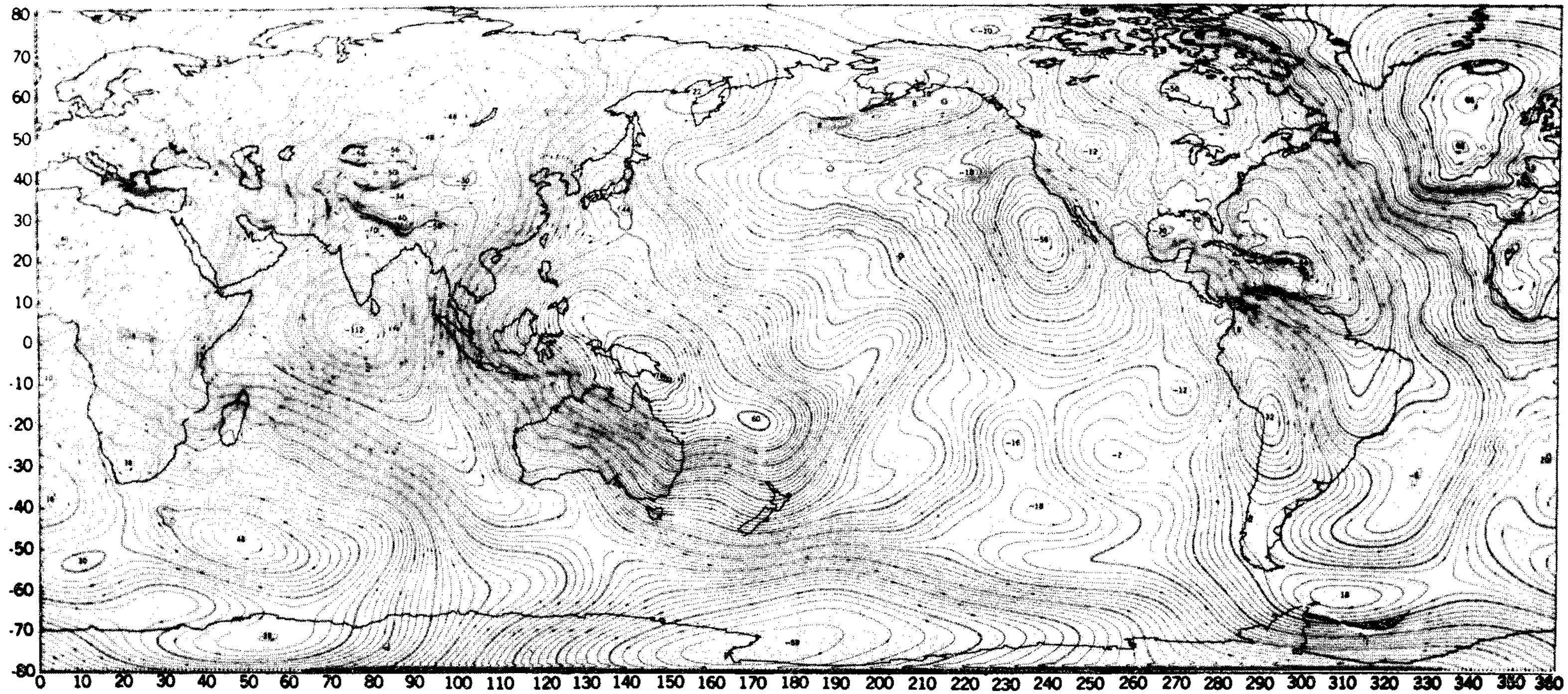

NASA/GODDARD SPACE FLIGHT CENTER

GLOBAL DETAILED GRAVIMETRIC GEOID BASED UPON A COMBINATION OF THE GSFC GEM-6 EARTH MODEL AND 1°×1° SURFACE GRAVITY DATA

CONTOUR INTERVAL 2 METERS, EARTH RADIUS: 6378.142 KM, 1/F=298.255, GM=398600.9 KM^3/SEC^2

562-2190
DEC. 73

7. Discussion

STOLZ: Why did you choose these particular sections for illustrating the comparisons?

VINCENT: The selection was arbitrary.

WALCOTT: What is the basis for the ±2 m figure quoted for accuracy?

VINCENT: It is based on values of dynamic station parameters and astro-geodetic geoid comparisons.

MATHER: Did you always go out to twenty degrees (from the point of computation)?

VINCENT: No; only when data was available. If there was no information, the value of the difference was put to zero.

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Proc.Symposium on Earth's Gravitational Field & Secular Variations in Position (1973), 172-187.

A 1973 GRAVIMETRIC GEOID OF AFRICA

ABSTRACT

Results of a preliminary gravimetric geoid computed on the Geodetic Reference System 1967 (GRS67) are given in the form of undulation and deflection maps.

For the computations, $1^o \times 1^o$ mean free air gravity anomalies formed the inner zone and five degree equal area values the outer zone around each computation point. Some of the required $1^o \times 1^o$ means, together with their accuracies, were obtained by statistical prediction methods using an autocovariance function derived from the known $1^o \times 1^o$ values.

The results confirm the general south-easterly down slope of the geoid undulations. The average computed standard errors are about 3.6 m in the undulations and 0"6 in the deflections.

1. Introduction

Research in geodesy on Africa within Africa is generally very frustrating, mainly because of lack of data (and funds). One usually starts off by writing to related departments in the various countries in the continent. Most of these departments never reply and the few that do, refer one to either Britain, France or the United States of America. With some luck, one may receive some data from Britain or (in most cases), from the USA several letters and months later!

Computation of the geoid (undulations and deflections) and the related accuracies from gravity anomalies is a straightforward application of well-known formulae. However, as far as we know, this has never been done for the whole African continent. This paper is thus a primary attempt at computing what is, in fact, a "free air co-geoid" which, it is hoped, would be improved upon as more gravity data becomes available.

2. Gravity Data Used

The free air gravity anomaly data used consisted basically of: United States Aeronautical Chart and Information Center $1^o \times 1^o$ mean values (ACIC 1971), Rapp's five degree equal area values (RAPP 1972) and Bureau Gravimetrique International Bouguer anomaly map and $5^o \times 5^o$ means (BGI 1971). A total of 5450 $1^\circ \times 1^\circ$ sub-blocks, enclosed within 197 5° equal area blocks, were required. 2500 of the $1^o \times 1^o$ values were given in the ACIC report and 210 of these were replaced by what were considered to be better values. Of the remaining 2950 $1^o \times 1^o$ values, 350 were obtained from Bouguer anomalies either given as point values or read off from several maps that were available to us. The rest, 2600 of them, were predicted using the minimum variance or contouring methods. Figure 1 shows the area covered by the $1^o \times 1^o$ mean values.

For the minimum variance method, an autocovariance function was developed from the 2850 known $1^\circ \times 1^\circ$ means. This function was then used to predict the unknown $1^\circ \times 1^\circ$ values in any area within the limits of the maximum range of the function. The prediction equations given by MORITZ (1969) were used.

Thus, using matrix notation,

$$\hat{\Delta g} = \bar{C}\,(C + D)^{-1}\,\Delta g \qquad (1),$$

and

$$m^2_{\Delta g} = \bar{\bar{C}} - \bar{C}\,(C + D)^{-1}\,\bar{C}^T \qquad (2),$$

where
$\hat{\Delta g}$ is the predicted $1^\circ \times 1^\circ$ value;
Δg is the column vector of known $1^\circ \times 1^\circ$ means;
m^2 is the variance of the predicted anomaly;
C is the matrix whose elements are the covariance between the known blocks;
$\bar{C}$ is a row vector whose elements are the covariances between the block p being predicted and the known blocks i used in the prediction; thus

Thus

$$\bar{C}_i = C_{pi} \qquad (3);$$

D represents the given variances of the known $1^\circ \times 1^\circ$ anomalies used in the predict prediction; and
$\bar{\bar{C}}$ is the variance of the $1^\circ \times 1^\circ$ anomalies.

Thus

$$\bar{\bar{C}} = C_{ii} \qquad (4).$$

However, there were some areas, particularly south of the West African coast, where contouring was used owing to the short range nature of the covariance function.

As a result of the new and predicted $1^\circ \times 1^\circ$ values, 168 five degree equal area means were replaced, 116 of them differing by 5 mgal or less from the Rapp values. The variance of each new value was obtained from equation 2 with appropriate definitions of the matrices.

Finally, the one degree and five degree anomalies were transformed into the GRS67 system to which the computed deflections and undulations were referred. The transformation equations were

$$\Delta g_{67} = \Delta g_R + 1.71 + 0.11 \sin^2\phi \text{ mgal} \qquad (5),$$

where Δg_R are the Rapp five degree anomalies referred to a differently defined reference system, and

$$\Delta g_{67} = \Delta g_I + 2.33 - 13.6 \sin^2\phi \text{ mgal} \qquad (6),$$

where Δg_I are the $1^\circ \times 1^\circ$ values referred to the International Gravity Formula.

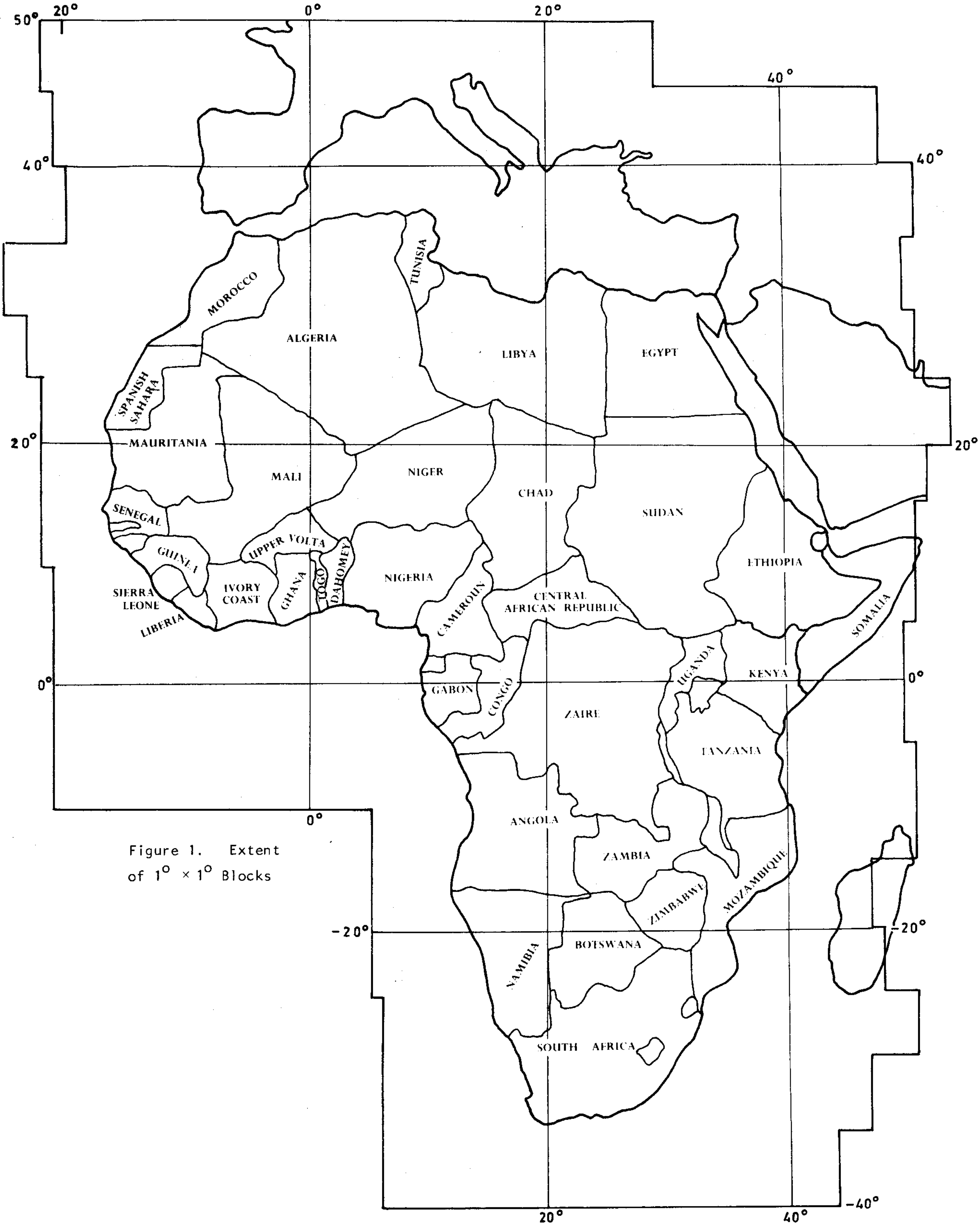

Figure 1. Extent of $1^{\circ} \times 1^{\circ}$ Blocks

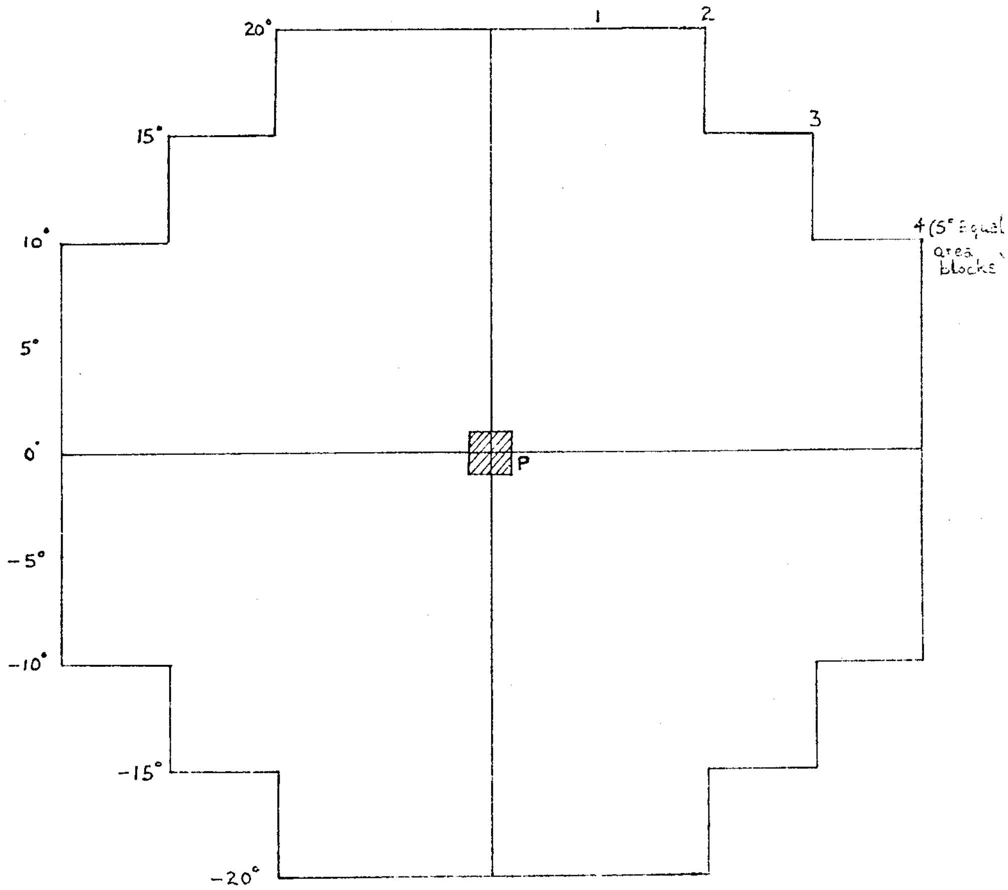

Figure 2. Limits of Inner Zone Around Each Computation Point P. The Four Shaded $1^\circ \times 1^\circ$ blocks were not included in the Computations

Both equations 5 and 6 imply a Potsdam correction of -14 mgal (IAG 1971) and an atmospheric gravity correction of -0.87 mgal (ECKER & MITTERMAYER 1969)

3. Computational Techniques

Stokes' and Vening Meinesz equations are the relevant equations for the computation of the undulations and deflections of the vertical respectively. Thus

$$\begin{Bmatrix} N \\ \\ \xi \\ \\ \eta \end{Bmatrix} = \frac{1}{4\pi G} \begin{Bmatrix} R \iint \Delta g \; S(\psi) \; d\sigma \\ \\ \rho \iint \Delta g \; \dfrac{dS(\psi)}{d\psi} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} d\sigma \end{Bmatrix} \qquad (7),$$

where $S(\psi)$ is Stokes' function;

G is the mean Earth gravity ;

R is the mean Earth radius;

Δg are free air anomalies;

$d\sigma$ is the element of area of the surface σ whose anomaly is Δg and at an azimuth α from the computation point; and

ρ = cosec 1''.

In practice the required values are computed from any of several numerical forms of equation 7. For example,

$$\begin{Bmatrix} N \\ \\ \xi \\ \\ \eta \end{Bmatrix} = \frac{1}{4\pi G} \begin{Bmatrix} R \sum\limits_m \Delta g \; S(\psi) \; \Delta\sigma \\ \\ \rho \sum\limits_m \Delta g \; dS(\psi) \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} \Delta\sigma \end{Bmatrix} \tag{8}$$

or

$$\begin{Bmatrix} N \\ \\ \xi \\ \\ \eta \end{Bmatrix} = \frac{1}{4\pi G} \begin{Bmatrix} R \sum\limits_m \Delta g \; \frac{1}{n} \sum\limits_n S(\psi) \; \Delta\sigma \\ \\ \rho \sum\limits_m \Delta g \; \frac{1}{n} \sum\limits_n dS(\psi) \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} \Delta\sigma \end{Bmatrix} \tag{9}$$,

where m is the number of large blocks covering the Earth's surface; and

n is the number of sub-blocks in each large block m.

Equation 8 was used for the computations done in this paper. Because of the behaviour of Stokes' function and its derivative as the computation point is approached, the computations are usually done for an inner zone using smaller blocks, and for an outer zone using larger blocks, 1° × 1° and 5° equalarea mean values respectively for this paper. Thus, symbolically,

$$\begin{Bmatrix} N \\ \xi \\ \eta \end{Bmatrix} = \begin{Bmatrix} N \\ \xi \\ \eta \end{Bmatrix}_{\text{Inner}} + \begin{Bmatrix} N \\ \xi \\ \eta \end{Bmatrix}_{\text{Outer}} \tag{10}$$.

Equations 8 and 10 can be written in matrix form as

$$E = K_I \, A_I + K_0 \, A_0 \tag{11}$$,

where E represents the computed undulations and deflections;

A represents the gravity anomalies for the inner zone I and the outer zone 0; and

K represents the coefficients of the anomalies, whose elements are given by

$$\begin{Bmatrix} k_{N_i} \\ \\ k_{\xi_i} \\ \\ k_{\eta_i} \end{Bmatrix} = \begin{Bmatrix} \frac{R}{4\pi G} \; S(\psi_i) \; \Delta\sigma_i \\ \\ \frac{\rho}{4\pi G} \; dS(\psi_i) \begin{Bmatrix} \cos\alpha_i \\ \sin\alpha_i \end{Bmatrix} \Delta\sigma_i \end{Bmatrix} \tag{12}$$.

The covariance matrix of the computed undulations and deflections Σ_E follows from equation 11 OBENSON (1973). Thus

$$\Sigma_E = K_I \Sigma_{A_I} K_I + K_0 \Sigma_{A_0} K_0 + K_I \Sigma_{A_I A_0} K_0 + K_0 \Sigma_{A_I A_0} K_I \qquad (13),$$

where

Σ_A is the covariance matrix of the given anomalies. Only the first two terms on the right of equation 13 were used in the evaluations done here since $\Sigma_{A_I A_0}$ is generally not known in practice.

3.1 Determining the Limits of the Inner Zone

The spherical cap defining the inner zone has been variously given as extending up to between 10° to 40° away from the computation point (UOTILA 1959; MATHER 1969; NEEDHAM 1970). It was however determined here by comparing the deflections and the undulations, computed separately, from a five degree equal area anomaly block and from $1^{\circ} \times 1^{\circ}$ values within the five degree block at various points. The location of this block was varied in distance and azimuth from the computation point. Though the spherical distance differed with the position of the computation point, it was found that the lengths, in terms of $\Delta\phi$ and $\Delta\lambda$ for any spherical range of differences between the one degree and five degree contributions to the deflections and undulations, were the same for all points. Table 1 illustrates this for a maximum difference of not more than 0.08 m for the undulation and 0".02 for the deflection components.

From these tests, the inner zone was then defined as a circle of radius $S = \sqrt{[\Delta\phi^2 + \Delta\lambda^2]} = 20^{\circ}$ around the computation point. This limit produces a total maximum average error of about 1.5 m in the undulation and 0".3 in the deflections. However, owing to the shape of the given anomaly blocks, the inner zone for each computation point was taken as shown in figure 2.

Since five degree equal area blocks were used, the east-west lengths were chosen in terms of whole blocks - hence a maximum of four blocks on either side of the point. For most points within the

Table 1

Sample Results Used to Determine the Limits of the Inner Zone

Computation Point				Differences $(5^{\circ} - 1^{\circ})$ Contributions		
ϕ (deg)	λ (deg)	ψ (deg)	$S = \sqrt{(\Delta\phi^2+\Delta\lambda^2)}$ (deg)	ξ (sec)	η (sec)	N (m)
70	0	6.1	17.3	0.003	0.004	0.01
45	5	12.3	17.3	0.010	0.018	0.08
20	0	16.5	17.3	0.001	0.001	0.01
0	15	17.7	17.3	0.001	0.001	0.01
-10	20	17.5	17.3	0.001	0.005	0.03
-35	25	14.7	17.3	0.000	0.003	0.01
-60	0	9.4	17.3	0.018	0.010	0.04

continent, the arrangement in figure 2 applied, but there were other points, especially near the coast where $\Delta\phi$ and $\Delta\lambda$ were as low as 10° or two 5° equal area blocks away. However, the average number of $1^\circ \times 1^\circ$ blocks inside the inner zone was about 1200. For lack of more detailed data, the four $1^\circ \times 1^\circ$ blocks immdeiately surrounding the computation points were not used in the computations.

4. Results

The undulations and deflections of the vertical were computed at 91 points at the intersection of the five degree equal area blocks within the continent. Figures 3, 5 and 7 show the contour maps of N, ξ and η respectively. Figures 4, 6 and 8 are their accompanying standard errors.

To obtain the total value of either the undulatión of deflections at any point, the contribution of the four $1^\circ \times 1^\circ$ blocks surrounding the point should first be computed from a more detailed anomaly field and simplified forms of equation 7, and then added to the value read off maps. For the undulations and their accuracies, these values are generally negligibly small. However figure 3 confirms the general south-easterly down slope of the undulations, decreasing from a maximum of 56 m in northern Morocco to -8 m in Somalia and Kenya. Figures 4, 6 & 8 indicate average standard errors of about 3.6 m in the undulations and $0\overset{''}{.}6$ in each deflection component throughout the continent.

5. Discussion

It is usual to compare free air undulations with equivalent astro-geodetic values in order to determine absolute accuracies; hence some information on the effect of the "indirect effect" and the reliability of the estimated anomalies and their accuracies. It was not possible to do this here because of the lack of numerical data on any astro-geodetic geoids computed in any part of the continent. A suitable comparison would have been the recently-completed 12° latitude astro-geodetic geoid profile connecting western Sudan, just north of the Chad/Central African Republic boundary with Dakar in Senegal (WALKER 1971; MCCALL 1970; YATER 1971). However this was not possible as none of the papers mentioned above contained any discrete values either of the profile or of the Adindan datum on which the profile was based. But a visual comparison between a profile taken approximately along latitude 12°N in figure 3 and that given by Walker shows that both "rise steeply from Sudan to central Nigeria and then drop smoothly from central Mali to Dakar" (WALKER 1971).

The results given here should be useful in the determination of various Earth parameters and transformations between the various geodetic datums on the continent.

6. Acknowledgments

The research for this paper was supported in part by a research grant for computer time from the Faculty of Engineering through the Dean, Professor I.O. Oladapo. The author is grateful and also wishes to thank all those who contributed advice and/or gravity data, in particular Professor R.H. Rapp, Elmer Hauer and Enilo Ajakaiye.

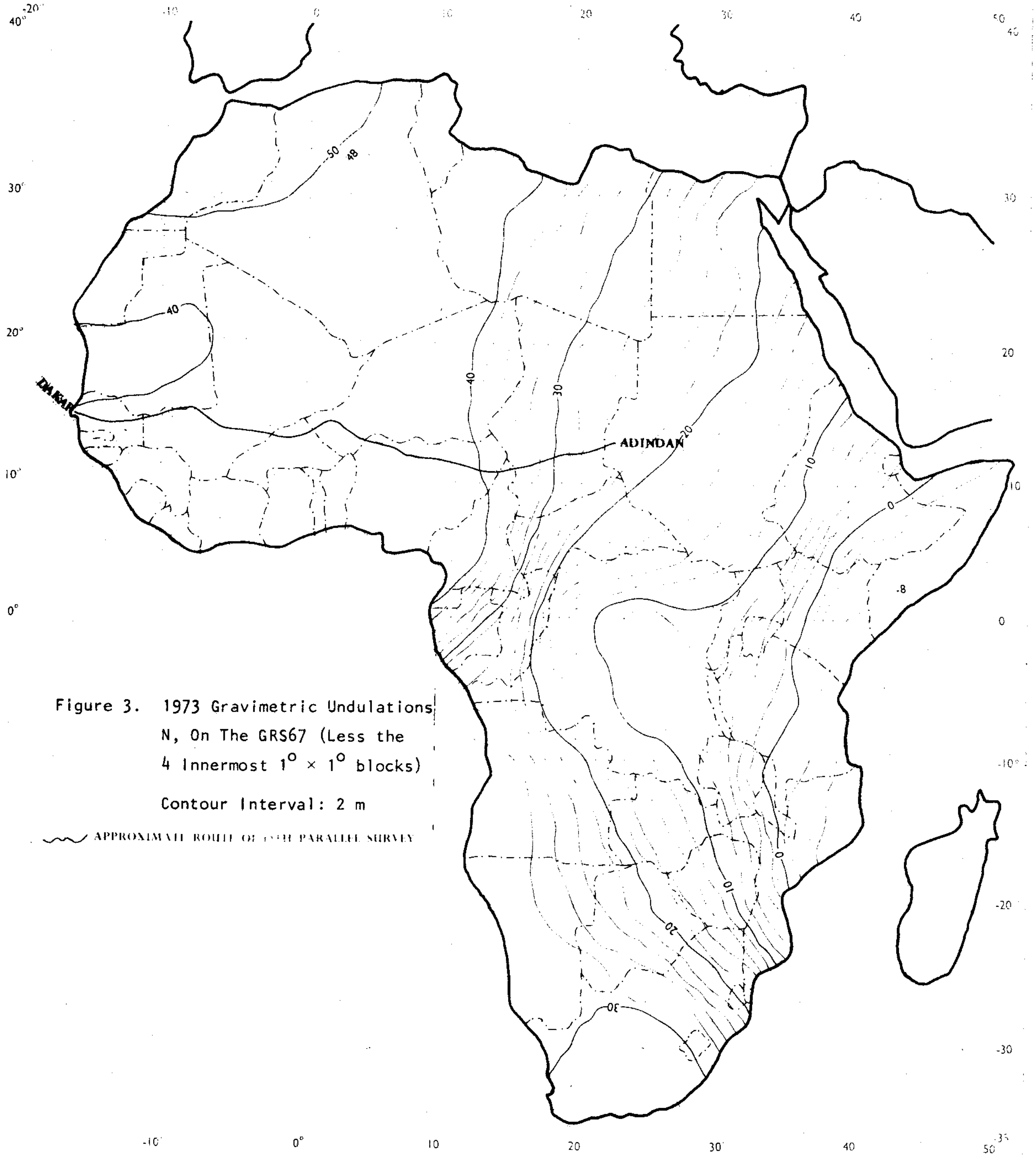

Figure 3. 1973 Gravimetric Undulations N, On The GRS67 (Less the 4 Innermost $1^{\circ} \times 1^{\circ}$ blocks)

Contour Interval: 2 m

APPROXIMATE ROUTE OF 12TH PARALLEL SURVEY

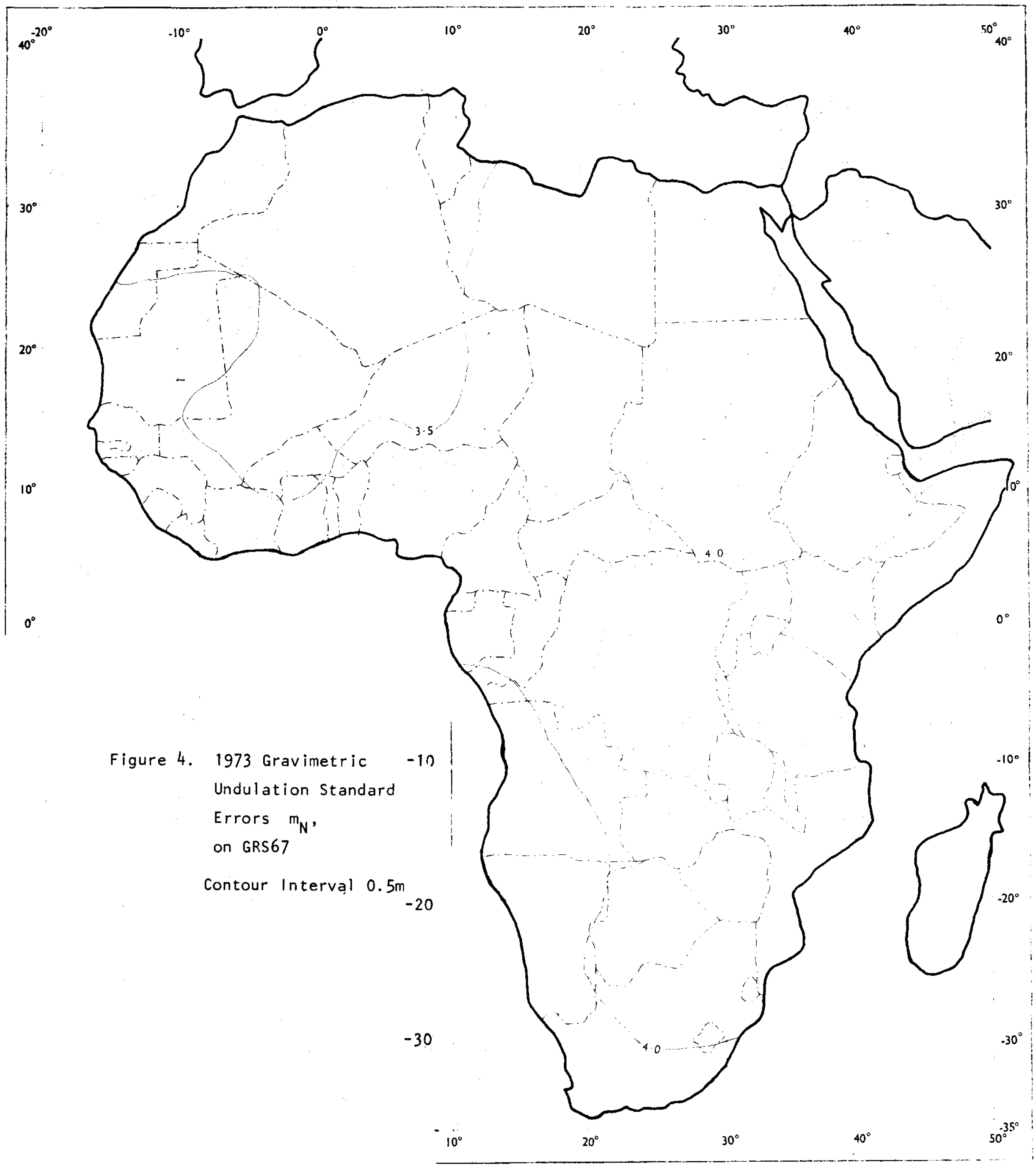

Figure 4. 1973 Gravimetric Undulation Standard Errors m_N, on GRS67

Contour Interval 0.5m

Figure 5. 1973 Gravimetric Deflections
Meridian ξ, on GRS67
(Less 4 Innermost Blocks)
Contour Interval- 1 sec.

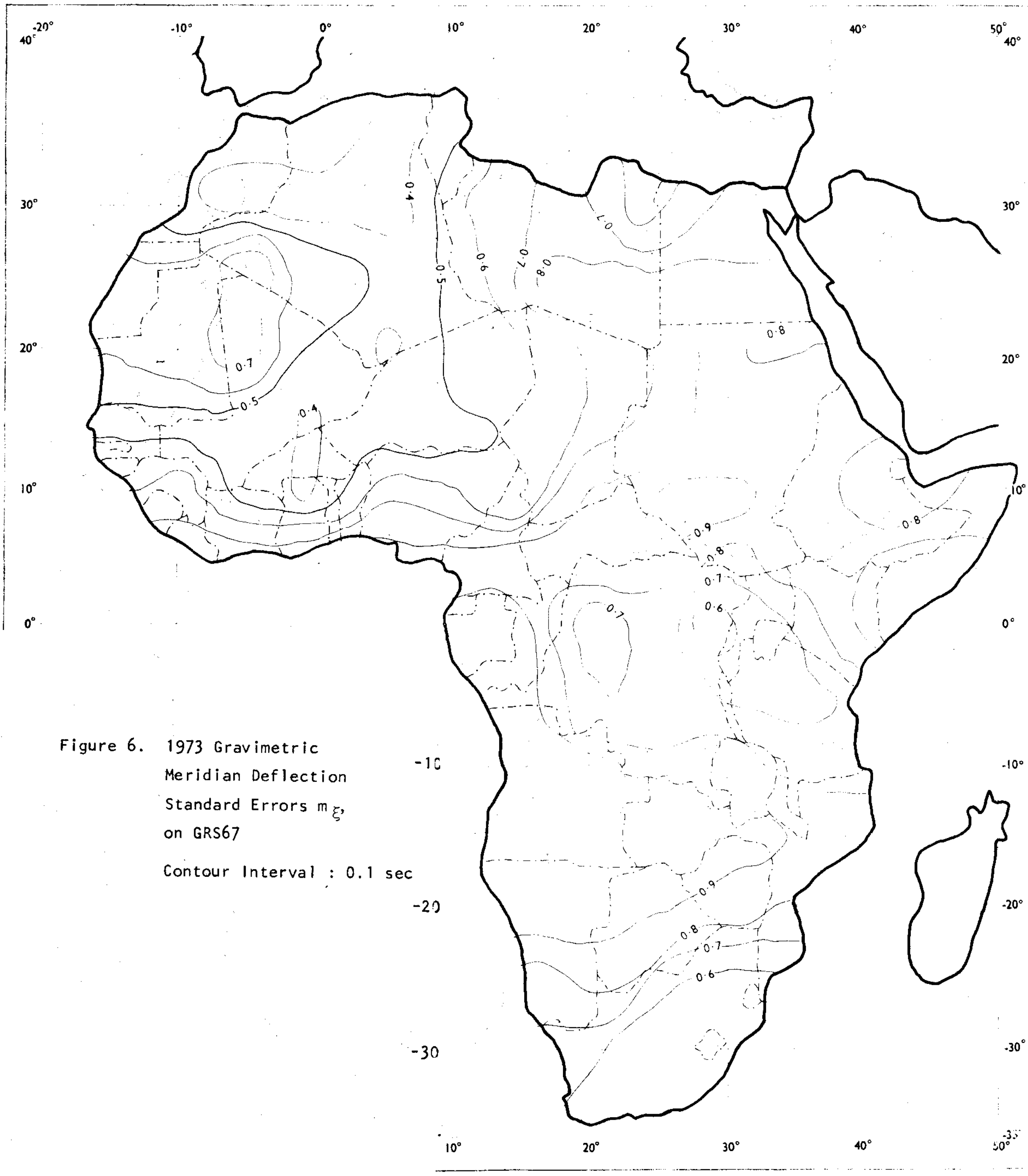

Figure 6. 1973 Gravimetric Meridian Deflection Standard Errors m_ξ, on GRS67

Contour Interval : 0.1 sec

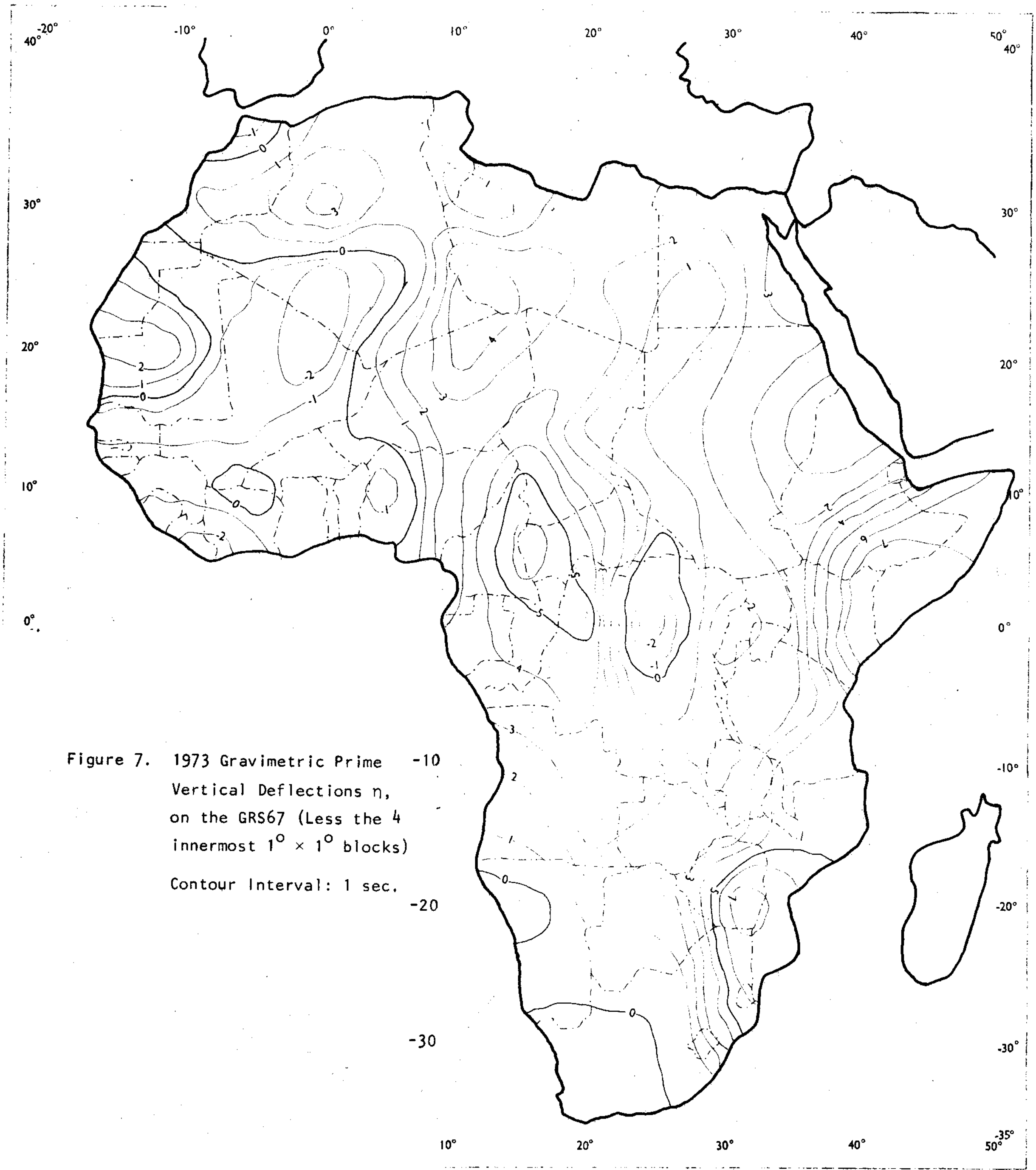

Figure 7. 1973 Gravimetric Prime Vertical Deflections η, on the GRS67 (Less the 4 innermost $1^{\circ} \times 1^{\circ}$ blocks)

Contour Interval: 1 sec.

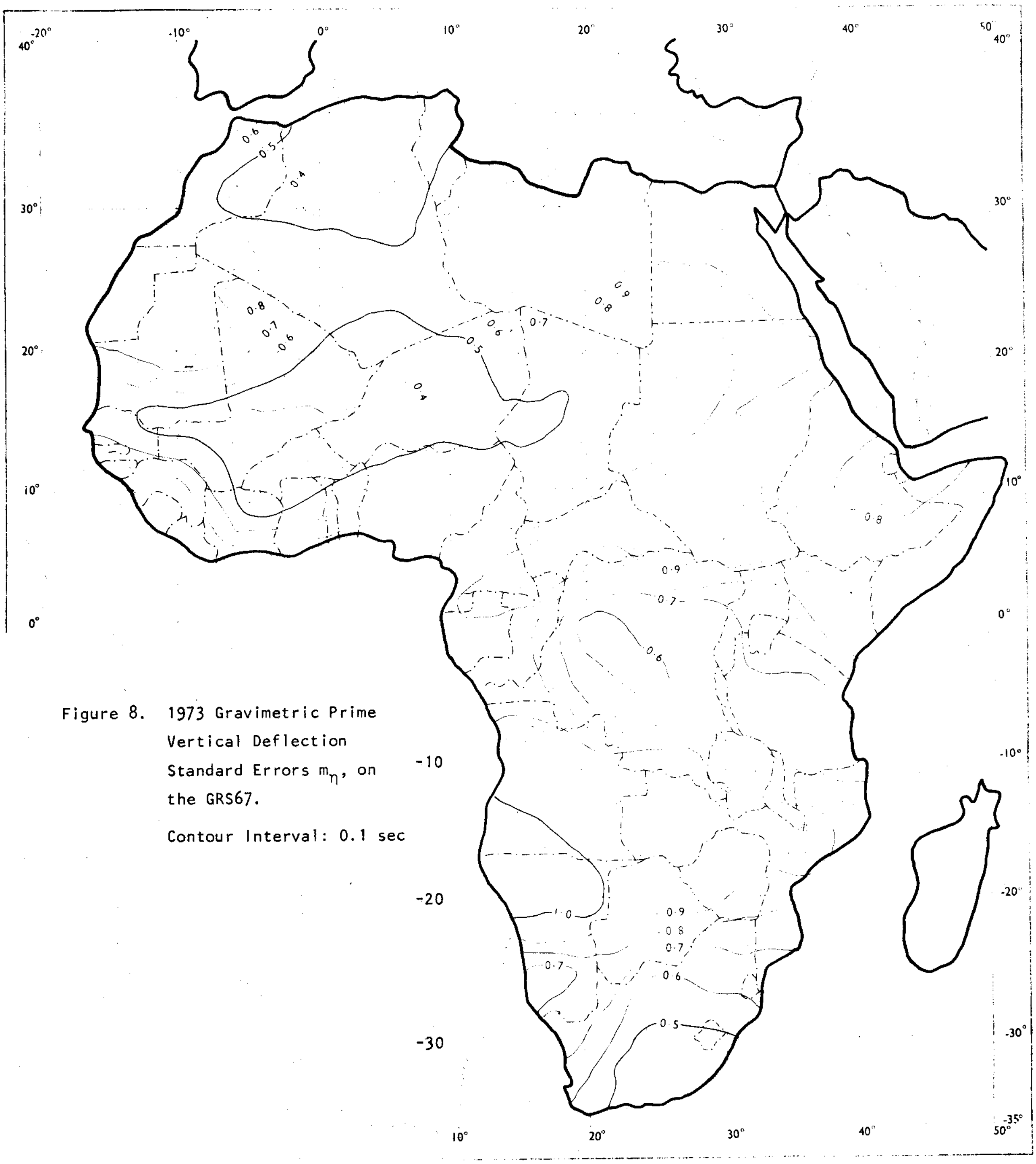

Figure 8. 1973 Gravimetric Prime Vertical Deflection Standard Errors m_η, on the GRS67.

Contour Interval: 0.1 sec

Dave Gordon of the University Computer Centre helped with programming problems and Rowland Asoegwu and Jimoh Ogunsanya (both post-graduate students in the department), carried out some of the prediction and map contouring.

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8. Discussion*

WALCOTT: As an outsider, I have a question. Gravity is measured with respect to the geoid .. ?

RAPP: Gravity is measured with respect to the Earth's surface. The question is what you should do with this gravity. In all solutions of the boundary value problem you work with some residual. The main question relates to how you formulate the boundary value problem. In the days of Stokes, you formulated it on the geoid by eliminating the masses exterior to it. This was incorrect and from developments between 1932 and 1945, it was accepted that we would try to compute the surface of the Earth (i.e., deflections of the vertical and height anomalies). The next question is deciding on the best way to do that. As Mather said (earlier) this morning, all sorts of complications arise.

WALCOTT: Is there a difference in datum between Δg_T and Δg_S?

RAPP: You have to be careful to make sure there is not a difference in the datum. To do that we have to define the reference ellipsoid (or the reference surface to which the undulations are to be referred) very carefully. If this is not done, for example, if the gravity anomalies were computed using a certain gravity formula which is not compatible with that used for satellite potential coefficient determinations, there will be a systematic error in the undulations which are computed. Some recent work (RAPP, R.H. 1973 . Accuracy of Geoid Undulation Computations. *J.geophys.Res.*78,7589) showed that an error of 1 mgal in defining the proper gravity formula can give a 1 - 3 m error in this type of formulation. It is really a critical issue.

KEARSLEY: Can you comment on the stated accuracy of the deflections of the vertical?

RAPP: They sound reasonable when you consider that the effect of the four inner blocks is excluded. A big inaccuracy comes from the inner area.

MATHER: I would like to comment on that. Once we were doing some work for (the Division of) National Mapping, gravimetrically checking astro-geodetic deflections of the vertical. Computation out to 150 km using a tenth degree grid enabled us to recover all but about 1½ arcsec in the long wavelength component and the effect of the innermost zone (within 5 km of the point of computation).

MORITZ: If the inner zones are not known, instead of excluding them altogether, it would be advisable to rather fit a polynomial to the four inner zone mean values.

ECKHARDT: We are discussing the separation of N_1, N_2 & N_3 and presumably hoping that N_3 is small. Another way of doing this for a limited area like Africa or North America is to modify Stokes' function. If we can say that what we measure on ground is error free, except that it doesn't cover the whole Earth. If we take the difference between (this field and) say, a (12,12) satellite model, the contribution of the difference will only be in the higher degree harmonics. Stokes' function can therefore be modified by subtracting the lower degree harmonics arbitrarily such that the outer zone effects become zero, on subtracting a set of Legendre polynomials $P_n(\psi)$ from Stokes' function $S(\psi)$ such that

$$S(\psi) - \sum h_n P_n(\psi)$$

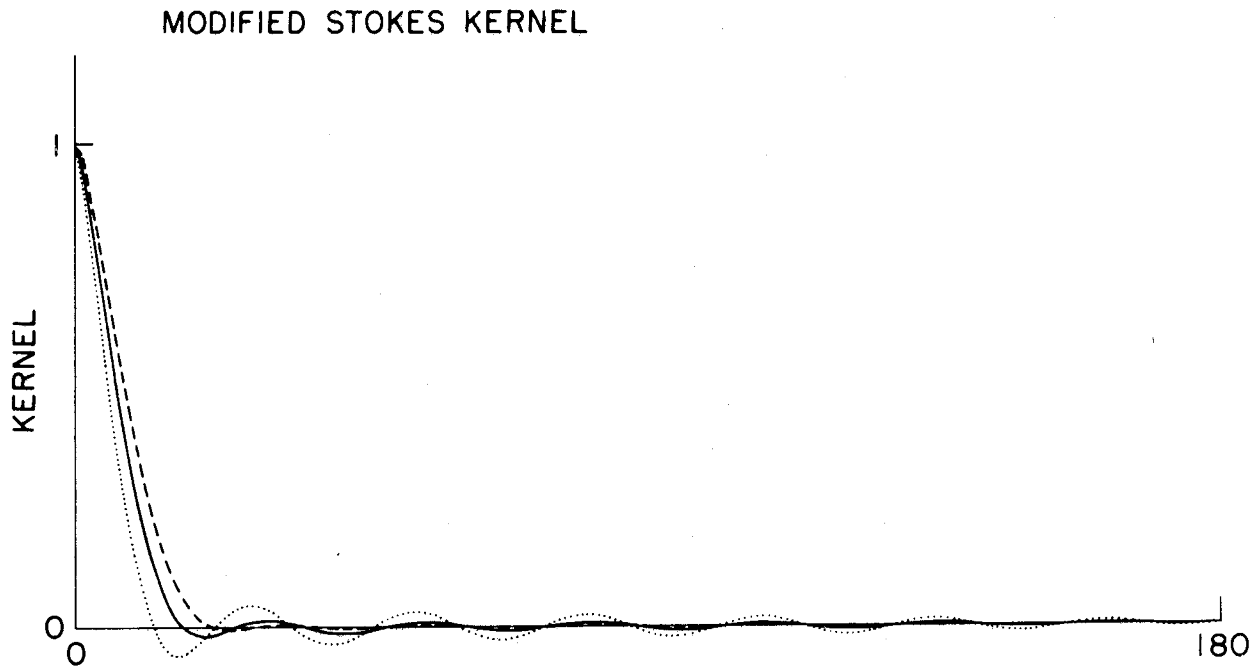
becomes small. I have done this for example, for a (12,12) solution and say 15°(radius of surface gravity). This is one way of modifying Stokes' kernel and ignoring the (outer zone) term. (For illustration see figure on p.187).

RAPP: This type of approach can be found in (MOLODENSKII, M.S. ET AL 1962. *Methods for Study of the External Gravitational Field & Figure of the Earth.* Israel Program for Scientific Translations,Jerusalem) where it is shown how you can modify Stokes' kernel to any particular degree and the idea of truncation theory was developed for many functions by DE WITTE (1967. *Geophys.J.R.astr.Soc.*12,449-464). The objection I have to this that there is no reason to believe that the anomalies computed from potential coefficients to say degree 12, are sufficiently accurate that we can justify forcing contributions from degree two upwards to be zero. We force the low degree terms to be zero by summing Stokes equation not from n = 2, but from $n = n_{max}$ which will force any information in the lower degree terms to be essentially zero. It will be necessary to guarantee that there is no error in these particular anomalies. I can show that N is generally small. A recent publication (RAPP

1973. *Op.cit.supra*) shows that the magnitudes from this region are quite small in relation to the errors in the gravity field.

MATHER: I would like to comment that Molodensky truncation functions were used to prepare a geoid map of Australia by GRUSHINSKY AND SAZHINA (1971. *J.geol.Soc.Aust.* 18,183-199) of the Soviet Union and we find that this technique gives good agreement with the Stokesian approach.

* This paper was presented on G. OBENSON'S Behalf by R.H. RAPP.



See comment by D. Eckhardt on p.186

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GRAVIMETRIC GEOID OF CANADA *

ABSTRACT

A gravimetric geoid over Canada has been calculated from Stokes' formula. The gravity data used in this computation are in the form of ½° block averages over Canada and a three degree belt beyond her southern boundary, 1° × 1° mean values over a large part of U.S.A. and 5° × 5° block averages over elsewhere.

The computed geoid refers to Reference System 1967 and indicates a marked depression of geoidal height at the western edge of Hudson Bay. While the position of the depression is in good agreement with similar studies made earlier, its magnitude of 60 m is significantly higher. Another local depression of the geoid is at the southern part of British Columbia-Alberta border.

The "local variations" of geoidal height due to ½° and 1° × 1° data set have been separately computed and their accuracy has been estimated to better than 1.0 m. The standard error of the regional contribution to the geoid height may, however, be as large as 5.0 m, when 20 mgal is assumed for the standard error of 5° × 5° block averages.

1. Publication Details

The text and diagrams of this paper have been published. The relevant details are:

Contribution 499, Earth Physics Branch, Department of Energy, Mines & Resources, Ottawa, Canada.

2. Discussion *

LAMBERT: I understood you wanted a solution of the geoid to reduce satellite observations. Won't satellite observations themselves give all the accuracy required for geoid determinations?

WALCOTT: The contribution from the local geoid is the part not included in the coefficients and is as large as 20 m in some areas. The geoid obtained is very smooth. The main point I want to make is that if I took out the difference between observed gravity and that computed from satellite geoids, it would give a contribution of N_2 which is about 20 m. We want an accuracy of about 1 m.

LAMBERT: I was under the impression you could get 1-2 m from X, Y and Z.

WALCOTT: The X, Y and Z have been obtained to 1-2 m, but to obtain elevation at the point, you need the geoid to better than 1-2 m.

---

* This paper was presented on behalf of D. NAGY & M.K. PAUL by R.I. WALCOTT.

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Proc. Symposium on Earth's Gravitational Field & Secular Variations in Position (1973), 189-201.

DEFLECTIONS OF THE VERTICAL FROM GRAVIMETRY IN THE NARRABRI REGION OF NEW SOUTH WALES

ABSTRACT

The Narrabri-Manilla region of New South Wales has been chosen as a suitable area for testing the theories used for the computation of the deflection of the vertical by gravimetric means. This is because:

i) the national astro-geodetic networks through the area reveal it as a disturbed region; and

ii) the region contains a cross-section of terrains which enables testing under greatly varying conditions.

The expressions used in the computation are examined to test their relevance, both as to basic concepts used in their derivation and also to see the limitations inherent in terms related to correction for topography.

1. Introduction

1.1 Aim of the Paper

The computation of the deflection of the vertical by gravimetric means has received a lot of attention. Most of this has sprung from the solution offered by Molodensky, and has aimed at modifying his original formulae to obtain a more convenient computation and easier interpretation. To date the formulae have been successfully applied to theoretical models (e.g. MOLODENSKII ET AL 1962, p. 217) but, to the author's knowledge, these formulae or their modifications have yet to have had unqualified success in application to a real-life situation. There are two aspects which are the main cause of such failures, and it is naturally, difficult to tell how much either aspect is contributing to this failure. One is the lack in the coverage of gravity data and the other is the uncertainty of the behaviour of the terrain correction terms in areas of steep terrain.

The aim of this paper is to look at two of the approaches which have been developed for the computation of the deflection at the surface, to discuss the assumptions made at the various stages of the derivation, to try and predict the points of weakness in the resulting expressions when these are applied in the test region and to consider their adaptability for computational purposes.

1.2 Symbols

$d\sigma$	=	element of surface area on unit sphere.
dS	=	element of surface area on earth's surface.
dz	=	increment in orthometric elevation
$\frac{d}{d\psi}\{f(\psi)\}$	=	Vening-Meinesz function

$f(\psi)$	=	Stokes function
g	=	observed gravity at the Earth's surface
h	=	normal height above ellipsoid
h_d	=	height anomaly
$M\{\Delta g\}$	=	global mean value of Δg
N	=	elevation of geoid above spheroid
N_f	=	free air geoid
r	=	distance between point at which computation is taking place (P) and element of surface area dS
r_o	=	r on the surface of the sphere
r_1	=	r on the earth's surface
R_m	=	mean radius of the earth
R_p	=	radius vector at P
S_ϕ	=	distance along the meridian

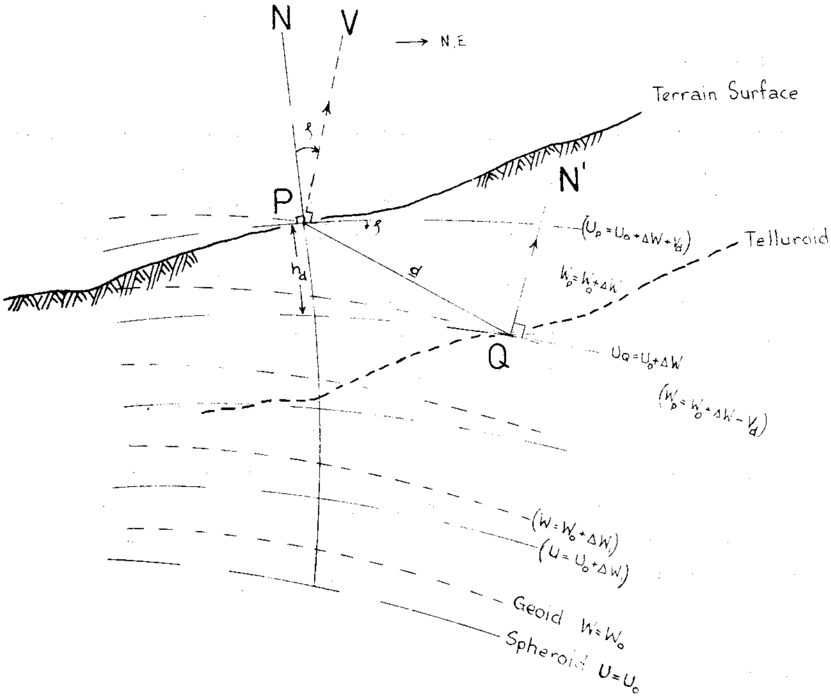

Figure 1. Deflections at the Surface

s_λ	=	distance along the prime vertical
U	=	potential due to the reference system (normal potential or spheropotential)
U_o	=	normal potential at the surface of the ellipsoid
V_d	=	disturbing potential
W	=	geopotential
W_o	=	potential of the geoid
x_i	=	local rectangular Cartesian coordinate system, where x_3 lies along the normal, and the x_1x_2 plane defines the local horizon (x_1 is North, x_2 is East)
α	=	azimuth
β	=	ground slope; subscripts 1, 2 refer to components in N, E directions respectively.
γ	=	normal gravity due to reference system.
Δg	=	free-air gravity anomaly at Earth's surface
ΔW	=	difference in potential between geoid and a generalised geop
λ	=	longitude, positive East
ξ	=	components of the deflection of the vertical; subscripts 1, 2 refer to component in N, E directions respectively; subscript p refers to deflection at P.
θ	=	deflection of the vertical; positive of outward normal lies North and East of the normal
Φ	=	a function of the density of the surface layer
ϕ	=	latitude, positive North
ψ	=	angle subtended at the geocentre between computation point P and element of the surface area, dS

2. The Test Area

2.1 Location and Nature

The area chosen for the test area is the Narrabri-Manilla region in the state of New South Wales, and lies 600 km north-west of Sydney and 300 km inland. It is bounded roughly by $-30^o07'$ to $-30^o52'$ in latitude and $+149^o45'$ in longitude. This region contains the junction of four sections of the astro-geodetic levelling network of Australia, and distributed through it are 12 astro-geodetic stations spaced about 30 km apart along the loops. These stations are reckoned to be fixed to an accuracy of about 0.4 arcsec in latitude and 0.8 arcsec in longitude (see MATHER ET AL 1971, p. 11).

There were a number of reasons for choosing this area. One was the abundance and distribution of control stations mentioned above. More importantly, computations already carried out showed large differences between the astro-geodetic values and those determined gravimetrically, indicating the shortcomings of the method used for computation in this disturbed region. The area encompasses a great variety of topographical types, ranging from the completely flat plains of the Wee-Waa district to the West to the very rugged and broken terrain of the Kaputar National Park. The control stations themselves are situated in terrain of varying types, grading from the flat, through the small symmetrical ad isolated hills, to foothills and finally rugged mountain ranges. It must be realised that the terrain itself is largely the reason for the anomalous deflections, nevertheless it is an advantage to have such a gradation as this should indicate at what stage the theory, especially that for inner and middle zones, will break down, and how much 'reinforcement' is needed to satisfy the limits of accuracy.

2.2 Gravity Data

Gravity in the area has been surveyed by the Bureau of Mineral Resources, Geology and Geophysics as part of their programme to provide a complete gravity coverage of the continent (see IBID, p. 7). This has resulted in a density of gravity readings of about 1 station per 16 km^2 in flat areas to 1 station per 8 km^2 in the mountains in the area of this test region. The density required for accurate computations is still somewhat a matter of conjecture, but has been variously thought to be (for mountainous areas) 1 point every 0.5 to 1 km within several kilometres of the control station (PELLINEN 1968) and "a .05 degree (5 km) grid within 0.5° (5 km) of the control point, in addition to the 0.01° grid within 0.1° when evaluating using a computer" (MATHER ET AL 1971, p. 27).

Several field trips have been made to intensify the existing gravity field. As a general rule the aim has been to encircle the station with 6 to 10 gravity readings at a radius of about 2-3 km, with the existing density doubled to a distance of 8 km (all of these being chosen with an overriding consideration that the points be chosen in critical positions, such as at the foot of the hill or mountain).

Gravity was measured by a Worden gravimeter (kindly loaned by the Bureau of Mineral Resources) with traverses terminating at the Isogal Stations at either Narrabri or Tamworth. Height was determined in a number of ways. In some cases it was possible to use the third-order control stations or level runs placed for the mapping programme in the area, but usually (especially for inner zone stations) height was determined by trigonometric heighting with distances observed direct by EDM or found by subtense methods. In some cases height was fixed by single base altimetry over short distances. Position was found either by radiation from a known station or by scaling from the excellent 1:31 680 (2 inches to a mile) maps which cover most of the area.

The final accuracy of gravity values, especially those determined from the later trips, is expected to be of the order of 0.3 mgal.

3. The Gravimetric Approach to Plumb-Line Deflections

3.1 The Classical Approach

In the gravimetric approach to this problem, one is trying to determine the tilt of the normal equipotential surface, the spherop, to the actual equipotential surface, the geop. The classical approach to this is to consider these two surfaces at the level of the reference surface. It then becomes a matter of finding the rate of change of the separation of the geoid from the spheroid, N, in the direction of the two axes defining the local coordinate system.

Hence, (e.g. HEISKANEN & MORITZ 1967, p. 112)

$$\xi = -\frac{dN}{dS_{\phi}} = \xi_1$$

$$\eta = -\frac{dN}{dS_{\lambda}} = \xi_2 \qquad \ldots (1)$$

which, when applied to Stokes Integral, result in the Vening-Meinesz expressions

$$\xi_j = \frac{1}{4\pi\gamma} \iint \Delta g \frac{d}{d\psi}\{f(\psi)\}\cos \alpha_j . d\sigma \quad \ldots j = 1, 2 \qquad \ldots (2)$$

where $\alpha_1 = \alpha$; and $\alpha_2 = 90 - \alpha$

It is worth emphasising that the problems associated with the solution of (2) are a direct consequence of the difficulties associated with Stokes solution and the evaluation of the separation of the geoid-ellipsoid system. This problem is well expressed by Brovar when he states " Geodesy can and must solve this problem (i.e.,of relating the geodetic survey at the surface to the ellipsoid - Author's comment) without involing any hypotheses concerning the internal structure of the earth" (BROVAR ET AL 1964, p. 91).

3.2 The Contemporary Approach

In 1945 Molodensky devised an approach which by-passed the need to reduce information to the elusive geoid. This approach has had a great impact on physical geodesy and methods are still being developed to provide efficient and practical means of solving the rather cumbersome expressions which derive directly from his formulation. Some of these methods will be treated later, but firstly it is necessary to show in simple terms the new geometrical concepts which evolved as a result of this new approach.

The surfaces of the earth (S) is thought to be represented by a second surface known as the telluroid. This was defined originally by HIRVONEN (1960, p. 39) as the locus of points whose positions were defined by the geodetic ϕ and λ of the surface point P and whose spheropotential was equal to the geopotential at the P. This (see figure 1) was the intersection of the normal through P with the spherop $U = W_p$. Definition in this way was inconclusive and a modified definition was suggested by DE GRAAFF-HUNTER (1960, p. 193). In this system the "Terroid" became the locus of associated points defined by the astronomically observed values of ϕ and λ for P on the spherop $U = W_p$ (see point Q, figure 1),

$$\text{i.e. } \phi_Q = \phi_{P_A} \quad , \quad \lambda_Q = \lambda_{P_A} \quad , \quad U_Q = U_o + \Delta W_P \qquad \ldots (3)$$

In terms of location and derivation this modification made little difference, but it did provide a more absolute definition for the position of the reference surface (hereafter called the telluroid) as its planimetric location was no longer relative to the local ellipsoid chosen for the geodetic computations. It can be seen that the telluroid is still a reflection of the terrain but is now slightly displaced (the displacement being a function of the deflection of the vertical) and that the normal to the spherop through Q has the same spatial orientation as the vertical through P. (For a more formal explanation, see MATHER 1968, pp. 34 and 42).

It is the height anomaly (h_d) between the geop W_p and the spherop $U = U_o + \Delta W$ which is the subject of the solution and, incidentally, which substitutes for N to produce a surface known as the "quasi-geoid" when referred to the reference surface, the ellipsoid. (See HEISKANEN & MORITZ 1967, p. 293; MOLODENSKII ET AL 1962, p. 76).

3.3 The Deflection at the Surface

The deflection of the vertical at P can now be defined as the small change in the separation between U_p and W_p for an increment in the distance along the surface W_p. This change in separation manifests itself as a change in the height anomaly (h_d) and hence the two components at the surface in the meridian and the prime vertical can be seen to be

$$\xi_{1_p} = \frac{dh_d}{dS_\phi}$$

$$\xi_{2_p} = \frac{dh_d}{dS_\lambda} \qquad \ldots (4)$$

The variable h_d is calculated at the telluroid, hence it is necessary to relate the separation of the two surfaces W_Q, U_Q at Q to the equivalent separation between W_p and U_p at P. This has been expressed as (MATHER 1971, p. 86; HEISKANEN & MORITZ 1967, p. 313)

$$\xi_{j_p} = - \frac{dh_d}{dx_j}\Big|_{W=W_p} = - \left(\frac{dh_d}{dx_j}\right)_{Tell} + \frac{dh_d}{dx_3} \cdot \frac{dx_3}{dx_j} \quad \ldots j = 1, 2 \qquad \ldots (5)$$

Where the second term is the correction to the differential at the telluroid and is seen to be the change in the height anomaly with height compounded with the change in the height of the telluroid in the direction being considered.

For completeness it should be recognised that the 3-dimensional axis system at Q does not have the same spatial orientation as the axis system at P, having suffered the small displacement ζ. The axis system at Q must therefore be resolved into the axis system at P.

$$\text{viz.} \quad \xi_j = \left\{ - \left(\frac{dh_d}{dx_j}\right)_{Tell} + \frac{dh_d}{dx_3} \frac{dx_3}{dx_j} \right\} \cos \xi_j \quad \ldots j = 1, 2$$

which, on expanding $\cos \xi_j$ and accounting for the magnitude of ξ in Australia, will degenerate to (5).

The second term on the right hand side of (5) evaluates as $- \frac{\Delta g}{\gamma} \cdot \tan \beta_j$ where β_j is the slope of the ground (this being equivalent to the telluroid slope at the associated point). This term, as has been shown in HEISKANEN & MORITZ (1967, p. 314) and MATHER (1971, p. 88) compensates with part of the first term in the expression being considered when this term is expanded.

The evaluation of this first term will obviously depend upon the approach adopted to find the height anomaly. Two approaches will be referred to (i) Molodensky's approach, using surface layer techniques and (ii) the approach which uses as a starting point Green's third identity. A summary of these approaches can be found in (MATHER 1973, pp. 21-28 and pp. 32-43), but for a more detailed development the reader is referred to MOLODENSKII ET AL (1962, pp. 118-124) and HEISKANEN & MORITZ (1967, pp. 300-312) for the former approach, and for the latter approach MATHER 1970, pp. 10-21 and MATHER 1971, pp. 78-86.

4. The Height Anomaly and its Differential

4.1 By Surface Layer Techniques

4.1.1 Derivation

This approach is based on the premise that the potential relating to a body can be expressed in terms of an attracting layer on the surface of that body. This is extended to express the disturbing potential (and not the potential) in terms of the layer

i.e. $$V_{d_p} = \iint_\sigma \frac{\Phi}{r} \, dS \qquad \ldots (6)$$

where Φ is related to the density of the surface layer on S which is held to be producing the disturbing potential V_{dp} at P.

The development of the theory is well known (see MOLODENSKII ET AL 1962, pp. 118-124) and will not be repeated here. The end product is an expression which gives both the height anomaly and (subsequently) the deflection in terms on a series of successive approximations of the density of the anomalous surface layer, in turn expressed as a function of the surface gravity anomalies and corrections to these at each point for the irregularities of the earth's surface.

Hence, to the second order, it is found that

$$h_d = \frac{R}{4\pi\gamma} \iint_\sigma (\Delta g + G_1 + G_2) \, f(\psi) \, d\sigma - \frac{R^2}{4\pi\gamma} \iint \frac{(h-h_p)^2}{r_o^3} \Delta g \, d\sigma \qquad \ldots (7)$$

and

$$\xi_j = \frac{1}{4\pi\gamma} \iint_\sigma (\Delta g + G_1 + G_2) \frac{d}{d\psi} (f\psi) \cos \alpha_j \, d\sigma$$

$$+ \frac{3R^2}{4\pi\gamma} \iint_\sigma \frac{\cos \frac{1}{2} \psi}{r_o^4} (h-h_p)^2 \Delta g \cos \alpha_j \, d\sigma - \frac{\Delta g + G_1}{\gamma} \tan \beta_j \qquad \ldots j = 1, 2 \qquad \ldots (8)$$

where

$$G_1 = \frac{R^2}{2\pi} \iint_\sigma \frac{h-h_p}{r_o^3} \Delta g \, d\sigma$$

and

$$G_2 = \frac{R^2}{2\pi} \iint_\sigma \frac{h-h_p}{r_o^3} G_1 \, d\sigma + \Delta g . \tan^2 \beta$$

α_j is defined in (1).

Some of the methods devised to modify the above are considered below.

4.1.2 Practical Evaluation

A summary of some of the approaches developed to evaluate equation (8) can be found in (PELLINEN 1968). Some use mathematical devices to simplify the expressions. Others, by modifying the original concept have used a different physical model as the starting point and have thus made modifications.

(a) In 1962, Pellinen suggested an approach which aims at removing the effect of the topography from the general solution and independently evaluating the effect the 'removed' topography will have on the deflection. This method as originally developed is given by PELLINEN (1962) and is also outlined by MORITZ (1969, pp. 27-30).

The free-air anomaly at the surface is adjusted to account for the contribution made to this anomaly by the topography above a stated reference surface.

i.e. $\Delta g_c = \Delta g - \Delta g_T$

where Δg_c is the anomaly corrected for topography
Δg is the free-air anomaly at the surface
and Δg_T is the contribution to g of the topography

Δg_c is substituted for Δg in the Equations (7) and (8) to find the parameters defining the anomalous field at the reference surface. Thus, to a first order approximation, we find

$$\xi_j = \frac{1}{4\pi\gamma}\iint_\sigma (\Delta g_c + G_{1_c}) \cdot \frac{d}{d\psi} \cdot \{f(\psi)\}. \cos\alpha_j . d\sigma + \Delta\xi_{j_T} \qquad \ldots (9)$$

where

$$\xi_{j_T} = -\frac{k\rho R^3}{\gamma}\iint \frac{h-h_p}{r_o^2 r_1} \cdot \sin\psi \cdot \cos\alpha_j \cdot d\sigma$$

$$G_{1_c} = G_1 \text{ with } \Delta g_c \text{ substituted for } \Delta g.$$

As mentioned by Moritz, an advantage in this approach is that the G_{1_c} values will be similar and smoother than the G_1 values in the original expression (8). However, Δg_c may attain large values being in essence the Bouguer anomaly, as might the corrections $\Delta\xi_{j_T}$. A device to alleviate this problem has also been suggested by PELLINEN (1968) (see also MORITZ 1969, pp. 30-33).

A spherical surface, concentric to the original reference surface at sea level and passing through the computation point P, is held to have a surface layer of density ρh which produces an anomalous potential field. The gravity anomaly resulting from this surface layer will have a compensating influence on the mass of the topography removed in the aforementioned approach. If the anomaly accumulating in this way (which is shown to be the Faye anomaly, the free-air anomaly plus the terrain correction only) is now used in place of Δg_c in the earlier expression (9), and due consideration given to the correction term which results, it is found that (MORITZ 1969, p. 31)

$$\xi_j = \frac{1}{4\pi\gamma}\iint_\sigma (\Delta g + C)\frac{d}{d\psi}\{f(\psi)\}\cos\alpha_j\, d\sigma$$

$$+ \sum_{n=1}^{\infty}\frac{1}{4\pi\gamma}\iint_\sigma \bar{g}_n \frac{d}{d\psi} f(\psi)\cos\alpha_j\, d\sigma + \delta\xi_j \qquad \ldots\ j = 1, 2$$

where $\Delta g + C$ is the Faye anomaly.

$\bar{g}_n$ are the correction terms computed using the Bouguer anomalies

$$\text{and } \delta\xi_j = \frac{k\rho R^3}{\gamma}\iint \frac{h-h_p}{r_o}\left(\frac{1}{r_o} - \frac{1}{r_1}\right)\sin\psi\cos\alpha_j\, d\sigma \qquad \ldots\ j = 1, 2$$

$$C = k\rho R^2 \iint_\sigma \left(\frac{1}{r_o} - \frac{1}{r_1}\right) d\sigma \qquad \ldots (10)$$

By using this device the gravity anomalies and the correction term $\delta\xi$ are both reduced in size. Also, as with the first approach, the uncertainty of the density of the sub-surface is overcome because it is compensated when the correciion term is added. The adoption of the reference surface passing through the computation point (as in the second approach) should also improve the convergence of the higher order terms, and improve the accuracy of the first order approximation.

(b) The other type of approach used in solving Molodensky's expression use mathematical devices in order to simplify them. In this respect it is worth mentioning a method developed by MORITZ (1969) who uses analytical continuation of the gravity anomaly from surface to sea level and thus gains a solution by means of successive approximations. While some doubts about the validity of this approach (i.e. continuation below the surface of the attracting body) are expressed, it is felt to be justified by the equivalence gained with matched terms of the modified Molodensky approach. In this way, MORITZ (1969, pp. 35-37) derives to second-order accuracy

$$\xi_j = \frac{1}{4\pi\gamma}\iint_\sigma (\Delta g + g_1 + g_2)\frac{d}{d\psi}\{f(\psi)\}\cos\alpha_j\, d\sigma \qquad \ldots\ j = 1, 2$$

where

$$g_1 = -(h-h_p)\, L_1(\Delta g)$$

$$g_2 = -\tfrac{1}{2}(h-h_p)^2 L_1\{L_1(\Delta g)\} - (h-h_p)\, L_1(g_1)$$

$$\text{with } L_1(f) = \frac{R^2}{2\pi}\iint \frac{f-f_p}{r_o^3}\, d\sigma \qquad \ldots (11)$$

Moritz claims that, though for all practical purposes this is identical with Molodensky's original expressions, it is an easier statement to evaluate. A big advantage from the computing viewpoint is that successive terms of the series are evaluated recursively, although it is probable that the 2nd order is as high an order as is needed for most cases.

Nevertheless, their expressions do appear simpler to evaluate than those in (6), although care must be taken to ensure the terms converge significantly. To assist in this, as Moritz states, it is possible to substitute the Faye anomaly for the free-air anomaly in (11) and apply the corrections as per (10). Now the g_1 and g_2 terms being computed from the Bouguer anomalies will be smaller and this should aid convergence.

Obviously for inner zone computations a device such as is used in 4.2(b), or as suggested in ARONOV ET AL (1971), using an intermediate step, will have to be adopted.

4.2 By Green's Third Identity

4.2.1 Derivation

The disturbing potential when expressed in terms of Green's Third Identity is used to obtain the height anomaly

$$h_d = \frac{W_o - U_o}{\gamma} - R_m\{\Delta g\} + \frac{R_m}{4\pi\gamma}\iint_\sigma \Delta g . f(\psi).d\sigma$$

$$+ \frac{R_m^2}{2\pi\gamma}\iint_\sigma \frac{1}{r_o}\left[\{(h_p - h) + R_m \sin\psi \frac{dh}{dr}\}\frac{V_d}{r_o^2} - \gamma\xi_j \tan\beta_j\right] d\sigma$$

(See MATHER 1971, pp. 78-86 for derivation)

Differentiating this with respect to the two axes defining the horizontal plane at P, the tilts of the two equipotential surfaces in these directions result as the sum of two components, that directly contributed by the Vening-Meinesz expressions, and a correction to this for the departure of the topography from the (ideal) reference surface, viz.:

$$\xi_{P_j} = \frac{1}{4\pi\gamma}\iint_\sigma \Delta g \frac{d}{d\psi}\{f(\psi)\}\cos\alpha_j \; d\sigma$$

$$+ \frac{R}{2\pi\gamma}\iint_\sigma \left[(\{(R.\sin\frac{dh}{dr_o} + h_p - h)\frac{-3\cos\frac{1}{2}\psi}{2\sin\frac{1}{2}\psi} + R\cos\psi\frac{dh}{dr_o}\}\frac{V_d}{8R^3\sin^3\frac{\psi}{2}}\right.$$

$$\left. + \frac{\gamma\cos\frac{1}{2}\psi}{4R\sin^2\frac{1}{2}\psi}\xi_j \tan\beta_j)\cos\alpha_j + (-1)^j\frac{R\sin\psi}{8R\sin^3\frac{1}{2}\psi}\frac{d}{dx_o}(\frac{dh}{dr_o})V_d\frac{\sin\alpha_j}{\sin\psi}\right] d\sigma$$

$$\ldots j = 1, 2 \qquad \ldots (12)$$

(see IBID, pp. 86-89 for full development)

4.2.2 Practical Evaluation

For computation purposes (12) can be greatly simplified by dividing the area involved into 3 zones, (see outer, central and inner zones) and assuming a planar approximation for the inner and central zones, with outer zone computations ($>3^o$) made on a spherical model. This reduces the expression (15) to

(a) *The Stokesian Term*

$$\xi_{f_{p_j}} = \frac{1}{4\pi\gamma}\int_{\psi o}^{\pi}\int_{0}^{2\pi} \Delta g \frac{\partial}{\partial\psi}\{f(\psi)\} \cos\alpha \, d\sigma \qquad \ldots j = 1, 2$$

$$- \frac{1}{2\gamma} r_i \left(\frac{d\Delta g}{dxj}\right) \left(1 + \frac{3x_i}{4R}\right)$$

The first term being the familiar Vening-Meinesz formula applied to central and outer zones, and the second term the inner zone contribution to this quantity.

(b) *The Terrain Correction Term*

$$\xi_{c_{p_j}} = \frac{1}{2\pi}\int_{\psi_o}^{\psi}\int_{o}^{2\pi} \{\frac{1}{\psi^2}\sum_{k=1}^{2} \xi_{f_k} \tan \bar{\beta}_k - \left[\left(2\frac{dh}{dr_o} + 3\frac{h_p-h}{R}\right)\cos\alpha_j + \right.$$

$$\left.(-1)^j \frac{\partial}{\partial A_c}\left(\frac{dh}{dr_o}\right)\sin\alpha_j\right] \frac{N_f - N_{fp}}{\psi^3}\} \, d\sigma \qquad \ldots j = 1, 2 \qquad \ldots (13)$$

where $\psi' = 3^o$, $\quad \psi_o = 0.01^o$ i.e. 1 km

and $\dfrac{dh}{dr_o} = \cos\alpha_c \tan\beta_1 + \sin\alpha_c \tan\beta_2$

$$\frac{d}{dx_c}\frac{dh}{dr_o} = - \sin\alpha_c \tan\beta_1 + \cos\alpha_c \tan\beta_2$$

$$\alpha_c = 180^o - \alpha$$

Here the introduction of the term $N_f - N_{f_p}$ is an equivalent to a zero-degree datum shift and is analogous to the introduction of a reference surface through P mentioned in 4.1 (a) of the surface layer solution. It entails a knowledge of the geoid-spheroid separation at all points involved in the middle zone computation and this can be obtained (if not already known) from the data which is to be used in the computations.

Similarly the term $\xi_\alpha \tan\beta_\alpha$ will generally be small, so zero-approximation values for these can be computed as a first step and stored ready for use as part of the programme.

5. Conclusions

Although the above expressions have not yet been tested for the region under investigation, it is worthwhile trying to predict how they will behave under the conditions which prevail there. It is, in this respect, useful to note the results of an investigation by DIMITRIJEVICH (1972) who calculated the affect of the terrain correction (equivalent to G_1 in expression (8)) on the deflection at a number of control stations in the United States.

The greatest topographic effect is down the west coast of the continent where mountains reach about 4 000 metres above sea level, and it was mainly in this region that the gravity anomalies received the correction. The results of the computation showed that for a station lying 4° east of the mountain range the terrain corrections introduced about .4 arcsecs into the prime vertical component (with, understandably little effect being felt in the meridional component). This effect attenuated as the computation point moved eastward. It also lessened when the computation point was taken in the mountains themselves, suggesting a compensatory effect due to the rough symmetry of the terrain about the control station.

In the test region under investigation a similar situation exists. On the western edge plains stretch north west and south with the Nandewar Ranges to the east. The inner zone will easily satisfy the planar assumption, but the central zone, being unsymmetrical in the east-west direction is expected to introduce large corrections into the value for ξ_2. This will be further complicated by the irregularity and steepness of the terrain to the east manifesting as an uncertainty in the β or $h-h_p$ and $N_f-N_{f_p}$ terms. There may also be problems in obtaining convergence in the G_1 and G_2 terms in this area, and the use of the Faye anomaly should help to alleviate this. There will be a further uncertainty in ξ_2 introduced by the fact that for the outer zone, the region to the East becomes ocean. This means that, although the spherical approximation will hold good, Δg values will be poorer. The use of the Stokesian approximation for the expansions of the outer zone in the other three directions will also be valid particularly to the west. The north/south extensions will probably not hold so well, as to the south at $\psi = 6^{\circ}$ one meets the Snowy Mountains region with mountains reaching 2 500 m and to the north at $\psi = 22^{\circ}$ lies New Guinea with its rugged mountain systems. Also, the use of the free-air anomaly in such large mountain passes as the Himalayas is a source of systematic error, and here the technique which applies terrain corrections should prove stronger.

Moving eastward into the mountains themselves one is confronted with more problems relating mainly to the assumptions of planarity in the derivation. It will be necessary to reduce the inner-zone radius, perhaps even to as little as 200 m, and it may well be that the intensified gravity field may still prove insufficient. It is fortunate that the area is well mapped andinterpolations may prove to be sufficient to strengthen the field. There will be some 'balancing' of the terrain effects due to a closer approximation to symmetry in the central zone. Even so it may well be that the assumptions prove inadequate and a more rigorous approach will have to be applied.

6. Acknowledgements

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8. Discussion

RAPP: About eight months ago a dissertation was submitted by a graduate student (EMRICK, H.) at Ohio State University that discussed, among other things, techniques for computing deflections of the vertical at Pike's Peak in the Colorado Rockies. The idea was to use the G_1 term of Molodenskii, the modification $\overline{G}_1$, the terrain correction and the two corrections you mention g_1 and g_2. He used blocks down to 200 m squares and carried out computations to about 150 km from the computation point at several points. He computed deflections of the vertical using four different techniques including the Bjerhammar iteration down to the reference sphere and compared the differences. The maximum difference was probably less than one second between the uncorrected deflections and the various correction terms. The differences between the various correction term models was a maximum of 0.5 sec.

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* See pp. 117-153 of these Proceedings

GENERAL DISCUSSION ON SESSION C - (H.MORITZ AS CHAIRMAN)

MUELLER: (To MATHER) How can you get a value of W_o from a global geoid solution like that produced by VINCENT ?

MATHER : Stokes' integral as commonly known is insensitive to terms of zero degree. The formulation of the boundary value problem in terms of gravity anomalies at the surface of the Earth can be made without making any assumptions on this account. It is not possible to draw any conclusions about the term of zero degree without external constraints as there is one term W_o which is unknown at *some* level of precision. We do not know the potential of the geoid to 1 part in 10^7 at the present time.

MUELLER: You therefore state that in addition to a solution of the boundary value problem, it is necessary to have an external constraint before a value can be obtained for W_o?

MATHER: Yes.

MORITZ: How does the geoid in Canada compare with the VINCENT-MARSH global geoid?

WALCOTT: I couldn't say at this time as I haven't seen the detail of their geoid.

MUELLER: You said that the N_3 term is in the order of 0 - 20 m. You also stated that it is negligibly small. Which statement is true?

WALCOTT: My statement is true. What may not be true is the data in the Bulletin Geodesique reference. RAPP might like to comment on that?

RAPP: I am trying to recall the Bulletin Geodesique article and I dont recall the global five degree anomaly field.

WALCOTT: The techniques are right but the data may be questionable.

MATHER : I would like to comment on the methods of solution used, assuming we are using the level of data dealt with in the preparation of the three geoid maps presented today. VINCENT mentioned that only an (8,8) solution was needed to adequately represent distant zone effects in such solutions. The use of Molodenskii truncation functions should adequately cover the contribution of outer zone effects, provided the inner zone were well represented by surface gravity. This technique has been used by some Russian groups to prepare geoid maps. There is a certain uncertainty in the present techniques as it is not clear where the surface gravity representation ends. A presentation later in this symposium by Y. HAGIWARA will illustrate the use of this technique.

WALCOTT: Surely the technical problem is that there is insufficient data?

MATHER: The practical consequences are marginal, but the technique is neater.

MUELLER: I would like to refer to VINCENT's map. From strictly geometrical satellite solutions, we compared his undulations at 158 stations around the globe and the average difference was 0.2 m. This is a pretty good geoid solution.

WALCOTT: A stepwise function was used to fit data across the US border. There are differences between ACIC data as compared with our (Canadian) data in contiguous positions.

MATHER: Has the discrepancy between the Ottawa datum and the Washington Datum been taken into account (when preparing the Canadian geoid map)?

WALCOTT: Yes they have. The differences between ACIC data and our data are not systematic.

MUELLER: As MATHER pointed out, the question is the extent of the wavelength of the systematic error. Hopefully they are not of long wavelength.

GRAFAREND: A comment to MATHER. You mentioned the approximation of your solution up to the order of e^3. I have some doubts because the basic formulation of the problem is only correct to the order of e^2. We have studied the linearity and non-linearity problems in some detail and have found that especially vertical deflection calculations are very sensitive to higher order terms in the formulation. A recent reference to the problem of a non-linear formulation of the geodetic boundary value problem is the contribution of M. PICK (1973. *Studia geod. et geophys.* 17,173) where the influence of the higher order terms on regional effects is shown for the region of CSSR and cannot be ignored, even for geoidal undulations. Thus, the accuracy of some solution is also dependent on the accuracy of the formulation of the problem. The central question is how accurate the linear formulation is. *

MORITZ: As far as the accuracy of the data goes, the present formulation should be sufficient and there are many other factors that come in, for example, problems of computing those second and third order terms in the linear formulation. There is also the question of numerical errors due to lack of data which may be much larger.

MATHER: Another important question is deciding on whether or not a uniform method should be adopted for processing gravity information so that the "best" solution can be obtained from irregularly distributed information. Take the three geoid maps presented today. VINCENT ignored higher degree effects in those areas where no one degree information was available (equivalent to representation by the spherical harmonic model); I presume NAGY & PAUL adopted a similar procedure, while it would appear that OBENSON did a covariance analysis for prediction purposes. These are different approaches. There is a case to be put forward for a uniform approach. I do not know whether there is a clear cut answer to this problem.

MORITZ: I don't quite know what you mean by uniform processing. There is a related topic which can be more fully discussed at the appropriate session (Session I) - the optimum treatment of existing data since this problem involves statistical considerations.

MATHER: Maybe something could be said in general terms as it is felt that the differences in the solutions is due in part, to the different methods used to process the data.

MORITZ: Yes. Since the data is largely the same, differences in answers will always be due to the different procedures used. The various procedures should be encouraged so long as they are proceeding on valid lines.

QURESHI: How do you define random and systematic error? We talk about different persons processing data differently. A systematic error in one set of data (processed by one person) may be random in the global context.

MATHER: Significant sources of systematic error in surface gravity anomalies are firstly those in the datum for observed gravity, and secondly those in the height datum. We assume that all datums are located on the same equipotential surface. The boundary value problem, as commonly formulated is dependent on the term ΔW with respect to the geoid. Irrespective of whether errors on this account are due to levelling errors (as maintained by oceanographers) or so-called sea surface topography, we still have the effect on the data. These are sources of systematic error of long wavelength. A numerical manipulation could be performed to get rid of these effects. An example of random effects are elevation errors at individual gravity stations.

QURESHI: How about marine gravity observations where we have large errors with an accuracy of ± 5 mgal?

MORITZ: There are two different sorts of errors. Firstly there are measuring errors and secondly there are interpolation errors in gravity observations which are taken along profiles. The latter errors may be larger.

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*Post-Symposium Reply from MATHER: Having since read PICK's paper, it can be stated that the basis for his development had already been taken into consideration in the solution to order e^3 referred to.