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THE ROLE OF STATISTICAL TECHNIQUES IN THE DETERMINATION OF THE EARTH'S GRAVITATIONAL FIELD

ABSTRACT

The paper discusses the statistical structure of the terrestrial gravity field and applications to physical geodesy. Among these applications are least squares prediction (interpolation and extrapolation) of gravity anomalies and deflections of the vertical, and least squares collocation; the collocation method permits an optimal simultaneous determination of geodetic positions and of the terrestrial gravity field by combining different data of any kind -- terrestrial angle, distance and gravity measurements as well as data from advanced satellite techniques.

1. Introduction

The classical statistical technique in geodesy is least-squares adjustment. Here the observational errors are treated on a statistical basis. R.A. HIRVONEN (1956) was the first to recognize that the irregularities of the anomalous gravity field make a statistical treatment of gravity anomalies meaningful and useful. W.M. KAULA (1959) made a comprehensive statistical analysis of the gravity anomaly field, which provided a basis for many subsequent investigations.

In the terminology of stochastic processes, the field of the gravity anomalies is treated as a two-dimensional stochastic process on a sphere. Therefore, least squares prediction techniques for stochastic processes were adapted to the interpolation and extrapolation of gravity (MORITZ 1962) and extensively applied (RAPP 1964; SOLTAU 1970).

Statistical methods may also be used for estimating the accuracy of the predicted gravity anomalies and of quantities derived from them, such as geoidal heights, deflections of the vertical, etc. A comprehensive presentation of these developments until 1966 is found in chapter 7 of (HEISKANEN & MORITZ 1967). The present paper attempts to review subsequent developments and to illustrate the methods by treating some aspects in more detail.

2. Statistics of the Gravity Field

Let the spherical harmonic expansion of the gravity anomaly Δg be written as

$$\Delta g = \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\bar{a}_{nm} \bar{R}_{nm}(\theta,\lambda) + \bar{b}_{nm} \bar{S}_{nm}(\theta,\lambda) \right) \qquad (1),$$

where $\bar{R}_{nm}$ and $\bar{S}_{nm}$ are fully normalized surface harmonics:

$$\bar{R}_{nm}(\theta,\lambda) = \bar{P}_{nm}(\cos\theta)\cos m\lambda \quad ; \quad \bar{S}_{nm}(\theta,\lambda) = \bar{P}_{nm}(\cos\theta)\sin m\lambda \tag{2},$$

$\bar{P}_{nm}(\cos\theta)$ being a fully normalized Legendre function (IBID,section 1-14); θ (co-latitude) and λ (longitude) are spherical co-ordinates. Then the covariance function of the gravity anomaly is given by (IBID.section 7-3)

$$C(\psi) = \sum_{n=2}^{\infty} c_n P_n(\cos\psi) \tag{3},$$

where $P_n(\cos\psi)$ are the conventional Legendre polynomials, ψ being the spherical distance, and the coefficients c_n are expressed in terms of the coefficients $\bar{a}_{nm}$ and $\bar{b}_{nm}$ by

$$c_n = \sum_{m=0}^{n} \left(\bar{a}_{nm}^2 + \bar{b}_{nm}^2 \right) \tag{4}.$$

This covariance function fully describes the (wide sense) statistical behaviour of the anomalous gravitational field. The coefficients c_n are called degree variances and frequently denoted by σ_n^2.

A comparison of recent values for σ_n^2 is given by RAPP (1973b,table 3). They conform remarkably well to Kaula's rule of thumb (KAULA 1966b): he suggests that the root mean square value of a potential harmonic coefficient of degree n could be estimated by

$$\pm \frac{10^{-5}}{n^2} \tag{5}.$$

This corresponds to degree variances

$$\sigma_n^2 = \gamma^2 (n-1)^2 (2n+1) \frac{10^{-10}}{n^4} \tag{6},$$

where γ is a mean value of gravity (980 gal).

RAPP (1972) investigated a series of analytical models to improve this rule of thumb, recommending

$$\sigma_n^2 = \frac{a(n-1)}{(n-2)(n+b+cn^2)} \tag{7},$$

with

$$a = 251.6 \text{ mgal} \; ; \quad b = 12.93 \; ; \quad \text{and} \quad c = 0.00071 \tag{8}.$$

LAURITZEN (1973) and TSCHERNING (1972b) used a closed analytical expression for the covariance function which amounts to putting

$$\sigma_n^2 = A\frac{n-1}{n-2} \tag{9}.$$

A review of statistical analysis of data distributed over a sphere, which is applicable to the gravity field, is given in (KAULA 1967).

The autocovariances of any other quantity of the anomalous gravity field, such as geoidal heights

or deflections of the vertical, and their cross-covariances are derived by covariance propagation. See (HEISKANEN & MORITZ 1967,section 7.7) - the theory given there for error covariance functions applies without change to other covariance functions - and (MORITZ 1970a,section 7).

A theoretical and numerical investigation of the covariance structure of geoidal heights, gravity anomalies, vertical deflections, and anomalous gravity gradients is found in (MEISSL 1971). TSCHERNING (1972b) made analytical and numerical evaluations of various covariance functions.

For local purposes, the Earth's surface may be approximated by a plane. The plane theory of the covariance structure of the gravity field is somewhat simpler than the spherical theory and was investigated by several authors; cf. (SHAW ET AL 1969; GRAFAREND 1971a; GRAFAREND 1971b; GRAFAREND 1973a; GRAFAREND 1973b).

Grafarend points out that for vector fields $s_i (i=1,2)$ such as deflections of the vertical, homogeneity and isotropy do not imply that all covariances depend only on the distance r; instead, the general form of a homogeneous and isotropic covariance tensor C_{ij} corresponding to the vector s_i is

$$C_{ij}(P,Q) = \Psi(r)\,\delta_{ij} + \{\,\Omega(r) - \Psi(r)\,\}e_i e_j \qquad (10),$$

where $\Psi(r)$ and $\Omega(r)$ are arbitrary functions and the direction enters through the unit vector e_i from point P to point Q; δ_{ij} denotes the components of the unit matrix. He also tested this structure in empirical deflection fields (GRAFAREND 1971c; GRAFAREND 1972a).

GROTEN (1972) investigated effects due to non-uniform distribution of data, incomplete elimination of trends (anisotropy, non-stationarity, etc) and their influence on linear regression prediction.

An important negative result concerning the interpretation of the anomalous gravity field as a stochastic process was obtained by LAURITZEN (1973). He proved that it is impossible to find a stochastic process, harmonic outside a sphere, which is both Gaussian and ergodic.

Therefore MORITZ (1972,section 8) proposed to use, instead of an interpretation as a stochastic process, an interpretation in terms of a covariance analysis of individual functions as rigorously elaborated already by WIENER (1930); cf. also (DOOB 1949). This also takes into account that in reality there is only one individual Earth's gravity field and not a phase space of many such fields.

3. Least-Squares Prediction and Collocation

Least-squares prediction was applied, not only to gravity anomalies, but also to deflections of the vertical (HEITZ 1969; GRAFAREND 1971a; GRAFAREND 1971b).

Least-squares filtering, which is closely related to prediction, was applied for removing inertial noise from airborne gravimeter data (MORITZ 1967; MEISSL 1970; SZABO & ANTHONY 1971).

The usual least-squares prediction is linear since the results s_i are obtained as linear functions of the data x_j, of form

$$s_i = B_{ij} x_j \qquad (11),$$

summation over subscripts occurring twice is implied. Optimum linear estimates are the best among all estimates, linear and non-linear, if the quantities under consideration are normally distributed. As we have just seen, however, the assumption of normal distribution for the anomalous gravity field leads to difficulties. Therefore KAULA (1967) and GRAFAREND (1972b) considered non-linear prediction in the form of a series

$$s_i = B_{ij} x_j + C_{ijk} x_j x_k + D_{ijk\ell} x_j x_k x_\ell + \ldots\ldots \qquad (12).$$

So far however, linear estimation is almost exclusively used.

The decisive generalization of linear least-squares prediction to a general theory of estimation of any element of the anomalous gravitational field (gravity anomalies, geoidal heights, deflections of the vertical, harmonic coefficients of the geopotential) from arbitrary, even heterogeneous data was given by KRARUP (1968; 1969); this method was called least-squares collocation. He also clarified the relation between least-squares collocation and least-squares adjustment, by exhibiting collocation as an adjustment in Hilbert space with a kernel function; for prediction of time series this had been done by PARZEN (1961). More on the Hilbert space aspect may be found in (TSCHERNING 1971; TSCHERNING 1973).

In a different but equivalent approach, which is simpler by avoiding the use of Hilbert space, least-squares collocation was applied to the solution of a number of problems in physical geodesy, such as the discrete geodetic boundary value problem, the application of aerial gravimetry, the geodetic use of gradiometer measurements, and the combination of gravimetry with satellite-determined harmonics and with astro-geodetic data (MORITZ 1970a).

A somewhat different line of development, already foreshadowed in (KAULA 1963), leads to the incorporation of parameters representing systematic effects. For the case of gravity this was done in (MONGET 1969), (MONGET & ALBUISSON 1971), (LAUER 1971), (MORITZ 1969) and (MORITZ 1970c), where also applications to other fields of geodesy are outlined. This line of development finally leads to a joint least-squares determination of geometric positions and of the gravity field ("Integrated Geodesy"). Cf. (KRARUP 1971; EEG & KRARUP 1973); here we shall follow the presentation given in (MORITZ 1972).

The basic model for least-squares collocation including systematic parameters is

$$x = AX + s + n \qquad (13).$$

Here x, the "measurement", is a vector formed by the observations; X is the vector of the non-stochastic parameters; A is the "sensitivity matrix" characterizing the influence of the parameters X on the measurement x; and s, the "signal", and n, the "noise", are two different random vectors of zero mean.

The noise n is nothing else than the random measuring error of the quantity x, and s represents the effect of the anomalous gravity field on this quantity. The part AX comprises, after linearization by Taylor's theorem:

1. the effect of the reference ellipsoid and of the normal gravity field;
2. geometric parameters such as point co-ordinates; and
3. systematic errors and trends (e.g., gravimeter drift).

Now it is easily seen that *every* geodetic measurement can be split up according to equation 13. As an example, consider gravity g

$$g = \gamma + \Delta g + n_g \qquad (14).$$

Here, normal gravity γ represents AX, the gravity anomaly Δg stands for s, and n_g signifies the measuring error.

For geometrical quantities such as azimuths, horizontal angles or zenith distances, AX represents the corresponding ellipsoidal ("geodetic") quantity, and s is nothing else than the "reduction to the reference ellipsoid" as described in (HEISKANEN & MORITZ 1967,section 5.4 & 5.5). In a similar way, satellite observations (variations of orbital elements, directions, distances, doppler and altimeter data, etc) may be treated.

The formulas for least-squares collocation may be derived from two different minimum principles:

1. From a suitable generalization of the well known adjustment principle

$$v^T P\, v = \text{minimum},$$

where P is the weight matrix, v is a random vector, and ^{T} denotes its transpose.

2. From the condition that the standard error of the result be minimized.

This is in full analogy to least-squares adjustment; again, the two different minimum principles turn out to be equivalent.

The basic computational formulas are as follows. The parameter vector X is given by

$$X = (A^T \overline{C}^{-1} A)^{-1} A^T \overline{C}^{-1} x \qquad (15),$$

then any signal s_P may be obtained from

$$s_P = C_P^T \overline{C}^{-1} (x - AX) \qquad (16).$$

The signal s_P may be any quantity of the anomalous gravity field; an anomalous potential T, a geoidal height N, a component of the deflection of the vertical (ξ or η), a gravity anomaly Δg, etc.

The matrix $\overline{C}$ is the covariance matrix of the vector x, it consists of a signal part C, and a noise part D:

$$\overline{C} = C + D \qquad (17).$$

D is the error covariance matrix, representing the effect of the random measuring errors n. In adjustment computations, only the matrix D is considered and it is called the variance-covariance

matrix.

The signal covariance matrix C is the characteristic novelty in least-squares collocation; it represents the effect of the anomalous gravity field on the quantities under consideration.

The essential point is that the signal covariances, for any measured element, must be derived from a single basic covariance function by covariance propagation, as we have seen in section 2.

The matrix C_P^T is the row vector

$$C_P^T = (C_{P1}, C_{P2}, \ldots, C_{Pq}) \qquad (18),$$

representing the signal covariances between s_P and the measurements x; it is likewise to be derived from covariance propagation.

Equation 15 is analogous to the equation determining the parameters in least-squares adjustment, with the important difference that in adjustment by parameters, we have the error covariance matrix D instead of $\overline{C}$, whereas in collocation, the signal covariances C enter as well through equation 17.

Equation 16 is analogous to the basic equation for least squares prediction; cf. (HEISKANEN & MORITZ 1967,p.268). In fact (LOC.CIT.,equation 7.63) is a special case of equation 16, if there are no measuring errors (n=0, D=0, $\overline{C}$ = C) and no systematic parameters (A=0), and if both the measurements x and the signal s_P to be computed are gravity anomalies.

The present method may thus be regarded as a combination of least-squares adjustment and least-squares prediction into a unified scheme. It gives a solution which is optimal in the sense that it gives the most accurate results obtainable on the basis of the available data.

It should be emphasized that the estimated quantity s_P and the measurements making up the vector

$$x^T = (x_1, x_2, \ldots, x_q)$$

may be of different nature; for example, x_1 may be a measured gravity anomaly, x_2 may be a terrestrial baseline, x_3 may be a component of the deflection of the vertical, x_4 may be a satellite range-rate, etc, whereas s_P may be a geoidal height to be computed.

The accuracy of the estimated quantities X and s_P may be evaluated by formulas which are generalizations of the corresponding formulas for least-squares adjustment and least-squares prediction.

An attractive feature of collocation is the fact that the signal s and the noise n, which often have a similar order of magnitude, are treated on an equal statistical basis.

We remark that all calculated quantities s, whatever they are, refer to one and the same anomalous gravity field, so that our method is indeed self-consistent.

This field, besides being optimal in the sense of highest accuracy, is also the smoothest gravity field that is compatible with the given data. Therefore it can be analytically continued down to sea level without any difficulties.

The usual solution of problems of physical geodesy by means of integral formulas may be considered as limiting cases of collocation solutions when the coverage by gravity measurements becomes continuous.

In fact however, the case of discrete measurements, which underlies least-squares collocation, is much more realistic than the case of continuous gravity coverage. Interpolation and vertical reduction, essential in the classical procedures are, so to speak, built in now; data of different kinds can be combined; and measuring errors are automatically adjusted.

As an idealization, we might assume that *all* geodetic measurements (from triangulation to advanced dynamic satellite techniques) obtained so far are combined into a single solution set out in equations 15 and 16, to give the best result for the Earth's geometry and gravity field. As a matter of fact, this cannot be fully and literally realized in practice because it would involve the inversion of an excessively large $\overline{C}$ matrix.

The inversion of the $\overline{C}$ matrix constitutes, in fact, the main computational problem with this method. On the other hand, the inversion of $\overline{C}$ must be performed only once for a given set of data; any desired quantity s_p, of the anomalous gravity field, and also its accuracy, may then be computed with the same matrix $\overline{C}^{-1}$.

This, in practice, the number of data to be combined is limited by the size of the matrix that can be inverted in the computer. This presupposes suitable representative selection of the data and some working "from the large to the small" in several steps; cf. (MORITZ 1973a).

BJERHAMMAR (1971) and KOCH & LAUER (1971) advocated the use of Kalman filtering and prediction; BJERHAMMAR (1973b) and MORITZ (1973a) investigated the relation between stepwise collocation and the Kalman method.

4. Applications of Collocation

TSCHERNING (1970) applied collocation to the determination of the geoid in Scandinavia from astro-geodetic deflections of the vertical and from gravity; for a comparison with least-squares prediction see (HEITZ & TSCHERNING 1972).

Rapp compared the collocation procedure of MORITZ (1970b) with the adjustment procedure of RAPP (1969); his results are given in (RAPP 1973a) and presented at this symposium (RAPP 1973b).

An application of collocation to the determination of zonal harmonics from satellite observations is described in (MORITZ & SCHWARZ 1973) and (SCHWARZ 1973). The mathematical formulation of this problem leads to a linear system of the form

$$Ms + n = x \qquad (19),$$

where the vector x comprises (essentially) the observed variations in the orbital elements such as $\Delta\Omega$ and $\Delta\omega$, n represents the measuring errors, and s consists of the unknown spherical harmonic coefficients (or more precisely, the deviations of these coefficients from the corresponding coefficients for the normal gravity field); since there are infinitely many such coefficients, the vector s is an infinite vector. The matrix M represents the influence of these coefficients on the

observations x; it has a finite number of rows and infinitely many columns.

The usual way of solving equation 19 is to truncate the infinite vector s so as to get an over-determined system, which is solved by the adjustment of parameters. It may however, also be solved by collocation, with the result

$$s = C M^T (M C M^T + D)^{-1} x \qquad (20).$$

Here C represents the covariance matrix of the vector s; it is an infinite diagonal matrix whose elements are closely related to the degree variances σ_n^2 (section 2), and D is the error covariance matrix for the observations x.

The collocation solution appears to have certain advantages over the conventional method. It maintains the proper balance between the order of magnitude of coefficients and of measuring errors by treating them on an equal statistical basis, thus taking seriously the fact that especially for higher coefficients, their error has almost the same order of magnitude as the coefficients themselves. Furthermore collocation separates the individual coefficients in an optimal way and gives values for all infinitely many coefficients - the best values obtainable on the basis of the given data; if a higher coefficient can no more be reliably determined, then collocation gives a value close to zero.

The improvement of the statistical model by including gravity field covariances also leads to more realistic accuracy estimates for the resulting coefficients.

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7. Discussion

TAPLEY: I have a question related to the original model

$$\mathbf{x} = \mathbf{AX} + \mathbf{s} + \mathbf{n}.$$

In the presentation you assumed that the signal **S** was a random variable with zero mean at the observation point and with a non-zero mean elsewhere. You combined the s' as a zero mean plus **n**. Is this a type of phenomenon you would ordinarily encounter in geodesy; e.g., one which is zero at the observation point and non-zero elsewhere?

MORITZ: Well, it is not zero at the observation point and non-zero elsewhere. It's just the average over the whole Earth. For example, if **s** is the gravity anomaly, the integral or mean of the gravity anomaly over the whole Earth is zero, which is equivalent to there being no zero degree harmonic.

TAPLEY: This is not the point. You separate the signal into two parts which you denote as **s** at points other than the observation point, and **s'** at the observation point. You then combine **s'** with **n** and make the assumption that this combination (s'+n) has zero mean and known covariance. Then the expected value of this quantity is always assumed to be zero at the observation point.

MORITZ: Yes; at the observation point and everywhere else.

TAPLEY: However, you obtain an estimate for **s** in your final results. The estimate cannot be forced to be below s_P which corresponds to an estimate at the point.

MORITZ: Going back to the original report, one has $M\{n\} = 0$ and $M\{s\} = 0$. The **s** is the predicted value. It is the expected value taking into account the given measurements.

TAPLEY: Your estimate is your bias; then it cannot be forced to be zero elsewhere.

MORITZ: Let us consider the simplest case of the gravity anomaly. Let us say you have a measured value Δg, and you want to predict it another point. The average at any point will be zero. If one has no observations, then the best estimate is the value zero. But as soon as one has obtained some sample values, the best estimate will no longer be the value zero.

TAPLEY: Now if you go to another point and have a sample at another point, and if it is forced to be zero at one point, this implies that the results will be zero at any other collocated point.

MORITZ: No. The thing in mathematical terms is the expectation and the conditional expectation on the basis of one observed sample. Here the expected value of Δg is zero. The expectation of Δg at P on the assumption that Δg has been observed, is just the prediction formula. The difference between conditional expectation and ordinary expectation is the basic point.

TAPLEY: That requires a knowledge of the covariance matrix C_P. Getting a value for C_P is halfway to having the problem solved. Will any C_P satisfy the paper's assumption that you can always force the mean to zero?

MORITZ: It will not, because if we can compute a covariance function, then one knows the gravity field over the whole Earth and then one does not need statistics. But this is the case in statistical applications everywhere.

TAPLEY: One other question related to the observation relation applied to estimation. You know there will be observation errors $\Delta\omega$ or random errors in the covariances D. Why not go ahead and tackle the problem by estimating the unknown parameters directly with standard minimum variance, maximum likelihood or least squares algorithms?

MORITZ: If we use this approach, then the quantities J_2 and J_3 are considered as parameters and not as measured quantities. But I just want to consider them as random quantities in order to treat signal quantities and measurements on an equal statistical basis.

TAPLEY: But they are apparently considered as random quantities. When you write down the sigma, you treat your covariance matrix associated with your estimate of the observations as random observation noise.

MORITZ: But if you use this approach, one doesn't get any C matrix, only a D matrix.

TAPLEY: You get a D', not a C or an a priori estimate associated with the expected value as

equal to zero.

GRAFAREND: It was a question of having a case of truncating the series; and if you truncate, you have aliasing. The defect to the presented solution is that you have a biased solution. You avoid truncation, but you introduce bias.

TAPLEY: But with the observation relationships being non-linear .....

MORITZ: I think it is unbiased. It depends on what you understand by bias. Grafarend and I disagree on this.

KAULA: This problem is identical with what the seismologists call the inversion problem. They come into it from an entirely different direction because they have a highly under-determined situation. The question is how much of a continuum the data can resolve. Moritz' equation has been derived by JACKSON (*Geophys.J.R.astr.Soc.* 28,p.97, 1972) and by WIGGINS (*Revs.geophys.* 10,p.251,1972). With regard to the matrix C, there is quite a variety of diverse possibilities apart from a simple diagonal matrix, e.g., optimizing a certain function such as minimizing the temperature at the bottom in thermal convection. A powerful tool applicable here is singular value analysis: finding the diagonal roots and the pre- and post-multiplying unitary matrices, to give the relationships to these roots of the parameters on the one hand and the data on the other. This is an interesting technique which could be used in gravity field problems, particularly in analyzing radar altimetry of the oceans, where the parameters are not so obvious as in the case of satellite perturbation analysis for spherical harmonics.

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DATA EVALUATION BY COVARIANCE ANALYSIS, EXERCISED ON PHOTOGRAPHIC SATELLITE OBSERVATIONS

ABSTRACT

As an example of a multiparametric stochastic process of vector-type, some plates obtained with a high accuracy sidereal driven satellite camera are studied.

Subject of analysis is the set of residuals, obtained after reduction with different models. If beside the white noise there is any statistical information left, it produces correlations between the residuals, which can be described using different characteristical functions.

Quantitative tests show, that with simple models it is possible to extract the whole information out of the plates, so that a directional accuracy of a few tenths of arcseconds can be expected.

In connection with precise Laser distances such direction measurements can contribute to geodynamical studies.

1. Introduction

The increasing accuracy of laser distance measurements to satellites reaching the order of decimeters (PLOTKIN et al 1973) opens new possibilities for the study of geodynamic effects (DUNN et al 1973). One method for the determination of secular variations in the relative position of two or more points is the measuring of base-chord lengths (KOLACZEK & WILSON 1973) using simultaneous distance and direction observations to satellites. The advantage of this combined method is that only two stations are involved in observation at the same time. The simultaneous operation of four lasers, which would be necessary for the geometrical solution of pure distance-measurements (CAMPBELL et al 1973) has to cope with great difficulties.

The disadvantage of the combined method is seen in the limited accuracy obtainable for direction observations, which generally is assumed to be 1" (KOLACZEK & WILSON 1973). Regarding the formula for the propagation of errors (equation 1) for the base-chord length b, which corresponds to figure 1 it is evident that best results are obtained with γ as large as possible; a small angle γ will nullify the advantage of the high distance accuracy.

$$m_b^2 = \cos^2\alpha\, m_{s_1}^2 + \cos^2\beta\, m_{s_2}^2 + h_S^2 \frac{m_\gamma^2}{\rho^2} \qquad (1)$$

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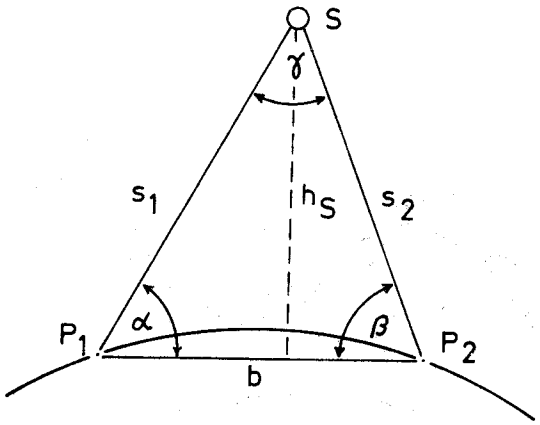


Figure 1. Determination of Base-Chord b with Simultaneous Range and Direction Observations to S

The method described could be much more powerful if it were possible to determine directions with an accuracy in the order of 0".1. The presented paper deals with the question whether such accuracy for camera observations is attainable and how it can be proved.

2. Review of theoretical foundations

In plate reduction we have to solve a transformation-and-interpolation-problem. Star positions in the star catalogue and on the photographic plate are considered as two fields of identical points which can be transformed into each other using a set of transformation parameters AX obtained in a least squares solution.

Using these parameters it is possible to transform the unknown information of the plate, i.e. the satellite positions, into equatorial directions. If the set of parameters AX - the functional model - is able to describe the whole transformation process, the residuals v after transformation contain only a pure random part n (white noise, resulting from the measuring errors). If the functional model is not sufficient, the residuals contain another part, the "signal term" s, hence

$$v = s + n$$

In general we can write according to figure 2

$$x = AX + s + n \qquad (2)$$

where x stands for the observations. Equation 2 is the general form of the collocation concept (MORITZ 1973).

The next step is the decide whether the "signal" term s exists, that means whether there is any information left in the set of residuals, *or* if the residuals contain only "white noise", which

would mean

$$v = n$$

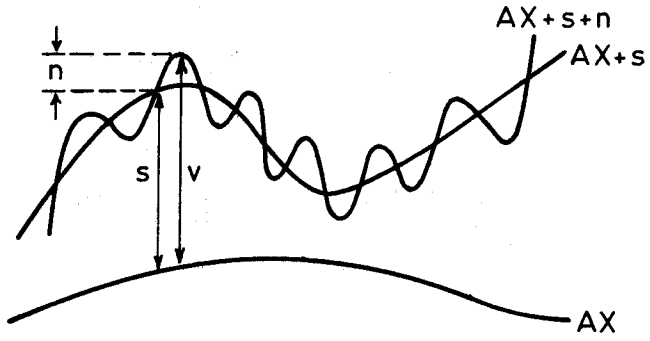


Figure 2. Concept of Collocation

If the term s exists, there are two possibilities to proceed:

a) the mathematical model AX can be supplemented with further parameters. (In plate reduction we have the important advantage that the basic concept of projection is known).

b) application of the collocation method using equation 2. In this case the unknown signal-term in the satellite point can be estimated - with respect to the signals of the surrounding star points - by means of least squares interpolation.

The answer on the question whether there are signals s left in the residuals or not can be given by estimating the two or more parametric correlation functions. These functions are also needed for the determination of the matrix C in the well known formula 3 for least squares interpolation (MORITZ 1973).

$$s_P = C_P^T C^{-1} v \qquad (3)$$

where P stands for the interpolated point.

In the case of plate reduction we find a stochastic process of vector-type in a two dimensional space (plate). The general form for the two-point-correlation-function of the residuals is

$$E\{\xi_i(\underline{r}_1)\xi_j(\underline{r}_2)\} = \Phi_{ij}(\underline{r}_1,\underline{r}_2) \qquad (4)$$

where $\underline{r}_1$, $\underline{r}_2$ describe the two points between which the correlations are to be determined and ξ_i, ξ_j are the components of the random variable $\underline{v}$, described in the coordinate system i, j (figure 3).

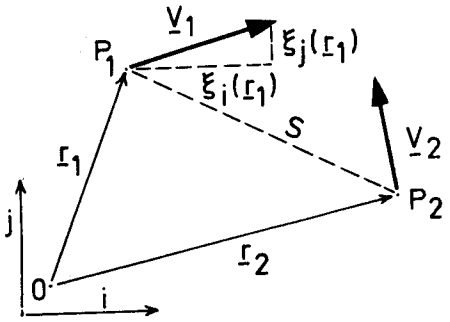


Figure 3. The Random Variable $\underline{v}$ (Residuals After Plate Reduction)

Practical work can be made much easier, if the stochastic process is, in a statistical sense, *homogeneous*, that means invariant against translation, and *isotropic*, that means invariant against rotation. In the homogeneous case equation 4 becomes

$$\Phi_{ij}(\underline{r}_1, \underline{r}_2) = \Phi_{ij}(\underline{r}_2 - \underline{r}_1) \qquad (5)$$

and in the isotropic case

$$\Phi_{ij}(\underline{r}_1, \underline{r}_2) = \Phi_{ij}(|\underline{r}_2 - \underline{r}_1|) = \Phi_{ij}(s) \qquad (6)$$

For more details see SEEBER (1972).

It can be shown that, in a statistical homogeneous and isotropic stochastic process, the two-point-correlation-functions of vector-type (2 dimensions) can be described by only two characteristical functions (OBUCHOW 1958). These functions have only one independent variable, the distance s between the two observation points, and are called "longitudinal" correlation function L(s) and "lateral" correlation function Q(s). They can be determined by dividing the residuals $\underline{v}$ in one component in the direction of s and another perpendicular to it (figure 4).

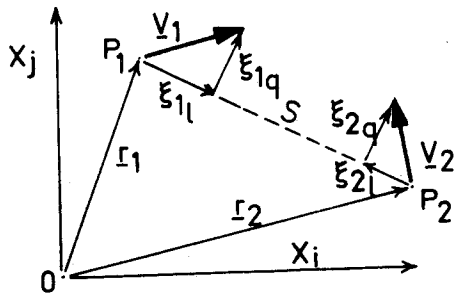


Figure 4. Components for L(s) and Q(s) in a Moving Coordinate System P_1P_2

Then we have

$$L(s) = E\{\xi_{1\ell}\xi_{2\ell}\} \qquad Q(s) = E\{\xi_{1q}\xi_{2q}\} \tag{7}$$

In this way $L(s)$ and $Q(s)$ are defined in a coordinate system which is moving with the observation points. In a fixed system $x_i x_j$ we find (SEEBER 1972), (OBUCHOW 1958), (GRAFAREND 1972)

$$\Phi_{ij}(\underline{r}_1,\underline{r}_2) = Q(s)\delta_{ij} + \frac{L(s)-Q(s)}{s^2} x_i x_j. \tag{8}$$

Equation 8 is named after the first investigators TAYLOR and KARMAN.

For correlation functions between three and more points the necessary number of such invariant distance functions increases considerably (IBID). These correlations functions are required for non-linear collocation problems, if the signal-level of the observation is very high (IBID).

For several problems, especially in plate reduction, it is better to refine the mathematical model AX in equation 2 until the linear formula 3 is sufficient and the correlation matrix C can be calculated by only two functions namely $L(s)$ and $Q(s)$. In addition these two functions enable us to see if there is any information left in the field of residuals after reduction.

3. Numerical Investigations

As observation material some plates were chosen, obtained with the new sidereal driven ballistic satellite camera BMK 46/18/1:4, 5 at the satellite station Bonn-Todenfeld (SEEBER 1972). This camera is equipped with a new type objective ASTRO-TOPAR of high geometrical accuracy. Distortion is smaller than 5 Micron.

Correlation analysis was done with the residuals after reduction using a 6-parameter-photogrammetric model. The set of residuals for one plate is shown in figure 5. The mean error of unit weight m_o for this plate was $\pm 2{,}6\mu m$.

The corresponding two-point-correlation functions $L(s)$ and $Q(s)$ are shown in figure 6. It is evident that no information is left in the residuals; hence the signal-term s is zero and only the noise-term n exists. Consequently any least squares estimation efforts are unnecessary in this case.

As a comparison, figure 7 shows residuals with a high content of information (neglected refraction correction) leading to strong correlations (figure 8). Furthermore it is evident that the field of residuals is neither homogeneous nor isotropic (figure 9). As a consequence the use of equations 2 and 3 requires a set of numerous functions $L(s)$ and $Q(s)$ for a single plate. Moreover, it cannot be expected that the linear method 3 is sufficient, therefore non-linear prediction methods with three, or more, point correlation functions have to be used.

A much simpler way is to use the physical knowledge about the whole process and correct the observations before the reduction. Doing that we obtain immediately "white noise" as is shown in figures 5

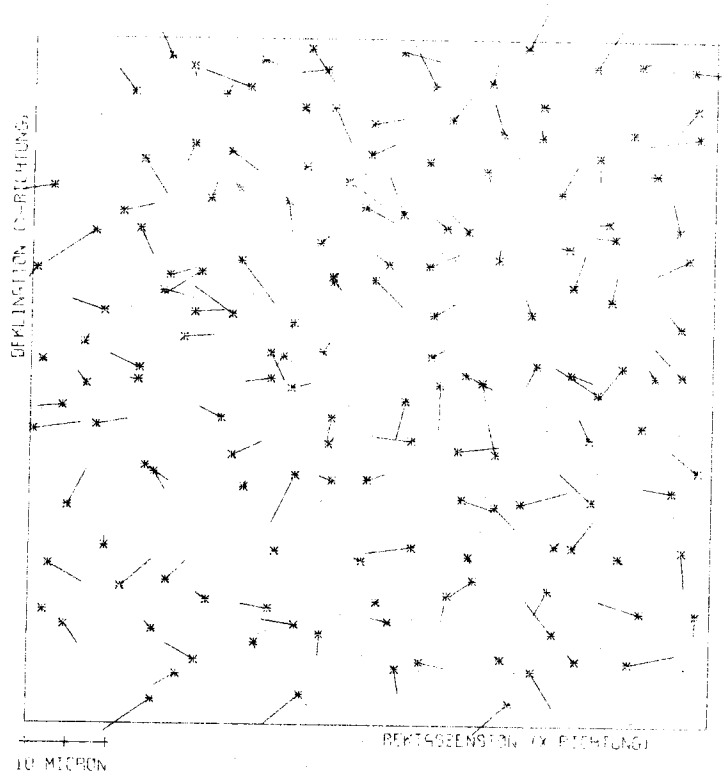


Figure 5. Residuals After Plate Reduction, 6 Parameters

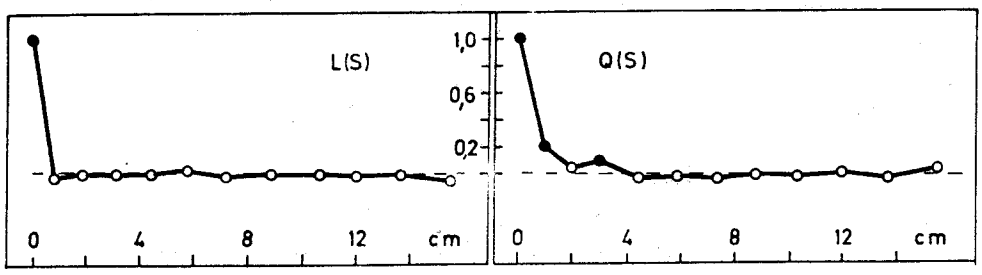


Figure 6. Longitudinal (left) and Lateral (right) Correlation Functions, Model with 6 Parameters

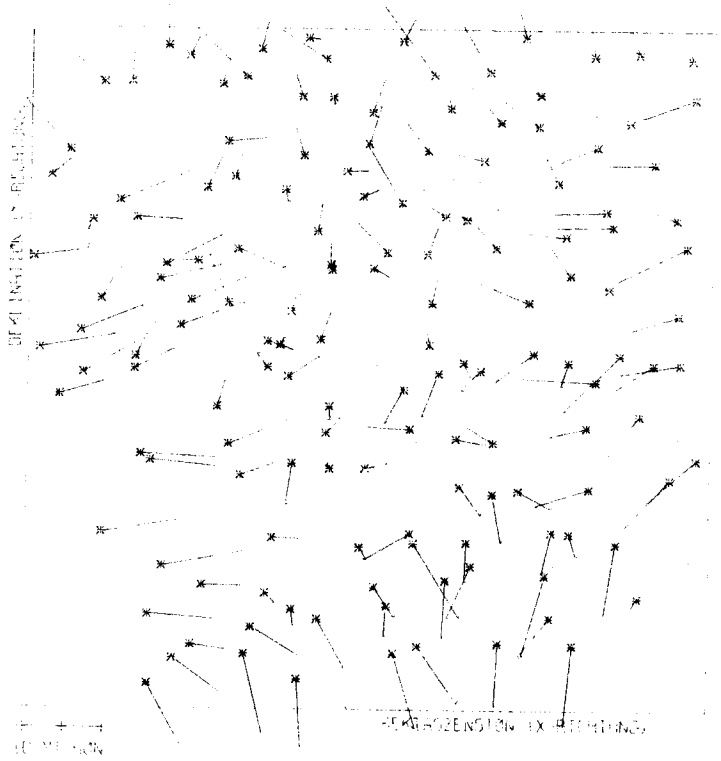


Figure 7. Residuals after Reduction with an Insufficient Model (Neglected Refraction Correction)

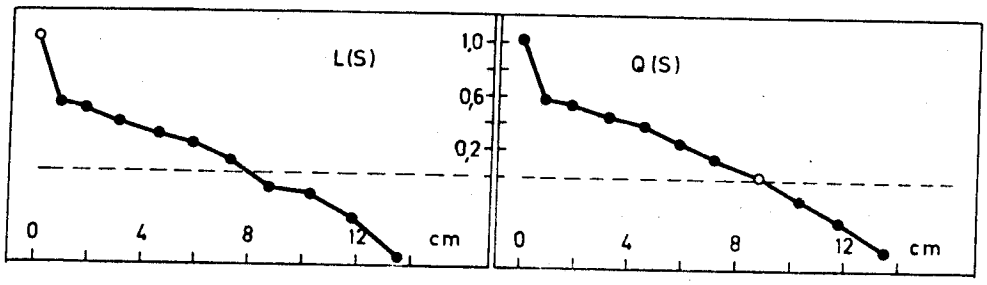


Figure 8. Longitudinal (L(s)) and Lateral (Q(s)) Correlation Functions for an Insufficient Model

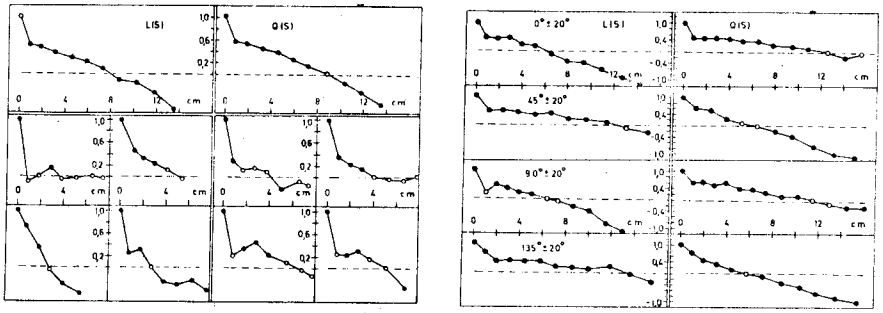
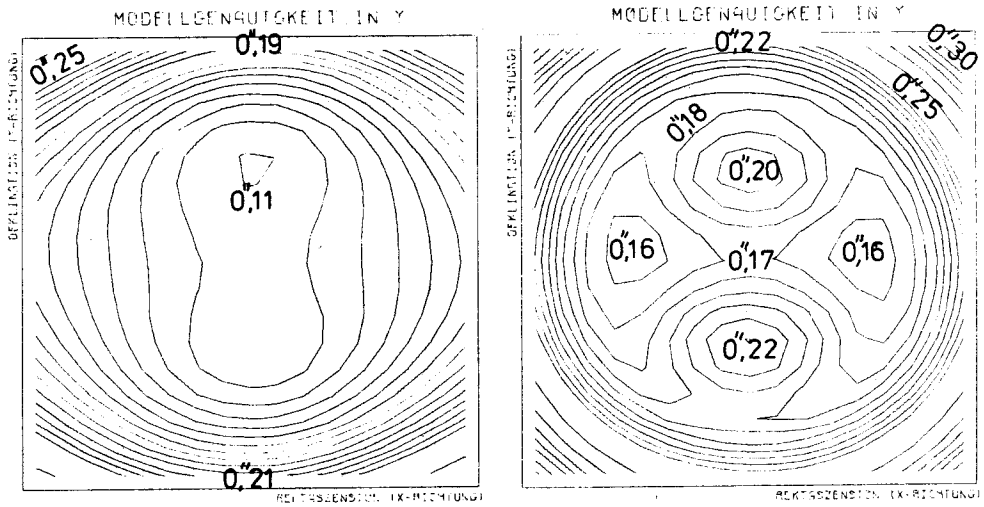


Figure 9. Test on Homogeneity (left, 4 Parts of a Plate) and Isotropy (right, 4 directional Classes)

and 6. This holds also for a simple astrometric interpolation model with 6 parameters, so that the use of simple models for the reduction of BMK-plates is sufficient if the physical knowledge about the projection process is applied.

As a consequence, the transformation error, resulting from a least squares solution, is very small. Figure 10 gives an impression of the transformation errors for the whole plate corresponding to a simple model (left) and a more sophisticated one (right).



6 parameters 12 parameters

Figure 10. Mean Error of an Adjusted Fictitious Point all over the Plate for a Simple Model (left) and a More Sophisticated One (right)

Apart from the inconsistencies in the star catalogues amounting to several tenths of arcseconds, the directional accuracy of 0".1 may be regarded as realistic since all information has been pulled out of the residuals.

4. Conclusion

Data evaluation by covariance analysis is a powerful tool for testing the realistic accuracy obtainable in directional observations to satellites. Using a camera like the BMK such directions can be determined with an accuracy of a few tenths of arcseconds. In combination with laser distance measurements it is possible to contribute to the study of secular variations in position.

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6. Discussion

GRAFAREND: Do you have an estimate of the confidence band of your correlation function for the isotropic case?

SEEBER: Yes, but they are small; of the order of 10%.

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TRUNCATION ERROR ESTIMATE AND TRUNCATED GRAVITY ANOMALY

ABSTRACT

Stokes' integration of the gravity anomaly is extended over the whole surface of the Earth in determining geoidal height. If the integration is made over an area restricted to a spherical cap, a truncation error in geoidal heights results from neglecting the remaining regions. A truncation error is also defined in the Vening Meinesz integration for deflections of the vertical. COOK (1951) was the first to investigate truncation errors for geoidal height and deflection of the vertical. MOLODENSKII ET AL (1962) later derived truncation formulas by using a truncation coefficient Q_n.

The present paper is devoted to obtaining a mathematical relationship between the Molodenskii coefficient Q_n and Cook's truncation function q_n by means of the "truncated gravity anomaly". Maps of the world-wide distribution of truncation errors are also presented.

The truncation procedure produces low-order terms of spherical harmonics. This implies that a truncated gravity anomaly is caused by an anomalous density distribution which is different from that of the "real" Earth. The centre of a geoid calculated from the truncated gravity anomaly may not coincide with one of the "real" geoid. In other words, the truncation procedure apparently removes or shifts a part of the Earth's masses. In the present paper, the 0-th and 1-st order terms that result from the truncation of the satellite gravity anomaly are estimated as a few mgal.

1. Introduction

Gravimetric geoidal height and deflection of the vertical are obtained at a point P on the Earth's surface by the Stokes and Vening Meinesz integrations, respectively, which are extended over the whole surface of the Earth. The Stokes and Vening Meinesz functions converge into zero with spherical distance from the point P, so that it may be natural to think that we can approximately calculate the geoidal height and deflection of the vertical by respectively integration the Stokes and Vening Meinesz functions over an area restricted to a spherical cap around P. The convergence of the above-mentioned functions however, is so slow that the effects of neglecting the remote zones on the integrations are not small. Such an effect is called "truncation error" by DE WITTE (1967).

The truncation error was first investigated by COOK (1951), who published tables of truncation errors for the geoidal height up to a truncation angle 30°, and for the deflection of the vertical up to 18°. Probably not aware of Cook's work, MOLODENSKII ET AL (1962) later derived truncation error formulas by using the truncation coefficient Q_n, which is called the "Molodenskii coefficient" by DE WITTE (1967). The primary motivation of Molodenskii et al. was not to obtain the truncation errors but functions of rapid convergence instead of the Stokes and Vening Meinesz functions.

HIRVONEN & MORITZ (1963) discussed the truncation errors by using Molodenskii coefficients Q_n, and a similar discussion appears in the famous textbook "Physical Geodesy" authored by HEISKANEN & MORITZ (1967). However, in their papers, there is no description of Cook's work and the truncation error formula for the Vening Meinesz integration derived by Molodenskii et al. The truncation error

formulas for the Vening Meinesz integration introduced in these papers are equal to neither Cook's nor Molodenskii's.

DE WITTE (1967) pointed out that Cook's method is valid for the treatment of Vening Meinesz truncation errors. In order to distinguish Cook's truncation coefficient from the Molodenskii coefficient Q_n, de Witte defined q_n anew, which he called the "Vening Meinesz kernel function", and "Cook's truncation function" by MATHER (1970). HAGIWARA (1972) obtained a mathematical relation between Q_n and q_n, showing that Cook's truncation formula for the Vening Meinesz integration is essentially identified with the one derived by Molodenskii et al.

DE WITTE (1967) obtained a table of Molodenskii coefficients Q_n up to order $n = 25$, actually estimating the truncation errors for the geoidal height and deflection of the vertical at Meade's Ranch, the North American Geodetic Datum Station, by applying Q_n to the satellite gravity anomaly obtained by GUIER (1965), and concluding that the truncation errors are not negligibly small.

It may be possible to obtain the gravimetric geoid by means of numerical calculation of the Stokes integral of terrestrial gravity anomalies extending up to a spherical distance, together with truncation error considerations based on satellite gravimetry. However, such a treatment brings forward a fundamental problem which needs to be solved. It corresponds to the difference between the shape of an ellipsoid on which the terrestrial gravimetry is based, and that of a satellite determined ellipsoid. The centre of the former ellipsoid may not coincide with that of the latter.

Such a non-coincidence of position also generates errors in the geoidal estimation in combining terrestrial gravity anomalies with satellite ones. Although the positional relation is absolutely impossible to be solved from the gravimetry only, this kind of problem still remains to be considered further in physical geodesy. MATHER (1970) discussed the zero-order term of geopotential on truncation errors as caused by the difference between the shapes of the ellipsoids. From the other point of view, the author also discusses the associated problem in the present paper, defining the "truncated gravity anomaly" in a fashion similar to the Molodenskii treatment in the truncated Stokes function.

2. Molodenskii Truncation

The geoidal height is obtained by calculating the Stokes integration on extension over the whole surface of the Earth. Letting α and ψ be the azimuth and angular distance respectively, the geoidal height is

$$N = \frac{R}{4\pi G} \int_0^{2\pi} d\alpha \int_0^{\pi} \Delta g \, S(\cos \psi) \sin \psi \, d\psi \qquad (1),$$

where R is the mean radius of the Earth, G is the mean gravity over the Earth's surface, Δg the gravity anomaly and S is Stokes' function, given by a series of Legendre polynomials as follows:

$$S(\cos \psi) = \sum_{n=2}^{\infty} \frac{2n + 1}{n - 1} P_n(\cos \psi) \qquad (2).$$

When the integration with respect to ψ is extended only upto an angular distance ψ_o, the effect of

neglecting distant regions beyond ψ_o on the integration, which was called "truncation error" by DE WITTE (1967), is

$$\Delta N = \frac{R}{4\pi G} \int_0^{2\pi} d\alpha \int_0^{\pi} \Delta g \, S(\cos \psi) \sin \psi \, d\psi \qquad (3).$$

Introducing a truncation function

$$\bar{S}(\cos \psi) = \begin{cases} 0 & \text{for } 0 \leq \psi < \psi_o \\ S(\cos \psi) & \text{for } \psi_o \leq \psi \leq \pi \end{cases} \qquad (4),$$

equation 3 can be re-written as

$$\Delta N = \frac{R}{4\pi G} \int_0^{2\pi} d\alpha \int_{\psi_o}^{\pi} \Delta g \, \bar{S}(\cos \psi) \sin \psi \, d\psi \qquad (5).$$

We then expand $\bar{S}(\cos \psi)$ in a series of Legendre polynomials in the form

$$\bar{S}(\cos \psi) = \sum_{n=0}^{\infty} \frac{2n+1}{2} Q_n P_n(\cos \psi) \qquad (6),$$

where Q_n, which was called the "Molodenskii coefficient" by de Witte (IBID), was originally tabulated by MOLODENSKII ET AL (1962). On substituting equation 6 in equation 5, we get

$$\Delta N = \frac{R}{2G} \sum_{n=2}^{\infty} Q_n \, \Delta g_n \qquad (7),$$

where Δg_n is the n-th order spherical harmonic of Δg.

3. Truncated Gravity Anomaly

It is convenient for treatments of the truncation of Stokes' integral to define a "truncated Legendre function" given by

$$R_n^m(\cos \psi) = \begin{cases} 0 & \text{for } 0 \leq \psi < \psi_o \\ P_n^m(\cos \psi) & \text{for } \psi_o \leq \psi \leq \pi \end{cases} \qquad (8).$$

If the above function is expanded in a spherical harmonic series, we have

$$R_n^m(\cos \psi) = \sum_{k=m}^{\infty} \frac{2k+1}{2} \frac{(k-m)!}{(k+m)!} R_{nk}^m P_k^m(\cos \psi) \qquad (9).$$

The coefficient R_{nk}^m is determined by the orthogonality condition of Legendre polynomials, so that

$$R^m_{nk} = \int_{\psi_o}^{\pi} P^m_n(\cos\psi)\, P^m_k(\cos\psi)\, \sin\psi\, d\psi \tag{10}.$$

Now, suppose the case of $m = 0$ in equation 10. For brevity, replacing R^0_{nk} by R_{nk}, and putting $\mu_o = \cos\psi_o$, we obtain

$$R_{nk} = \int_{\psi_o}^{\pi} P_n(\cos\psi)\, P_k(\cos\psi)\, \sin\psi\, d\psi$$

$$= \begin{cases} \frac{1}{2n+1}\left\{1 + \mu_o\left(P_n(\mu_o)\right)^2 + 2\sum_{m=1}^{n-1} P_m(\mu_o)\left(\mu_o P_m(\mu_o) - P_{m+1}(\mu_o)\right)\right\} & \text{for } n=k \\ \frac{1}{(n-k)(n+k+1)}\left\{kP_n(\mu_o)\left(P_{k-1}(\mu_o) - \mu_o P_k(\mu_o)\right) - nP_k(\mu_o)\left(P_{n-1}(\mu_o) - \mu_o P_n(\mu_o)\right)\right\} & \text{for } n\neq k \end{cases} \tag{11}.$$

Especially

$$R_{00} = 1 + \mu_o \quad ; \qquad \text{and} \qquad R_{11} = (1 + \mu_o^3)/3 \tag{12}.$$

Applying equation 2 to equation 11, we obtain

$$\int_0^{\pi} R_n(\cos\psi)\, S(\cos\psi)\, \sin\psi\, d\psi = \sum_{k=2}^{\infty} \frac{2k+1}{k-1} R_{nk} \tag{13}.$$

Furthermore, supposing $m = 1$, equation 10 becomes

$$R^1_{nk} = \int_{\psi_o}^{\pi} P^1_n(\cos\psi)\, P^1_k(\cos\psi)\, \sin\psi\, d\psi$$

$$= n(n+1)\, R_{nk} + P^1_n(\cos\psi_o)\, P_k(\cos\psi_o)\, \sin\psi_o \tag{14}.$$

The global distribution of the gravity anomaly is usually expressed by spherical harmonics as

$$\Delta g = G \sum_{n=2}^{\infty} (n-1) \sum_{m=0}^{n} \left(C_{nm}\cos m\alpha + S_{nm}\sin m\alpha\right) P^m_n(\cos\psi) \tag{15}.$$

If a "truncated gravity anomaly" is defined in a fashion similar to 4 by

$$\overline{\Delta g} = \begin{cases} 0 & \text{for} \quad 0 \leq \psi < \psi_o \\ \Delta g & \text{for} \quad \psi_o \leq \psi \leq \pi \end{cases} \tag{16},$$

equation 16 can be expressed as

$$\overline{\Delta g} = G \sum_{n=2}^{\infty} (n-1) \sum_{m=0}^{n} \left(C_{nm}\cos m\alpha + S_{nm}\sin m\alpha\right) R^m_n(\cos\psi) \tag{17}.$$

This is a spherical harmonic expression of the truncated gravity anomaly, where attention should be paid to the fact that the Legendre function in equation 15 has been replaced by the truncated

Legendre function in equation 17.

The truncation of the geoidal height can readily be treated by using the truncated gravity anomaly. Instead of equation 5, the truncation error of the geoidal height is given by

$$\Delta N = \frac{R}{4\pi G} \int_0^{2\pi} d\alpha \int_0^{\pi} \overline{\Delta g}\, S(\cos\psi) \sin\psi\, d\psi \qquad (18).$$

Substitution of equations 2 and 17 into equation 18 gives

$$\Delta N = \frac{R}{2G} \sum_{n=2}^{\infty} \Delta g_n \sum_{k=2}^{\infty} \frac{2k+1}{k-1} R_{nk} \qquad (19).$$

On comparing equation 19 with equation 7, we obtain the Molodenskii coefficient as a series in R_{nk}:

$$Q_n = \sum_{k=2}^{\infty} \frac{2k+1}{k-1} R_{nk} \qquad (20).$$

By the way, Q_n is also obtained directly from the relation 13.

4. Truncation of the Vening Meinesz Integral

In a fashion similar to the truncation of the geoidal height in the Stokes integration, the truncation error of the Vening Meinesz integration is defined as

$$\begin{pmatrix} \Delta\xi \\ \Delta\eta \end{pmatrix} = \frac{1}{4\pi G} \int_0^{2\pi} \begin{Bmatrix} \cos\alpha \\ \sin\alpha \end{Bmatrix} d\alpha \int_{\psi_o}^{\pi} \Delta g \frac{dS}{d\psi} \sin\psi\, d\psi \qquad (21).$$

If we rewrite the above by using the truncated gravity anomaly, we have

$$\begin{pmatrix} \Delta\xi \\ \Delta\eta \end{pmatrix} = \frac{1}{4\pi G} \int_0^{2\pi} \begin{bmatrix} \cos\alpha \\ \sin\alpha \end{bmatrix} d\alpha \int_{\psi_o}^{\pi} \overline{\Delta g} \frac{dS}{d\psi} \sin\psi\, d\psi \qquad (22).$$

Differentiating both sides of equation 2 with respect to latitude ϕ and longitude λ, and taking the following relations into account,

$$\left.\begin{aligned} \frac{\partial P_n(\cos\psi)}{\partial\phi} &= \frac{dP_n(\cos\psi)}{d\psi} \frac{\partial\psi}{\partial\phi} = P_n^1(\cos\psi)\cos\alpha \\ \frac{\partial P_n(\cos\psi)}{\partial\phi} &= \frac{dP_n(\cos\psi)}{d\psi} \frac{\partial\psi}{\partial\lambda} = P_n^1(\cos\psi)\sin\alpha\cos\phi \end{aligned}\right\} \qquad (23),$$

and

$$\int_0^{\pi} R_n^1(\cos\psi) \frac{dS}{d\psi} \sin\psi\, d\psi = -\sum_{k=2}^{\infty} \frac{2k+1}{k-1} R_{nk}^1 \qquad (24),$$

we obtain

$$\begin{pmatrix} \Delta\xi \\ \Delta\eta \end{pmatrix} = -\frac{1}{2G} \sum_{n=2}^{\infty} \frac{1}{n(n+1)} \begin{pmatrix} \partial\Delta g_n/\partial\phi \\ \partial\Delta g_n/(\cos\phi\ \partial\lambda) \end{pmatrix} \sum_{n=2}^{\infty} \frac{2k+1}{k-1} R^1_{nk} \tag{25}.$$

Similarly to equation 7, we assume the following formula by using a newly defined coefficient Q^*_n:

$$\begin{pmatrix} \Delta\xi \\ \Delta\eta \end{pmatrix} = -\frac{1}{2G} \sum_{n=2}^{\infty} Q^*_n \begin{pmatrix} \partial\Delta g_n/\partial\phi \\ \partial\Delta g_n/(\cos\phi\ \partial\lambda) \end{pmatrix} \tag{26}.$$

Comparing the above with equation 25, we prove

$$Q^*_n = \frac{1}{n(n+1)} \sum_{n=2}^{\infty} \frac{2k+1}{k-1} R^1_{nk} \tag{27}.$$

Cook's truncation function q_n is similarly given by

$$q_n = -\frac{1}{2} \sum_{k=2}^{\infty} \frac{2k+1}{k-1} R^1_{nk} \tag{28}.$$

Taking equations 14 and 20 into account, Q^*_n can be expressed as

$$Q^*_n = Q_n + \frac{1}{n(n+1)} S(\cos\psi_o)\, P^1_n(\cos\psi_o) \sin\psi_o \tag{29}.$$

A table of Q^*_n was presented by HAGIWARA (1972). Figure 1 shows low-order terms of Q_n and Q^*_n.

Similarly we obtain the relation between Q_n and q_n as

$$q_n = -\frac{n(n+1)}{2} Q_n - \frac{1}{2} S(\cos\psi_o)\, P^1_n(\cos\psi_o) \sin\psi_o \tag{30}.$$

Figure 2a shows the truncation error in the Stokes integration in the case of $\psi_o = 20^o$, using spherical harmonic coefficients given by GAPOSCHKIN & LAMBECK (1970). It is noticed in the figure that the error of the geoidal heights amounts to ±30 m or so. The truncation error of the meridian deflection of the vertical amounting to 3.0 sec or so arranges zonally as shown in figure 2b; on the contrary, that of the prime vertical deflection indicates a sectorial distribution (see figure 2c)

5. Zero and First Order Terms Caused by Truncation

The gravity anomalies are caused by subterranean distribution of anomalous masses. When we define a truncated gravity anomaly, we must bear in mind that such a gravity anomaly corresponds to a mass distribution which may be different from that of the real Earth. The truncation procedure apparently removes or shifts a part of the Earth's masses. The geoidal height is based on the assumption that the spherical functions of zero and first order terms of the disturbing potential are equal to zero.

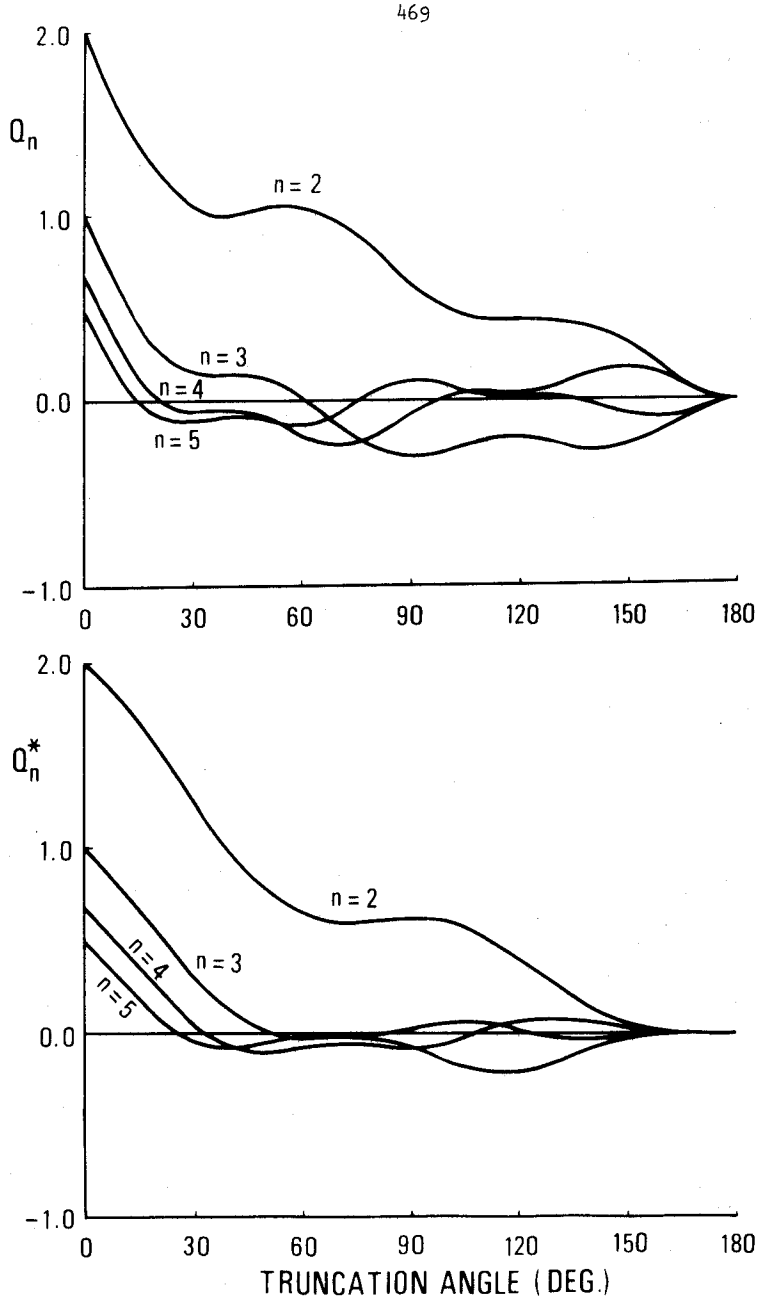

Figure 1. Q_n and Q_n^*

The assumption of the zero-order term implies that the global mean of the gravity anomaly is adopted as zero, and that of the first order terms is equivalent to setting the centre of the reference ellipsoid at the centre of the Earth's mass. If the centre of a geoid calculated from the truncated gravity anomaly does not coincide with that of the real geoid, the problem of the forbidden terms may come up.

We assume that the truncated gravity anomaly is expressed in the form

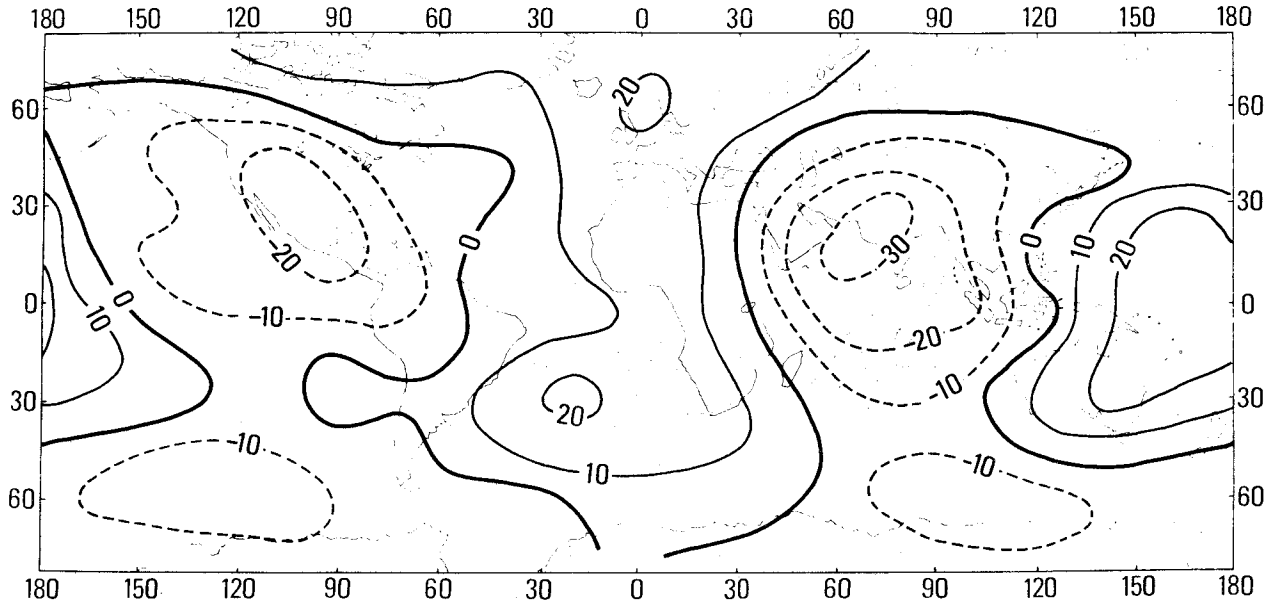

Figure 2a. Truncation Error Distribution for the Geoidal Height (Metres)

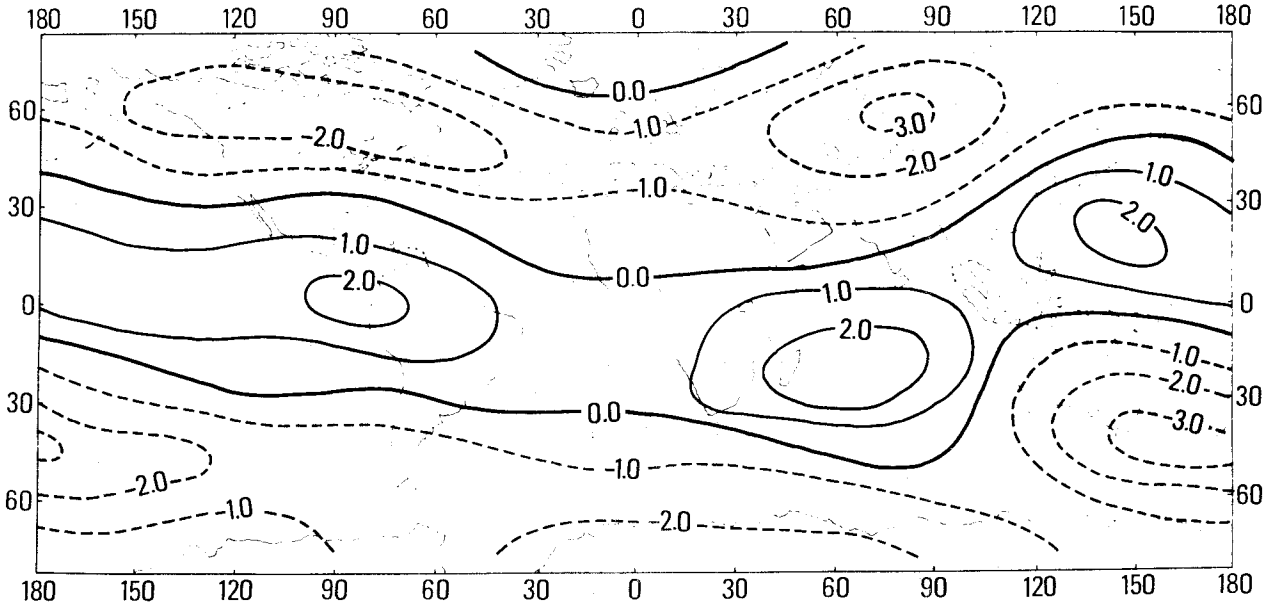

Figure 2b. Truncation Error Distribution for the meridian Deflection of the Vertical (sec.)

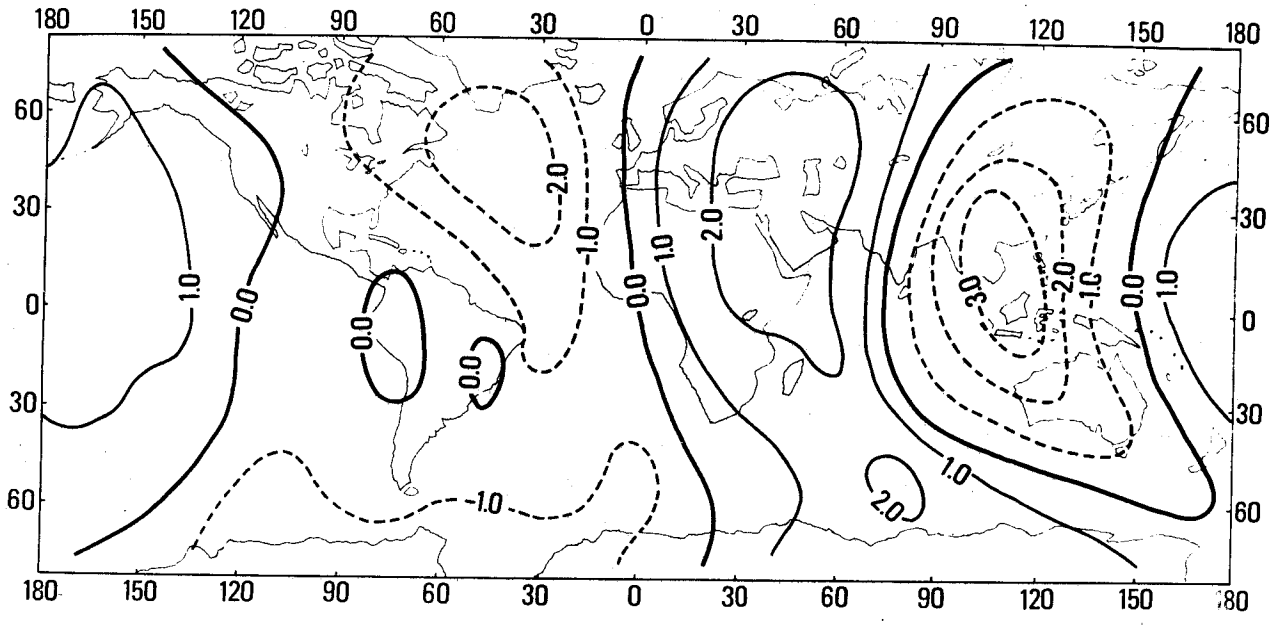

Figure 2c. Truncation Error Distribution for the Prime-Vertical Deflection of the Vertical (sec.)

$$\overline{\Delta g} = G \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(A_{nm} \cos m\alpha + B_{nm} \sin m\alpha\right) P_n^m(\cos \psi) \qquad (31).$$

In this case, attention is required to ensure that the series includes zero and first order terms. If there is no change in the masses due to the truncation procedure, these terms become zero, and

$$A_{nm} = (n-1)\, C_{nm} \qquad \text{and} \qquad B_{nm} = (n-1)\, S_{nm} \qquad \text{for} \quad n \geq 2.$$

From equations 17 and 31, together with the orthogonality condition of spherical harmonics, the following relations are derived.

$$A_{00} = \frac{1}{2G} \sum_{n=2}^{\infty} R_{n0}\ \Delta g_n \quad ; \quad A_{10} = \frac{3}{2G} \sum_{n=2}^{\infty} R_{n1}\ \Delta g_n$$

and

$$\begin{pmatrix} A_{11} \\ B_{11} \end{pmatrix} = \frac{3}{2G} \sum_{n=2}^{\infty} \frac{R^1_{n1}}{n(n+1)} \begin{pmatrix} \partial \Delta g_n / \partial \phi \\ \partial \Delta g_n / (\cos \phi\ \partial \lambda) \end{pmatrix} \qquad (32),$$

where A_{00}, A_{10}, A_{11} and B_{11} are functions of ϕ and λ.

Figures 3a to 3d show the global distributions of these terms, i.e., $A_{00}G$, $A_{10}G$, $A_{11}G$ and $B_{11}G$ in mgal

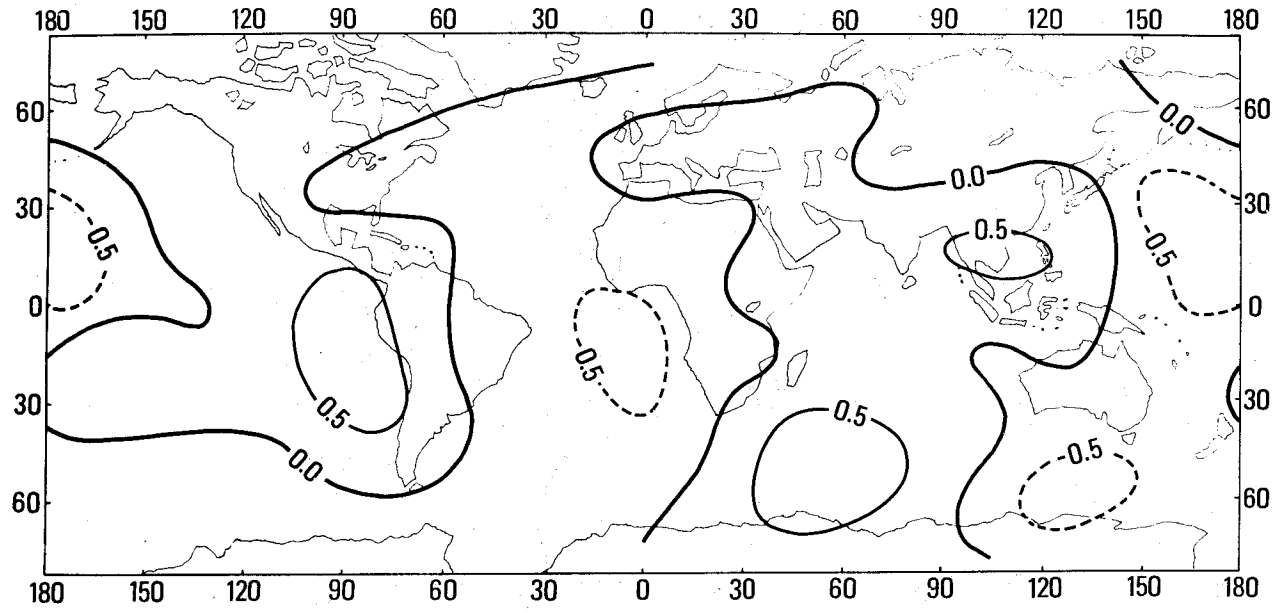

Figure 3a. Global Distribution of $A_{00}G$ (mgal)

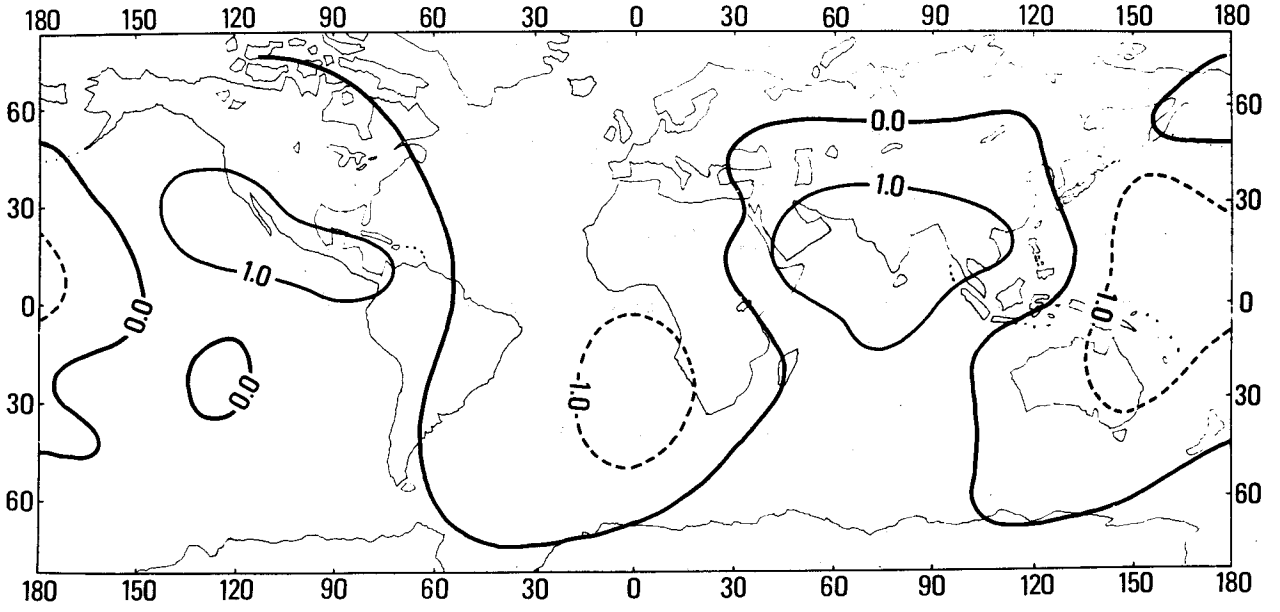

Figure 3b. Global Distribution of $A_{10}G$ (mgal)

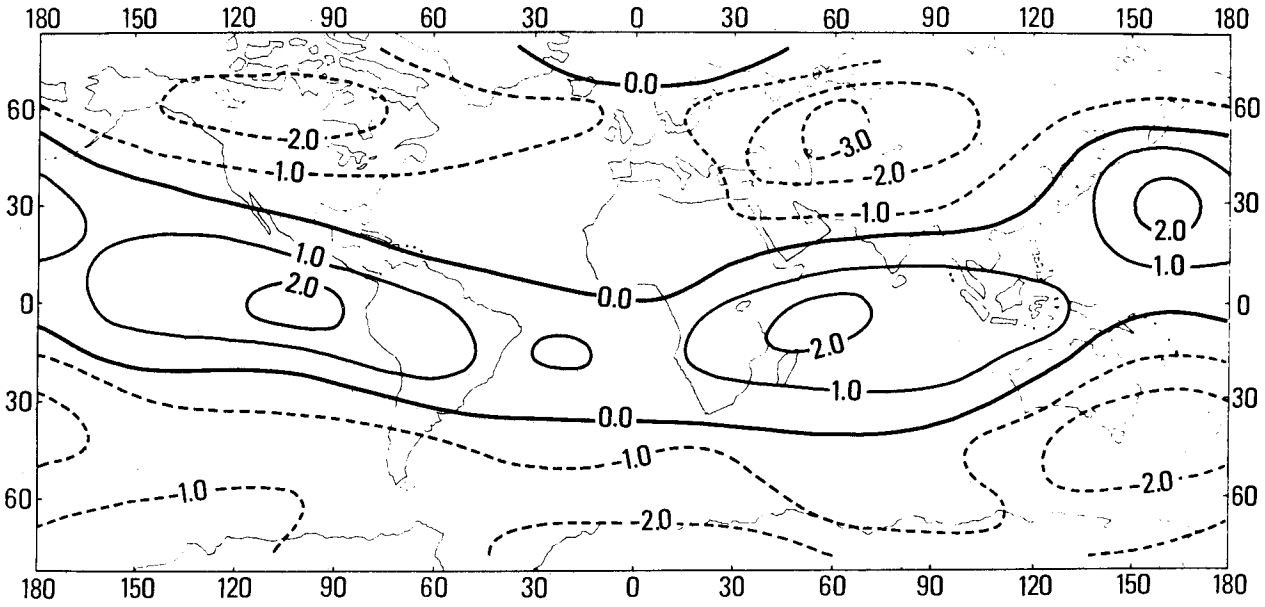

Figure 3c. Global Distribution of $A_{11}G$ (mgal)

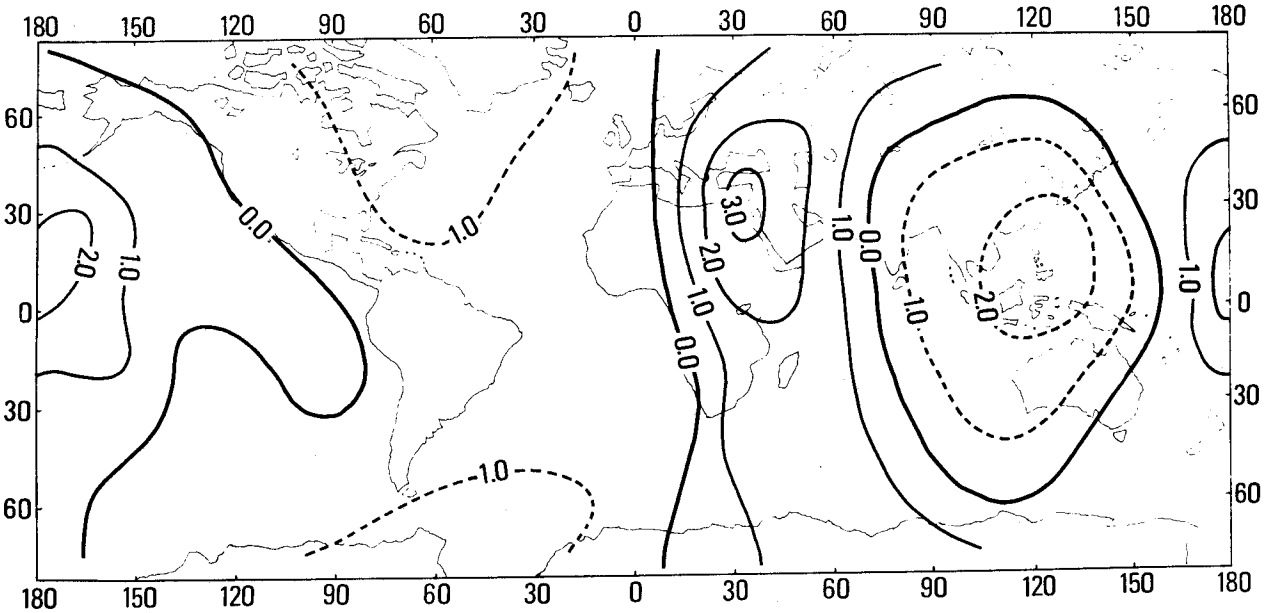

Figure 3d. Global Distribution of $B_{11}G$ (mgal)

for the case of $\psi_o = 20^o$, respectively, which are computed by using the coefficients given by GAPOSCHKIN & LAMBECK (1970). We see in the figures that $A_{00}G$ and $A_{10}G$ take small values below ±1 mgal, but $A_{11}G$ and $B_{11}G$ amount to ±3 mgal or so. The pattern of $A_{10}G$ resembles that of the truncation error of the geoidal height shown in figure 2a. $A_{11}G$ and $B_{11}G$ represent zonal and sectorial effects like the truncation errors in the meridian and prime vertical deflections of the vertical as seen in figures 2b and 2c respectively.

Although the determinations of these zero and first order terms are made, the problem of whether the centre of the reference ellipsoid coincides with that of the Earth's mass or not, still remains, because it is absolutely impossible to solve such an uncertainty from gravity data only. It should be emphasized however, that these zero and first order terms may correspond to the mass removal caused by the truncation procedure. Consideration of the truncation effect of zero and first order terms may be necessary when estimating a geoidal height and a deflection of the vertical from the terrestrial gravity anomaly distribution over an area within an angular distance of ψ_o, together with the satellite gravity anomalies over the remaining regions of the Earth's surface.

6. Acknowledgments

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ON THE DISCRETE BOUNDARY VALUE PROBLEM

ABSTRACT

The author made in (1963) and (1964) a re-definition of the principal problem of physical geodesy. The solution of this problem gives all quantities needed in modern geodesy as disturbance potentials, geoidal heights, vertical deflection and predicted values of the gravity anomaly. The solutions are valid for points in space and at the surface of the Earth. Various numerical solutions of this problem have been presented and this paper gives a summary of the present situation with references to the work of PICK (1965), FORSTNER (1966), MORITZ (1964), (1970), (1972), (1972a), REIT (1967), BJERHAMMAR (1968b), (1969), HOTINE (1969), KRARUP (1969), GROTEN (1970), BARLIK (1970) and HEIT & TSCHERNING (1971).

1. Formulation of the Problem

In classical geodesy, the principal problem was considered as one where gravity was given all over the physical surface of the Earth. This continuous approach was first treated in a more general way by MOLODENSKY (1948) and the general problem is normally called the problem of Molodensky.

For most geodetic applications we only know gravity at discrete points and the boundary value problem was later defined in the following way:

> A finite number of gravity data is given for a non-spherical surface, and it is required to find such a solution that the boundary values for the gravity data are satisfied in all given points (BJERHAMMAR 1963; BJERHAMMAR 1964). (*Collocation* in the classical mathematical sense).

2. Solution

An internal sphere was introduced for the solution of the previous problem. All boundary values of the physical surface were reduced down to this internal sphere using an integral equation for harmonic functions. See BJERHAMMAR (1963; 1968b), REIT (1967), KRARUP (1969), MORITZ (1970), TSCHERNING (1971) and LAURITZEN (1973).

This new approach was first received with considerable scepticism. For example PICK (1965) questioned the general stability of the solution. J.J. Levallois also expressed his opinion that he disagrees with the general approach and he has also given a "proof" for the impossibility of downward continuation. MORITZ (1964) was the first scientist who gave the new procedure full support and the author is grateful for many valuable discussions concerning this subject. MORITZ (1970) writes :

> "Solution of the Bjerhammar's Problem
>
> Let the gravity anomalies Δg_1, Δg_2 ..... Δg_n at n points on the Earth's surface, at elevations h_1, h_2, ... h_n, be known from observations; to determine a gravity field consistent with these observations.
>
> This formulation of the main problem of gravimetric geodesy, due to Bjerhammar is indeed more realistic than the usual formulation in terms of a boundary-value problem, since we observe at discrete points only. The solution of

> this problem is not unique. It may therefore be restricted by the additional requirement that the computed gravity field be as smooth as possible. This requirement makes sense because then there will be no spurious irregularities that do not correspond to reality.
>
> In a way, the non-uniqueness of the solution of Bjerhammar's problem corresponds to the non-uniqueness of the problem of interpolation. Thus we may expect that optimum interpolation - in the sense of least standard error, will correspond to an optimum solution of Bjerhammar's problem - in the sense of an optimally smooth gravity field. The remark at the end of section 3 indicates the possibility for such a correspondence to exist, and the solution to be described now will bear this out.
>
> In the solution we shall use another idea of Bjerhammar, analytical continuation of the free air anomaly Δg down to sea level. For the usual case, that the measured gravity anomalies are assumed to be known at every point of the Earth's surface, this method has been described in (HEISKANEN & MORITZ 1967,section 8-10)"

HOTINE (1969) made a detailed presentation of the method and also gave his support for the new technique. In these new early descriptions of the method, there was always included a direct computation of the reduced gravity anomaly Δg^* at the internal sphere. The numerical presentation of the reduced Δg was given with the use of power series or grids having constant gravity inside each surface element. In order to test the validity of the method, several test models were analyzed. A numerical model with several thousand tons of mass in one single point outside the internal sphere was properly evaluated without any serious difficulties (BJERHAMMAR 1969). The accuracy of the solution was improved with increasing power of the applied polynomial giving a bias smaller than the rounding errors. For this giant mass concentration in one single point outside the sphere, all vertical deflections had errors less than 0".3. The study showed that the final results improved considerably when adding a correction for residuals at the observation points, using the classical formulas for the computation of the vertical deflections. In practical application we can consider all external mass to be located in a very large number of points with, for example, one million times smaller mass. For each such point, we can make a corresponding gravity reduction and the residual errors will be less than 1 part in 10^6 for each point. The individual errors will be positive as well as negative and the total error will have a tendency to cancel. It can easily be proved that the errors cancel exactly when the external mass can be "focused" into points below the sphere.

KRARUP (1969) was the first mathematician who studied this technique in a more advanced way and he stated (IBID,p.9)

> "As far as I can see, the most important point of view introduced in physical geodesy since the appearance of Molodensky's famous articles, is Bjerhammar's idea of calculating an approximation of the potential by collocation, at the points where gravity anomalies have been measured, of potentials that are regular down to a certain sphere inside the surface of the Earth.
>
> From the classical theory this idea looks very venturesome, because we know that the actual potential of the Earth is not regular down to the Bjerhammar sphere. Nor does the evidence Bjerhammar produces in support of his idea seem convincing to me at all. As we shall see later on from the point of view of the new potential theory, very much can be said in favour of the determination of the potential by interpolation methods as well as the use of potentials that are regular outside the Bjerhammar sphere.
>
> There is a very close connection between the problem of the Bjerhammar sphere and the problem of the convergence of series in spherical harmonics or better the question of approximation of potentials by series in spherical harmonics, and here I believe we have reached the very core of the foundation of physical geodesy."

In studying this new technique, Krarup makes use of the following theorem which he calls the Runge theorem (IBID,p.54) :

> " Given any potential regular outside the surface of the Earth and any sphere in

the interior of the Earth. For every closed surface surrounding the Earth (which may be arbitrarily near the surface of the Earth) there exists a sequence of potentials regular in the whole space outside the given sphere and uniformly converging to the given potential on and outside the given surface.

This theorem is extremely important. In fact it permits a mathematical treatment of physical geodesy .....".

In the Appendix of his book, Krarup gives the following proof.

"Proof of Runge's theorem.
I want to prove that any potential in an open bounded region Ω can be approximated by potentials regular in an open sphere Σ containing Ω in its interior. The region Ω is supposed to be bounded by a surface ω which is sufficiently regular, i.e., by having finite curvature all over. This condition could be weakened very much, but I do not think it would be of interest in this connection. It is, however important that $\Sigma - \Omega$ is connected. "

It shouldn't perhaps be necessary to deal further with these questions, but Krarup's contribution has gained exceptional interest among geodesists. Furthermore, this very important work is based upon advanced studies with the use of functional analysis, and most geodesists will probably have difficulties when reading it. We will therefore try to explain some of the operations and also give some additional comments.

First we note that the Runge theorem as defined on page 54 of Krarup's publication is not quite identical with that which he wants to prove in the Appendix. It is, for example, added that $\Sigma - \Omega$ should be connected. There is no explanation for this restriction and we simply note that the Runge theorem is valid for harmonic polynomials if and only if the complement $(C\Omega)$ is connected. However, Krarup instead uses the notation $\Sigma - \Omega$. We also note that Krarup has not anticipated any harmonic polynomial and therefore his restriction is not mandatory.

It is of interest to note that Krarup on page 54, quite correctly restricts his Runge theorem with the use of a closed surface surrounding the Earth "which can come arbitrarily near the surface of the Earth." This restriction is essential for the Runge theorem and it must be required that the given potentials are harmonic in the neighbourhood of this region Ω. This means we cannot come down all the way to the actual surface of the Earth when using this theorem. The potentials could very well misbehave in the space between our region Ω and the physical surface of the Earth. This we have to remember in our further analysis. Krarup uses two lemmas for this proof and we quote:

" Lemma 1. For every function f continuous in $\Omega+\omega$ and 0 on ω we have that from

$$\int_{\Omega} f \cdot \psi \, d\Omega = 0 ,$$

for all potentials ψ regular in Σ follows

$$\int_{\Omega} f \cdot \phi \, d\Omega = 0,$$

for all potentials ϕ regular in Ω."

In the proof, Krarup defines a Hilbert space H consisting of functions, not necessarily potentials, f defined in Ω so that

$$\int_{\Omega} f^2 d\Omega \quad \text{is finite.}$$

Then Krarup writes

" Since functions of the type f in Lemma 1 are dense in H we have Lemma 2. In the Hilbert space H any element orthogonal to every potential ψ regular in Σ and restricted to Ω is orthogonal to every element ϕ of H that is a regular potential in Ω. "

However if lemma 1 goes to lemma 2 for a class of f, dense in H, then this is not true for all f in H, and this is unfortunately necessary for the proof used by Krarup. (The theorem can be proved using the technique outlined below.)

The next step could be a re-formulation of the Krarup proof, but it seems more natural to look into the basic definition of our problem. We easily see that if we want a strict proof for the geodetic approach, then we cannot accept the Runge theorem because the potential can very well misbehave between the compact set and the physical surface. However, the Runge theorem is a step in the correct direction, but it is not the complete answer. Fortunately we can use a well known theorem for potential theory by *Keldych* which we give in a geodetic version (see KELDYCH & LAVRENTIEFF 1937).

> *Let E be a compact set with a connected complement CE. Then every function continuous on the boundary E* can be uniformly approximated on E* with harmonic polynomials if and only if there are no unstable points at the boundary (i.e., there are no points on E* where the complement CE is thin).***

For the geodetic use of this theorem we put a sphere in the interior of the Earth and make an inversion of our model. The infinity point now comes in the interior of the sphere.

A proof can be outlined in the following way. We note that non-stable points cannot be approached from the complement by a cone. The stable points are dense in the boundary. The space of functions continuous on E* is denoted C and the space of measures μ on E* with the potential in the open complement of E equal to zero is denoted M. F is the linear subspace of C generated by elementary harmonic functions $\Phi_n^a(x)$

$$\Phi_n^a(x) = H_n(x-a) \cdot |x-a|^{-(2n+1)}, \quad n \geq 0,$$

where $H_n(x)$ is a harmonic polynomial of order n, a is the given point and x the actual point. Here F is dense in C (and thus uniform approximation is possible) if and only if zero is the only measure orthogonal to F. This is obvious from the well-known Hahn-Banach theorem. (The measures μ are orthogonal to F if and only if

$$\int \Phi_n^a(x)\, d\mu = 0 \qquad \text{for all } n.)$$

We have a connected complement CE and the measure μ is orthogonal to F if and only if it is in M. Furthermore M contains only the zero measure if and only if there are no unstable points on E* (#)

This problem has been carefully studied by several authors and we make reference to the work of KELDYCH & LAVRENTIEFF(1937), DENY (1939) and LANDKOF (1972). (After small modifications we also prove the Krarup-Runge theorem.)

This very general theorem can be called the *central theorem of physical geodesy*. As a special case, we obtain the Runge theorem which we re-formulate in the following way.

** Furthermore we also note that the potential is determined all over in the space when it is defined as a continuous function over the surface of the Earth.

(#)Note that the theorem is not restricted to uniform approximations with the use of a sphere.

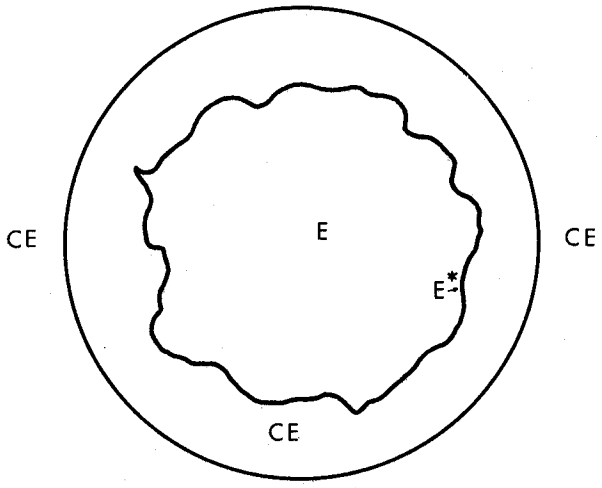

Figure 1. The Compact Set

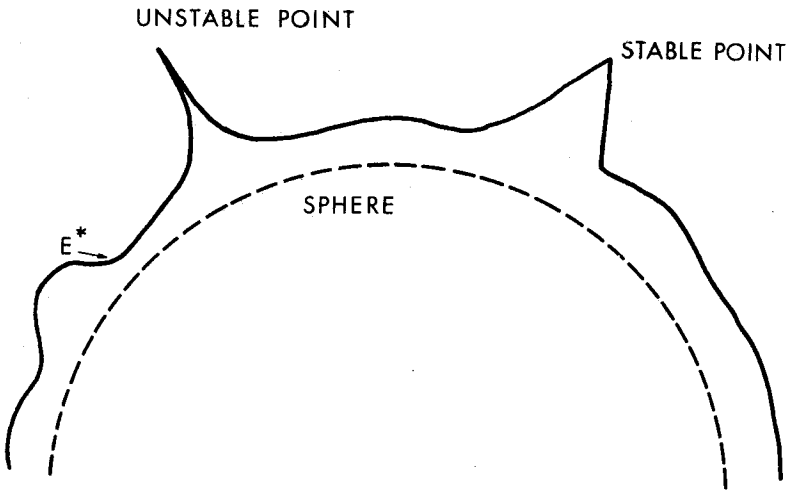

Figure 2. After Inversion

Let the compact E have the boundary E*. Then every function harmonic on E and in the neighbourhood of E U E* can be uniformly approximated on E. The approximating functions can be harmonic polynomials if and only if the complement of E is connected.

We note that the Runge theorem includes no restrictions on the surface. The Keldych theorem is more general but includes restrictions on the surface. These restrictions are without practical importance for geodetic application.†

We now go back to our original formulation of the boundary value problem (BJERHAMMAR 1963). It is obvious that Krarup didn't accept our discrete approach without using the Runge theorem of uniform approximation. A uniform approximation is of course no necessary tool for the acceptance of a discrete solution that fits in all given points. We are fully entitled to use any solution that fits all given points. We will always have sufficient harmonics for a fit of this type if we restrict the study to geodetic applications. The next question will be if our selected estimator will be better if the Keldych theorem is satisfied. Still we cannot give a limit for the maximum error of our selected estimate, but we know that the errors of some selected integral norm can be arbitrarily good. By increasing the number of observations, we can perhaps approach this limiting value.

The theorems of Keldych have a limited impact upon our discrete approach because it postulates a continuous approach. We have only observed a small subset of the original domain of boundary values and an infinite number of degenerations will be included in our solutions. In the practical applications, we have perhaps upto one million observations but we cannot handle more than a few thousand in a simultaneous computer run. In such situations we will find it useful to give solutions that minimize some additional norms for the residuals. *The contribution of Krarup has been extremely rewarding for the study of all these problems.*

LAURITZEN (1973) writes:

> "We shall not consider these difficulties in detail, but a way to overcome them (KRARUP 1969) is to search for the potential among functions, regular and harmonic, outside a sphere which is contained in the Earth's interior, the so called *Bjerhammar sphere*. Of course, this does not solve the problem, but because of the very important theorem of Runge (see IBID) we can in this way find an approximation to the potential which in some sense is arbitrarily good. "

The Runge theorem plays an important role in the present theory but to some extent it seems to be misinterpreted and we make the following remarks:

1. The errors in our geodetic approach cannot be made arbitrarily small because the effect of the Earth tide (sun and moon) is not negligible.
2. If observations are only made at discrete points, then the Runge theorem says nothing about the correctness of the predictions for intermediate points. (Probably uniform approximation will require that the measured points are uniformly distributed.)**

Continuous approach: A completely unbiased solution does not exist in the general case. However, the bias can be made arbitrarily small with the use of uniform approximation.

Discrete approach: An infinite number of unbiased estimators can be found. The unbiasedness is normally restricted to linear functions of the observations and cannot generally be extended to the continuous case.

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† Note that already the Dirichlet problem will have no solution when unstable points are included. Therefore the Runge-Krarup approach suffers from the same limitations with respect to the surface.

** In a private note, L. Hörmander has verified the uniform convergence for the two-dimensional case when using equidistant spacing.

3. Observation Errors

Our uniform approximation is only possible if no observation errors are included. However, in all physical models we find observation errors. For various reasons we cannot avoid observation errors ε of the following two categories:

1. $E\{\varepsilon_1\} = 0$ (accidental) $\varepsilon_1 \in \varepsilon$
2. $E\{\varepsilon_2\} \neq 0$ (systematic) $\varepsilon_2 \in \varepsilon$

We will also recognize combinations of these errors. It is obvious that the ε_2-type of errors will give an additional bias in our final answer; for example, calibration errors. Our solution will of course be influenced by the model bias ε_3 which however, can be made arbitrarily small by uniform approximation. We have

3. $\varepsilon_3 \geq 0$ (systematic).

This means that we always will be exposed to systematic errors. It can therefore be justified to accept a more simple physical explanation for our approach.

4. Numerical Solution

The following integral equation was used for the numerical solution of the gravity reduction to an internal sphere (BJERHAMMAR 1962; BJERHAMMAR 1963; BJERHAMMAR 1964).

$$\Delta g_j = \frac{r_j^2 - r_o^2}{4\pi r_j} \iint \frac{\Delta g^*}{r_{ji}^3} \, ds \qquad (1),$$

or in matrices

$$\Delta g = K \, \Delta g^*.$$

Any element k of the matrix K is obtained with the use of the formula

$$K_{ji} = \sum_{n=0}^{\infty} (2n + 1) \left(\frac{r_o}{r_j} \right)^{n+2} P_n(\cos \omega_{ji})$$

or

$$K_{ji} = \frac{(r_j^2 - r_o^2) r_o^2}{r_{ji}^3 r_j} .$$

where ds = surface element of the sphere, Δg_j = gravity anomaly at the fixed point,
Δg^* = gravity anomaly at the internal sphere, r_o = radius of sphere,
r_{ji} = distance between the fixed point at the surface of the Earth (or in space) P_j and the moving point on the internal sphere,
r_j = geocentric distance to the actual point, and
r_i = distance between the centre of the Earth and the intersection of the surface of the Earth and a radius vector through the moving point.

4.1 Iterative Approach

In the first approach the measured gravity anomalies were used as approximate Δg^* values, and the residuals were used for corrections

$$\Delta g^* = \Delta g_o + \left\{ \Delta g_o - \frac{r_j^2 - r_o^2}{4\pi r_j} \iint \frac{\Delta g}{r_{ji}^3} \, ds \right\} .$$

For further details see (BJERHAMMAR 1962; BJERHAMMAR 1963; BJERHAMMAR 1969). MORITZ (1964) presented a "trick" (also given by BJERHAMMAR (1964))

$$\Delta g^* = \Delta g_o + \frac{r_j^2 - r_o^2}{4\pi r_j} \iint \frac{\Delta g_o - \Delta g}{r_{ji}^3} \, ds .$$

It is obvious that this solution is only an approximation of the previous solution above. For large values of $(r_j - r_o)$ this type of solution gives a considerable loss of accuracy. A number of different iterations are described in (BJERHAMMAR 1969).

4.2 Power Series (Least Squares Solution)

See also FORSTNER (1966), REIT (1967) and GROTEN (1970).

$$\Delta g^* = \Delta g + c_1 h + c_2 h^2 + c_3 h^3 + \ldots\ldots + c_n h^n \qquad (2),$$

where h is the height above the sphere.

4.3 Spherical Harmonic Solution

$$\Delta g = \frac{GM}{r_j} \sum_{n=2}^{\infty} \left(\frac{r_o}{r_j} \right)^n (n-1) \sum_{m=0}^{n} P_{nm}(\sin \phi) \left(C_{nm} \cos m\lambda + S_{nm} \sin m\lambda \right) \qquad (3),$$

where

G = Newtonian constant and M is the mass of the Earth

4.4 Matrix Solution (non-singular case)

$$\Delta g = K \, \Delta g^*, \quad \text{with} \qquad \Delta g^* = K^{-1} \Delta g \qquad (4).$$

4.41 Matrix Solution with Singular K-matrix

$$(\Delta g - K \Delta g^*)^T (\Delta g - K \Delta g^*) = \min \; ; \; \Delta g^* = (K^T K)^{-1} K^T \Delta g = K_{01}^{-1} \Delta g$$

See(BJERHAMMAR 1964).

4.42

$$(\Delta g^*)^T (\Delta g^*) = \min \; ; \qquad \Delta g^* = K^T (K K^T)^{-1} \Delta g = K_{10}^{-1} \Delta g$$

(BJERHAMMAR 1968a)

4.43

$$(\Delta g - \hat{\Delta g})^T (\Delta g - \hat{\Delta g}) = \min, \quad \text{where} \qquad \hat{\Delta g} = K \, \Delta g^*,$$

with the constraint

$$(\Delta g^*)^T \Delta g^* = \min \quad ; \quad \Delta g^* = K^T(KK^T)^{-1}K(K^TK)^{-1}K^T \Delta g = K_{11}^{-1} \Delta g \quad \text{(IBID)}.$$

These three solutions are very general and can be used with *weights* in the classical way.

4.44

With an infinite dimensional matrix K(∞×n) we transcribe 4.42 for a finite number of observations

$$\frac{1}{4\pi} \iint (\Delta g^*)\, d\sigma = \min ,$$

where σ is the unit sphere. The elements of KK^T are here obtained after integration over the sphere. An arbitrary element is then for the points P_j and P_i

$$(KK^T)_{ji} = \frac{1}{4\pi} \iint \sum_{n=0}^{\infty} (2n+1) \left(\frac{r_o}{r_j}\right)^{n+2} P_n(\cos \omega_j) \sum_{\underline{n}=0}^{\infty} (2\underline{n}+1) \left(\frac{r_o}{r_i}\right)^{\underline{n}+2} P_{\underline{n}}(\cos \omega_i)\, d\sigma.$$

After an expansion in spherical harmonics, we obtain (also in 4.46:5)

$$(KK^T)_{ji} = \sum_{n=0}^{\infty} (2n+1) \left(\frac{r_o^2}{r_j r_i}\right)^{n+2} P_n(\cos \omega_{ji}) \; ; \quad \text{thus } K_{10}^{-1} \text{ is also defined.}$$

4.45

We replace Δg all over by T in equation 1. Then we have the true Poisson formula

$$T_j = \frac{r_j^2 - r_o^2}{4 r_o} \iint \frac{T^*}{r_{ji}^3}\, d\sigma; \qquad K_{ji} = \sum_{n=0}^{\infty} (2n+1) \left(\frac{r_o}{r_j}\right)^{n+1} P_n(\cos \omega_{ji}),$$

K_{ji} being the elements of the K matrix.
Using the technique of 4.44, we obtain

$$\frac{1}{4\pi} \iint (T^*)^2\, d\sigma = \min; \qquad \text{or} \quad ||T^*||^2 = \min.$$

The elements of KK^T are obtained in the corresponding way (also in 4.46:1)

$$(KK^T)_{ji} = \sum_{n=0}^{\infty} (2n+1) \left(\frac{r_o^2}{r_j r_i}\right)^{n+1} P_n(\cos \omega_{ji}).$$

The solutions of 4.44 and 4.45 can also be obtained with the theory of Hilbert spaces with reproducing kernels.

4.46 Mean Values

The spherical harmonics of order n will be denoted T_n for the disturbance potential and Δg_n for the gravity anomaly. Then we have the relations

$$T_j = \sum T_n^* \left(\frac{r_o}{r_j}\right)^{n+1} \qquad ; \qquad \Delta g_j = \sum \Delta g_n^* \left(\frac{r_o}{r_j}\right)^{n+2}$$

$$T_j = \sum r_o \, \Delta g_n^* \, (n-1)^{-1} \left(\frac{r_o}{r_j} \right)^{n+1} \quad ; \quad \Delta g_j = \sum \frac{1}{r_o} \, T_n^* \, (n-1) \left(\frac{r_o}{r_j} \right)^{n+2} .$$

With the use of these basic harmonics, we compute the following mean values after an integration over the internal sphere. The integrated means are denoted $E(T_j T_i)$, $E(\Delta g_j, \Delta g_i)$ etc., when the two points in question are P_i and P_j. Furthermore we introduce the special notations

$$E(T_n^* \, T_n^*) = s_n^2 \qquad \text{and} \qquad E(\Delta g_n^* \, \Delta g_n^*) = \sigma_n^2 .$$

$$||T|| = \min$$

4.46:1 $$E(T_j T_i) = \sum s_n^2 \, (2n+1) \left(\frac{r_o^2}{r_j r_i} \right)^{n+1} P_n(\cos \omega_{ji})$$

4.46:2 $$E(\Delta g_j \Delta g_i) = \frac{1}{r_o^2} \sum s_n^2 \, (2n+1) \, (n-1)^2 \left(\frac{r_o^2}{r_j r_i} \right)^{n+2} P_n(\cos \omega_{ji})$$

4.46:3 $$E(T_j \Delta g_i) = \frac{1}{r_i} \sum s_n^2 \, (2n+1) \, (n-1) \left(\frac{r_o^2}{r_j r_i} \right)^{n+1} P_n(\cos \omega_{ji}).$$

$$||\Delta g|| = \min$$

4.46:4 $$E(T_j T_i) = r_o^2 \sum \sigma_n^2 \, (2n+1) \, (n-1)^{-2} \left(\frac{r_o^2}{r_j r_i} \right)^{n+1} P_n(\cos \omega_{ji})$$

4.46:5 $$E(\Delta g_j \Delta g_j) = \sum \sigma_n^2 \, (2n+1) \left(\frac{r_o^2}{r_i r_j} \right)^{n+2} P_n(\cos \omega_{ji})$$

4.46:6 $$E(T_j \Delta g_i) = \frac{r_o^2}{r_i} \sum \sigma_n^2 \, (2n+1) \, (n-1)^{-1} \left(\frac{r_o^2}{r_j r_i} \right)^{n+1} P_n(\cos \omega_{ji})$$

The summation can be taken from 3 to infinity for all Legendre polynomials. Order one must be excluded in two of the expressions. Order two is anyway eliminated with the use of the reference ellipsoid. In our following study, we consider σ_n^2 and s_n^2 given constants (degree variances).

The presented means have a special meaning in the theory of stochastic processes. Furthermore equation 4.46:1 can be used as a reproducing kernel in Hilbert space. Our study is simply a least squares application. (The stochastic process approach has been questioned by Lauritzen who proved that ergodicity is not prevailing. See (LAURITZEN 1973).)

4.47

For points on the sphere and outside the sphere we can estimate gravity, disturbance potentials and vertical deflections in a general system which is valid for the singular as well as the non-singular case. (Estimates for points below the physical surface have no physical meaning.)

Gravity :

$$\Delta g_j = k_g \, \Delta g_i^* = k_g \, K_g^T \, (K_g K_g^T)^{-1} \, \Delta g_i = k_{gg} K_{gg}^{-1} \, \Delta g_i$$

Disturbance Potential:

$$T_j = f_T \Delta g^* = f_T K_g^* (K_g K_g^T)^{-1} \Delta g_i = k_{Tg} K_{gg}^{-1} \Delta g_i$$

or

$$T_j = k_T T^* = k_T K_T^* (K_T K_T^T)^{-1} T_i = k_{TT} K_{TT}^{-1} T_i .$$

The individual elements are defined by

$$K_{gg} = \sum_{n=0}^{\infty} k_n (2n+1) t^{n+2} P_n(\cos \omega_{ji}) = k_{gg} \rightarrow \frac{t^2 - t^4}{\rho^3} \quad \text{for } k_n = 1 \text{ and } \rho = (1+t^2 - 2t \cos \omega_{ji})^{\frac{1}{2}}$$

$$K_{TT} = r_o^2 \sum_{n=2}^{\infty} k_n (2n+1) (n-1)^{-2} t^{n+1} P_n(\cos \omega_{ji}) = k_{TT} ; \quad \text{and}$$

$$r_j \sum_{n=2}^{\infty} k_n (2n+1) (n-1)^{-1} t^{n+2} P_n(\cos \omega_{ji}) = k_{Tg} \rightarrow r_j t^2 \left(\frac{2}{\rho} - 3\rho + 1 - 5t \cos \omega_{ji} - 3t \cos \omega_{ji} \ln \phi\right) \text{ for } k_n = 1 \text{ where } \phi = (1 + \rho - t \cos \omega_{ji})^{\frac{1}{2}}.$$

Vertical Deflections:

$$\xi_j = -\frac{1}{\gamma_j} \frac{\partial T_j}{\partial x} = -\frac{1}{\gamma_j} f_x K_{gg}^{-1} \Delta g \qquad \begin{Bmatrix} f_x \\ f_y \end{Bmatrix} \rightarrow \begin{Bmatrix} \cos \alpha_{ji} \\ \sin \alpha_{ji} \end{Bmatrix} t^3 \sin \omega_{ji} \left(\frac{2}{\rho^3} - 8 + \frac{3(\rho+1)^2}{2\rho\phi} - 3 \ln \phi\right) \text{ for } k_n = 1.$$

$$\eta_j = -\frac{1}{\gamma_j} \frac{\partial T_j}{\partial y} = -\frac{1}{\gamma_j} f_y K_{gg}^{-1} \Delta g$$

Three important types of solutions are obtained as special cases

I: $k_n = 1$ and $t = r_o/r_j$		non singular (BJERHAMMAR 1964)
II: $k_n = \sigma_n^2$ and $t = r_o^2/r_j r_i$		$\|\|\Delta g^*\\| = \min$ (cf.MORITZ 1970)
III: $k_n = s_n^2 (n-1)^2 (1/r_o^2)$ and $t = r_o^2/r_j r_i$		$\|\|T^*\\| = \min$ (cf.KRARUP 1969)

The "non-parametric solution" of BJERHAMMAR (1964) minimizes the $||\Delta g^*||$ norm with $\sigma_n^2 = (r_i/r_o)^{n+2}$. The "covariance" matrix is here asymmetric. High order harmonics are given low weights. The solution by Bjerhammar (IBID) gives the solution of MORITZ (1970) if r_j is replaced by $r_j r_i / r_o$. Furthermore this solution is identical with the solution of KRARUP (1969) because $r_o \Delta g_n^* = T_n^* (n-1)$. Krarup and Moritz excluded potentials that are not rotation invariant. Here we accept the symmetric as well as the non-symmetric cases. No probabalistic meaning of the covariances is anticipated. The Moritz solution is obtained as a limiting value from our solution 4.42 above (BJERHAMMAR 1968b) when using infinite points at the "sphere".

4.48 Antireduction

If a very large number of unknowns are used, then we cannot use the methods available now for the inversion of the corresponding matrices. For most cases, it is unnecessary to compute the inverse and the methods of Gauss or Cholesky are useful for the back

substitution. Still it will be very difficult to include all observations and therefore we use the method of antireduction in order to find the best solution. The technique is based upon the following procedure. With our discrete approach we find a solution that satisfies the boundary values at n given points. The total set of measured points includes N observations. Gravity is now computed for all N points with the use of our previous solution for n points. The residual gravity anomaly is then obtained for all these points. The final solution is then of the following type for the disturbance potential

$$T = \frac{1}{4\pi r_j} \iint \Delta g \sum_{n=2}^{\infty} (2n+1)(n-1)^{-1} \left(\frac{r_o}{r_j} \right)^n P_n(\cos \omega) ds + \frac{1}{4\pi r_j} \iint \Delta G \sum_{n=2}^{\infty} (2n+1)(n-1)^{-1} P_n(\cos \omega) ds,$$

where

ΔG = residual gravity anomaly.

If still higher accuracy is wanted, then we continue with iterative methods. Molodensky's technique will probably be useful.

The choice between a least squares solution (or stochastic processes) and the non-singular approach is not critical when antireduction is included.

Finally it should be noted that our non-singular solution in all cases gives a better condition number than the corresponding solution with rotation invariant potentials. However, only the least squares types of solution have a quite symmetrical K_{gg}-matrix. A somewhat simpler computational technique can be used in the symmetrical case. This computational gain is obtained at the cost of a general loss of the non-stationary parts of the solution.

4.5 Filtering

Our solution with generalized inverses is re-written

$$\Delta g^* = QK^T(KQK^T)^{-1} K(K^T R^{-1} K)^{-1} K^T R^{-1} \Delta g = \lim_{\delta \to 0} QK^T(KQK^T + \delta R)^{-1} \Delta g,$$

with

$$(\Delta g - k \Delta g^*)^T R^{-1} (\Delta g - K \Delta g^*) = \min \qquad \text{and} \qquad (\Delta g^*)^T Q^{-1} \Delta g^* = \min.$$

This solution gives an *unbiased* estimate of Δg^* for any Δg.

If we restrict our estimate to observations with $E\{\Delta g\} = 0$ then we have an *unbiased* estimate also for $\delta \neq 0$ and we can write

$$\Delta g^* = QK^T(KQK^T + R)^{-1} \overline{\Delta g} \qquad \text{for} \qquad E\{\overline{\Delta g}\} = 0.$$

Here KQK^T is positive definite. We can now use this solution for a prediction of Δg anywhere at the surface of the Earth or in space. The final prediction $\hat{x}$ is then (for $Q = I$)

$$\hat{x} = u \Delta g^* = u K^T(KK^T + R)^{-1} \overline{\Delta g}.$$

This expression can be re-written with the obvious notation

$$\boxed{\hat{x} = k(\overline{K} + R)^{-1}\,\overline{\Delta g}} \qquad \text{where} \qquad \overline{\Delta g} = \Delta g - E\{\Delta g\}\ .$$

It is easily verified that this type of solution satisfies the condition

$$(\Delta g^*)^T\ \Delta g^* + (\Delta g - K\,\Delta g^*)^T\ R^{-1}\ (\Delta g - K\,\Delta g^*) = \min.$$

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7. Discussion*

QURESHI: You are inverting upward continuation. Why don't you call this downward continuation?

MORITZ: It is a downward continuation. Downward continuation cannot be done in an elementary way. So the obvious way is to use the integral approach, via upward continuation.

DOOLEY: Referring to this theorem; that we modify the gravity field by a small amount and get an analytical solution at any depth. This is in conflict with what we have always been taught that if you have a mass at a certain depth - a point mass - it gives a very sharp anomaly and this simply cannot be continued any further without divergence.

MORITZ: This is the familiar case of a mass point causing singularity on analytical continuation. But it is still true that by modifying the external gravity field by a small amount, one can continue it downward. The downward continuation will be rather irregular, but it will exist. It will be something like approximating a delta function by an analytical function.

* Paper presented by H. MORITZ.

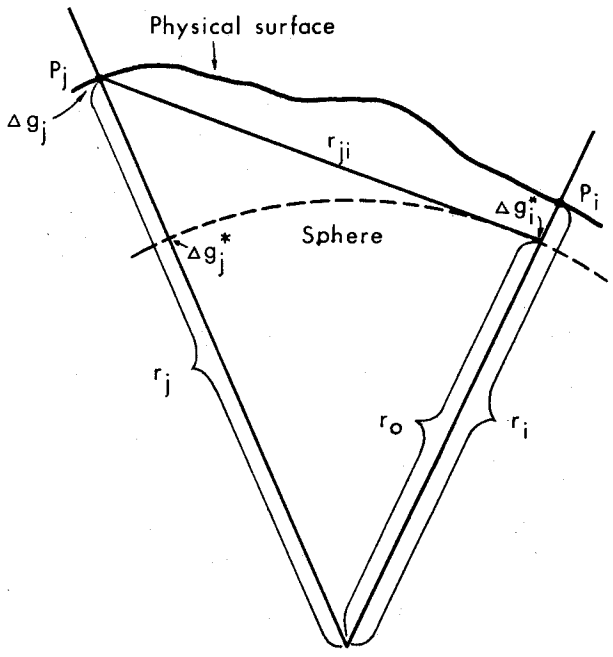

Figure 3. The Physical Surface and the Sphere

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A COMPARISON OF ESTIMATION METHODS FOR THE REDUCTION OF LASER RANGE OBSERVATIONS **

ABSTRACT

The accuracy, stability and convergence characteristics of the classical batch estimation algorithm is compared with the characteristics of the sequential estimation algorithm. The relationship of each of these methods to the least squares collocation algorithm is described. The accuracy of the estimate obtained with each of these methods is compared by examining the converged state estimate and the observations from the Beacon Explorer-C satellite. It is concluded that, for the same estimation accuracy, the rectified sequential estimation algorithm has a faster convergence and a larger initial radius of convergence. Furthermore the results indicate that it can be used to study the evolution of the estimates of constant parameters such as tracking station co-ordinates.

1. Introduction

In recent years, the accuracy of the observations of near-Earth satellites has undergone dramatic improvement. For instance, laser ranging systems currently measure the range between a tracking station and an orbiting satellite to within 50 cm. It is expected that such accuracy will improve to within 5 to 10 cm. Satellite observations with this precision are of interest in the study of various phenomena such as the determination of the shape and physical composition of the Earth as well as studies of solid Earth tides, tectonic plate motion, and polar motion. However, to employ observations of such accuracy in the study of these phenomena, estimates of the position and velocity of the satellite must be obtained to an accuracy commensurate with the observation precision.

There are two fundamental factors which limit the accuracy with which the orbit of a satellite can be determined and predicted. These two factors are:

1. The computational procedure used to reduce the observations and to predict the future motion of the satellite; and
2. The accuracy with which the mathematical model used to describe the satellite's motion is known.

There are two basic procedures for processing satellite observations to obtain an estimate of the orbit of a space vehicle. The methods are referred to as batch processors and sequential processors. The sequential orbit determination procedure processes each observation at the time point at which it is received and an estimate of the state is obtained at that time. On the other hand, the batch processor , which obtains the estimate at some reference epoch, requires that the entire sequence of observations be processed before the estimate of the state can be made. Both batch and sequential estimation algorithms are founded on the assumption that the non-linear equations of motion and observation-state equations can be linearized with reference to some a priori solution. Then the deviation from the reference solution can be determined by the application of linear estimation

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techniques.

Errors of four basic types influence the accuracy of linear estimators:

1. errors due to linearization assumptions;
2. errors introduced in the computational procedures;
3. errors which occur in the observation process; and
4. errors due to inaccuracies in the mathematical model used to describe the dynamical process.

While it is anticipated that the last two error sources will influence both the batch and the sequential estimation algorithms in the same manner, errors due to the first effects may lead to differences in the estimate obtained by these algorithms, especially during the initial iterations.

The least-squares batch estimation algorithms are widely used in most geodetic work and in the majority of the real-time operational orbit determination programs. There have been few applications of sequential orbit determination algorithms to real time data reduction and even fewer detailed numerical comparisons between the characteristics of the batch type algorithms and the sequential algorithms in their applications to the reduction of real tracking data. While the formal mathematical equivalence between the sequential estimation algorithm and the batch estimation algorithm can be shown, the extended sequential estimation algorithm which rectifies the reference trajectory at each observation point, will yield the same numerical estimate as the batch algorithm only if the solution is iterated to convergence. Furthermore, the well known problem of "divergence" of the sequential estimate in the presence of a small state error covariance matrix has led to the convention of propagating the state error covariance matrix in a state noise compensated mode. The observation residuals obtained with this algorithm are significantly different from the residuals obtained with the batch algorithms, and this fact has lead to some reluctance to use the sequential estimation algorithm for many orbit determination applications.

In the following discussion, the derivation of both the batch and the sequential estimation algorithms are reviewed, and their computational characteristics, as determined by application to the problem of reducing precise laser ranging observations to a near-Earth satellite, are compared. The question of the nature of the convergence as well as the radius of convergence for each algorithm is also considered. The number of iterations, a pertinent quantity in determining the computing time required for convergence by both methods, is also considered.

The mathematical foundations for the batch estimation algorithms are reviewed by TAPLEY (1973). The derivations for the sequential estimation algorithm reside in the classic works of KALMAN (1960), and KALMAN & BUCY (1961). Additional references on estimation theory, which discuss both batch and sequential methods, include BRYSON & HO (1969), DEUTSCH (1965), JAZWINSKI (1970) and LIEBELT (1967).

2. Problem Formulation

The equations which govern the motion of a satellite in orbit about a central body can be expressed as

$$\dot{\vec{r}} = \vec{v}, \qquad \dot{\vec{v}} = -\frac{\mu \vec{r}}{r^3} + \vec{R}(\vec{r},\vec{v},\vec{\alpha},t) \tag{2.1},$$

where $\vec{r}$ is the position vector measured from the mass centre, $\vec{v}$ is the velocity vector, μ the gravitational parameter, and $\vec{R}$ is a vector of perturbations which depend on $\vec{r}$, $\vec{v}$, and the set of model

parameters $\vec{\alpha}$. For example, $\vec{R}$ may be due to the effects of atmospheric drag, the non-central components of the central body gravitational force, the gravitational attractions of other bodies, etc. The parameters $\vec{\alpha}$ represent any unknown parameters whose value must be estimated during the estimation procedure, e.g., the mass of a third body, the spherical harmonic coefficients, $C_{\ell m}$ and $S_{\ell m}$, for the central body gravitational field, etc. Equations 2.1 can be expressed in first-order form as follows

$$\dot{\eta} = f(\eta,\alpha,t) \qquad \dot{\alpha} = 0 \tag{2.2}$$

where the six-dimensional vectors η and $f(\eta,\alpha,t)$ are defined as follows:

$$\eta^T = [\vec{r}^T : \vec{v}^T], \quad f^T(\eta,\alpha,t) = [\vec{v}^T : (-\mu\vec{r}/r^3 + \vec{R})^T] \tag{2.3}$$

The state vector X consists of all time dependent variables or constant parameters required to determine uniquely the evolution of the satellite's motion. With this definition, the n-dimensional state vector X can be expressed as follows:

$$X^T = [\eta^T : \alpha^T] \tag{2.4}$$

and the state equations become

$$\dot{X} = F(X,t), \qquad X(t_o) = X_o \tag{2.5}$$

where $F^T(X,t) = [f^T(\eta,\alpha,t) : 0]$. With X_o specified, the evolution of the dynamic system will be determined uniquely by the solution to equation 2.5. In the usual orbit determination problem, X_o will not be known perfectly and observations of the motion must be made to determine the best estimate of X_o.

Usually the state vector cannot be observed directly. Instead, the observation will be a non-linear function of the state. Since the observations are usually influenced by random observation error, the observation-state relationship can be expressed as

$$Y_i = G(X_i,t_i) + \varepsilon_i \tag{2.6}$$

where Y_i is a p-vector of observations of the state X_i at the epoch t_i, $G(X_i,t_i)$ is a non-linear function relating the state and the observations, and ε_i is a p-vector of observation errors.

Examination of equations 2.5 and 2.6 indicate that the basic equations of interest are non-linear. To apply linear estimation theory, equations 2.5 and 2.6 must be replaced by an equivalent set of linear equations. To accomplish this, a Taylor series expansion about some reference trajectory at each point in the time interval of interest can be used. If the definitions

$$\begin{aligned} x(t) &= X(t) - X^*(t), \qquad t_o \leq t \leq t_\ell \\ \text{and} \\ y_i &= Y_i - G(X_i^*,t_i), \qquad i = 1, \ldots\ldots \ell \end{aligned} \tag{2.7}$$

are used, where $X^*(t)$ indicates the reference trajectory, then the deviation from the reference trajectory and the observation associated with this deviation can be expressed as

$$\dot{x}(t) = A(t)\,x(t), \qquad x(t_o) = x_o, \qquad t_o \leq t \leq t_\ell$$

and (2.8),

$$y_i = \tilde{H}_i x_i + \varepsilon_i , \qquad i=1, \ldots\ldots \ell$$

where terms of $O\{(x_i - x_i^*)^2\}$ are neglected, and where $A(t) = [\partial F/\partial X]^*$ and $\tilde{H}_i = [(\partial G/\partial X)]^*$. The symbol $[\ ]^*$ indicates that the quantity in the brackets is evaluated on the reference solution.

By noting that the first of equations 2.8 has the solution

$$x_i = \Phi(t_i,t_k)x_k \tag{2.9}$$

where $\dot{\Phi}(t_i,t_k) = A(t)\,\Phi(t_i,t_k)$, $\Phi(t_k,t_k) = I$, the observation sequence y_i, $(i=1,\ldots,\ell)$, can be reduced to a single epoch. If

$$y = \begin{pmatrix} y_1 \\ y_2 \\ . \\ . \\ y_\ell \end{pmatrix}, \qquad H = \begin{pmatrix} \tilde{H}\,\Phi(t_1,t_k) \\ \tilde{H}_2\Phi(t_2,t_k) \\ . \\ . \\ \tilde{H}_\ell\Phi(t_\ell,t_k) \end{pmatrix}, \qquad \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ . \\ . \\ \varepsilon_k \end{pmatrix},$$

then the observation sequence can be represented as

$$y = H x_k + \varepsilon \tag{2.10}$$

The m-vector y, where $m = p \times \ell$, will represent the entire set of observations used in the estimate. Generally, $m > n$, but if a few observations are to be combined with an a priori estimate, $m \leq n$ is permissible. The observation error ε_i is assumed to satisfy the a priori statistics:

$$E\{\varepsilon_i\} = 0, \qquad E\{\varepsilon_i\varepsilon_j^T\} = R_i\,\delta_{ij} \tag{2.11}$$

where δ_{ij} is the Kronecker delta and where R_i is a $p \times p$ positive definite matrix.

Given an estimate of the state $\hat{x}_j = E\{x_i|y_i, \ldots, y_j\}$, and the associated state error covariance matrix $P_j = E\{(x_j-\hat{x}_j)(x_j-\hat{x}_j)^T|y_1, \ldots, y_j\}$, the estimate and the covariance matrix at a future time t_k is given by the expressions

$$\bar{x}_k = \Phi(t_k,t_j)\,\hat{x}_j, \qquad \bar{P}_k = \Phi(t_k,t_j)\,P_j\,\Phi^T(t_k,t_j) \tag{2.12}$$

The predicted estimate of the state at the time, t_k, can be treated as an observation whose associated error is $\eta_k = x_k - \bar{x}_k$. It is a straightforward procedure to show that, for an unbiased estimate $\hat{X}_j$,

$$E\{\eta_k\} = 0 \quad , \quad E\{\eta_k\eta_k^T\} = \bar{P}_k \tag{2.13}$$

Hence, combining equation 2.12 with equation 2.10, the following observations and a priori estimates of the state at a time t_k are available:

$$y_k = H_k x_k + \varepsilon_k \quad ; \qquad \bar{x}_k = x_k + \eta_k \tag{2.14}$$

If the state vector x_k is an n-vector of constant parameters, then the state transition matrix will

reduce to the identity matrix for all values of time, i.e., $\Phi(t_i,t_k) = I$. Equations 2.14 can be used to combine a priori estimates x_k with new observations y_k where the observation errors are assumed to satisfy equations 2.11 and 2.13.

3. Minimum Variance Estimate

If the best linear, unbiased, minimum variance estimate of x_k is sought, given equations 2.11, 2.13 and 2.14, the estimate will be determined by the following expression

$$\hat{x}_k = \left(H_k^T R_k^{-1} H_k + \bar{P}_k^{-1}\right)^{-1} \left(H_k R_k^{-1} y_k + \bar{P}_k^{-1} \bar{x}_k\right) \tag{3.1}$$

In equation 3.1, the vector y_k may be only a single observation or it may include an entire batch of observations. The a priori estimate $\bar{x}_k$ may represent the estimate of the initial conditions propagated to the time t_k, or it may be based on the reduction of a previous batch of data. The computation of the estimate $\hat{x}_k$ requires the inversion of an $n \times n$ matrix and if the dimension n is large, this inversion can lead to computational difficulties.

If the a priori estimate is zero, i.e., if $\bar{x}_k = 0$, then equation 3.1 reduces to

$$\hat{x}_k = \left(H_k^T R_k^{-1} H_k + \bar{P}_k^{-1}\right)^{-1} H_k^T R_k^{-1} y_k \tag{3.2}$$

Note that equation 3.2 still retains the covariance matrix $\bar{P}_k$ associated with the estimate $\bar{x}_k = 0$ and the estimate $\hat{x}_k$ will be different than the estimate obtained if no a priori information is available. The no a priori information case is characterized by the limit as $\bar{P}_k \to \infty$. For this case, equation 3.1 reduces to

$$\hat{x}_k = \left(H_k^T R_k^{-1} H_k\right)^{-1} H_k^T R_k^{-1} y_k \tag{3.3}$$

Note that if the observation noise covariance matrix R_k is replaced by an observation weighting matrix W_k, where $W_k = R_k^{-1}$, the expressions in equations 3.1, 3.2 and 3.3 give the weighted least squares estimation algorithms. Furthermore, if the observation errors and the a priori values for the initial conditions satisfy Gaussian distributions, then equation 3.1 is also the maximum likelihood estimate.

The primary differences between the different estimation algorithms lie in the nature of the a priori information. In this regard, the least squares collocation algorithm described by MORITZ (1970) can be shown to be equivalent to the algorithm given by equation 3.2, i.e., the minimum variance estimate for the case $\bar{x}_k = 0$. This equivalence is discussed in a subsequent paragraph.

In describing the computational algorithm for implementing equation 3.1, it is convenient to define the following matrix expressions

$$L = H^T R^{-1} H + \bar{P}_o^{-1} = \sum_{i=1}^{\ell} H_i^T R_i^{-1} H_i + \bar{P}_o^{-1}$$

and

$$M = H^T R^{-1} y = \sum_{i=1}^{\ell} H_i^T R_i^{-1} y_i + \bar{P}_o^{-1} \bar{x}_o \tag{3.4}$$

where $H_i = \tilde{H}_i\ \Phi(t_i,t_o)$. With these definitions, equation 3.1 becomes

$$\hat{x}_o = L^{-1}\ M \tag{3.5}$$

Then the procedure for computing the batch estimate of x can be summarized as follows:

Given: X^*_o, $\bar{x}_o$, $\bar{P}_o$ and y_i, i=1, ...ℓ,

Batch computational algorithm

1. Compute $L_o = \bar{P}_o^{-1}$, $M_o = \bar{P}_o^{-1}\ \bar{x}_o$, set $t_k = t_1$, $t_{k-1} = t_o$.

2. Integrate: From t_{k-1} to t_k

$$\dot{X}^* = F(X^*,t),\quad X^*(t_{k-1}) = X^*_{k-1}$$

$$\dot{\Phi}(t,t_{k-1}) = A(t)\ \Phi(t,t_{k-1});\qquad \Phi_{k-1} = \Phi(t_{k-1},t_o) \tag{3.6}$$

3. Compute:

$$\tilde{H}_k = \partial G(X^*_k,t_k)/\partial X_k,\quad H_k = \tilde{H}_k\ \Phi(t_k,t_o),\quad N_k = H_k^T R_k^{-1}$$

$$L_k = L_{k-1} + N_k H_k \tag{3.7}$$

$$M_k = M_{k-1} + N_k y_k$$

4. Test: if $k = \ell$, go to 5; if $k < \ell$, increase k by 1 and return to 2.

5. Compute

$$\hat{x}_o = L_\ell^{-1}\ M_\ell \tag{3.8}$$

Note that in the algorithms summarized by equations 3.7, L_k is a symmetric $n \times n$ matrix and consequently computation of only the upper half of the matrix is necessary. If it is assumed that the vector y_k is a p-vector, and that the observations are correlated, i.e., the $p \times p$ matrix, $E[\varepsilon_k \varepsilon_k^T] = R_k$ has non-zero off-diagonal elements, then the elements in L_k, which must be computed, can be expressed as follows :

$$(L_{ij})_k = (L_{ij})_{k-1} + \sum_{\ell=1}^{p}\sum_{m=1}^{p} \left(H_{\ell i}\ \sigma^{-1}_{\ell m}\ H_{mj}\right)_k,\ i=1, \ldots ,n;\ j=1, \ldots , n \tag{3.9}$$

If the observations are uncorrelated, i.e., if $\sigma_{\ell m} = 0$, $\ell \neq m$, then each observation can be included independently and equation 3.9 reduces to the following convenient form

$$(L_{ij})_k = (L_{ij})_{k-1} + \sigma_k^{-1}\ (H_i H_j)_k,\ i=1, \ldots ,n\ ;\ j=1, \ldots ,n \tag{3.10}$$

The remaining elements are obtained by noting that $L_{ij} = L_{ji}$ for all i and j. Using the symmetric property of L_ℓ can have a significant effect on the computation speed. Furthermore, by forcing symmetry through requiring that $L_{ji} = L_{ij}$, the matrix L_ℓ is guaranteed to be symmetric and positive definite.

4. Sequential Estimation Algorithms

The algorithms given in equations 3.1, 3.2 and 3.3 are referred to as batch estimation algorithms since all of the observations are reduced before the estimate can be obtained. Equation 3.1 can be used as the starting point for developing a sequential estimation algorithm in which the observations can

be included sequentially as they are obtained.

The following matrix identity is a fundamental relationship between the batch and sequential estimation algorithms (see TAPLEY (1973))

$$P_k = \left(H_k^T R_k^{-1} H_k + \bar{P}_k^{-1}\right)^{-1} = \left(I - K_k H_k\right)\bar{P}_k \qquad (4.1),$$

where

$$K_k = \bar{P}_k H_k^T \left(H_k \bar{P}_k H_k^T + R_k\right)^{-1} .$$

The $n \times n$ matrix P_k in equation 4.1 is the covariance matrix associated with the estimate $\hat{x}_k$. Note, in examining equation 4.1, that the first expression requires an $n \times n$ matrix inversion, while the second expression requires the computation of K_k. As can be seen by examining equation 4.2, the computation of K_k requires inverting a $p \times p$ matrix, where p is the dimension of the observations. If a single scalar observation is processed, then the computation of P_k would be obtained by a single scalar division. If the identity given in equation 4.1 is substituted into equation 3.1, it is a straightforward algebraic procedure to rearrange the results to obtain the following expression for the estimate of x

$$\hat{x}_k = \bar{x}_k + K_k\left(y_k - H_k \bar{x}_k\right) \qquad (4.3),$$

where

$$\bar{x}_k = \Phi(t_k, t_{k-1})\, \hat{x}_{k-1} ,$$

and

$$\bar{P}_k = \Phi(t_k, t_{k-1})\, P_{k-1}\, \Phi^T(t_k, t_{k-1}). \qquad (4.4).$$

Note that equation 4.3 obtains the estimate $\hat{x}_k$ as a linear combination of the predicted estimate $\bar{x}_k$ plus a weighting factor multiplied by the difference between the observation y_k and the predicted observation $\bar{y}_k = H_k \bar{x}_k$. The a priori or predicted value of the estimate and the associated a priori covariance matrix are given by equations 4.4. Equation 4.3 can be used to obtain the updated estimate when only a single observation is included by a single scalar division. Whereas, to include the single observation using the batch estimation algorithm would require that an $n \times n$ matrix be inverted. The procedure for computing the estimate with a sequential estimation algorithm can be summarized in the following equations:

Sequential Estimation Algorithm

Given : $X_{k-1} = X^*_{k-1}$, $\bar{x}_{k-1} = \hat{x}_{k-1}$, $\bar{P}_{k-1} = P_{k-1}$, y_i, $(i=k, \ldots, \ell)$

Compute :

1.

$$\dot{X}^* = F(X^*, t), \quad X^*(t_{k-1}) = X^*_{k-1}, \quad t_{k-1} \leq t \leq t_k$$

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(t_{k-1}, t_{k-1}) = I \qquad (4.5).$$

2.

$$\bar{x}_k = \Phi(t_k, t_{k-1})\hat{x}_{k-1}$$

$$\bar{P} = \Phi(t_k, t_{k-1})\, P_{k-1}\, \Phi^T(t_k, t_{k-1}) \qquad (4.6).$$

3.

$$K_k = \bar{P}_k H_k^T (H_k \bar{P}_k H_k^T + P_k)^{-1}$$

$$y_k = Y_k - G(X_k^*, t_k) \tag{4.7}$$

4.

$$\hat{x}_k = \bar{x}_k + K_k (y_k - H_k \bar{x}_k)$$

$$P_k = I - K_k H_k \bar{P}_k \tag{4.8}$$

5.

$$\hat{X}_k = X_k^* + \hat{x}_k \tag{4.9}$$

6. If $k < \ell$, index k and return to 1.

Examination of the above algorithm indicates that as $P_k \to 0$, the estimation procedure will become insensitive to the observations and the estimate will diverge due either to errors introduced in the linearization procedure, computational errors, or errors due to an incomplete mathematical model. Modifications of the algorithm which account for the model errors are discussed by TAPLEY & INGRAM (1971; 1973).

In order to minimize the effects of the errors due to linearization assumptions, the extended form of the sequential estimation algorithm can be used. The extended or rectified sequential estimation algorithm can be obtained by noting that the best estimate of the state of the vehicle will usually be closer to the true trajectory than the initial reference trajectory. As a consequence, the current best estimate can be used to linearize the non-linear equations and to propagate the estimate between observation points. If the current trajectory is updated or rectified to include the local estimate of the error, i.e., if $X_{k-1}^* = \hat{X}_{k-1}$, then the value of x_{k-1} in equation 4.4 will be zero, and as a consequence, the predicted value $\bar{x}_k$ will always be zero. This follows since the state deviation is governed by a system of linear differential equations and if the initial conditions are zero, the solution, or the predicted value, will always be zero for any subsequent time point. Hence, if $\bar{x}_k$ is zero, the sequential estimation algorithm can be expressed in the following extended or rectified form:

Rectified Sequential Estimation Algorithm

Given : $\hat{X}_{k-1}, P_{k-1}, Y_k, \quad (k=1, \ldots, \ell)$

Compute :

1.

$$\dot{\bar{X}} = F(\bar{X}, t), \quad \bar{X}_{k-1} = \hat{X}_{k-1}$$

$$\dot{\Phi} = A(t)\Phi, \quad \Phi(t_{k-1}, t_{k-1}) = I \tag{4.10}$$

2.

$$\bar{P}_k = \Phi(t_k, t_{k-1})\, P_{k-1}\, \Phi^T(t_k, t_{k-1})$$

$$K_k = \bar{P}_k H_k^T (H_k \bar{P}_k H_k^T + R_k)^{-1} \tag{4.11}$$

3.

$$\hat{X}_k = \bar{X}_k + K_k [Y_k - G(\bar{X}_k, t_k)]$$

$$P_k = I - K_k H_k \bar{P}_k \tag{4.12}$$

4. If $k < \ell$, increase k by one and return to 1.

Note that the linear terms $\bar{x}_k = \Phi(t_k,t_{k-1})\hat{x}_{k-1}$ and $H_k\bar{x}_k$ which appeared in equations 4.6 and 4.8 are not present in equations 4.11 and 4.12. All higher order terms which were neglected in the Taylor series expansion used to linearize the original non-linear problem are zero also.

The symmetric property of the covariance matrix $\bar{P}_k$ can be used to reduce the computations in the same manner as described in the batch estimation algorithms. Note that the a priori covariance matrix $\bar{P}_k = \Phi(t_k,t_{k-1})\, P_{k-1}\, \Phi^T(t_k,t_{k-1})$ will satisfy the following matrix differential equation

$$\dot{\bar{P}} = A(t)\,\bar{P} + \bar{P}\,A^T(t), \qquad \bar{P}(t_{k-1}) = P_{k-1} \tag{4.13},$$

where $A(t) = \partial F(\bar{X},t)/\partial X$. Equations 4.13 can be used to replace the n × n matrix integration expressed by the second of equations 4.10 as well as the two n × n matrix multiplications implied by the first of equations 4.11. Furthermore, since $\bar{P}$ is symmetric, only $\{n\times(n+1)\}/2$ of the n × n equations in 4.13 must be integrated. The remaining elements in $\bar{P}$ are obtained by requiring that $(\bar{P})_{ij} = (\bar{P})_{ji}$, for all i,j. Note however, that the $\{n \times (n+1)\}/2$ system of equations must be integrated as a coupled system and this can lead to numerical difficulties. The solution for $\Phi(t_k,t_{k-1})$ can be obtained as n-independent solutions to a system of n-differential equations. Further discussions of these questions are given by TAPLEY (1973).

5. Least Sqaures Collocation Estimate

Consider the special case where a priori statistics on the initial state are given as $x_o \sim \{0,\bar{P}_o\}$; that is, the initial state estimate is $\bar{x}_o = 0$ and the associated state error covariance matrix is $\bar{P}_o$. Then the least squares estimate with a priori data, as given in equation 3.1, can be expressed as follows

$$\hat{x}_k = \left(H_k^T R_k^{-1} H_k + P_k^{-1}\right)^{-1} H_k^T R_k^{-1} y_k = P_k H_k^T R_k^{-1} y_k \tag{5.1}.$$

Note that in this expression, an n × n matrix must be inverted in order to obtain the estimate x_k. The state error estimate covariance matrix P_k can be written in the following equivalent form

$$P_k = \left(H_k^T R_k^{-1} H_k + \bar{P}_k^{-1}\right)^{-1} = \bar{P}_k - \bar{P}_k H_k^T \left(H_k \bar{P}_k H_k^T + R_k\right)^{-1} H_k \bar{P}_k \tag{5.2}$$

Note that the last expression in equation 5.2 requires inverting a p × p matrix where p is the dimension of the observation data vector which is to be reduced. Equation 5.2 is referred to as the fundamental identity, the Schurr identity, or the inside-out rule in various places in the literature. If the last expression in equation 5.2 is substituted for P_k in equation 5.1, the following result is obtained.

$$\hat{x}_k = \left(\bar{P}_k - \bar{P}_k H_k^T \{H_k \bar{P}_k H_k^T + R_k\}^{-1} H_k \bar{P}_k\right) H_k^T R_k^{-1} y_k \tag{5.3}$$

Algebraic re-arrangement leads to the following expressions

$$\begin{aligned}\hat{x}_k &= \bar{P}_k H_k^T \left(I - \{H_k \bar{P}_k H_k^T + R_k\}^{-1} H_k \bar{P}_k H_k^T\right) R_k^{-1} y_k \\ &= \bar{P}_k H_k^T \left(H_k P_k H_k^T + R_k\right)^{-1} \left(H_k \bar{P}_k H_k^T + R_k - H_k \bar{P}_k H_k^T\right) R_k^{-1} y_k \end{aligned} \tag{5.4}$$

Then, cancelling the two identical terms in the second bracket in equation 5.4, and noting that $R_k(R_k)^{-1} = I$, the following equivalent expression for the estimate of x_k is obtained

$$\hat{x}_k = \bar{P}_k H_k^T (H_k \bar{P}_k H_k^T + R_k)^{-1} y_k \tag{5.5}.$$

Equation 4.6 is identical to the expression given by MORITZ (1970) and by MORITZ & SCHWARZ (1972). This equivalence can be seen by noting the following similarities in notation

$$\bar{P}_k = C_{ss}, \quad H_k = A^T, \quad R_k = C_{nn}, \text{ and } \hat{x}_k = s \tag{5.6}.$$

Note that in these investigations Moritz & Schwarz point out the difficulty in obtaining $\hat{x}_k$ using equation 5.5 because the matrix to be inverted is of the same dimension as the observation data vector, i.e., it is a $p \times p$ matrix where $p >> n$. As a consequence, if the dimension of the data vector is much larger than the parameters to be estimated, equation 5.5 is an extremely inconvenient algorithm to use. On the other hand, the equivalent estimate for $\hat{x}_k$, given by equation 5.1, requires the inversion of only an $n \times n$ matrix, where n is the size of the parameter vector to be estimated. Note that either equations 5.1 or 5.5 may be used, depending on the size of the matrix to be inverted. For the case $p = n$, the computational requirements involved using either equation 5.1 or 5.5 will be identical. If $p < n$, then equation 5.5 should be used and if the a priori estimate of x_k, i.e., $\bar{x}_k$ is not zero, then for $p < n$, equation 4.3 should be used.

As a final remark, it should be noted that the expressions 5.1 through 5.6 indicate the equivalence between the least squares with a priori data and the least squares collocation proposed by Moritz & Schwarz, and the expressions demonstrate that the least squares collocation is a special case of the general least squares method. It is well known in studies related to minimum variance, maximum likelihood, and least squares estimation theory, that the estimate obtained will be dependent on the a priori data, and if a priori estimates of the initial values of the parameters are available, complete equivalence between methods can be established only when identical a priori data are used. Furthermore, the least squares estimate with a priori data will always give a different answer from that obtained with the conventional least squares estimate, as given by equation 3.3, i.e., the case when $\bar{P}_k = \infty$, since a different set of a priori statistics will be used in these two approaches.

6. Comparison of Batch and Sequential Estimation Algorithms

In the previous section, the algebraic equivalence of the sequential, the batch estimation and the least squares collocation algorithms have been shown. The extended sequential estimation algorithm which rectifies the trajectory at each observation point will be equivalent to the estimate obtained by these algorithms only if the data is iterated to obtain a converged solution for the trajectory. It is anticipated that the rectified sequential estimation algorithm will converge more quickly because the effects of linearization are minimized and, furthermore, that the estimate on any given iteration will be more accurate. As a corollary, the region of convergence for the extended or rectified sequential estimation algorithm will be larger than for the regular sequential estimation algorithm or for the batch estimation algorithm. It should be noted further that the batch estimation algorithm requires the inversion of an $n \times n$ matrix, where n is the dimension of the state vector, while for uncorrelated observations, the sequential estimation algorithm requires only a sequence of scalar divisions.

In order to compare the numerical performance of both the batch and sequential algorithms, a set of high-accuracy laser ranging observations of the Beacon Explorer-C satellite obtained by the Goddard Space Flight Center was used. The Beacon Explorer-C (BE-C) or Explorer 27 was launched with a Scout vehicle on April 29, 1965, from Wallops, Virginia, to study the ionosphere and the shape of the Earth. The orbit characteristics are 940 km perigee height, 1315 km apogee height, an eccentricity of 0.025, an inclination of 41.2°, and a period of 107.8 minutes. The BE-C was equipped with a laser retro-reflector array as well as beacons for atmospheric studies. The satellite is aligned with the Earth's magnetic field. The observations were obtained with the Goddard Space Flight Center tracking station (GODLAS) and a mobile station SENLAS located at Seneca Lake, New York, approximately 400 km due north of GODLAS. The set of observations was originally obtained in conjunction with an experiment conducted by D.E. Smith of Goddard Space Flight Center, aimed at measuring the Earth's polar motion (SMITH ET AL 1972b). In addition, the data have been used to evaluate the effect of solid Earth tides on the BE-C satellite (KOLENKIEWICZ ET AL 1973b). Of the approximately six months of data obtained by GODLAS and SENLAS, a sequence of four consecutive passes beginning September 2, 1970, was used in this investigation. This is the densest set of data obtained during the entire tracking period due mainly to favourable weather during these dates. The station locations are shown in figure 1, and the ground track for a typical four-pass sequence is shown in figure 2. The station co-ordinates for the study reported in this paper were those determined by SMITH ET AL (1972a), shown in table 1, where the height is that above the reference ellipsoid, where a_e = 6,378,155 m and f = 1/298.255.

T a b l e 1

Station Co-ordinates

Station	Geodetic Latitude ° ' "	Geodetic Longitude ° ' "	Height m
GODLAS	39 01 13.880	283 10 18.500	18.8
SENLAS	42 42 04.881	283 10 17.203	200.0

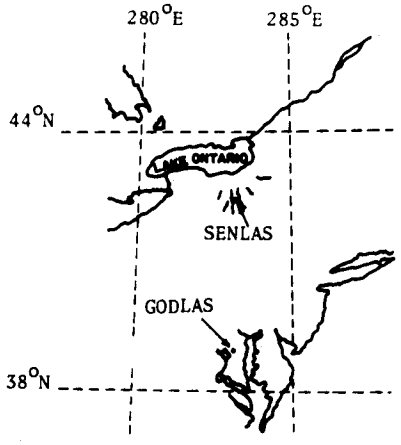
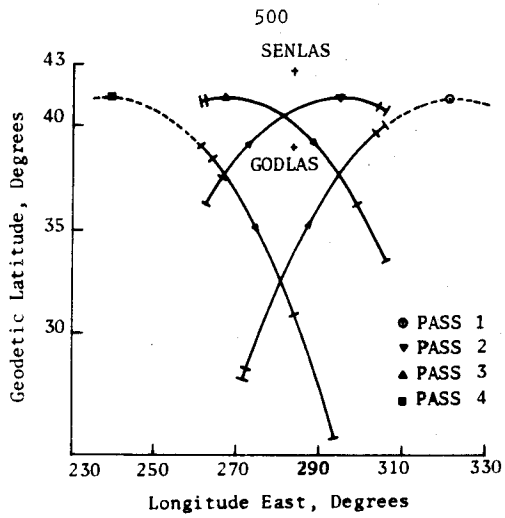


Figure 1. Tracking Station Locations (from SMITH ET AL 1972a)



Solid Line Indicates Satellite Being Observed

Figure 2. Beacon Explorer-C Ground Tracks (From SMITH ET AL 1972a)

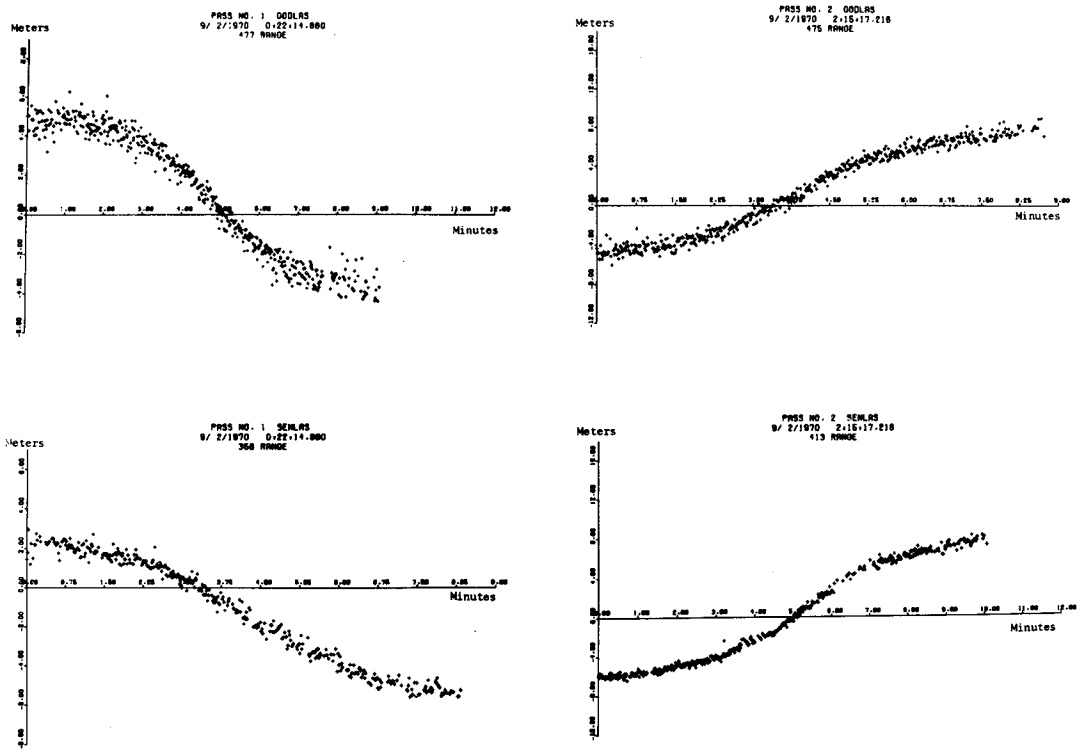


Figure 3a. Batch Processor Range Residuals for Passes 1 and 2

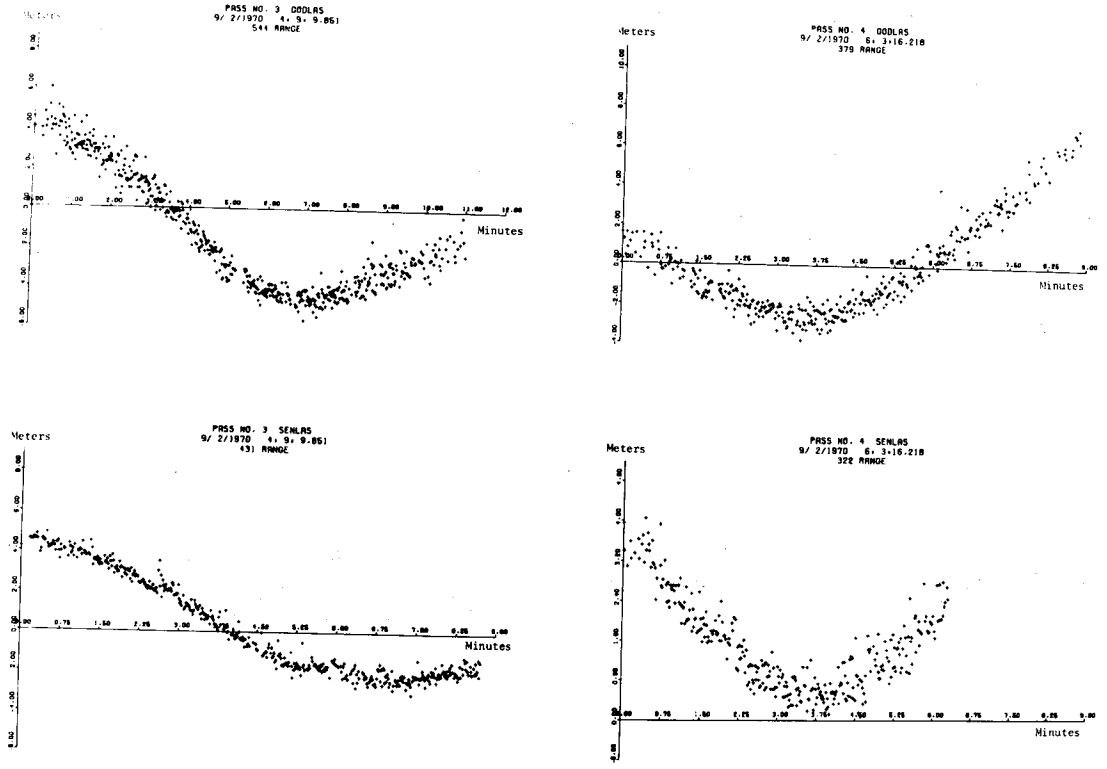


Figure 3b. Batch Processor Range Residuals for Passes 3 and 4

Figures 3a and 3b show the observation residual patterns obtained when the sequence of BE-C range observations are reduced using a batch processor which iterates to convergence. The residuals from the GODLAS station are shown at the top of each figure, while the SENLAS residuals are shown at the bottom. Note that the scatter has an amplitude of approximately 50 cm. Note also that the observation residual has a bias of approximately 8 m in amplitude. This bias is most likely due to the gravitational model used for the computation. The SAO Standard Earth II (GAPOSCHKIN & LAMBECK 1970) was used for this study. KOLENKIEWICZ ET AL (1973a), using the Goddard Earth Model I (GEM I) (LERCH ET AL 1972), obtained somewhat smaller residual patterns. However, since the specific accuracy is not of interest in this study, and since only the qualitative nature of the residuals are of interest, further consideration will not be given to the geopotential models. In addition, the effects of atmospheric drag were not included in this study. It has been determined in independent studies (SMITH ET AL 1972b) that drag does not significantly influence the motion of the BE-C satellite during the period studied. This fact is not surprising in view of the high altitude of the BE-C satellite and the short data arc examined (four revolutions). Further details on the nature of the numerical computations are given by SCHUTZ ET AL (1973).

In figures 4a and 4b, the observation residuals obtained during the first pass of the sequential estimation algorithm are shown. Due to the rapid convergence of the extended sequential estimation algorithm, the converged results are essentially identical to the results shown on the first iteration through the data. Note that the basic character of the observation residuals are quite different from the batch results during the first three passes, but they show a close agreement during the fourth pass. For instance, in the initial phase of the third pass, the model bias error is on the order of

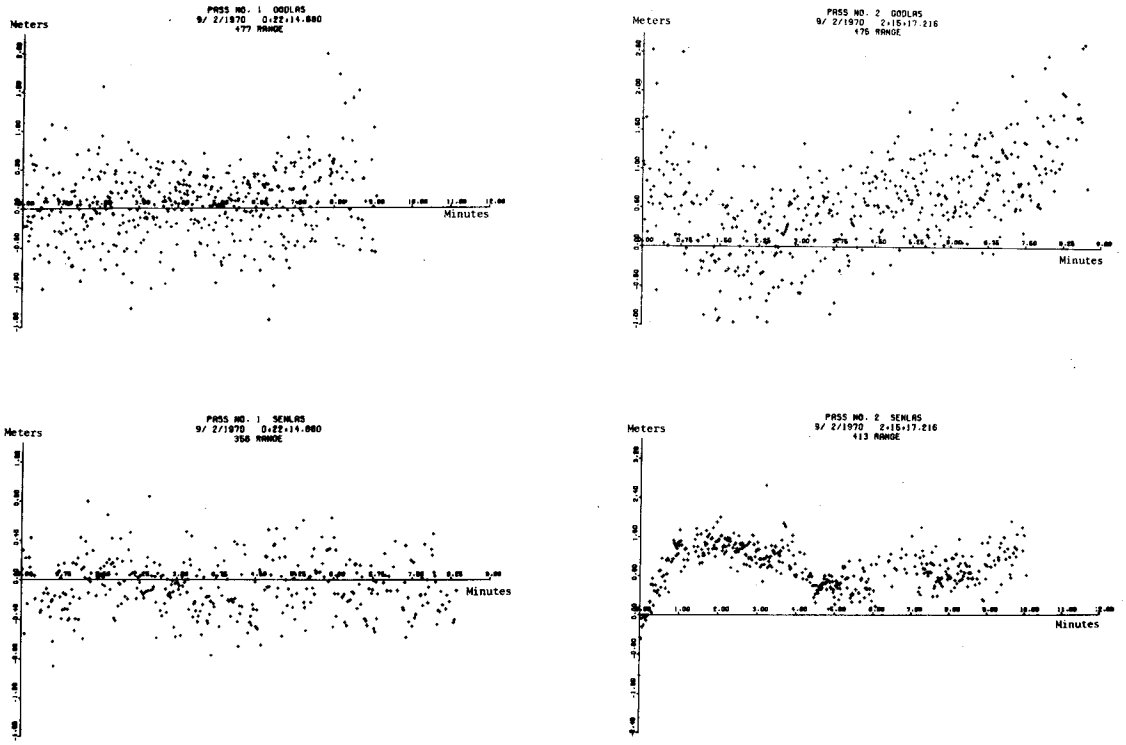


Figure 4a. Sequential Processor Range Residuals for Passes 1 and 2

33 m. The scatter about the bias is on the order of 50 cm for the GODLAS data and on the order of 35 cm for the SENLAS data, indicating that the actual observation accuracies are somewhat better than the formal observation accuracy of 1 m which is assigned to the data. It should be noted that the initial state estimate obtained on the first iteration agreed with the converged iteration estimate to less than 0.1m in position and less than 0.1 mm per second in velocity.

Although the observation residual patterns generated by the sequential and the batch estimation algorithms are quite different, as shown in figures 3 and 4, the agreement between the estimates can be evaluated by converting the estimates to a common epoch. The epoch used for the batch estimation algorithms was the initial time and a comparison can be made by taking the final sequence estimate, i.e., the estimate at the final time on the fourth pass, and integrating backwards through four passes to the initial time. The resulting state obtained by backward integration can then be compared with the estimate of the initial conditions obtained by the batch estimation algorithm. Table 2 shows the converged estimates of the conditions at the initial epoch obtained by the batch and sequential estimation algorithms. The a priori values for the initial covariance matrix $\overline{P}_o^1$, assumed elements of of 250,000 m^2 for the position component elements of the covariance matrix and 100 m^2s^{-2} respectively, for the elements of the velocity components. The matrix was assumed to be diagonal with no off-diagonal elements. Note that two iterations are required by the batch algorithm for less than 10cm accuracy in the converged solution for an a priori state which differs from the converged value by 75 m (case 1), whereas the extended sequential requires one iteration. If the a priori state error is larger, e.g. 600 m (case 2), then four iterations are required by the batch estimation algorithm to achieve convergence to this accuracy, while the extended sequential estimation algorithm still

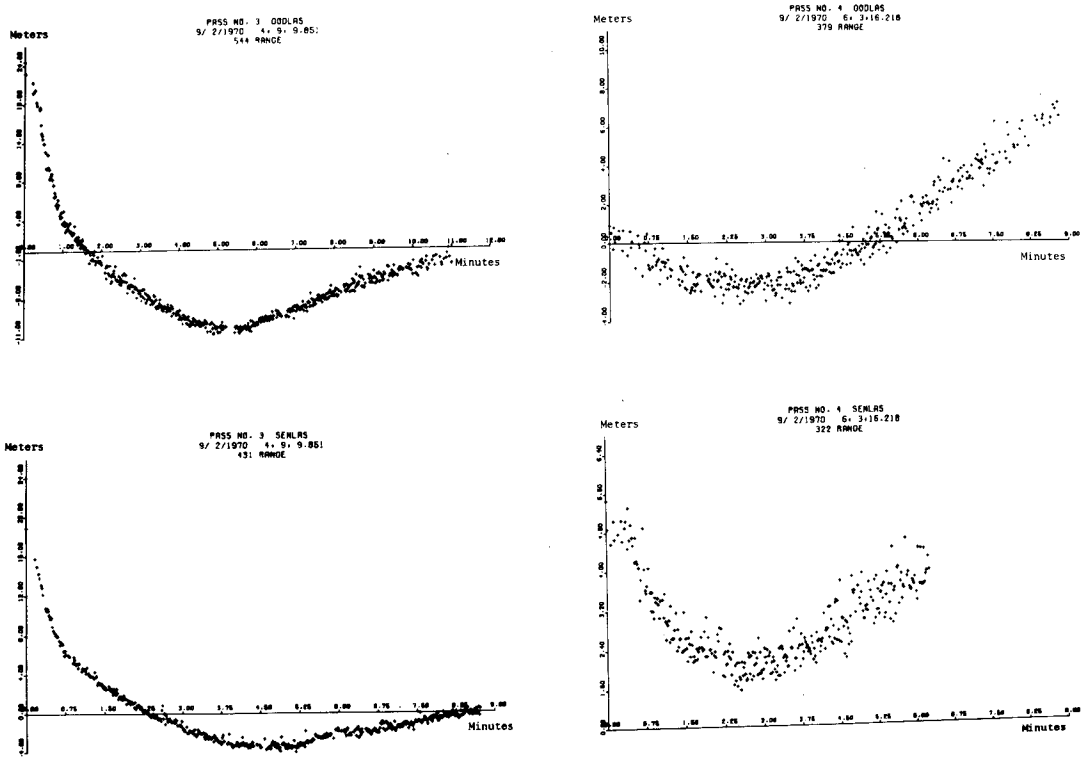


Figure 4b. Sequential Processor Range Residuals for Passes 3 and 4

achieves the same accuracy as before in one iteration. Further details on these computations are given by SCHUTZ ET AL (1973).

Comparison of the converged batch and extended sequential estimates of the initial state shows agreement to less than 10 cm in position and to less than 1 mm per second in velocity. A further question which should be considered relates to the accuracy of the predicted state; i.e., whether the disagreement in the epoch state estimates have a significant influence on the magnitude of the predicted error. An examination of the residual patterns during the first four revolutions shows no discernible difference in the residual pattern. Table 3 shows a comparison of the predicted residuals during the 42nd revolution using the predicted state obtained with the estimated sequential and batch initial conditions, as given in table 2. In addition, table 3 shows the difference in the predicted states during the 42nd revolution. Note that the predicted positions agree to less than 1 m, while the velocity components agree to less than 1 m sec^{-1}. From these results, it can be seen that the batch estimation algorithm and the extended sequential estimation algorithm yield converged estimates for the initial state which agree to less than 10 cm in position. Since the observations being reduced have a formal accuracy of 1 m, the estimates obtained by the two algorithms agree to well within the formal accuracy assigned to the observations.

The extended sequential estimator has a distinct advantage over the batch algorithms in problems

Table 2

Converged Epoch Estimates for Case 1 and Case 2 Initial Conditions

Estimator	CONVERGED SOLUTIONS X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$	No.of Iterations for< 10 cm Converged Values
Case 1							
Batch $(\bar{P}_o = \infty)$	-1531394.255	-6310301.372	3397639.175	6610.939326	338.148095	3443.581299	2
Batch $(\bar{P}_o = \bar{P}_o^{-1})$	.255	.372	.175	.939326	.148095	.581299	2
Extended Sequential $(\bar{P}_o = \bar{P}_o^{-1})$	.201	.383	.280	.939336	.148192	.581172	1
Case 2							
Batch $(\bar{P}_o = \infty)$	-1531394.255	-6310301.372	3397639.175	6610.939326	338.148095	3443.581299	4
Batch $(\bar{P}_o = \bar{P}_o^{-1})$	.255	.372	.175	.939326	.148095	.581299	4
Extended Sequential $(\bar{P}_o = \bar{P}_o^{-1})$	.201	.383	.277	.939336	.148190	.581175	1

Table 3

Prediction for 42nd Revolution

Pass Begins:September 5 1970 2:8:31.6693 UTC; Estimation Epoch: September 2 1970 0:22:14.8805 UTC

Estimators	Time From Beginning of Pass (sec)	Batch Residual	Sequential Residual	Predicted State Differences (No Filtering) Based on Filtering 6 Hours Data Commencing Sept. 2 1970 ΔX	ΔY	ΔZ	$\Delta\dot{X}$	$\Delta\dot{Y}$	$\Delta\dot{Z}$
Converged Batch	0.	-49.9	-50.3	0.473	-0.186	0.063	0.000	0.000	0.000
- 1 Iteration	179.93271	-56.0	-56.0	0.452	-0.248	0.006	0.000	0.000	0.001
Extended Sequen-	335.93261	-4.7	-4.4	0.420	-0.296	-0.043	-0.001	0.001	0.000
tial									
Converged Batch	0.	-49.9	-50.7	1.020	-0.367	0.090	0.000	0.000	0.000
-Converged Exten-	179.93271	-56.0	-56.1	0.072	-0.506	-0.021	0.000	0.001	0.001
ded Sequential	335.93261	-4.7	-4.0	0.903	-0.615	-0.117	-0.001	0.001	0.000

Units: m & m sec^{-1}

involving the determination of constant parameters. If the state vector includes the satellite position and velocity as well as constant parameters, e.g., station co-ordinates, then the state estimate at each observation time provides the evolution with time of the estimate of those parameters. To illustrate this fact, figure 5 shows the evolution of the height above the reference ellipsoid for GODLAS obtained with the laser range observation of the BE-C satellite described previously. It can be seen that if the last half of data in, say, the third pass had

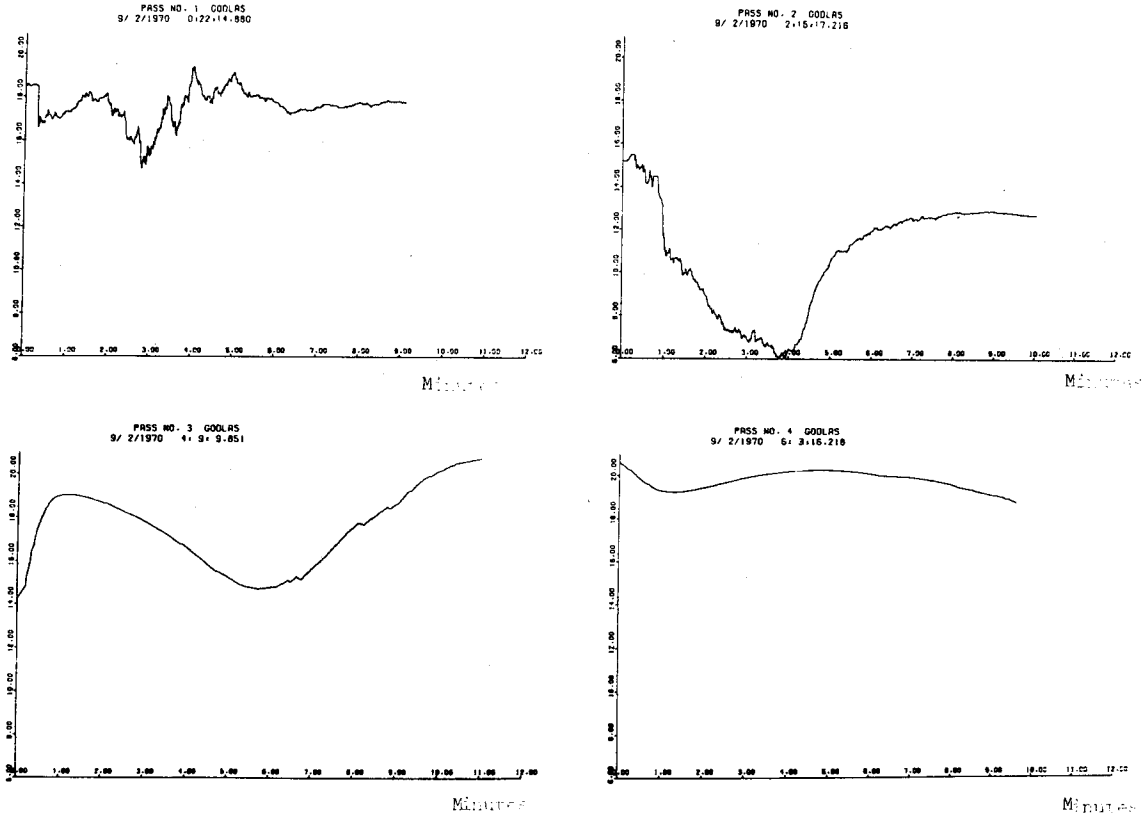


Figure 5. GODLAS Height Estimate in Metres from Extended Sequential Filter

not been included, the estimated height would have been considerably different from that at the end of the pass. The wide variations as well as the large observation residuals are, in fact, due to geopotential model errors. It can be shown that the second pass estimate is less affected by the model error and consequently the best estimate of the station co-ordinates will be obtained using data from two consecutive passes. Further details are given by SCHUTZ ET AL (1973b).

Finally, the question of the convergence properties of the estimation algorithms were studied. The results obtained do not show absolute bounds for the radius of convergence, but are intended to simply ascertain whether any significant differences exist. To generate specific initial conditions, each component of the converged batch estimate of the epoch was multiplied by a percentage factor. The a priori batch estimate with a priori information, diverges at 0.2%, whereas the extended sequential estimate diverges at 2.5%. It can be seen from figure 6 that converged sequential solutions are obtained with greater initial state errors It is important to note that the extended sequential estimate converges for a 3% initial error using the values adopted for $\overline{P}_o$ in this investigation, but diverged for larger values of $\overline{P}_o$ (SCHUTZ ET AL 1973a). This indicates the importance of the a priori covariance matrix. The results shown here indicate that the extended sequential algorithm has a significantly larger radius of convergence than the batch algorithms. The results also indicate the importance of the a priori information on the initial state errors in increasing the radius of convergence of both the batch and the sequential estimation algorithms.

7. Conclusions

The conclusions to be drawn from examination of the results presented in the previous section can be summarized as follows:

1. The least squares batch, sequential and least squares collocated algorithms are equivalent if the same a priori statistical information is used by each algorithm.
2. Using laser range observations of a near-Earth satellite obtained over four revolutions for which the observation accuracy is on the order of 50 cm, the estimates obtained with the batch and the extended sequential estimation algorithms are numerically equivalent to an order of 10 cm in position and 0.1 mm per second in velocity.
3. The extended sequential estimation algorithm which rectifies the trajectory at each observation point, converges to within 10 cm of the converged solution in one iteration, while the number of iterations required by the batch estimation algorithm is more dependent on the a priori data used.
4. The region of convergence for the extended sequential estimation algorithm appears to be approximately 10 times larger than the region of convergence for the batch estimation algorithm for the two station BE-C data examined in this study.
5. The extended sequential estimation algorithm provides the evolution of constant parameter estimates, such as tracking station co-ordinates and the effect of model error on the estimates can be determined directly.

From these above conclusions, the general conclusion can be reached that the sequential estimation algorithm is a useful approach to a number of problems in precision orbit determination for near-Earth and near planetary satellites. Where identical a priori data and identical modeling are included, the estimation accuracies agree to an extremely high precision with the batch results. However, the estimates are usually obtained with fewer iterations.

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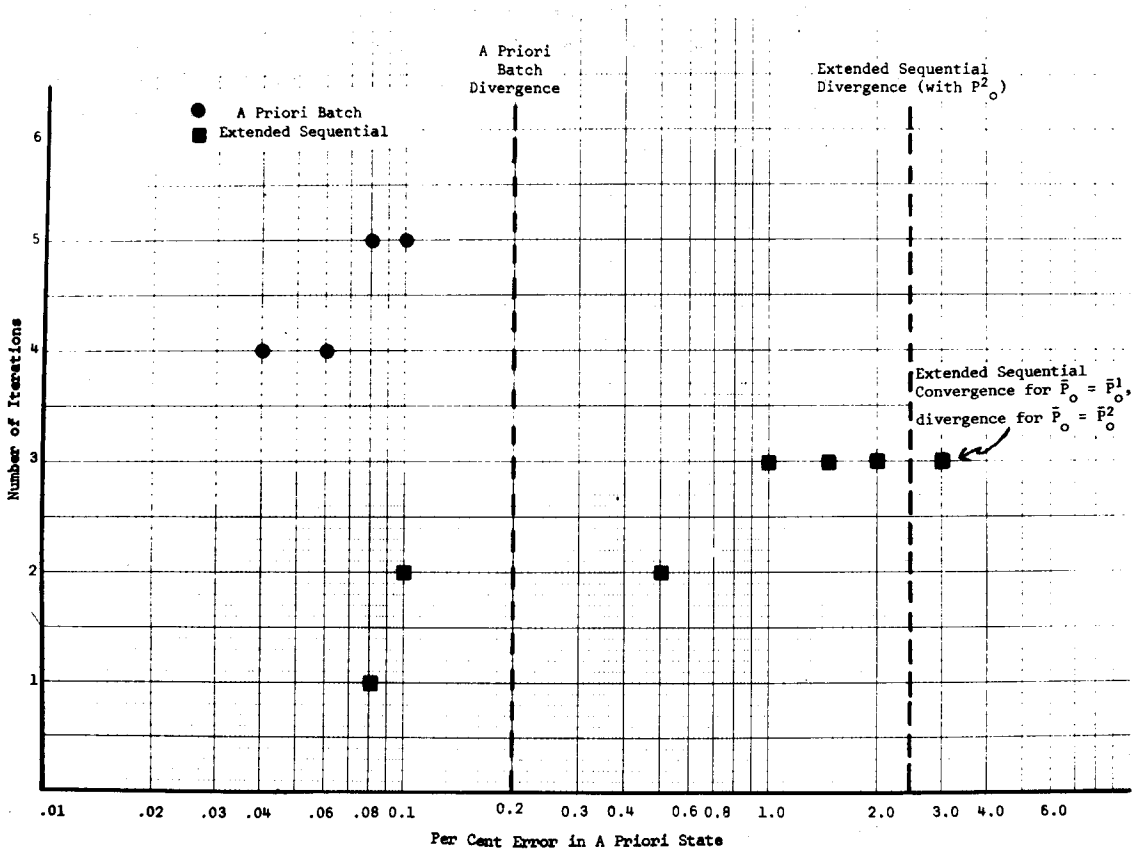

Figure 6. Radius of Convergence

9. Discussion

GRAFAREND: May I comment on some work I have done. I calculated the covariance function for the gravity anomalies, vertical deflections and the height anomalies, and assumed I had a Markov process for the gravity field. The procedure worked well for the gravity field and the vertical deflection field, but I had difficulties with the undulation fields as I had a singularity problem. So one basic assumption in the concept of approaching the Kalman-Bucy type of filter is that one has a Markov process. Did you assume a Markov process?

TAPLEY: No; as a matter of fact, the slide I used was a least squares algorithm. This made no assumption on the statistical nature. All I had really assumed was that the covariance matrices associated with the observation was a specified weighting matrix; and we get the least squares recursive formulation by just doing the manipulation shown. The Markov assumption is not essential. It is an essential assumption if one wants to use the Kalman-Bucy filter. But we are not required to assume that model. If one wants to do batch recursive computation, it is not an essential assumption.

KAULA: What type of covariance do you use in trying to model error sources which are not homogeneous but dying out with distance? It seems to me some sort of overlapping batch processing has to be done to get covariances more realistic than simple exponentials.

TAPLEY: Yes; we have considered several alternate representations. The exponential representation dies with time. It has the disadvantage that it is a rather artificial type approximation and will not predict very well. Alternate models which overcome this difficulty have been studied. But with the exponential model, if you want to go from one batch to another, then the exponential model has to be modified.