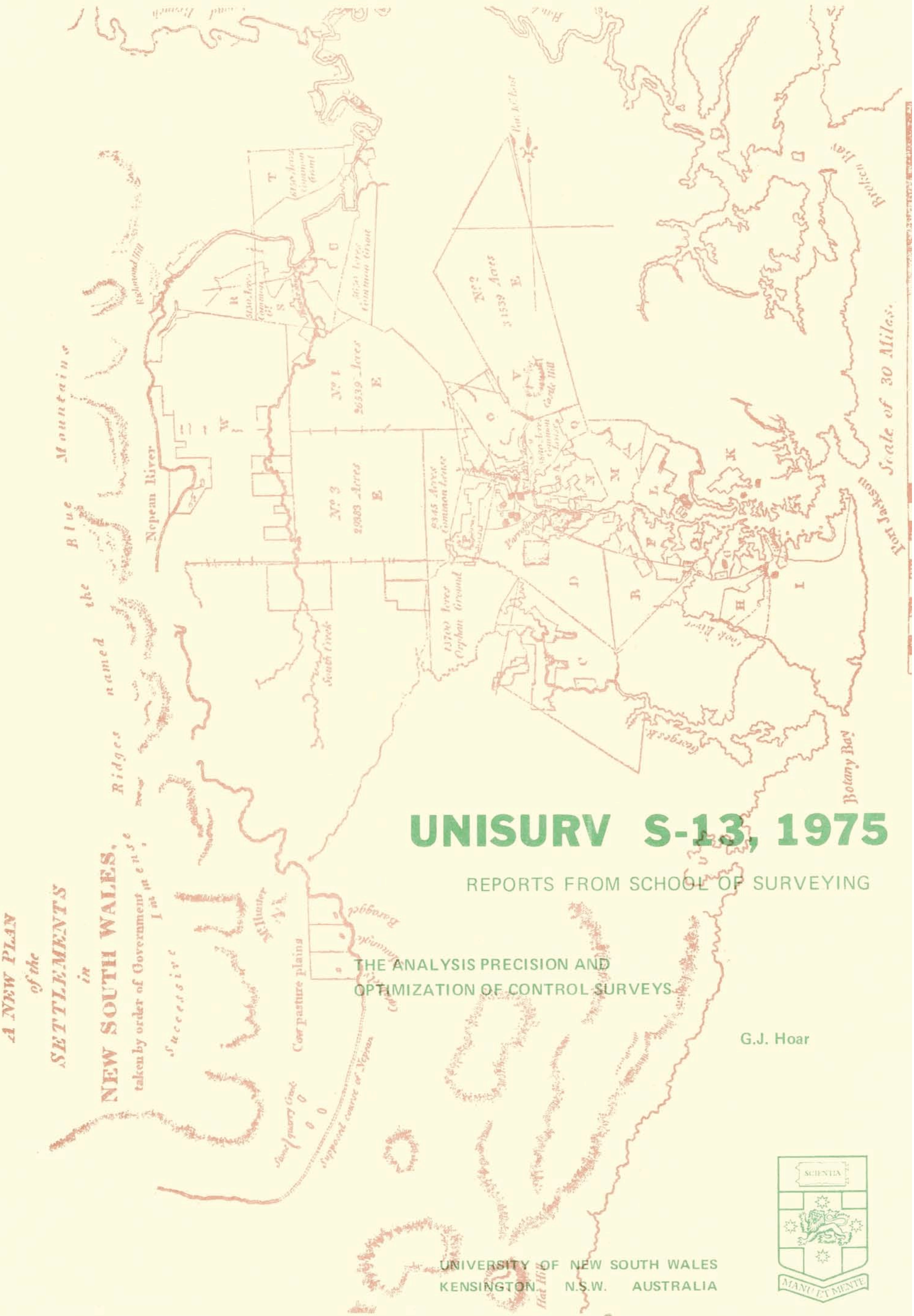

UNISURV S-13, 1975

REPORTS FROM SCHOOL OF SURVEYING

THE ANALYSIS PRECISION AND OPTIMIZATION OF CONTROL SURVEYS

G.J. Hoar

UNIVERSITY OF NEW SOUTH WALES
KENSINGTON N.S.W. AUSTRALIA

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S Green Hills

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Received September, 1974

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THE ANALYSIS, PRECISION AND OPTIMIZATION OF CONTROL SURVEYS

SUMMARY

The problem of optimization is basically one of estimating the precision of station coordinates in a proposed survey. The measure for coordinate precision used in this study is the error ellipse. Error ellipses may be obtained from the given optimization procedure, their accuracy being chiefly dependent on the precision of the variances of the proposed observations which will form the survey in question. The factors affecting the precision of both angular and distance observations are analysed and estimates of their magnitude, based on literature, past experience and experimental work are made. These individual estimates are combined by the Law of Propogation of Variances to predict the observational variance that is likely to be achieved in the field.

Methods of analysing the precision of angular and linear observations are examined and suitable, theoretically valid, methods are formulated for each type. These methods are applied to a number of actual networks and the variances of the observations forming these networks are estimated. The agreement between these variance estimates and those from the analysis of contributing factors is generally good provided that the differences between laboratory and field conditions are taken into account.

The variances obtained from the analysis of the actual networks were broken down to basic components, taking into account the equipment and techniques used, and the conditions under which the observations were taken. The agreement between the basic components of the observational variances in the large networks tested shows that it is practical to estimate the observational variances from these basic components for use in the optimization of surveys, given knowledge of equipment, techniques and conditions.

Such estimates are used in two examples designed to show some practical techniques which are of use in the optimization of surveys.

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CHAPTER 1

INTRODUCTION

Control surveys are of many types. They range from geodetic surveys , covering many hundreds of kilometers, to the relatively small scale control surveys required for an integrated cadastral survey system. They include engineering surveys which range in precision from those required for dam deflection surveys to those required for approximate setting out of engineering works. All control surveys irrespective of size or intended precision, are of limited use until an estimate can be made of the precision to which the stations have been, or will be, fixed. It is important that the survey should have adequate precision to fulfil the purpose for which it was designed.

The error ellipse is the accepted criterion of point precision and the factors on which it depend are:

1. The approximate coordinates of the network stations.
2. The type of the proposed observations, (Directions, Distances, Azimuths etc.), and their position in the network.
3. The precision of each observation forming the network.

The statistical theory and techniques needed to calculate error ellipses has been developed over the years but there has been little investigation into the precision of the ellipse. It will be seen that this precision is almost entirely dependent on the precision to which the observational variances have been estimated.

One approach to the problem of estimating observational variance is to analyse the factors that contribute to it by empirical and experimental means. While the results of such an analysis are useful, the method is not satisfactory for estimation of variance due to the difficulty in simulating field conditions in laboratory experiments.

Attempts have been made in the past to overcome this difficulty by analysing field observations. Most of these attempts have been based on the fact that observations in a network must fulfil certain geometric conditions and the variance estimates obtained were derived from the condition misclosure. These methods suffer from the major disadvantages that they use only a percentage of the observations, that they can only be used with angular observations, and that they usually require fairly extensive calculations which would not otherwise be necessary.

Some relatively recent work has been done on the application of Variance Factor analysis to the estimation of observational variances, Ashkenazi *(1970)*. Such an approach overcomes all the problems of the condition closure methods as it is applicable to both angular and linear observations, it uses all the observations and it involves virtually no superfluous calculation. A significant part of this report is devoted to improving the techniques associated with Variance Factor analysis, and several observed networks are analysed to show the application of the improved technique (vide Chapters 5, 6 and 7).

Once variance estimates for observations taken using given instruments, given techniques and under given observing conditions have been obtained, it is possible to break these estimates down into the basic components of variance. Using these components, the variances of observations taken, or to be taken, using specified equipment and observing techniques and under specified observing conditions may be assessed, or predicted. These predicted variances may be used in optimization studies, aimed at calculating the optimum configurations of stations, and type, position and precision of observations to achieve a specified precision of point fixing. Two examples of such studies are given in order to demonstrate some of the practical techniques involved (vide Chapter 9).

Standard metric units and their abbreviations, as given in Australian Standard 1000 (1970), will be used in this report. Exceptions to this are the abbreviation of "sec" for second of arc and "sec^2" for seconds of arc squared. Abbreviations for mathematical terms will be explained when they are first used.

CHAPTER 2

THE THEORY OF OPTIMIZATION

2.1 Introduction

The optimization of a network involves the pre-analysis of that network to ascertain the optimum configuration of the stations, and the optimum placement and precision of the observations between the stations. As the word pre-analysis implies, optimization is carried out before the network is observed, and the actual values of the proposed observations are not required.

In the past, the planning of surveys has been based on empirical standards, experience and intuition. The validity of this procedure depends on the pattern of new work conforming closely to that of past experience. Unfortunately, no two surveys are the same and any deviation from standardised design can only be assessed intuitively. This intuitive assessment may have adverse effects on the precision of the survey.

The empirical standards set down by the various authorities tend to be fairly similar in form. Two examples of these standards are the requirements for horizontal control surveys set down by the United States Coast and Geodetic Survey *(Gosset, 1959)* and by the National Mapping Council of Australia (1966).

The requirements of the Coast and Geodetic Survey were written before the wide-spread use of electronic distance measurement became common. Scale control was obtained from bases measured mainly by invar bands. Measurement by invar band is a very time consuming procedure and hence these bases were spaced as widely as possible. This spacing was calculated by a trial and error method testing the strength of figure of different configurations of stations between the bases. The formula used was explained as follows; *(Gosset, Ibid)*. "Strength of Figure formula is an expression of the comparative precision of computed lengths in a triangulation net as determined by the size of angles, the number of conditions to be satisfied and the distribution of base lines and points of fixed position".

The formula used to calculate strength of figure was:

$$S = \frac{D-C}{D} \Sigma(\delta_A^2 + \delta_A\delta_B + \delta_B^2) \qquad \ldots \quad (2.1)$$

where S is the strength of figure,

D is the number of directions observed in the network, less the directions observed along the fixed line,

C is the number of conditions to be satisfied in the network,

and δ_A, δ_B are the respective logarithmic differences of the sines, expressed in units of the sixth decimal place, and corresponding to a change of one second in the distance angles, A and B of a triangle. The distance angles of a triangle are the angles opposite the side that is known and the side that is required.

Gosset *(ibid, p.268)* gives a table of values for $(\delta_A^2 + \delta_A\delta_B + \delta_B^2)$, in which the arguments are the two distance angles in a particular triangle. The summation, Σ, is taken for all the triangles used in computing the value of the side in question from the side taken to be absolutely known.

As an example of the application of the formula, consider the network shown in Fig. 2.1.

Side AB is the fixed side, EF is the side in question, and directions are measured, both ways, along all lines in the network. The sizes of the angles, in degrees, are shown in Fig. 2.1. The number of conditions to be satisfied, (C), is eight, and the number of observed directions, (D), excluding those over the fixed line is twenty-two. Therefore:

$$\frac{D-C}{D} = 0.64$$

Two possible ways of calculating through the network are:

1. Through triangles ABC, BCD, CDF and CEF.
2. Through triangles ABD, ACD, CDE and DEF.

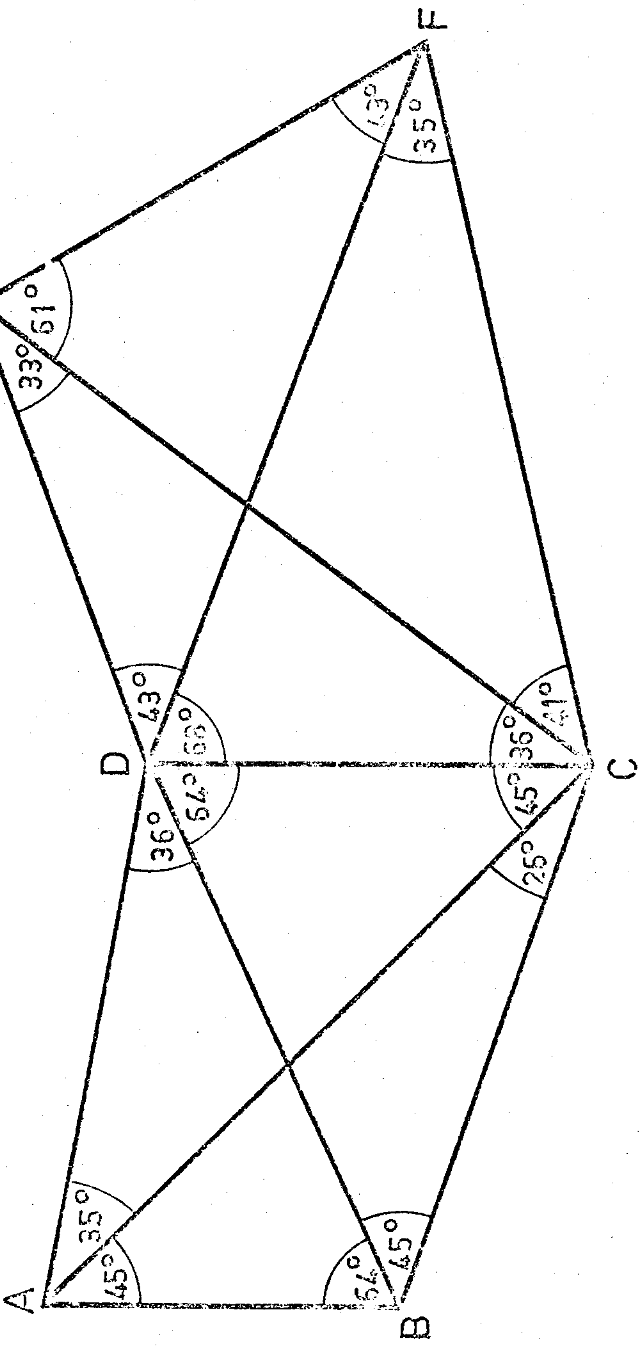


FIG. 2-1

Evaluation of (2.1) for these chains gives values for strength of figure of:

1. S = 0.64 (32 + 7 + 12 + 11) = 40
2. S = 0.64 (13 + 20 + 27 + 13) = 47

All possible chains would be tested, and limits were placed on the values of S obtained. If, for a certain configuration, the values fell inside these limits, the spacing of bases in that configuration was regarded as being satisfactory.

The usefulness of the method, and others like it, falls down when the network does not take the form of a chain. In the case of triangulation covering a broad area, there will be many ways of calculating the strength factors using the many different chains of triangles between the bases.

The National Mapping Council criterion is far less stringent, in that it is only suggested that strength of figure lie within certain limits. It was realised that this formula can only apply to a limited number of networks and hence it is only offered as a suggestion.

In both sets of standards, the only other guidelines for the design of the configuration of networks are very vague regulations stating that figures be well conditioned, that single triangle chains not be used and specifying the spacing between stations etc. Even if it was possible to conform to these regulations in every case, they certainly provide no guarantee that the accuracy specifications will be met.

Some specifications were given for the accuracy of angle and distance measurement, but the surveyor had to rely mainly on experience in deciding what field measurements and procedures were required to obtain these accuracies. He could only be sure that he was getting these accuracies by testing his observations after the survey was completed. In the majority of cases this situation led to the gross "over observation" of the network, usually with little improvement in positional precision. However, this was more economical than finding that the standards had not been reached after completion of the fieldwork, and thus having to re-visit the field stations for further observations.

The implicit requirement of a survey is that stations be coordinated to some specified positional precision. Common practice is to assess a survey only on the observational precision, acting on the assumption that provided the scheme is composed of figures of sufficient strength and without any abnormalities, the observational precision will reflect the positional precision. The observational precision is usually assessed by examining triangle or other misclosures, or by examining the size of corrections to the observations after an adjustment. However, the over-riding weakness of such a method is that the estimate of positional precision can only be obtained after the scheme has been observed and adjusted.

Positional precision may be assessed by means of the error ellipses at each of the stations in question. These ellipses may be calculated from the weight coefficient matrix for the station coordinates. This weight coefficient matrix is not dependent on the actual values of the observations and may be obtained from an optimization procedure.

Contemporary thought on the interpretation of error ellipses is that they give the precision of coordinates with respect to the fixed control, or in the case of relative ellipses the precision of the differences of coordinates between the terminal points. Some research projects within the School of Surveying, University of New South Wales, have recently thrown doubt on the validity of this interpretation.

In an effort to resolve this doubt, a number of people are working on projects designed to test various hypotheses. It is not anticipated that any final answer to this problem will be found before 1974. One research project along this line is a Ph.D thesis to be written by R. Lister and readers after 1974 should refer to that thesis.

Since the qualitative aspect is not in doubt, the current interpretation has been adhered to in this report.

2.2 Concepts of Optimization

To optimize a network, knowledge of three components is required.

1. The approximate coordinates of the stations.
2. The type of the proposed observations (Directions, Distances, Azimuths etc.) and their position in the network.
3. The anticipated precision for each type of observation.

With this information, it is possible to calculate the error ellipse for the stations and verify that the desired positional precision can be attained. The effect of additional or fewer observations and changes in observational precision can be assessed by varying components two and three. The effect of additional stations, or stations in different locations can be assessed by modifying component one. In this way the most economical method of achieving the desired positional precision may be determined before any field measurements are taken.

2.3 Theory of Optimization

The Parametric or "Variation of Coordinates" method of adjustment is now the most commonly used for survey adjustments. This adjustment procedure has been given by many authors *(e.g. Allman, 1967 and Madcour, 1968)* but will be repeated here to show how the method may be used for the optimization of a survey.

The estimates of the variates (the coordinates) are given by the parametric equations:

$$P = p + V = AX + C \qquad \text{... (2.2)}$$

or more simply:

$$AX + T = V \qquad \text{... (2.3)}$$

where

P is the matrix of adjusted observations.

p is the matrix of observations.

V is the matrix of corrections to the observations.

A is the matrix of coefficients, in which the individual elements are dependent on the coordinates and the type of observation.

X is the matrix of parameters.

C is the matrix of constants .

and T is the matrix of absolute terms such that $T = C - p$.

Consider observations made in a plane coordinate system. The parametric equation for a direction observation from station i to station j can be shown *(Clark, Vol. 2, 1966)* to be of the form:

$$V_{\alpha_{ij}} = \alpha^{c}_{ij} - p_{\alpha_{ij}} - \overline{O}_{i} - \Delta O_{i} + a_{ij}(\Delta X_{i} - \Delta X_{j}) + b_{ij}(\Delta Y_{i} - \Delta Y_{j}) \quad \ldots (2.4)$$

where $V_{\alpha_{ij}}$ is the correction to the observation,

α^{c}_{ij} is the calculated bearing from station i to station j, and is derived using the approximate coordinates of those stations.

$p_{\alpha_{ij}}$ is the observed direction.

$\overline{O}_{i}$ is the approximate orientation of the direction observations from station i, with respect to the coordinate system.

ΔO_{i} is the least squares parameter for the orientation at station i, so that,

$$O_{i} = \overline{O}_{i} + \Delta O_{i}$$

where O_{i} is the least squares estimate of the orientation at station i.

Similarly,

$$X_i = \bar{X}_i + \Delta X_i$$

$$Y_i = \bar{Y}_i + \Delta Y_i$$

where (X_i, Y_i) are the least squares estimates of the plane coordinates of station i.

$$a_{ij} = \frac{\rho . \sin \alpha^c_{ij}}{D^c_{ij}}$$

$$b_{ij} = \frac{-\rho \cos \alpha^c_{ij}}{D^c_{ij}}$$

where ρ is the number of seconds in one radian,
and D^c_{ij} is the calculated distance between stations i and j, and is derived from the approximate coordinates of those stations.

Relating the terms of equation (2.4) to the general form of the parametric equation (2.3):

-1, a_{ij}, $-a_{ij}$, b_{ij} and $-b_{ij}$ are elements of the A matrix as they are the coefficients of the parameters ΔO_i, ΔX_i, ΔX_j, ΔY_i and ΔY_j, (elements of the X matrix). V_{ij} is an element of the V matrix. The constant term, $(\alpha^c_{ij} - p\alpha_{ij} - \bar{O}_i)$ is an element of the T matrix.

The parametric equation of a bearing observation from station i to station j, in a plane coordinate system is identical to (2.4) except that the orientation terms are omitted.

$$V\beta_{ij} = \beta^c_{ij} - p\bar{\beta}_{ij} + a_{ij} (\Delta X_i - \Delta X_j) + b_{ij} (\Delta Y_i - \Delta Y_j)$$

... (2.5)

The parametric equation for an angle observation at station i to stations j and k is the difference between two direction equations.

$$V_{\gamma ijk} = \alpha^c_{ik} - \alpha^c_{ij} - p_{\gamma ijk} + a_{ik}(\Delta X_i - \Delta X_k) + b_{ik}(\Delta Y_i - \Delta Y_k)$$

$$- a_{ij}(\Delta X_i - \Delta X_j) - b_{ik}(\Delta Y_i - \Delta Y_j) \qquad \ldots (2.6)$$

The parametric equation for a distance measured between stations i and j is of the following form:

$$V_{\Phi ij} = D^c_{ij} - p_{Dij} + \text{Cos}\ \alpha^c_{ij}(\Delta X_j - \Delta X_i) + \text{Sin}\ \alpha^c_{ij}(\Delta Y_j - \Delta Y_i) \qquad \ldots (2.7)$$

The corresponding parametric equations for a spherical coordinate system are similar in form.

From these equations the normal equations may be formed.

$$A^T G^{-1} A X + A^T G^{-1} T = 0 \qquad \ldots (2.8)$$

where G is the matrix of weight coefficients of the observations. The normal equations may be restated as:

$$NX + A^T G^{-1} T = 0 \qquad \ldots (2.9)$$

where

$$N = A^T G^{-1} A \qquad \ldots (2.10)$$

The solution of these normal equations is given by:

$$X = N^{-1} A^T G^{-1} T \qquad \ldots (2.11)$$

The matrix of weight coefficients of the adjusted parameters X is given by the inverse matrix N^{-1}, whence:

$$Q_{xx} = N^{-1} = \begin{vmatrix} Q_{x_1x_1} & Q_{x_1x_2} & \cdots & Q_{x_1x_u} \\ Q_{x_2x_1} & Q_{x_2x_2} & \cdots & Q_{x_2x_u} \\ \vdots & \vdots & & \vdots \\ Q_{x_ux_1} & Q_{x_ux_2} & \cdots & Q_{x_ux_u} \end{vmatrix} \qquad \ldots (2.12)$$

The error ellipses of the adjusted points are given by the equation:

$$\frac{1}{Q_{x_i x_i} Q_{y_i y_i} - Q^2_{x_i y_i}} \left(Q_{y_i y_i} X_i^2 - 2Q_{x_i y_i} X_i Y_i + Q_{x_i x_i} Y_i^2 \right) = S^2 \qquad \ldots (2.13)$$

Where X_i and Y_i are the adjusted coordinates of the i^{th} point and S^2 is the variance factor.

From equation (2.13), the semi-major axis, the semi-minor axis and the orientation may be derived as:

Semi-major axis =

$$S^2 \left\{ \tfrac{1}{2}(Q_{xx} + Q_{yy}) + \{\tfrac{1}{4}(Q_{yy} - Q_{xx}) + (Q_{xy})^2\}^{\frac{1}{2}} \right\}^{\frac{1}{2}} \qquad \ldots (2.14)$$

Semi-minor axis =

$$S^2 \left\{ \tfrac{1}{2}(Q_{xx} + Q_{yy}) - \{\tfrac{1}{4}(Q_{yy} - Q_{xx}) + (Q_{xy})^2\}^{\frac{1}{2}} \right\}^{\frac{1}{2}} \qquad \ldots (2.15)$$

and the orientation of the semi-major axis

$$\phi = \tfrac{1}{2}\tan^{-1} \frac{2Q_{xy}}{Q_{xx} - Q_{yy}} \qquad \ldots (2.16)$$

Consider the weight coefficient matrix G. This matrix may be arrived at as follows:

Generally the variance of the observation (σ_i^2) will be either unknown or determined by an estimator S_i^2 so that,

$$\sigma_i^2 \doteqdot S_i^2 \qquad \ldots (2.17)$$

Some suitable dimensionless number may be taken as a variance factor S^2 such that,

$$S_i^2 = S^2 g_{ii} \qquad \ldots (2.18)$$

where g_{ii} is the weight coefficient of the particular observation.

This may be expressed in matrix form as:

$$\begin{vmatrix} \sigma_1^2 & \sigma_{12} & \sigma_{13} \cdots \sigma_{1n} \\ & \sigma_2^2 & \sigma_{23} \cdots \sigma_{2n} \\ & & \sigma_3^2 \\ & & \ddots \\ & & & \sigma_n^2 \end{vmatrix} \div \begin{vmatrix} S_1^2 & S_{12} & S_{13} \cdots S_{1n} \\ & S_2^2 & S_{23} \cdots S_{2n} \\ & & S_3^2 \\ & & \ddots \\ & & & S_n^2 \end{vmatrix} = S^2 \begin{vmatrix} g_{11} & g_{12} & g_{13} \cdots g_{1n} \\ & g_{22} & g_{23} \quad g_{2n} \\ & & g_{33} \\ & & \ddots \\ & & & g_{nn} \end{vmatrix} \quad \ldots (2.19)$$

or Variance/covariance Matrix of Observations $\div$ Estimate of Variance/covariance Matrix of Observations $= S^2 G$

An estimate $\overline{S}^2$ of S^2 may be obtained after the adjustment.

$$\overline{S}^2 = \frac{M}{r} \qquad \ldots (2.20)$$

where r is the number of redundant observations in the network and M the minimum $(= V^T G^{-1} V)$ is given by:

$$M = (A^T G^{-1} T)^T X + T^T G^{-1} T \qquad \ldots (2.21)$$

The mathematical model for any adjustment is formed by the observations, their variances and the geometric configuration of the stations in the network. If the model is validly chosen then the value of $\overline{S}^2$ will approach the value of S^2.

In other words, if the observations and their estimated variances satisfy the mathematical model, then $\overline{S}^2 \doteqdot S^2$.

The S^2 in equation (2.13) is therefore nothing more or less than a scaling factor on the size of the ellipse in the same way as it is a scaling factor on the G matrix.

The error ellipses may therefore be calculated when the weight coefficient matrix , (Q_{xx}), and the variance factor, (S^2), are known.

The weight coefficient matrix of the adjusted coordinates Q_{xx} is not dependent on the values of the observations:

$$Q_{xx} = (A^T G^{-1} A)^{-1}$$

It depends only on the weight coefficient matrix of the observations (the variance matrix if $S^2 = 1$) and the geometric configuration of the network.

2.4 Factors Affecting The Accuracy of Optimization

It was stated in Section 2.2 that the error ellipses obtained from an optimization are dependent on three factors.

1. The approximate coordinates of the station.
2. The type of the proposed observations (Directions, Distances, Azimuths etc.) and their position in the network.
3. The anticipated precision for each type of observation.

The accuracy of the error ellipses will obviously not be greatly dependent on the accuracy of the approximate coordinates, except where points are very close together or where very thin triangles and other ill-conditioned figures are involved. Special care must be taken under such conditions, but in general the problem can be overcome by calculating the coordinates of the crucial points to ensure that they are in the intended relationship to each other. In the majority of cases, no significant accuracy will be lost if the coordinates are carefully scaled off maps, sympathetic in scale to the area of the survey.

No approximations are involved in choosing the types of observation to be used so there is no loss of precision in the error ellipses due to this source.

The accuracy of the error ellipses is critically dependent on the accuracy of the variances used in the optimization. However, it seems reasonable to assign a variance to observations taken with a certain make and model of instrument using specified techniques and with a knowledge of observing conditions. This conclusion has been confirmed by the analysis of independent networks using similar instruments. The details of this analysis are given in Chapter 7.

2.5 Conclusions

Error ellipses may therefore be calculated without any knowledge of the adjusted coordinates of the network points or of the actual values of the observations.

There is no need to use simulated observations as Schmutter and Adler *(1971)* have done.

In addition, their set of simulated observations is only one sample from an infinite population. Surely it is better to examine the whole population rather than a single sample.

Further, in using simulated systematic components, it may be assumed that these follow a predictable pattern in that their magnitude at each station could be determined within specified limits. In carrying out the adjustment, the predictable portion of the systematic error must be applied to the observation to reduce the observation to the reference surface. The remaining portion of the error can only be assumed as random and should be taken into account when assessing the estimate of the variance of the observation. The significance of this is that in the adjustment and in any statistical analysis of the network, only the random component of the systematic error should be considered. This component, taken with an estimate of the effects of plumbing errors, lateral refraction, instrument dislevelment etc., forms the external component of the observational precision.

Undoubtedly the most important aspect of the above is that the error ellipses can be obtained before the field work is carried out. An optimization procedure such as this enables the surveyor to make an estimate of manpower and expenditure required to obtain a given precision.

CHAPTER 3

FACTORS AFFECTING THE PRECISION OF ANGULAR OBSERVATIONS

3.1 Introduction

All angular observations may be expressed in terms of a basic element, the direction. For example, an angle may be deduced by taking the difference between two directions and astronomical azimuth observations consist of a direction to a star and a direction to a reference object. This being the case, the factors that affect the precision of direction observations will also affect the precision of angles or azimuths derived from them.

3.2 Definitions

Angular Observations include directions, angles and observed astronomical azimuths.

It is not proposed to enter the controversy about whether angles (repetition), or directions, (iterative), should be observed, and these terms will merely be defined as they are understood by the author and as they will be used in this report.

A Direction. If a theodolite is set up at a station A and pointed to a target B on face left, and the horizontal circle is read, a *semi-direction* is obtained. If another semi-direction is read to target B on face right, the mean of the two semi-directions will give the *direction* from station A to station B.

A single semi-direction should not be considered as an observation as it contains many of the systematic errors of the theodolite. If the observation is to be treated statistically, the observational errors should always be random.

An Angle. If the directions from a station A to stations B and C are observed, the angle BAC is the difference between the two directions. If the direction to a third station, D, is also read, the angles BAD and CAD may also be deduced. It should be noted that these angles are not observations in their own right, but are combinations of the direction observations from which they are derived. Further, the angles are correlated, as each direction observation is used in the derivation of more than one angle, and extreme care must be exercised if they are to be used in an error analysis. If the angles are obtained by the repetition method of observations, *(Clark, 1966)*, although they are still derived quantities, any particular direction is not used in the derivation of more than one angle, so such angles are independent and uncorrelated quantities and can be treated as normal observations in an error analysis.

As the observations taken in the field are directions rather than angles, the analysis set out later in this Chapter will be mainly concerned with directions.

Arc, Semi-Arc and Set. A *semi-arc* on stations B, C and D would be made up of the semi-directions, on one face, to stations B, C and D. An *arc* observed to these stations would be the mean of two semi-arcs observed on opposite faces, at the same circle setting or zero. A *set* of directions is the mean of a number of arcs.

3.3 Observing Procedure for Direction Observations

As Richardus *(1968, p.162-168)*, points out, there are two ways of observing a set of directions. These are, the observation of independent semi-arcs, and the observation of complete arcs. He favors the observation of independent semi-arcs on the grounds:-

1. The periodic or systematic part of circle graduation error is more completely eliminated as there are twice the number of circle settings. By changing the circle setting after each semi-arc, there are 2n independent samples, where m is the number of complete arcs. By changing the setting after each complete arc, there are only n independent samples.

2. If the circle setting is only changed after each complete arc, it is quite possible for the observer to mentally predict the readings on the second face from those taken on the first. If the values he actually reads do not coincide closely with his predicted readings, he may be tempted to reobserve, hence disturbing the distribution of the observations. If the circle setting is changed after each semi-arc, the readings in the first and second semi-arcs will be totally different and it is improbable that the observer will be able to predict the readings of the second from the results of the first.

However, the individual semi-arcs will contain some systematic errors, which can be minimised if the arc is treated as the unit of measurement. The observing procedure in this case is to change the circle setting only after the completion of each full arc.

3.4 Factors Affecting Precision

(a) Non-Verticality of the Vertical Axis.

If it is assumed that there are no manufacturing flaws in the theodolite, and that any non-verticality of the vertical axis is due to dislevelment of the instrument, then the error in horizontal circle reading caused by the vertical axis not being vertical can be shown *(Clark, 1969, p.78; Cooper, 1971, p.56-57)* to be:

$$e_v = i_v \tan h \sin \alpha \qquad \ldots (3.1)$$

where e_v is the error in horizontal circle reading, expressed in seconds,

i_v is the dislevelment, also in seconds

h is the angle of elevation to the target being sighted, and

α is the difference in direction between the target and the line of intersection of the horizontal plane and the plane defined by the perpendicular to the vertical axis.

It seems reasonable to make the following assumptions:-

1. That with careful levelling, the non-verticality of the vertical axis will not be greater than half a division of the plate bubble.
2. That even in mountainous regions, h for geodetic work would seldom be more than 10 degrees.

Given the plate bubble sensitivity of a particular theodolite, it is possible to estimate the maximum likely error in circle reading due to dislevelment. If this estimate is considered to be three standard deviations, then the standard deviation of horizontal circle reading due to this cause may easily be calculated. This was done for two typical theodolites;

Wild T2 - σ_v = 0.6 sec

Wild T3 - σ_v = 0.2 sec

It must be emphasised that this error is not cancelled by taking the mean of face left and face right semi-directions.

(b) The Line of Collimation not being Perpendicular to the Trunnion Axis.

The error in horizontal circle reading due to this cause may be shown, *(Clark, 1969, p.73; Cooper, 1971, p.58-60)*, to be:-

$$e_c = i_c \sec h \qquad \ldots (3.2)$$

where e_c is the error in horizontal circle reading, expressed in seconds,

i_c is the inclination of the line of collimation of the telescope to the normal to the trunnion axis, also in seconds, and

h is the elevation angle to the target being sighted.

i_c will depend on the position of the intersection of the cross-hairs with respect to the optical axis of the telescope.

The error e_c will be equal and opposite in sign for a face left and a face right pointing to the same target, and therefore can be eliminated by taking the mean of the two pointings.

(c) Lack of Precision in Manufacture.

Errors due to lack of precision in the manufacture of the theodolite, such as the trunnion axis not being perpendicular to the vertical axis, could be examined at this stage. However, theodolites used for geodetic and other precise work have reached such a standard that these errors tend to be negligible in the vast majority of instruments. This being so, it would seem invalid to analyse observations taken with an arbitary theodolite and attribute some of the estimated variance to errors caused by manufacturing inaccuracies. Further, simple field checks can be employed to verify that particular instruments are significantly free from such defects.

(d) Circle Graduation Errors.

Circle graduation errors may be divided into two classes; periodic errors and accidental errors. There are two types of periodic errors; long period and short period errors. The long period error is caused by irregularities in the spur gear of the circle dividing machine. In a theodolite such as the Wild T2, where two diametrically opposed graduations are used in reading the position of the circle, the period of this error is 180 degrees, or half the circumference of the circle. The short period errors arise from irregularities in the concave screw, (globoid worm), and in the cog wheel of the circle dividing machine to which it connects. The length of the period depends on the number of teeth on the spur gear of the dividing machine and on the smallest interval of division to be produced on the circle.

Accidental errors of circle graduation are caused by momentary changes in the circle dividing machine, which may be due to a speck of dust or oil, a small change in temperature, etc. The accidental errors do not repeat themselves in the same way as the periodic errors.

The wave form of the periodic errors is rather complex and, mathematically, is best represented by a Fourier series. It may be shown, (see Appendix A), that the periodic error in the mean result of four arcs, observed at zeros of 0°, 45°, 90°, and 135°, is free from the fundamental term and the first six harmonics of this series.

In early work, *(e.g. Ackerl, 1926)*, it was considered unnecessary to go beyond this, on the basis that periodic errors were of long enough a period to be eliminated at this stage. Evidence to the contrary has been found by Jochmann *(1956)* who has examined terms up to the twelfth harmonic. However, the general opinion is that these systematic errors are negligible after the sixth harmonic. Accepting this, the implication is that four arcs is sufficient to remove systematic errors of circle graduation, and that the mean of four arcs, is free of such errors. However, accidental errors still remain. Heuvelink *(1925)* gives a formula to calculate the variance of accidental errors, assuming that all systematic errors have been removed by the sixth harmonic.

Over the years, this formula has been evaluated for a number of instruments in a number of different tests. Zeiss Jena obtain a standard deviation of 0.18 seconds for their Theo 010. Lovell *(1964)* obtained 0.35 seconds for a similar instrument. Ackerl *(1926)* obtained 0.07 seconds for a Wild T2. Fondelli *(1956)* obtained values of 0.06, 0.09 and 0.08 for three Kern DKM3 theodolites.

It is difficult to make any reasonable estimates, using only these few results, but it would appear that for a single second theodolite of the Wild T2 or Theo 010 type, the standard deviation of accidental circle graduation errors is of the order 0.1 to 0.2 seconds, and for a geodetic theodolite of the Kern DKM3 type, of the order 0.1 seconds.

Systematic or periodic circle graduation errors should not have any affect on mean directions as long as at least four arcs are observed on zeros increasing by 180/ n for each zero, where n is the number of arcs to be observed.

(e) Other Systematic Errors Due to Instrumental Factors.

Factors such as drag and backlash in the theodolite, and twist in the tripod, can cause systematic errors in direction observations. Fortunately, these systematic errors can nearly always be eliminated or minimised by proper observing techniques, such as always taking the mean of face left and face right observations and always approaching targets from a clockwise direction in the first semi-arc and an anticlockwise direction in the second.

On the part of the target, factors such as assymmetry of the target and of its background can cause systematic errors. Surveying targets are usually symmetrical in shape, but the apparent centre of the target may move due to the phenomonon of phase. This occurs when the sun illuminates different parts of the target as it moves across the sky. The observer will tend to bisect the illuminated portion, (the portion he can see), and not observe to the physical centre of the target. However, this error will not be significant in the vast majority of cases.

(f) Pointing Error

Pointing errors are of two types, accidental and systematic. The accidental pointing errors arise partly from the atmospheric conditions prevailing along the optical path, and partly from limitations on the part of the observer and the instrument. Errors due to atmospheric conditions are, under most conditions, greater than those due to the observer and the instrument. The errors due to the instrument are partly a function of the optical conditions of the telescope (viz. Magnification and aperture), and are partly dependent on the efficiency and manufacture of the clamps and tangent screws.

Systematic errors in pointing will not, in general, show up in an analysis of internal variance (see Chapter 5). The reason being that most systematic errors due to instrumental factors are effectively cancelled by the use of a proper observing procedure and that those due to atmospheric factors will not be detectible, as in the short time over which the observations are made, the atmospheric conditions will not significantly change.

Investigations on pointing accuracy fall into two groups. Firstly, those dealing with indoor pointing i.e. free of atmospheric effects, and secondly, those dealing with outdoor pointing.

The investigation of Washer (1947) is representative of the first group. Recognising that the principal variables affecting the precision of telescope pointing at indoor targets are magnification, aperture and vernier acuity of the observer's eye, he performed a series of experiments in which magnification was varied, aperture and target brightness being held constant.

As the experiments were done by one observer, the variation in vernier acuity was kept to a minimum. He arrived at the relationship:

$$PE_s = \frac{4.962}{M} + 0.068 \qquad \ldots (3.3)$$

where PE_s is the probable error of a single pointing in seconds, and M is the magnification of the telescope in diameters.

Richardus *(1966, pp.51-52)* solves the probability equation for the normal distribution.

$$\text{prob } (x_1 < x < x_2) = \frac{1}{\sigma\sqrt{2\pi}} \int_{x_1}^{x_2} e^{\frac{-x^2}{2\sigma^2}} \, dx = \tfrac{1}{2} \qquad \ldots (3.4)$$

where $x_1 = -PE$

and $x_2 = +PE$

to give

$$PE = 0.6745\sigma$$

or

$$\sigma = \frac{PE}{0.6745} \qquad \ldots (3.5)$$

Using this relationship, equation (3.3) may be expressed as a standard deviation:

$$\sigma_{s.o.} = \frac{1}{0.6745} \left(\frac{4.962}{M} + 0.068 \right) \qquad \ldots (3.6)$$

or

$$\sigma_{s.o.} = \frac{7.357}{M} + 0.101 \qquad \ldots (3.7)$$

where $\sigma_{s.o.}$ is the standard deviation of a single pointing.

The constant 7.357 is the vernier acuity of the observer's eye. This figure is within the range of values obtained by different observers listed in a discussion of vernier actuity by Walls *(1943)*. Washer is of the opinion that the constant 0.101 is almost entirely due to turbulence in the short air column separating the telescope objective and the target.

If equation (3.7) is accepted, the standard deviation of a single pointing under indoor conditions with an experienced observer may be calculated for different instruments. For example the Wild T2 and T3 theodolites may be assessed as follows:

The Wild T2 has a magnification of 28 diameters, therefore

$$\sigma_{s.o.} = 0.364$$

The Wild T3 has three alternative eyepieces with magnification of 24, 30 and 40 diameters, therefore

$$\sigma_{s.o.} = 0.408 \quad (X24)$$

$$\sigma_{s.o.} = 0.346 \quad (X30)$$

$$\sigma_{s.o.} = 0.285 \quad (X40)$$

The investigations of Washer and Williams *(1946 &1947)* and Washer and Scott *(1947)* are fairly representative of work done on outdoor pointing accuracy. The work is of prime interest in surveying. In their investigation, the precision of a single pointing was measured for a single telescope with a variety of targets over distances ranging from 100 to 13,500 metres. A total sample of 4,700 pointings was taken by two observers on several different days, under a variety of weather conditions. Their results give the standard deviation of a single pointing as 0.92 sec. Washer and Williams *(1947)* also give a table of results of earlier investigations. Some of these results are so optimistic that it seems certain they have been obtained under indoor conditions. If such results are not considered then the average standard deviation calculated from the apparantely valid results is 0.89 sec, a figure not significantly different from Washer and Williams' own result. The fact that all the earlier formulae are a function of magnification and that their formula is independent of magnification, adds strength to the argument that, over geodetic distances and for magnifications in excess of twenty diameters, there is no significant correlation between magnification and outdoor pointing accuracy. *(Also found by Washer and Scott, 1947).* This conclusion could have been anticipated due to the dominant influence of atmospheric effects.

In addition, there seems to be no significant correlation between aperture and outdoor pointing accuracy. The results of Washer and Williams show some increase in standard deviation with decrease in aperture. However, it is quite possible that this effect may be caused by the decrease in illumination and contrast that accompanies a decrease in aperture.

Washer and Williams *(1946 and 1947)* also attempted to obtain some relationship between pointing accuracy and distance. The relationship obtained seems fairly meaningless when the scatter of the observations is considered and also when the greater influence of factors such as weather and visibility conditions are considered. It therefore seems reasonable to assume that there is no significant correlation between pointing accuracy and distance, once the atmospheric effects are taken into account.

It is of interest that, in this experiment, no appreciable difference in standard deviation between observers was found, and that the standard deviation fluctuated considerably from day to day, depending on the weather (atmospheric) conditions. The implication is that, under outdoor conditions, which are the conditions that prevail in surveying, the standard deviation of pointing should be reasonably constant for all experienced observers under similar atmospheric conditions.

The experiments previously mentioned *(Washer & Williams, 1946 & 1947; Washer, 1947, and Washer and Scott, 1947)* were virtually free from mechanical errors such as inefficiency of the clamps and tangent screws of the telescope, as the telescope was rigidly fixed and pointing was accomplished by the rotation of aweak prism in front of the telescope. A mirror was attached to this prism to reflect a beam of light on to a scale which was placed so that a small angular rotation of the prism would produce quite a large shift of the beam on the scale. This system effectively eliminated the mechanical part of pointing error. In the case of a theodolite, it is important to achieve the correct gearing relationship between the tangent screw and the reactions of the observer. *(Anon, 1947, p.29).*

Other sources of error would be faulty clamps and loose bearings in the theodolite. However, the art of the instrument designer and manufacturer has advanced to the extent that the influence of all these mechanical sources of error should be negligible, providing the instrument is properly maintained and adjusted.

It is worthy of note that Washer and Williams *(1946)* found a long period error or drift, which was usually superimposed over the short period errors. This drift was apparently due to changing atmospheric conditions (lateral refraction) as it was not found in the indoor pointing studies. The presence of such a drift emphasises the need for the pointings in individual sets of directions to be taken over a reasonably short period of time so that the individual means are not affected, and for the sets to be taken at intervals over a fairly long period of time to obtain a sample under as wide a range of conditions as possible.

The standard deviations of pointing, obtained by Washer and Williams *(1947)* and by the authors to whom they refer have been obtained under idealised conditions. Although their observations are under outdoor conditions in that the line of sight passes through varying atmospheric conditions, they are not subject to the full rigors of field conditions in that the observer is seated indoors and thus much more comfortable than the field surveyor would normally be. In addition, the telescope is not subject to the effects of sun and wind, and the targets used in the experiment are probably superior to the average surveying target. Further, an observer pointing to the same targets, thousands of times, will become more consistent than an observer under normal field conditions.

Therefore, while the experimental result is of interest in that it is a good indication of the possible precision of pointing, it cannot be assumed to be representative of the pointing precision that will be achieved under surveying field conditions. The figure $\sigma = 0.9$ sec will therefore be adopted as the standard deviation of a single pointing under idealised conditions.

(g) Reading Error

Reading error is an observer error and is the error made in reading the circles of a theodolite. For a given theodolite, this error will vary from observer to observer. As the purpose of this study is to make an estimate of the precision that the field surveyor is likely to obtain, experienced observers should be used as subjects. In the following analyses, two of the four observers are very experienced, while the other two have only a limited amount of experience. A field surveyor would probably fall somewhere between these two groups, so that a standard deviation derived from consideration of both these groups is probably representative.

The precision of reading is of course dependent on the type of instrument used as the different reading systems vary in accuracy, and for this reason the analysis was carried out for two instruments, the Wild T2 and T3 theodolites. The analysis consisted of each observer aligning the circle graduation marks and reading the micrometer fifty times. To eliminate the effects of backlash, the circle graduation marks were always aligned from the same direction.

The choice of fifty as the number of observations to be taken was rather arbitrary. It was necessary that the number of observations be large enough to give a statistically meaningful sample, yet not so large as to induce observer fatigue. In order to test for fatigue, it was decided to calculate the standard deviation of the first and second twenty five observations as well as the standard deviation of the total fifty observations. The standard deviation of the first twenty five observations was not consistently lower than that of the second twenty five, and therefore gave no definite evidence of fatigue. However, in some cases the differences between the three calculated standard deviations suggested that further analysis was warranted.

A programme was written, for a keyboard programmable calculator, to calculate and plot the progressive standard deviation as each observation was entered. In other words, when the n-th observation was entered, the standard deviation of the first n observations was calculated and plotted. Plots of standard deviation against number of observations were run for each set of observations. These are shown in figs. 3.1. to 3.8.

(i) Wild T3 Analysis

Four observers were tested in this analysis. The standard deviation plots for these observers are given in Figs. 3.1 to 3.4. The standard deviation of the total fifty observations is given, for each observer, in Table 3.1.

Observer	Standard Deviation (sec)
A. C.	0.28
H. M.	0.26
G. H.	0.35
A. K.	0.22

TABLE 3.1

In the cases of observers A. C. and A. K. this standard deviation does not seem to be valid. The plots for these observers (Figs. 3.1 and 3.4) level off fairly early and then after running level for a number of observations begin to rise again. This rise would appear to be due to observer fatigue and therefore the value of standard deviation where the plot is level would seem to be the more valid figure. This point may indicate the onset of fatigue, but the small number of sets taken does not allow any measure of certainty.

The plot of the observations taken by observer H. M. (Fig. 3.2) is fairly level from about observation twenty right through to the end. In this case the standard deviation of observation fifty appears to be the valid figure.

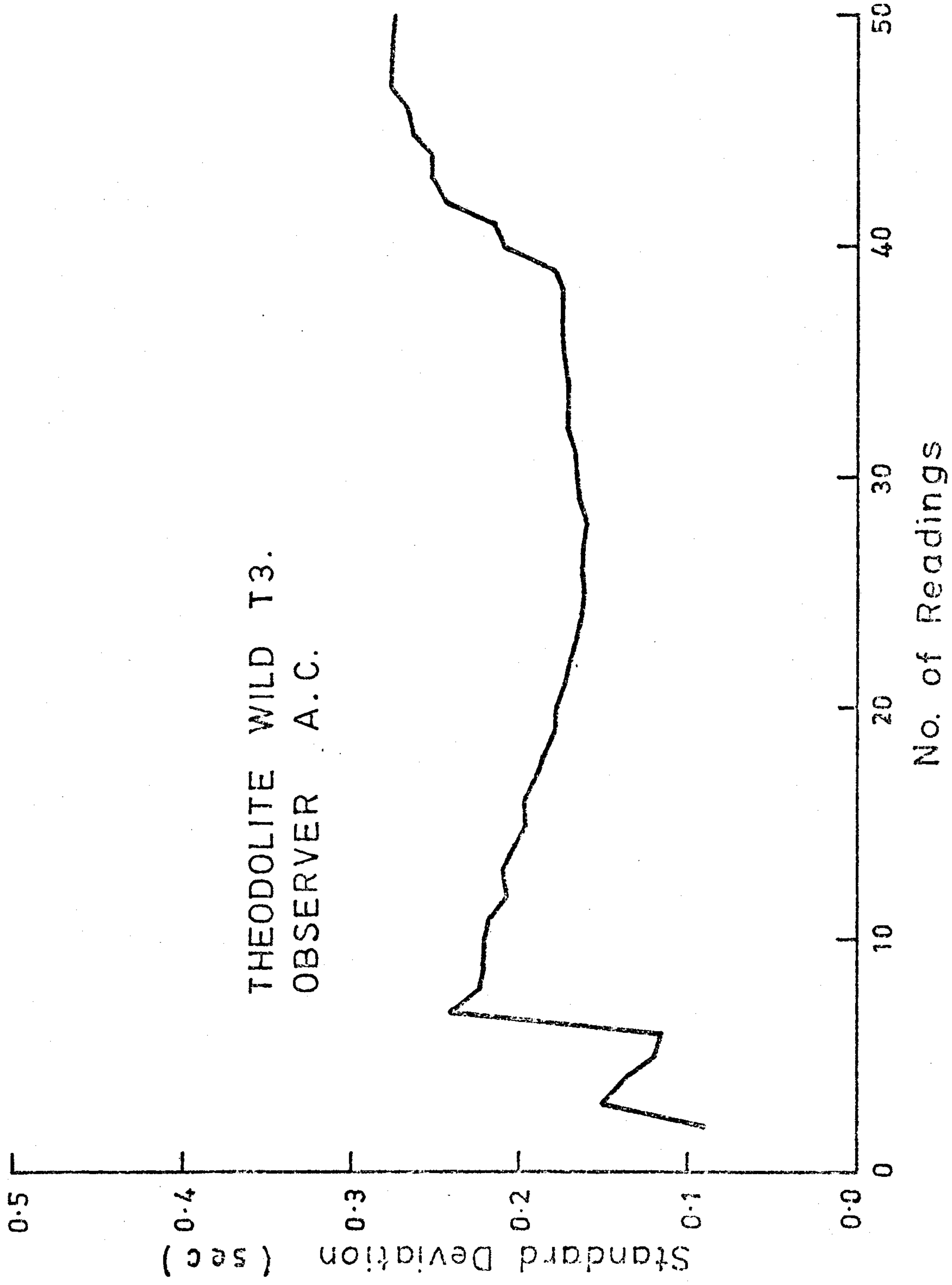

THEODOLITE WILD T3.
OBSERVER A.C.
Standard Deviation (sec)
0·0
0·1
0·2
0·3
0·4
0·5
0
10
20
30
40
50
No. of Readings

FIG. 3·1

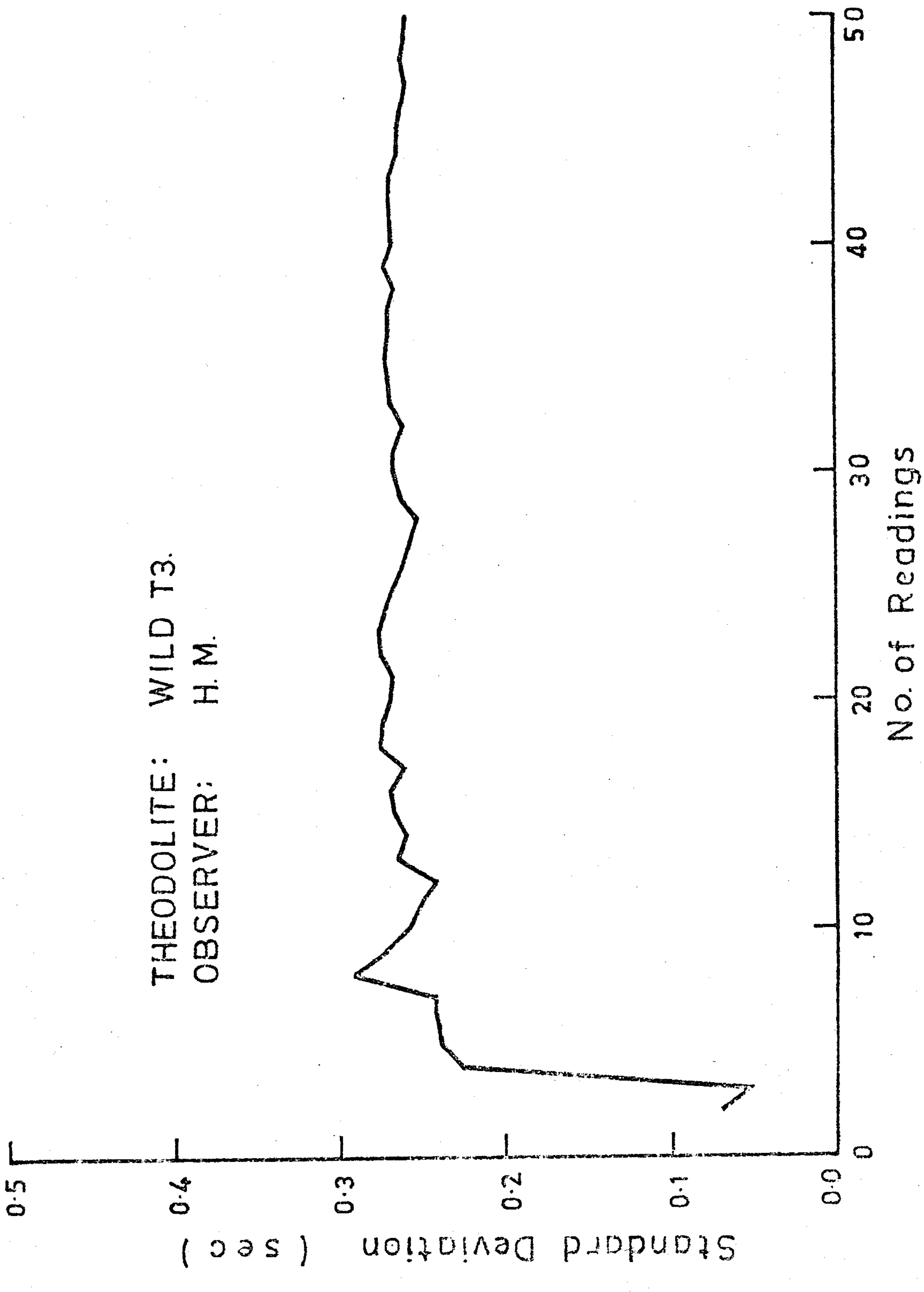

THEODOLITE: WILD T3.
OBSERVER: H. M.
Standard Deviation (sec)
0·0
0·1
0·2
0·3
0·4
0·5
0
10
20
30
40
50
No. of Readings

FIG. 3·2

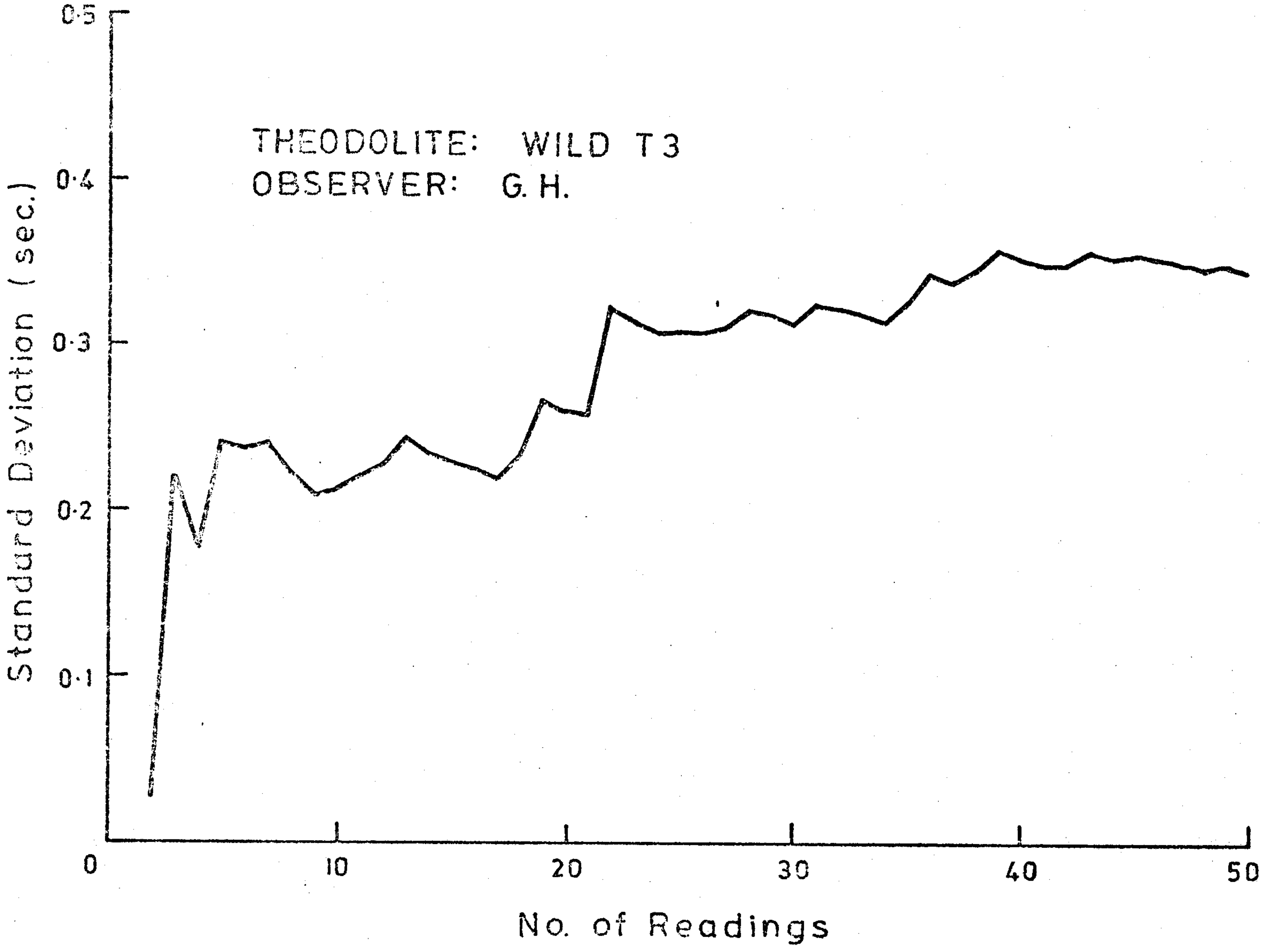
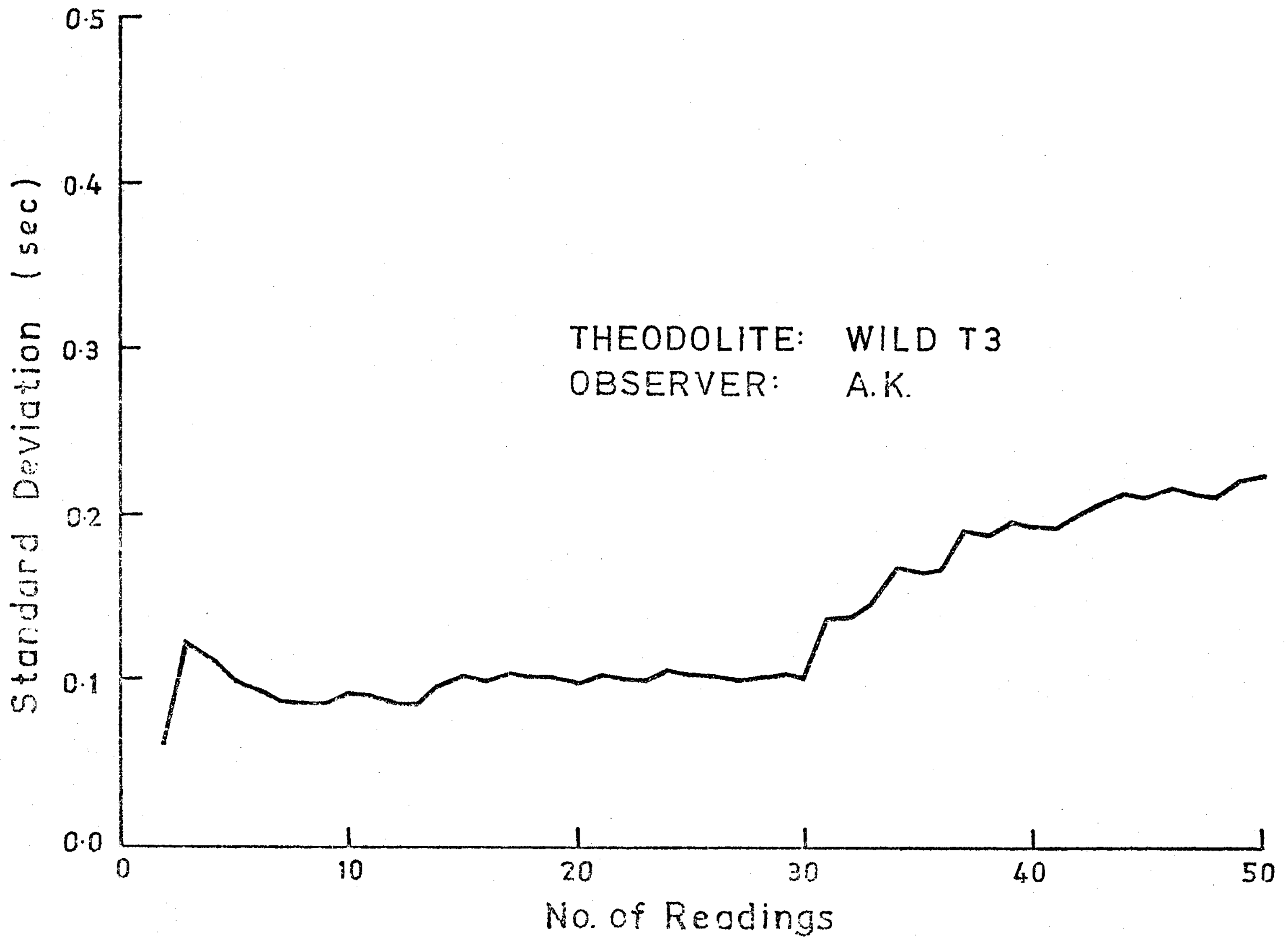

THEODOLITE: WILD T3
OBSERVER: G.H.
0·5
0·4
0·3
0·2
0·1
0
Standard Deviation (sec.)
10
20
30
40
50
No. of Readings

FIG. 3·3.



0·5
0·4
0·3
0·2
0·1
0·0
Standard Deviation (sec)
THEODOLITE: WILD T3
OBSERVER: A.K.
0
10
20
30
40
50
No. of Readings

FIG. 3·4

The observations of observer G. H. (Fig. 3.3) show a different pattern again. There are two level periods, one between observations twenty two and thirty four and the other between observations forty and fifty. In this case, it may have been warranted to take more than fifty observations to see if the plot started to rise again or remained level. This plot is rather hard to interpret as one is not sure which of these level periods indicates the more valid standard deviation. The first level period could be interpreted as the valid indicator, and the rise after observation thrity four interpreted as due to fatigue. It is of interest that this rise, thought to be due to fatigue, occurs at observations thirty nine and thirty for observers A. C. and A. K. (Figs. 3.1 and 3.4) respectively. If this is assumed to indicate a pattern, the rise in the plot for observer G. H. at observation thirty four, would seem to fit this pattern. On this basis, the level period between observation twenty two and thirty four, will be taken as indicating the more valid value of standard deviation for observer G. H. Even if the level period between observations forty and fifty is the true indicator, it will not cause a significant error in the final result as the difference in standard deviation between the two level periods is only about 0.04 sec.

The adopted standard deviations are given in Table 3.2.

Observer	Standard Deviation (sec)
A. C.	0.10
H. M.	0.26
G. H.	0.32
A. K.	0.17

TABLE 3.2

The problem of finding an estimate of the standard deviation of reading for the Wild T3 still remains. A "mean" standard deviation may be obtained from the standard deviations given in Table 3.2 using the Law of Propagation of Variances.

$$\sigma_{Reading} = \left((\sigma^2_{AC} + \sigma^2_{HM} + \sigma^2_{GH} + \sigma^2_{AK}) \;/\; 4 \right)^{\frac{1}{2}} \qquad \ldots (3.8)$$

Substituting the values of Table 3.2 into 3.8

$$\sigma_{Reading} = \left((0.10^2 + 0.26^2 + 0.32^2 + 0.17^2) \;/\; 4 \right)^{\frac{1}{2}}$$

$$= (0.052)^{\frac{1}{2}}$$

$$\sigma_{Reading} = 0.23 \text{ sec}$$

Rounding to the nearest 0.05 of a second,

$$\sigma_{Reading} = 0.25 \text{ sec}$$

The standard deviation of reading a Wild T3 theodolite, (or similar), is therefore estimated to be 0.25 sec.

(ii) Wild T2 Analysis

A similar analysis was carried out for the Wild T2 theodolite. Once again four observers were used. The standard deviations of the total fifty observations are given in Table 3.3. The plots for these observations are given in Figs. 3.5 to 3.8.

Observer	Standard Deviation (sec)
G. H.	0.74
L. B.	0.56
A. C.	0.61
H. M.	1.05

TABLE 3.3

Unlike the Wild T3 observations, there was no evidence of fatigue in these observations. The plots had a general pattern of initial fluctuations followed by a level period which extended to the end of the observations. The figures given in Table 3.3 with one exception were very close to those adopted. The plot of observations by observer H. M. (see Fig. 3.8) departed from the general pattern described above. After some initial fluctuations there was a drop followed by a steady rise that shows no real indication of levelling off. The standard deviation of these observations was rather arbitarily taken as 0.9 sec. This was the value of the standard deviation about half way along the steady rise.

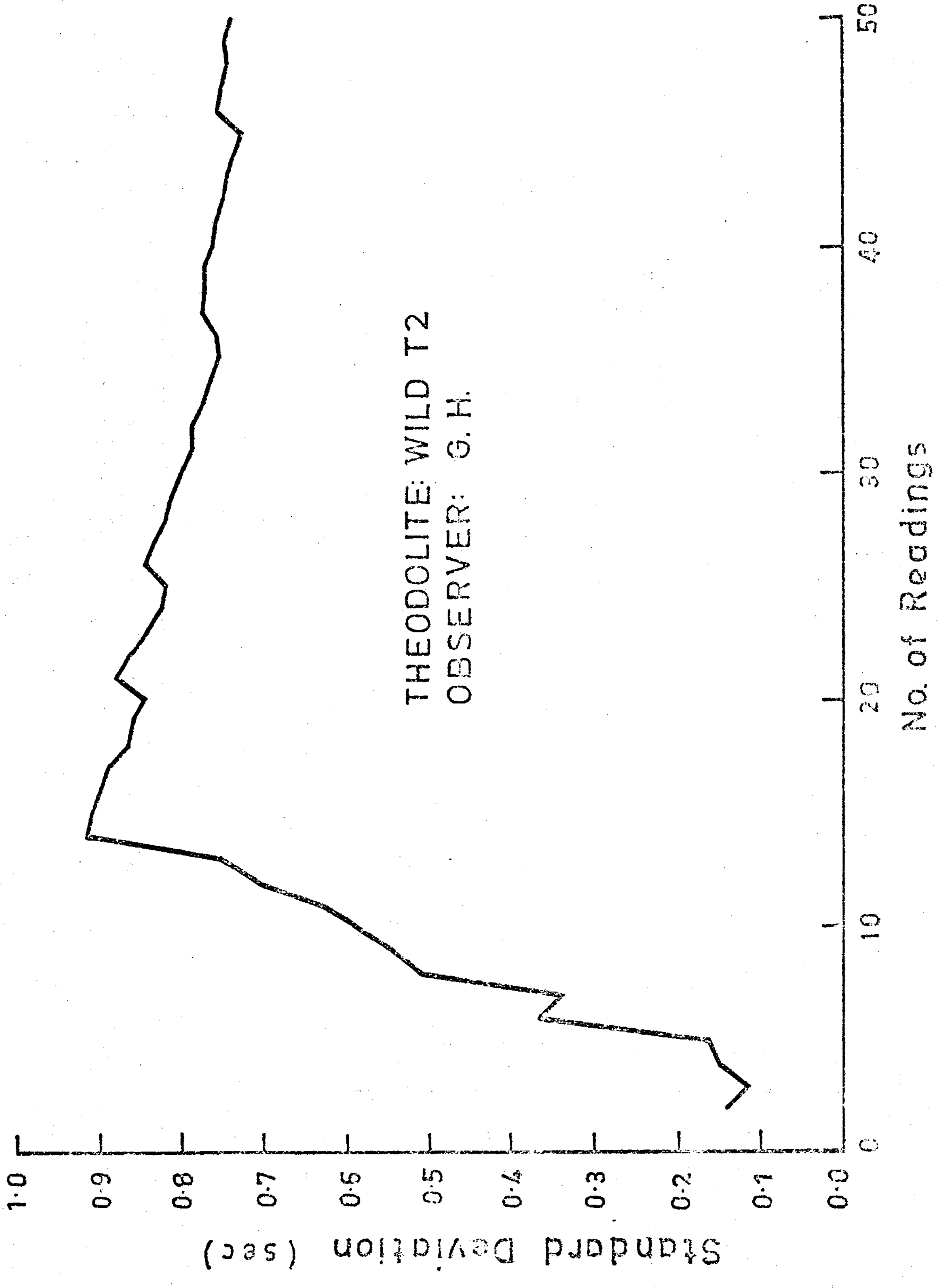
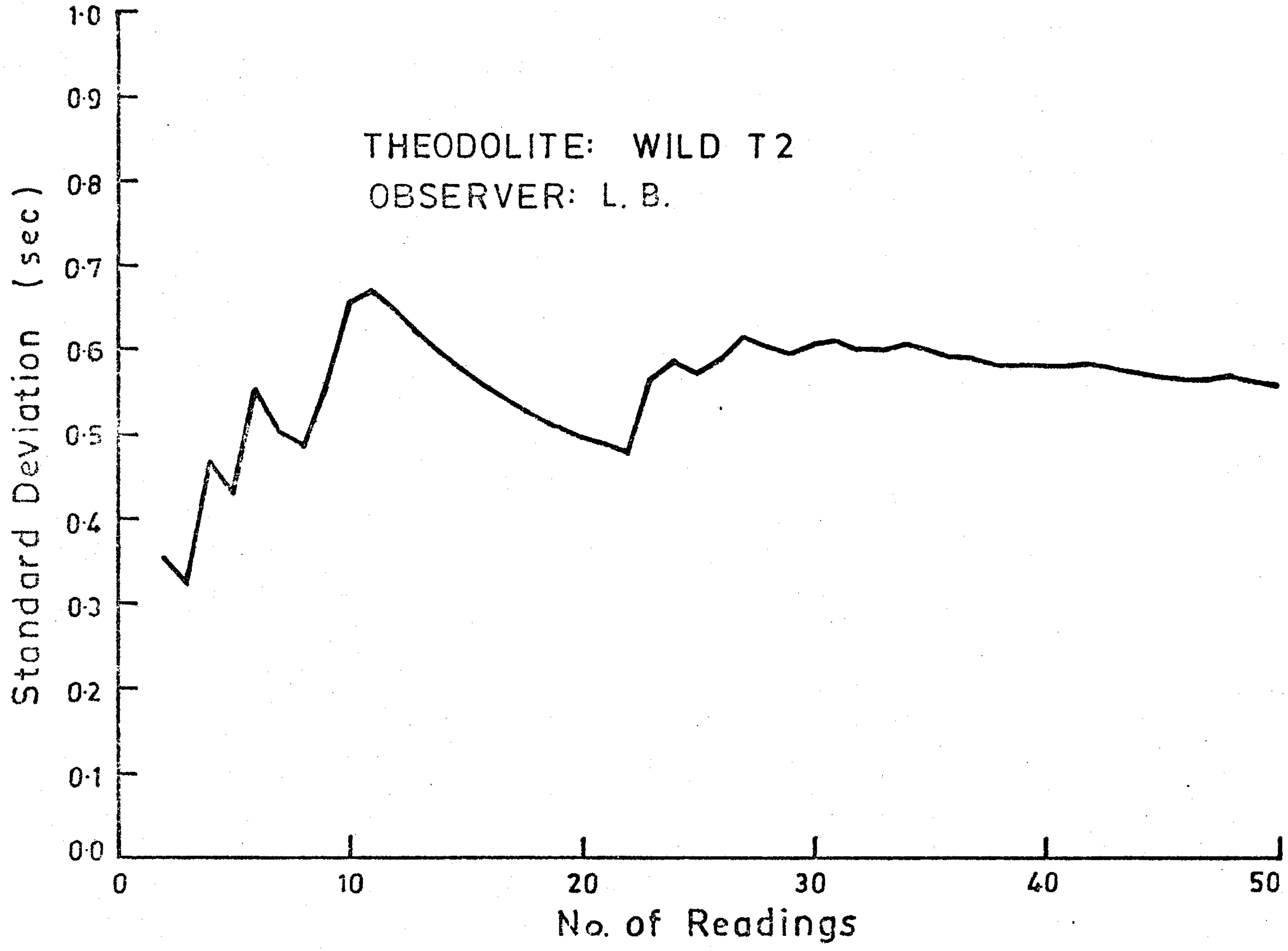

THEODOLITE: WILD T2
OBSERVER: G.H.
Standard Deviation (sec)
No. of Readings
0
10
20
30
40
50
0·0
0·1
0·2
0·3
0·4
0·5
0·6
0·7
0·8
0·9
1·0

FIG. 3·5



THEODOLITE: WILD T2
OBSERVER: L. B.
Standard Deviation (sec)
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0·0
0
10
20
30
40
50
No. of Readings

FIG. 3·6

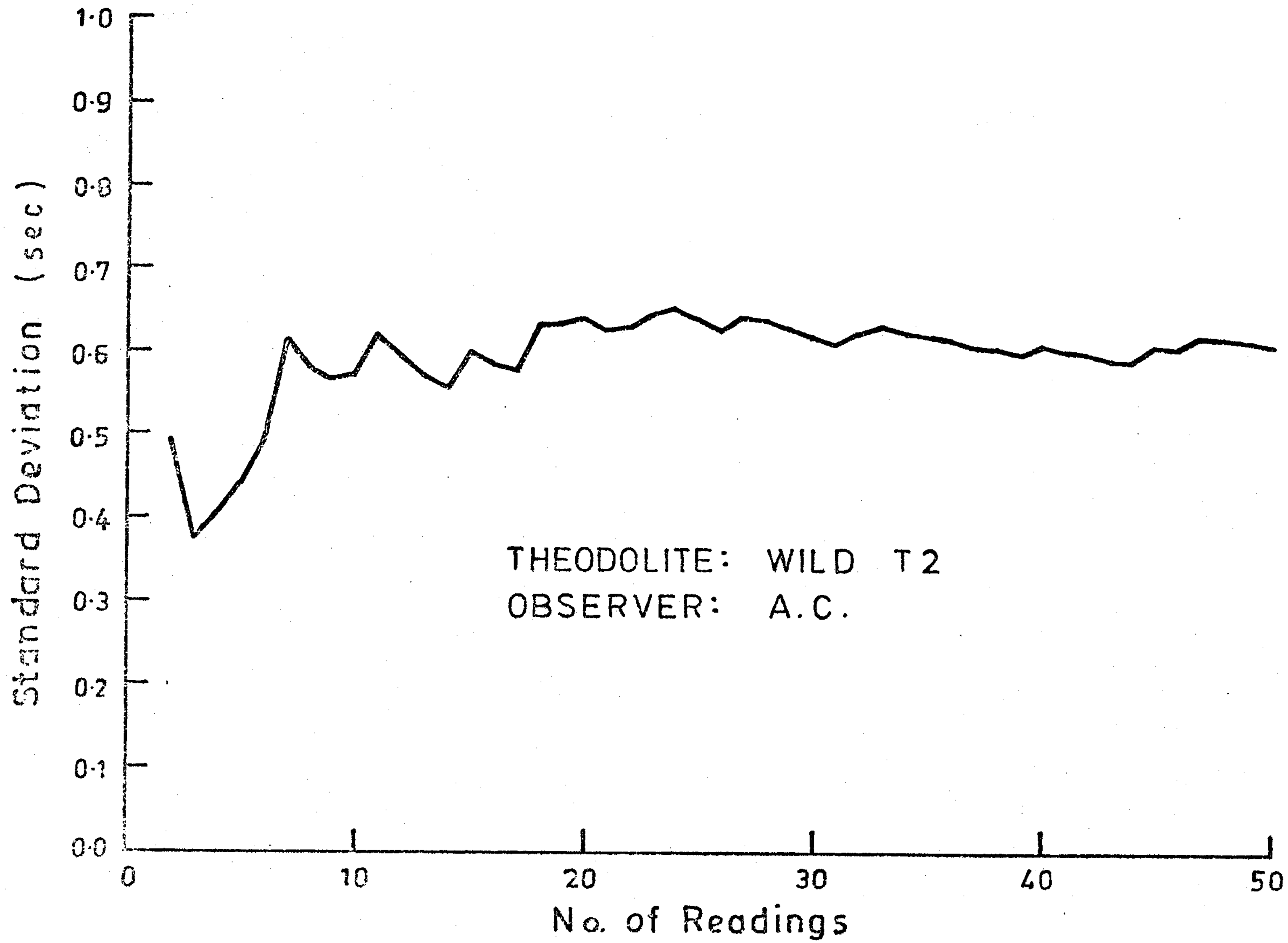
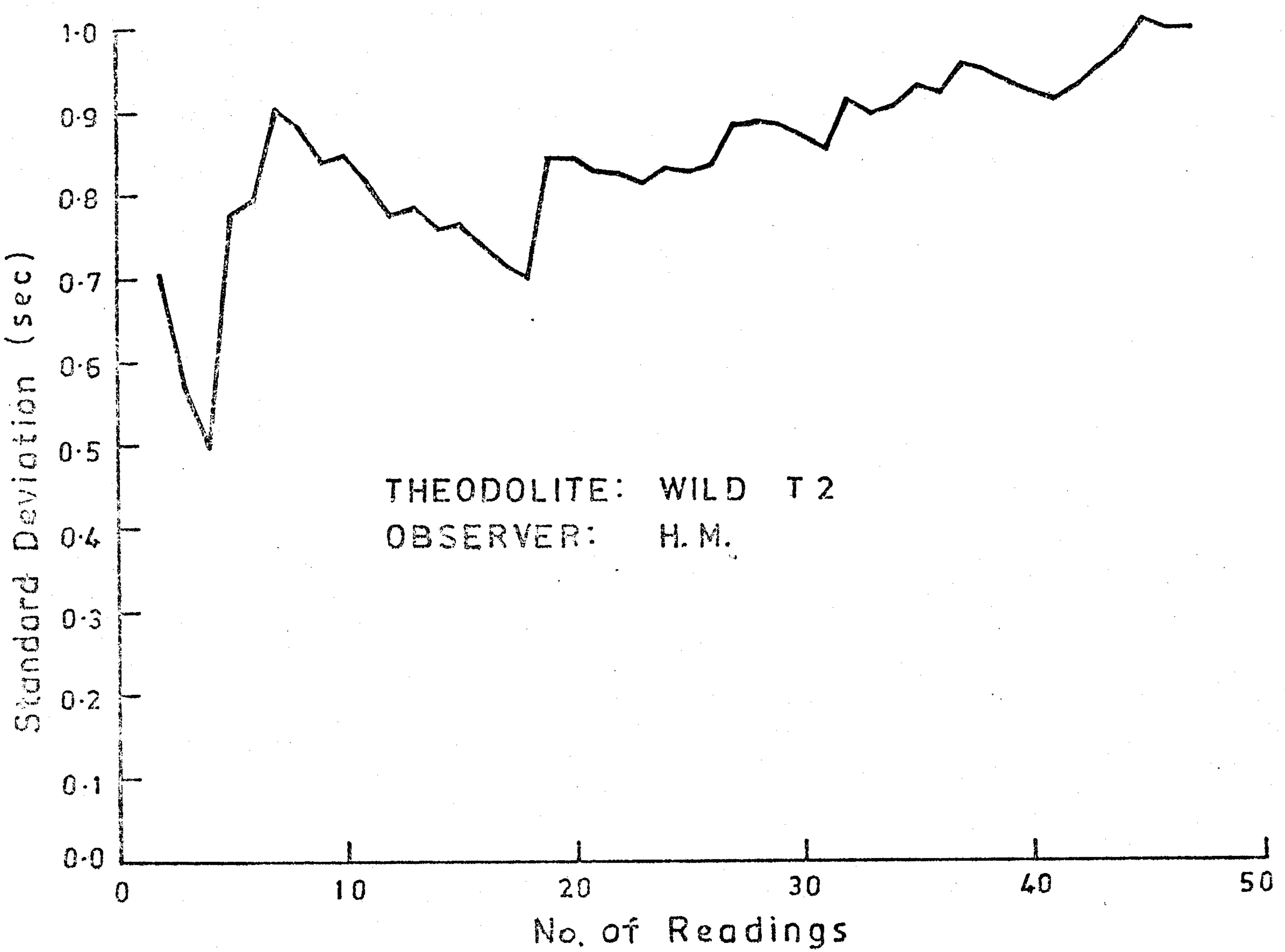

1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0·0
Standard Deviation (sec)
THEODOLITE: WILD T2
OBSERVER: A.C.
0
10
20
30
40
50
No. of Readings

FIG. 3·7



THEODOLITE: WILD T 2
OBSERVER: H. M.
Standard Deviation (sec)
1·0
0·9
0·8
0·7
0·6
0·5
0·4
0·3
0·2
0·1
0·0
0
10
20
30
40
50
No. of Readings

FIG. 3·8

The adopted standard deviation for each observer is given in Table 3.4.

Observer	Standard Deviation (sec)
G. H.	0.75
L. B.	0.58
A. C.	0.61
H. M.	0.90

TABLE 3.4

The values of Table 3.4 were substituted into equation 3.22 to give the estimate of standard deviation.

$$\sigma_{Reading} = 0.72 \text{ sec}$$

Rounding off to the nearest 0.05 of a second;

$$\sigma_{Reading} = 0.70 \text{ sec}$$

Therefore, the standard deviation of reading a Wild T2 theodolite, (or similar), given by this investigation is 0.70 sec.

This result may be compared with the results of a previous investigation by Watt *(1963)* . Watt gives standard deviations due to reading error, for a number of single second theodolites. His results are set out in Table 3.5.

Theodolite	Standard Deviation (sec)
Kern	1.05
Askania	0.40
Wild	0.56
Tavistock	0.43
Watts	0.42

TABLE 3.5

His standard deviation of 0.56 sec compares favourably with the figure of 0.70 sec obtained in this investigation. Undoubtedly his figure would have been obtained by an experienced observer, (or observers). In the present investigation only two of the observers (L. B. and A. C.) could be regarded as really experienced observers. The observers G. H. and H. M. are research workers with a limited amount of field experience. One would expect a standard deviation obtained from the observations of observers L. B. and A. C. to compare more closely with the figure obtained by Watt than the standard deviations of G. H. and H. M. This standard deviation was calculated by means of formula (3.22) as 0.58 sec. This is so remarkably close to the result obtained by Watt that it tends to confirm the method of interpretation (of the plots) used to obtain it.

The results obtained in both the T2 and the T3 analysis as well as those of Watt *(1963)* are not representative of the precision likely to be obtained in the field as they are obtained in laboratory tests under idealised conditions. They are more a measure of the best possible precision obtainable using the particular reading system. The significant differences between laboratory and field conditions are discussed in the previous section on pointing error.

3.5 Summary and Conclusions

It was stated in Section 3.2 that a direction observation is considered to be the mean of a face left and a face right semi-direction. This being the case, some of the factors discussed in the preceeding sections will not affect the precision of such an observation, as the effect of these factors is cancelled by taking the mean of the face left and face right pointings. Only the factors which affect the mean direction are summarized below.

The effect of errors in levelling, (non-verticality of the vertical axis), is not eliminated by taking the mean of face left and face right pointings. The error in the mean direction will be the same as the error in the individual pointing.

$$e_v = i_v \text{ Tan } h \text{ Sin } \alpha \qquad \ldots (3.1)$$

This formula was evaluated in the form of a standard deviation as $\sigma_v = 0.6$ seconds for a single second theodolite of the Wild T2 type, and as $\sigma_v = 0.2$ sec for a geodetic theodolite of the Wild T3 type.

It should be noted that errors due to dislevelment are components of external variance while the other errors given below are components of internal variance. (See Chapter 5 for definitions of internal and external variance).

The periodic or systematic part of circle graduation error was shown to be negligible when at least four arcs at zeros increasing in steps of 180/n degrees, where n is the number of arcs, were taken. Random errors still remain. The standard deviations of these random errors for single second and geodetic theodolites were estimated from the results of experiments found in a literature search. For a single second theodolite, σ_g was estimated as 0.2 sec, and for a geodetic theodolite, σ_g was estimated as 0.1 sec.

The pointing error in outdoor conditions, was seen to be independent of distance and of magnification, as long as the magnification was in excess of twenty diameters. Theodolite designer's and manufacturer's expertise appears to have reached the stage where mechanical sources of pointing error, such as inefficient clamps and the inappropriate relationship between tangent screw gearing and the reactions of the observer, produce negligible errors. There seems no reason to assume that geodetic theodolites have a smaller pointing error than single second theodolites, and the standard deviation of both, under ideal conditions, is estimated as $\sigma_p = 0.9$ sec. As a direction observation consists of a face left and a face right pointing, the standard deviation of a direction due to pointing will be taken as;

$$\sigma_p^2 = \left((0.9)^2 + (0.9)^2\right)/4 = 0.41$$

$$\sigma_p = 0.64 \text{ sec}$$

Reading error is the error made in reading the circles of the theodolite. This error seems to be reasonably constant for experienced observers on a given instrument under ideal conditions. It will vary from instrument to instrument depending on the precision of the reading system of the instrument. The results of the tests carried out and their agreement with the results found in the literature led to the estimation of the standard deviations due to reading error of the Wild T2 and T3 theodolites of 0.7 and 0.25 sec respectively. As two readings are taken for each direction observation, the standard deviation of a direction due to reading error may be calculated as follows;

$$\text{Wild T2: } \sigma_r^2 = \left((0.7)^2 + (0.7)^2\right)/4 = 0.25$$

$$\sigma_r = 0.50 \text{ sec}$$

$$\text{Wild T3: } \sigma_r^2 = \left((0.25)^2 + (0.25)^2\right)/4 = 0.03$$

$$\sigma_r = 0.18 \text{ sec}$$

An estimate of the internal standard deviation of a direction observation under idealised conditions may be calculated using the law of propagation of variances.

$$\sigma_d^2 = \sigma_g^2 + \sigma_p^2 + \sigma_r^2$$

where σ^2 is the variance of a direction observation, and σ_g^2, σ_p^2, σ_r^2 are as defined above.

For a theodolite of the Wild T2 type,

$$\sigma_d^2 = 0.04 + 0.41 + 0.25 = 0.70$$

$$\sigma_d^2 = 0.84 \text{ sec}$$

For a theodolite of the Wild T3 type,

$$\sigma_d^2 = 0.01 + 0.41 + 0.03 = 0.45$$

$$\sigma_d^2 = 0.67 \text{ sec}$$

The estimates of internal variance, calculated above, will tend to be quite optimistic as the component variances of pointing and reading are derived from laboratory experiments carried out under idealised conditions. It is not really possible to derive the component variances of field precision in an experimental environment as many of the more nebulous contributing influences are not likely to be taken into account. However, such experimental results are still important in that they indicate the proportional influences of the various contributing factors of observational variance. An analysis of actual field observations will give a more valid estimate of the total magnitude of internal variance but will not give the proportional influence of the contributing factors.

The results given above, will be discussed further in Chapter 8 when the results of analyses of field observations are given.

CHAPTER 4

FACTORS AFFECTING THE PRECISION OF LINEAR OBSERVATIONS

4.1 Introduction

The contempory surveyor has many tools and methods available to him for the measurement of distances. Only those systems which are in common use for geodetic surveys, the more precise types of engineering surveys and cadastral surveys will be considered in this report. Therefore, distance measurement by steel band and by short and medium range electronic equipment will be considered, whilst systems that are not in common use such as stadia, subtense bar, Loran, Shoran, Hiran, Aerodist and Hydrodist will not.

Factors affecting the precision of distance measurement arise from three sources:-

1. The instrument.
2. The observer.
3. The atmosphere.

Each of these sources will be considered in the following discussion.

4.2 Measurements by Steel Bands

(a) Introduction

"According to the Law of Propagation of Errors, the cumulative effect of constant errors will be proportional to the length of the line, while that of variable errors will be proportional to the square root of the length of the line." *(Clark, 1966, p.166).* Biesheuvel *(1962)* gives a formula for the standard deviation of a single span measurement by steel tape which is in agreement with the above statement, viz.

$$\sigma = (aS + bS^2)^{\frac{1}{2}} \qquad \ldots (4.1)$$

where a is the coefficient representing the variable or accidental errors of measurement,

b is the coefficient representing the constant or systematic errors of measurement, and

S is the length of the span.

Formula 4.1 may be modified using the Law of Propagation of Errors, to give the standard deviation for a distance, d, measured in n spans.

$$\sigma_d = (\sigma_1^2 + \sigma_2^2 + \sigma_3^2 + \ldots\ldots + \sigma_n^2)^{\frac{1}{2}} \qquad \ldots (4.2)$$

where σ_n is the standard deviation of the n-th span.
If all spans are of equal length, then

$$\sigma_1 = \sigma_2 = \sigma_3 = \ldots\ldots = \sigma_n$$

and

$$\sigma_d = (n(\sigma_1^2))^{\frac{1}{2}}$$

$$\sigma_d = (n)^{\frac{1}{2}}.\sigma_1 \qquad \ldots (4.3)$$

or

$$\sigma_d = [n(aS + bS^2)]^{\frac{1}{2}} \qquad \ldots (4.4)$$

If the lengths of the spans, S_i, are not equal, then

$$\sigma_d = [a(S_1+S_2+\ldots\ldots+S_n) + b(S_1^2+S_2^2+\ldots+S_n^2)]^{\frac{1}{2}} \qquad \ldots (4.5)$$

The constant or systematic errors of measurement include those due to alignment, to the band not being stretched straight, either horizontally or vertically, and those due to standardisation errors, errors in tension and temperature due to faulty or non-calibration of the spring balance and the thermometer. The variable or accidental errors include those due to the observer, the weather conditions such as wind and those due to random errors in tension and temperature measurements.

(b) Error Due to Alignment

These errors are always of one sign and will always tend to make the measured length too long. If d is the error in alignment of the end of the tape, (distance perpendicular to the line being measured), then the error in measured length, e, is

$$e = \frac{d^2}{2S} \qquad \ldots (4.6)$$

where S is the length of the band. Johnson *(1954)* points out that for a 100 metre band, with one end off line by 1 metre and the other end off line by 0.5 metres on opposite sides of the true line, an error of measurement of 11 mm will result. Errors due to this source should therefore be negligible if reasonable care is taken.

(c) Error Due to Horizontal Deformation

This error is due to the band not lying in a vertical plane between the end points. Once again this error is always of the same sign and will always make the measured length too long. If the error is due to the middle of the band (of length S) being D off line, then the error in the measured distance, e, is

$$e = \frac{2D^2}{S} \qquad \ldots (4.7)$$

If catenary measurement is being used, so that the band is off the ground for the whole of the length being measured, then errors due to this source will only arise if wind deforms the band.

(d) Error Due to Vertical Deformation

This error is due to the band not lying in a straight line in the vertical plane between the end points because of bumps and irregularities in the ground. Once again the error is always of the same sign and always making the measured length too long. If the error is due to a rise h at the centre of every 100 m length, the error, e, in a single band length measured by a band s metres long will be

$$e = \frac{2sh^2}{(100)^2} \qquad \ldots (4.8)$$

This error will not apply if catenary measurement is used.

(e) Error Due to Slope Measurement Errors

Errors from this source are of two kinds. Firstly, those of a systematic nature due to the maladjustment of the measuring instrument, and the consistent bisection and reading errors of a particular observer. These errors should be negligible if reasonable care is taken, if the slope is measured with a theodolite and if the slope does not exceed about 5 degrees. Secondly, there will be errors of an accidental nature. If a theodolite is used, these errors will be due to the factors discussed in Chapter 3. The error, e, in measured lengths, s, due to these errors may be expressed as:

$$e = \frac{s}{\rho} \sin \theta . \delta\theta \qquad \ldots (4.9)$$

where $\delta\theta$ is the error in slope measurement, in seconds,
and ρ is the number of seconds in one radian.

(f) Error Due to Error in Temperature Measurement

Temperature measurement for steel band chaining is usually by mercury thermometer. This method of measurement is unsatisfactory for two reasons, *(Campbell, 1971, p(v)):*

1. The mercury thermometer is not being used in the manner for which it was designed. The object is to measure the temperature of the band, but most thermometers are designed for total or partial immersion in a gas or liquid to find the temperature of that gas or liquid.

2. The band and the thermometer will not have identical heat capacities and therefore will reach different temperatures in a given period of time.

These factors can cause errors in measurement of the band temperature up to $10^{\circ}C$. The error in length measurement, e, due to an error δT in temperature measurement is:

$$e = s.c.\delta T \qquad \ldots (4.10)$$

where c is the coefficient of thermal expansion of the band and s is the measured length.

(g) Error Due to Error in Tension

Errors in length measurement due to errors in tension are of two types. Systematic errors due to faulty standardisation of the spring balance, and accidental errors resulting from slight variations in tension from bay to bay. The error in measured length, e, for an error in tension δF may be expressed as:

$$e = \frac{s.\delta F}{A.E} \qquad \ldots (4.11)$$

where A is the cross-sectional area of the band,
E is Young's modulus of elasticity, and s is the measured length.

Another appreciable error caused by errors in tension is the error in the sag correction. Once again this may be either systematic or accidental or both. The error in measured length due to the error in sag correction caused by an error in tension δF is:

$$e = 2.C.\frac{\delta F}{F} \qquad \ldots (4.12)$$

where C is the calculated sag correction for the particular distance, number of bays and assumed tension.

(h) Errors Due to Faulty Standardisation

These errors will be constant and of the same sign for the same length of the same band, but will be different in sign and magnitude for different bands and for different standardisations of the same band. The error is proportional to the length of band used for a measurement. If the standardisation error for the full length, S, of a band is K, then the error in a measured length, s, is:

$$e = \frac{s}{S}.K \qquad \ldots (4.13)$$

(i) Errors Due to Centering and Reading

Systematic errors from this source should be negligible, but there will be accidental errors, the magnitude of which depend on the method used in setting and reading.

(j) Evaluation of These Errors

Clark *(1966, p.170)* attempts to evaluate these errors. In doing this he places, what may only be seen as arbitary, values on the slope, temperature and tension measurement, and on the standardisation and reading errors.

For a 100 m steel band, 3.2 mm wide and 0.4 mm thick weighing 0.0115 kg per metre, used in catenary and supported at the 30 m and 60 m marks, he arrived at the formula:

$$P.\ E. = \pm\left((5.9 \times 10^{-6})N^2 + (10.4 \times 10^{-6})\ N\right)^{\frac{1}{2}} \qquad \ldots (4.14)$$

for the probable error of the measured length of a line, N tape lengths long.

For a single 100 m measurement this formula may be evaluated by placing N equal to 1, then:

$$P.\ E. = \pm(16.3 \times 10^{-6})^{\frac{1}{2}}$$

$$P.\ E. = \pm\ 0.004\ m$$

or by using the relationship between probable error and standard deviation, given in formula (3.5),

$$\sigma = 0.006\ m$$

This value does not sound unreasonable for cadastral and third or fourth order survey work. Experience on a large adjustment, recently carried out by Dr. J.S. Allman of the University of New South Wales, which involved approximately 1000 distances measured by two surveyors using different bands over a six year period, has shown that a standard deviation of 6 to 7 mm is a satisfactory figure for a measurement of this kind. This agreed very well with Clark's estimate.

The agreement between this estimate and Clark's estimate is surprising when firstly, Clark's rather arbitary estimation of the errors in slope, temperature, tension, etc., and secondly, the fact that Clark includes systematic errors in those measurements in his estimate of standard deviation, are taken into account.

In the normal course of events, the steel band would be standardised with the same thermometer and spring balance with which it will be used in the field. Therefore, any systematics in the spring balance will have no effect as the same reading is pulled during standardisation and in the field. Most bands are standardised on a fully supported base and in this case, the error in the sag correction of a field measurement due to a systematic error in tension will still remain. This error could be quite serious. In a single span measurement of 100 m, where 67N tension is being pulled on a 3.2 mm x 0.4 mm band, an error of 5N in tension will result in an error in measured distance of 0.015 m.

Fortunately, the error would seldom be of this magnitude as the vertical sag in an unsupported band of length 100 m, and of the same cross-section as above, will be 1.92 m. If the band was supported at every 30 m, as would more usually be the case, then the error in sag correction due to an error in tension of 5N will only be 1.5 mm.

The systematic errors in temperature measurement are very hard to evaluate and are quite variable in nature. Although they are systematic errors by definition, the only practical way to treat them is as random errors. Therefore, as long as due care is taken and sound field procedures are used, all temperature errors are best considered as accidental errors and should not be included in consideration of systematic error.

So, from a consideration of the factors affecting the precision of distances measured by steel band, it seems reasonable to suggest 40 mm^2 as a close approximation of the variance of chained distances up to 100 m, as long as sound field procedures have been used in the measurement.

4.3 Electronic Distance Measurement

The introduction of electronic distance measurement has had a very dramatic effect on the methods and organisation of surveying practice. Take as an example, a chain of geodetic triangulation. In the past, scale in the chain would be obtained by the measurement of base lines at intervals along the chain, using invar bands. This was a very time consuming process if any reasonable precision was to be obtained. With the advent of electronic distance measurement, such a baseline can now be measured in a matter of minutes as opposed to the days or even weeks previously required. The same or better accuracy is being obtained and because of the time saving, many more baselines may be measured. A significantly stronger network results.

Similar gains in time and precision have been made available to all other aspects of surveying.

The dependence of modern surveying on electronic distance measurement is such that a knowledge of the factors affecting its precision is vital.

Three classes of electronic distance measurement instruments will be considered in the following sections.

1. Instruments with carrier frequencies in the microwave portion of the electromagnetic spectrum.

2. Instruments with carrier frequencies in the visible light portion of that spectrum.

3. Instruments with carrier frequencies in the infrared portion of that spectrum.

(a) The Variance of an Electronically Measured Distance

The variance of an electronic distance measurement may be expressed, *(Chrzanowski and Derenyi, 1967)*, as:

$$\sigma^2 = (a + bs)^2 \qquad \ldots (4.15)$$

The first term 'a' is the result of changes in the electronic centre of the instrument, due to factors such as zero error and ground swing. It also includes the effect of errors due to reading and to the limits of phase resolution of the instrument.

The second term 'bs' is partly due to the uncertainty in the knowledge of the atmospheric conditions, and party due to the uncertainty in the value for the velocity of light. In 1957, the XIIth General Assembly of the International

Scientific Radio Union recommended that the best available value for the velocity of light was 299792.5 ±0.4 km/sec. This value was also accepted by the International Union for Geodesy and Geophysics. However, it now appears that 299792.46 km/sec may be a better value. *(Anon, 1972)*. The effect of the uncertainty in the velocity of light will not usually show up in variance, although it is present as a constant scale error in the measured distance.

Atmospheric measurements are normally taken at each end of the line and are seldom representative of the conditions along the whole length of the line. The longer the line, the less representative the readings at the terminals of the line will be, so the 'bs' term is therefore dependent on the length of the line, and is usually expressed as parts per million (ppm) of that length.

(b) Microwave Instruments

Microwave distance measurements use wavelengths of the order of a few centimetres. This system of measurement was first suggested by T.L. Wadley and is covered in his paper, "Electronic Principles of the Tellurometer" *(1958)*. The first commercially produced instruments, the Tellurometer models MRA1 and MRA2 had a carrier wavelength of 10 cm. The advantage of this wavelength was its ability to penetrate rain fog etc. and hence the MRA1 and MRA2 had a very long range, to the extent that the horizon tended to be the limiting factor. Burnside *(1971)* quotes a range of 150 km. The disadvantages of this wavelength were its very wide beam, (approximately 20° to the half power points) and its consequent susceptibility to ground reflection effects. While these instruments were very good for long range work, their lack of accuracy as well as their susceptibility to ground swing made them unsuitable for short range work.

These problems were, to a large extent, overcome by the "3 cm wavelength" generation of instruments. Tellurometer firstly produced two models, the MRA3 and MRA101 for military and commercial use respectively, and later another model, the MRA301; Wild produced the DI-50 Distomat; Cubic Corporation produced the DM-20 Electrotape; and Ertel produced the Distameter. These instruments did not have the same degree of fog and haze penetration and hence the range of the 10 cm instruments, but were quite superior in accuracy. The higher carrier frequency and smaller beam width (approximately 6° to 8° to the half power points) made them less susceptible to ground swing. The increased accuracy made this type of instrument suitable for all medium range distance measurement and all but the most precise short range distance measurement. This class of instrument was, and is, the most successful of the microwave instruments as it filled a definite need, that for a reasonably priced and accurate instrument, and has not really been superseded by more recent instruments, microwave or otherwise.

However, the demand for yet more accuracy led to the production of the Tellurometer MRA4. This instrument operates on an 8 mm carrier wave length and gives significantly better accuracy than the 3 cm instruments. The higher frequency and very narrow beam ($1\frac{1}{2}^{\circ}$ to the half power points) virtually eliminate the effects of ground swing. The shorter carrier wave length means a slightly shorter range and hence a more specialised use. The instrument is not widely used because of its specialisation, weight, bulk and cost. Its use is for very accurate geodetic work, consisting of lines of short to medium range.

The principles of operation of the microwave system will not be discussed in this report but are readily obtainable in a number of publications. (e.g. *Saastamoinen (1967)*, *Wadley (1958)*).

The factors affecting the precision of microwave distance measurement will be discussed in two groups, according to the two terms in the variance formula, (4.15):

1. Those contributing to the 'a' term.
2. Those contributing to the 'b' term.

In essence, these are instrumental and propagation errors respectively.

(i) Instrumental Errors

(a) Errors in Crystal Frequencies

The modulation frequencies of a microwave distance measurement instrument are produced by oscillating quartz crystals. An error of X parts per million (ppm) in the fine frequency will give an error of X ppm in the distances. The oscillation frequency of these crystals is dependent on temperature and can vary as the crystals age.

A drift in frequency due to temperature change will occur if the thermostat of the crystal oven fails to maintain the working temperature within the necessary limits. However, these ovens tend to be quite reliable and are capable of maintaining the frequency to one ppm. *(Marshall, 1967).*

Burnside *(1971, p.72)* quotes the rate of frequency drift with time as approximately 50 cycles per year. The results of the measurement of the modulation and carrier frequencies of two MRA101 Tellurometers are given in Hoar *(1969, p.183)* together with the values set by the manufacturer. It can be seen that the drift over two years is not more than 10 or 20 cycles away from the manufacturer's values. For the MRA4 Tellurometer, Bobroff *(1968, p.216)* and Yaskowich *(1968, p.231)* quote drifts of one to two cycles per month.

This drift with time tends to be fairly uniform and if the frequency is measured from time to time, distances measured between calibrations can be corrected for the drift by a linear interpolation depending on the date of measurement.

If the instruments are calibrated at fairly regular intervals, then the error due to variations in crystal frequencies must be considered negligible.

(b) Zero Error

Zero error is the variation of the difference in position, along the length of the line, of the electrical centre and the plumbing centre. The error is a composite error and is variable over the available range of carrier frequencies in the instrument and also over the cycle of the phase resolver. Zero error is due to spurious phase shifts from a number of sources. These sources can be placed into two groups; those due to circuitry components and those due to stray reflections from the small parabolic reflector in the Cassegrain reflector system. (See Marshall *(1967)* for a description of this system).

The phase shifts due to circuitry components are of two types. Firstly, shifts due to contamination between the AM and FM channels. The final phase resolution is between two 1 or 1.5 Khz. signals. (Depending on the make and model of the instrument.) One of these signals is amplitude modulated (AM) and the other is frequency modulated (FM). Contamination between these signals can occur and will result in an erroneous phase comparison, the magnitude of which Marshall *(ibid)* quotes to be of the order of a few centimetres for the MRA3 Tellurometer. Fortunately, the error is virtually eliminated by taking the mean of forward and reverse readings. The presence of contamination is usually indicated by a difference between forward and reverse readings. With improved technology the error has been eliminated on the MRA4.

The second source of phase shifts because of circuitry components is due to imperfections in the tuned circuits of the instrument. This source of error has been eliminated in Tellurometers (since and including the MRA3) by the addition of an extra circuit to lock the frequency of the phase comparison waves to their nominal values. If this had not been done, the error, which is partly a function of temperature, could amount to 30 mm in the MRA3 and 3 mm in the MRA4.

The phase shift due to antenna reflections would be quite serious if it were not for the fact that it is cyclic and is nearly cancelled out over the range of carrier frequencies. According to Marshall (*ibid*) this error for the MRA3 can vary from 50 mm up to, sometimes, 200 mm. However, the mean appears to be consistently accurate to within about 20 mm. With the improved antenna design of the MRA4, the error is only of the order 6 or 7 mm, the mean being correct to 1 or 2 mm.

Work by Robinson (*1968*) and Yaskowich (*1965, p.208*) on the model MRA101 and MRA3 Tellurometers respectively, indicate that the variations from the mean are of the order 15 mm, a figure significantly the same as that suggested by Marshall (*ibid*). This is also the figure given in the manufacturer's specifications. Therefore, for instruments with a 30 mm carrier wave length, the error due to zero correction will be taken as having a standard deviation of 15 mm.

Both Burnside (*1971*) and Yaskowich (*1968, p.230*) quote the error due to zero correction as having a standard deviation of 3 mm for the Tellurometer MRA4. This seems reasonable in comparison to the figure accepted for 30 mm carrier wave instruments when it is considered that zero error decreases with an increase in carrier frequency.

(c) Error Due to the Limit of Phase Resolution

Tellurometers up to and including the early versions of the MRA3 used a cathode ray tube readout system. The 1Khz. frequency was displayed as a circle and the relative phase of the returned signal as a break in this circle. This system suffered from many possible sources of error and Marshall (*ibid*) quotes the obtainable phase resolution as of the order one part in one hundred. For the MRA1 and MRA2 this corresponded to one nano second, or approximately 150 mm.

Later instruments used a phase resolver system. Marshall *(ibid)* describes the phase resolver as a device which produces a linear shift in phase with rotation of the rotor. In the Tellurometer system this resolver has been inserted in the 1.5KHz. channel. The resolver is geared to a dial or counter on the front panel of the instrument and can be rotated until the two 1.5KHz. signals are in phase. This condition is indicated by a null metre on the front panel of the instrument. Marshall *(ibid)* quotes an accuracy of one part per thousand for such resolvers. Burnside *(ibid)* quotes the same figure.

For the 30 mm carrier wavelength instruments, using a fine pattern frequency of 7.5KHz, one cycle of the phase resolver is equivalent to 10 m in distance. Therefore, phase resolution of one part per thousand is equivalent to distance resolution of 10 mm. For the MRA4, with a pattern frequency of 75KHz, one cycle of the resolver is only 1 m and therefore the distance resolution is 1 mm.

(d) Reading Error

As was pointed out in the above section, the more recent microwave distance measurement instruments have a readout consisting of a null meter and a digital scale or counter. There are two types of reading error that can be made using these instruments: the error in zeroing the null meter, and the error in reading the scale, or counter.

The results of an investigation made to determine the effects of errors made in zeroing the null meter of an MRA101 Tellurometer on the measured distance are given in Hoar *(1969, p.24)*. The investigation indicated a change in distance of approximately 20 mm for each scale division of the null meter. In normal conditions, the null meter is reasonably steady and the sensitivity of setting is such that it can be set to an accuracy better than one division with little difficulty. The error in a single reading due to an error in zeroing the null meter was accepted as having a standard deviation of 15 mm.

The error in reading the scale of this instrument is negligible as there is vernier reading to 10 mm.

The figure of 15 mm is in agreement with the figure quoted by Burnside *(ibid)*. This being the case, it would seem reasonable to accept Burnside's figure of 3 mm for the standard deviation of reading the Tellurometer MRA4.

In a normal measurement, forward and reverse readings would be taken on between 5 and 20 different carrier frequencies. As these readings are all meaned, the effect of reading error on a distance, measured with either a 30 mm or 8 mm carrier wavelength instrument, will be negligible.

(ii) Propagation Errors

(a) The Uncertainty in Refractive Index

It is normal practice in microwave distance measurement to take meteorological readings at both ends of the line, before and after measurement of the distance. A mean refractive index for the measurement is calculated from these values. The uncertainty in refractive index is due to two factors.

Firstly, the errors in the actual meteorological readings, and secondly the inability of the refractive index,

sampled at the two ends of the line, to accurately represent the mean refractive index along the whole length of the line. Burnside *(1971, p.73)* says that to ensure an error of less than 1 ppm from the first factor, the pressure reading should be correct to ± 2 milli-bars, the air temperatures should be correct to ± 0.1°C. These tolerances do not seem unreasonable but a previous investigation *(Hoar, 1969, p.26-28)*, suggests that they will give an error in refractive index of the order 3 ppm.

The degree of non-representation of the mean refractive index along the line, by meteorological readings taken at the terminals, is much harder to evaluate. Studies by Richards *(1965)* suggest that very little improvement is to be obtained by taking multiple meteorological readings along the line, between the terminals. However, this conclusion is for a particular line and cannot be extrapolated, as the variation in refractive index between the two terminal stations is very much dependent on the topographic nature of, and the prevailing conditions along the new line. My own experience on average lines tends to support Burnside's *(1971)* estimate of 3 ppm of the distance, for the average magnitude of this error.

Some work has been done on the use of atmospheric dispersion as a means of measuring refractive index, (e.g. *Thompson and Wood, 1965 and Owens, 1968)*, but systems using this technique are not yet available to the field surveyor and will not be discussed here.

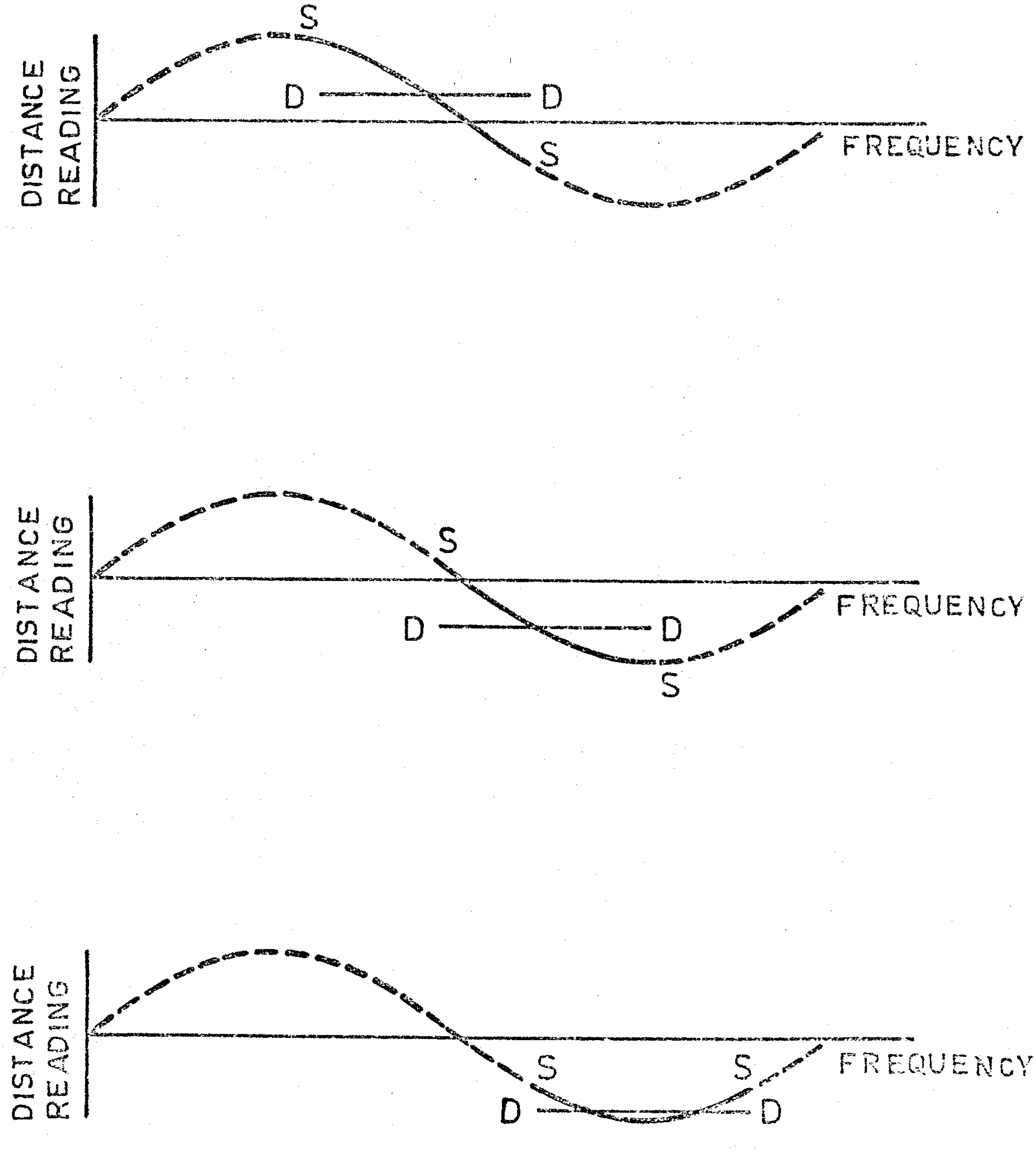
Considering all factors, the error in measured distance, due to the uncertainty in refractive index, appears to be 5 to 6 ppm of the distance, for microwave instruments under average conditions.

(b) Ground Swing

Ground swing is the term used to describe the error in measured distance caused by ground reflections of the microwave beam between the two instruments. The signal received is a vector combination of the direct signal and that which is reflected from the ground or other objects which may be in the path. The error is cyclic and variable. It depends on the carrier and modulation frequencies, on the length of the line and the height of the instruments, and on the reflectivity of the ground surface. The mechanics of ground swing are explained in Poder *(1962)*, Kupfer *(1967)* and Hoar *(1969)*.

The actual ground swing error is the difference between the mean of distance readings taken over a range of carrier frequencies and the true distance. The problem is that the cyclic form of ground swing is frequently of quite long wavelength and only a small part of the wave may be sampled over the range of carrier frequencies. This is illustrated in figure 4.1. The correct distance is the mean of the total swing curve and is represented by the frequency axis in Fig. 4.1. The full line part of the swing curve, SS, is the part that is sampled over the range of carrier frequencies. The measured distance is represented by the line DD. It can be seen that if the swing is large, the mean of an incomplete curve could introduce a considerable error into the distance.

For 30 mm carrier wave instruments, this error is usually taken as having a standard deviation of 15 mm. This value is derived from numerous measurements obtained under varied conditions in all parts of the world. The work done by Cabion *(1965, p. 184)* on prototype models of the Tellurometer MRA4 included a theoretical examination of the possible ground swing errors. The probable value of the error appears to be of the order, 3 mm. Yaskowich *(1968, p.226-230)* found ground swings of the order 50 to 100 mm using the MRA4 on some very extreme lines. These lines were extreme as ground swings of 7 m were being obtained with MRA3 instruments. On less extreme lines, ranging in length from 2 km to 9 km, Hall *(1967)* observed swings of the order ± 9 mm. Of this, about half is due to "antenna swing"



SS - OBSERVED PART OF SWING CURVE

DD - MEASURED DISTANCE

FIG. 4·1

(phase shift due to antenna reflections), and about half is due to actual ground swing. Hall's findings are in line with more general experience which has shown the figure of 3 mm to be what is normally expected for the MRA4.

Ground swing errors have been included in this section on Propagation Errors as ground swing is actually a propagation problem, i.e. the problem of propagating the beam directly between the two instruments. It is dependent on the length of the line as this is a contributing factor to the excess path distance of the reflected beam as referred to the direct beam. This excess path distance is also dependent on instrument height, and can be altered by changing instrument height alone. Hence, ground swing error should not be included in the 'b' term of the variance formula (4.15) as this term is reserved for factors dependent mainly on distance. Therefore, this error will be considered with the instrumental errors that comprise the 'a' term of the variance formula.

(iii) Summary

In the foregoing discussion estimates were made of the magnitudes of the various errors affecting the precision of 30 mm and 8 mm carrier wavelength microwave instruments. These estimates may be combined, using the Laws of Propagation of Variances, to evaluate the 'a' and 'b' terms of the variance formula for these instruments.

The 'a' term will be made up of zero error, the error due to phase resolver limitations and ground swing errors.

$$\sigma_S^2 = \sigma_{ZE}^2 + \sigma_{PR}^2 + \sigma_{GS}^2 \qquad \ldots (4.16)$$

where σ_{ZE}^2 is the variance due to zero error,

σ_{PR}^2 is the variance due to the limitations of the phase resolver, and

σ_{GS}^2 is the variance due to ground swing errors.

In the previous discussion these variances were estimated to be;

	30mm Carrier (mm)	8 mm Carrier (mm)
σ^2_{ZE}	225	9
σ^2_{PR}	100	1
σ^2_{GS}	225	9

By formula (4.16),

		30mm Carrier	8 mm Carrier
	σ^2_s	550	19
and	σ_s	23 mm	4.4 mm

Therefore, the estimates of the 'a' term are 23 mm and 4.4 mm for the 30 mm and 8 mm carrier wave instruments respectively.

The 'b' term for microwave distance measurement is the uncertainty in refractive index, and will not vary significantly between 30 mm and 8 mm carrier wave instruments as they both use microwaves. Therefore, from the previous discussion, b = 6 ppm.

Hence, the estimates of the variance formulae for a single measurement are:

$$\sigma^2_s = (23 \text{ mm} + 6 \text{ ppm})^2 \text{ mm}^2 \qquad \text{... (4.17)}$$

for microwave distance measurement instruments using a 30 mm carrier wave, and

$$\sigma^2_s = (4.4 \text{ mm} + 6 \text{ ppm})^2 \text{ mm}^2 \qquad \text{... (4.18)}$$

for an instrument using an 8 mm carrier wave.

The manufacturers (1966) claim a standard deviation of $\sigma^2_s = (15 \text{ mm} + 3 \text{ ppm})^2 \text{ mm}^2$ for a single measurement using MRA101 Tellurometers under favourable conditions and subject to zero error calibration. Webley *(1965)* points out that "favourable conditions" implies:

1. Instruments which have been specially calibrated.

2. Optimum line conditions, i.e. a line which has minimum reflections and uniform refractive index along its length.

3. Optimum adjustment and performance of the instruments.

To obtain more realistic estimates of accuracy, Webley *(ibid)* carried out a series of tests under average field conditions, using instruments that were neither zero calibrated nor specially aligned or adjusted. For medium and long line traverses, he obtained the variance $\sigma_s^2 = (25 \text{ mm} + 4 \text{ ppm})^2 \text{ mm}^2$. However, as Jones *(1968)* points out, this result is far from conclusive because of the rather invalid way in which it was obtained. Robinson *(1971)* obtained the same value for the 'a' term, (25 mm), in a study involving many measurements of a short line, under a variety of conditions. Such research indicates that the result obtained from the present analysis is probably fairly close to the correct variance.

The manufacturer's estimate of the variance of the Tellurometer MRA4 is $\sigma_s^2 = (5 \text{ mm} + 3 \text{ ppm})^2 \text{ mm}^2$. The "a" term of this expression agrees well with the same term of formula (4.18). Field tests carried out by Hall *(1967)* indicate that the value of the "a" term is approximately 7 mm and is virtually independent of the number of fine readings taken and of whether the line was measured in both directions or not. Hall's findings were not based on a large number of determinations (15) and, hence, do not warrant any modification of the "a" term of equation (4.18). However, the value of the term must be regarded with caution as the above does indicate that it could be slightly optimistic.

(c) Instruments Using Visible Light as a Carrier Wave

Chronologically, distance measuring instruments using visible light as a carrier wave preceeded those using microwaves. Bergstrand designed the first experimental instruments in the nineteen-forties, and the first survey orientated instrument, the Geodimeter NASM-2 became available in 1950. This instrument was intended for the measurement of base lines to a very high order of accuracy. It is of interest to note that the accuracy obtained by this instrument is yet to be bettered. The range of the NASM-2 in darkness and under good conditions was up to 50 km. However, the instrument suffered from a number of problems, in that it was extremely heavy and bulky and required a truck to transport it, it was not designed for use with a tripod and had to be mounted on a bench type arrangement, making accurate plumbing extremely difficult. It's power consumption, (around 150 Watts), was high and a generator was required to run the instrument.

The NASM-4 series of Geodimeter followed. These were much more practical instruments as far as surveying was concerned. They were of a more practical size and weight and could be mounted on a tripod. The manufacturer's lack of understanding of surveying problems was still in evidence, as the NASM-4 mounting system was not really of good design and much time was wasted in centering and pointing the instrument. The accuracy of the NASM-4 was inferior to that of the NASM-2, but still was better than any other contempory means of distance measurement. The range of the standard NASM-4 was approximately 1500 m in daylight and up to 15 km at night in good conditions. The optional mercury lamp extended the range in darkness to about 35 km.

The NASM-4 series was superceded by the NASM-6 series, introduced in 1965. This series was a vast improvement from every point of view. With the standard tungsten lamp, the NASM-6 had a range of up to 5 km in daylight and up to

15 km in darkness. The optional mercury lamp extended the daylight range to 10 km and the darkness range to 25 km. The NASM-6 series was improved over a period of time with the introduction of the NASM-6A and the NASM-6B. The latest model in the series is the NASM-6BL. The light source in this instrument is a laser instead of the tungsten and mercury lamps used in previous models. The main advantage of the laser is that the daylight and darkness ranges are practically the same at about 25 km.

In an attempt to cater for long range geodetic work, the NASM-8 Geodimeter was introduced. Like the 6BL, a laser is used as the light source. However, this laser is considerably more powerful than that used in the 6BL and the range of the NASM-8 is extended to approximately 60 km. in good conditions. The accuracy of the instrument is the same as that of the NASM-6 series.

A model 700 Geodimeter has also been introduced. This model is a combined theodolite and distance meter giving readouts of horizontal circle, vertical circle and distance. The precision of angle measurement is similar to that of a single second theodolite, and the precision of distance measurement is the same as that of the other recent Geodimeter models. This instrument is designed for engineering type work and the range at 3 km is relatively short. The light source used is once again a laser.

The quantitative information contained in this section has been taken from a number of pamphlets, put out by the manufacturers of Geodimeter, the AGA Company of Sweden. These pamphlets are too numerous to list and hence are not included in the bibliography.

(i) Advantages and Disadvantages of the Geodimeter System of Measurement

The visible light used as a carrier wave in the Geodimeter instruments has a wavelength of the order 560 nm. The beam is therefore able to be collimated closely.

Ground reflections are uncommon because of the narrow beam and because of the fact that there are very few natural surfaces that will give a strong reflection at this wavelength. Hence ground-swing is not a significant source of error.

The optical system of the Geodimeter allows the internal path of the light beam to be more closely defined than the microwave beam of the Tellurometer system. Hence, the instrumental zero error of Geodimeter instruments is only a few millimetres. This small zero error is also partly due to the passive reflector used at the remote end of the line. Retro-reflective prisms, as their name implies, return the light beam along the same path that the incident beam followed. They are designed so that their calibration constant is not dependent on the angle of incidence of the beam. Hence, no zero error is introduced by the reflector.

The range of this type of instrument is limited by the fact that visible light tends to be absorbed by the atmosphere and by the fact that under daylight conditions, unwanted light may enter the receiving optics and give rise to considerable noise which reduces the sensitivity of the measurement process. Therefore, with the exception of the laser NASM-8, the range of Geodimeter instruments is very much less than that of microwave instruments. Even with the NASM-8, it is impossible to measure a line longer than about 35 km, if there is the slightest haze present.

The refractive index of visible light waves is relatively insensitive to atmospheric conditions. Because of this, the 'b' term of the variance formula is usually stated as 1 ppm if meteorological readings are taken at one end of the line.

The insensitivity of refractive index, the low zero error and the virtual lack of ground swing make the Geodimeter system extremely accurate. For most models, the manufacturers quote a standard deviation of a single measurement of

$$\sigma = 5 \text{ mm} + 1 \text{ ppm} \qquad \ldots (4.19)$$

It will be seen in Chapter 7 that although this figure may be slightly optimistic, it is significantly correct.

(ii) Factors Affecting the Precision of Instruments Using Visible Light Carrier Waves

The factors that were shown to affect the precision of microwave distance measurement systems also affect the precision of instruments using visible light carrier waves. The nature and properties of these factors were discussed in detail in the microwave section and this section will therefore be confined to a brief discussion of their effect on visible light carrier wave instruments.

The errors in crystal frequencies have the same effect as in microwave instruments. Once again, the crystals are kept at constant temperature by means of an oven, and the variation of frequency with age is not a source of error as long as this variation is monitored at fairly regular intervals.

It was pointed out above, that the zero error of visible light instruments is much smaller than that of microwave instruments as, by virtue of the optical system used, the geometric path of the beam is more closely defined. It is only of the order of a few millimetres.

Errors due to the limit of accuracy of the phase resolver, in this case a delay line system, still apply. The readout system in the later models is the digital tumbler system. Burnside *(1971, p.72)* points out that this system seems to be capable of better than one thousandth of a cycle resolution. As the delay line system operates over half a wavelength, a full wavelength of the fine frequency being 10 m, this phase resolution is equivalent to a distance resolution of better than 5 mm.

Reading errors in the Geodimeter are of the order 1 or 2 mm, but as each measurement consists of several phase readings on several frequencies, the effect of reading errors on the mean distance is negligible.

The refractive index of visible light waves is not as sensitive as the refractive index of microwaves to variations in atmospheric conditions. For the same errors in meteorological readings as were used in the microwave discussion, the error in refractive index is about 1 ppm. If, as was done in the microwave discussion, a similar error is added for the uncertainty in refractive index along the line, it may be estimated that the total effect of errors in refractive index is probably less than 2 ppm of the distance measured.

It was pointed out above that ground swing errors do not have a significant effect on measurements by this system.

Therefore, from a rather cursory examination of the errors affecting visible light distance measurement, the manufacturer's standard deviation of 5 mm + 1 ppm seems quite reasonable for the Geodimeter instruments. The only exception to this is the NASM-4 series which may be slightly less accurate. For these instruments, the manufacturers quote a standard deviation of "less than 10 mm + 5 ppm". However, tests by Jones *(1964, p.383)*, on the NASM-4D, indicated a standard deviation of 4.5 mm + 0.5 ppm, for a single observation using the short delay line technique of observing. Therefore it seems reasonable to accept the one variance, of $(5 \text{ mm} + 1 \text{ ppm})^2$, as being applicable to all Geodimeter instruments.

(iv) Instruments Using Infra-Red Light as a Carrier Wave.

The final class of electronic distance measuring equipment to be considered is the class which uses infra-red light as a carrier wave. These instruments are very accurate and are designed for cadastral and precise engineering surveys. In general, their range is short, seldom exceeding 3 or 4 km. Most instruments in the class use a carrier wave that lies in the portion of the infra-red spectrum adjacent to visible light, and has a wavelength of approximately 0.9 nm.

The chief reason for using infra-red light as a carrier wave is the ease with which the Gallium-Arsenide (Ga-As) diode, which is normally used as the source of radiation, can be modulated. Unlike incandescent or gas-discharge lamps, the Ga-As diode can be directly modulated in the high frequency range by varying its supply current. *(McCullough, 1972)*. As far as the effects of atmospheric absorbtion and dispersion are concerned, infra-red waves, in general, are only slightly better than visible light waves. However, the wave-lengths used for distance measurement are in the "infra-red window", a band of wavelengths which have high atmospheric penetration properties.

The first infra-red distance meters were released in 1968. At the present time there are about a dozen types of instrument available. A few typical instruments are the Wild DI10, the Hewlett-Packard 3800B and the Tellurometer MA100.

Wild quote an accuracy of ±1 cm, irrespective of distance, for the DI10. This figure is acknowledged as being pessimistic, as at ranges less than the maximum range, the effects of crystal frequency variations and changes in atmospheric conditions will be diminished. If the accuracy was quoted in terms of the variance formula (4.15), then a better knowledge of the actual precision would be available. Indications are that the standard deviation of the instrument is about (5mm + 5 ppm). The range is quoted as up to 2 km, depending on the atmospheric conditions and the number of prisms used.

Hewlett-Packard quote a standard deviation of (3 mm + 10 ppm) for the 3800 series. Once again this figure is pessimistic, and trials by Robinson *(1972)* have shown that a standard deviation of (3mm + 5 ppm) is more appropriate. The range of the instrument is in excess of 3 km and in ideal conditions, Robinson *(ibid)* states that measurements of over 4.5 km have been made.

Tellurometer quote a standard deviation of (1.5 mm + 2ppm) for the MA100. This standard deviation seems close to the correct figure. The range of the instrument is quoted as of the order 2 km. However, the number of prisms required to obtain this range is approximately nine. Distances up to 1.5 km can be measured using 6 prisms.

(a) Factors Affecting the Precision of Instruments Using Infra-Red Light Carrier Waves.

As with visible light instruments, these factors will not be discussed in depth as their effect on infra-red instruments is very similar to their effect on microwave and visible light instruments.

Errors in crystal frequencies are once again a factor affecting precision. Most instruments in this class do not incorporate an oven to keep the crystals at constant temperature. Therefore, the variation of crystal frequency with temperature is a significant source of error. This error is usually taken to be of the order 3 ppm of the distance measured. One exception is the Tellurometer MA100, which does incorporate an oven and errors in frequency due to temperature variations in this instrument should be negligible. The disadvantage of an oven is that a warm-up period is required before any measurement can be made. Errors in crystal frequency due to aging should be negligible as long as the variation with age is monitored from time to time.

The zero error of these instruments is extremely small. Most incorporate an internal path calibration system by which the path distance inside the instrument may be monitored. This factor combined with the factors mentioned in the discussion on visible light instruments, combine to give only a very small zero error, usually of the order of a few millimetres.

Errors due to the limits of accuracy of the phase resolver are in this case negligible. The phase resolvers used in this class of instrument are of such a quality that distance resolution better than 1 mm can be obtained.

Many infra-red instruments use an automatic readout system. In this case there is of course no reading error. In the instruments using a manual system, the reading mechanism tends to be sufficiently sensitive for reading errors to be negligible.

The error due to the uncertainty in refractive index is the same as for visible light instruments. Gort *(1970)* has analysed the refractive index formula and found that errors of $10^{\circ}F$ in temperature and 1 inch in pressure cause an error in refractive index (or distance) of about 15 ppm. Under field conditions it is fairly simple to obtain temperature readings to $1^{\circ}C$ and pressure readings to 2mb. The graphs given by Gort *(ibid)* may be used to show that these accuracies in meteorological readings will give an error in refractive index of about 1.5 ppm. Burnside *(1971)* quotes a different figure. He quotes an error in refractive index of 1 ppm for an error in temperature reading of $1^{\circ}C$ and an error in pressure reading of 2 mb. Averaging these figures and making an allowance for the changes in conditions along the line, the total effect on distance of uncertainty in refractive index seems to be of the order 2 ppm. There is provision on some instruments (for example the Hewlett-Packard) for the presetting of a refractive index factor, so that the readout is corrected for atmospheric conditions. The system simply changes the modulation frequency slightly to compensate for the difference between the nominal and the measured refractive index.

As was the case with visible light instruments, ground swing has no significant effect on measurements with infra-red instruments.

It is not possible to estimate a representative variance for all infra-red instruments, as the range of accuracies in this class is quite considerable. For example, the Tellurometer MA100 has a resolution of 0.1 mm while the Wild D110 has a resolution of 3 mm. From a consideration of the factors above, in particular the ovened crystals, the MA100 manufacturer's claim of $(1.5 \text{ mm} + 2 \text{ ppm})^2$ seems realistic. For most other instruments in this class, a variance of $(5 \text{ mm} + 5 \text{ ppm})^2$ is not greatly in error.

CHAPTER 5

THE ANALYSIS OF ANGULAR PRECISION

5.1 Introduction and Definitions

The purpose of this Chapter is to examine methods of analysing the precision of angular observations. Chapter 3 examined the factors contributing to angular precision and used the results to predict the precision that is likely to be achieved in the field. The present Chapter does the reverse, in that it is concerned with the analysis of observations that have already been made, in order to assess the precision that was achieved. Distance observations will be similarly treated in Chapter 6.

The total variance of an angular observation may be broken up into internal variance and external variance.

Internal variance takes into account the random errors due to the instrument, the observer using the instrument and the sighting conditions. It does not take into account errors due to discrepancies in levelling the instrument, and in plumbing it over the ground station. The errors due to the instrument include random circle graduation errors, (the observing procedure should be such as will cancel out systematic graduation errors, or at least reduce the final effect to the magnitude of random errors), effect of residual errors after adjustment of the instrument, and so on. The errors due to the observer include pointing and reading errors. In summary, internal variance is the variance found when the direction observations at a station are analysed by examination of the field books.

External variance is due to random errors which show up when the observations are used to form a network, as is done in an adjustment. Such random errors include those due to the plumbing and levelling of the instrument, random changes in the deflections of the vertical, asymmetry of opaque targets, and lateral refraction *(Bomford, 1952)*. Of these, the major contributing factor is lateral refraction. A consideration of these errors will help clarify the distinction between internal and external variance.

When an instrument is set up at a station, it is plumbed over the station and levelled. This initial plumbing and levelling is usually not altered during the course of the observations taken at that set-up. The deflection of the vertical is a constant at any given station. The random changes occur only from station to station. The error due to erroneous intersection of asymmetric opaque targets will not vary during observations taken at a single set-up except when the influence of phase is present. However, as was pointed out in Chapter 3, the influence of phase will not be significant in the vast majority of cases. The only factor which can change is lateral refraction. Experience has shown that, in nearly every case, as long as the observations are not spread over too great an interval of time, lateral refraction will not change significantly.

Therefore, the influence of these factors on observations taken in one set-up at a given station will be significantly constant. Hence, they will not show up when the internal variance or consistency of the observations is calculated, but they are still present in the observations as systematic errors of that set-up and station. When the consistancy or variance of all observations at all stations is considered, (as, in effect, is done in an adjustment), they become randomized and therefore will show up in the calculation of total variance. The external variance may then be found by subtracting the internal variance from the total variance.

TOTAL VARIANCE = INTERNAL VARIANCE + EXTERNAL VARIANCE ... (5.1)

The methods of analysing variance considered below are not consistent in that some test for total variance, some for internal variance, and some for a variance between these two. (i.e. Internal variance plus the variance due to some of the factors giving rise to external variance.) The nature and properties of each method will be discussed.

A number of methods of analysing the precision of angular observations have been put forward over the years. As they apply mainly to angles and directions, these two types of observation will be treated first.

5.2 Methods Based on Condition Closure

(a) Ferrero's Formula

This is probably the best known of the classical methods for estimating the variance of angular observations. The estimate of variance of an observed angle is given by:-

$$\bar{\sigma}_\alpha^2 = \frac{\sum\limits_n (\Delta^2)}{3n} \qquad \ldots (5.2)$$

where Δ is the misclose of an individual triangle,

n is the number of triangle miscloses considered,

and $\bar{\sigma}_\alpha^2$ is the estimated variance of an observed angle.

This formula may be proved as follows *(Ashkenazi, 1971 and Von Forstner, 1933)*:

Consider a triangle with angles α, β and γ.

$$\alpha + \beta + \gamma = (180 + \varepsilon) + \Delta$$

where ε is the spherical excess of the triangle.

If the angles are not correlated and have the same estimated variance $\bar{\sigma}^2_\alpha$.

then $\bar{\sigma}^2\ (180 + \varepsilon + \Delta) = \bar{\sigma}^2\ (\alpha + \beta + \gamma) = 3\bar{\sigma}^2_\alpha$

so $\bar{\sigma}^2_\Delta = 3\bar{\sigma}^2_\alpha$

If the individual triangles are not correlated,

$$\bar{\sigma}^2\ (180 + \varepsilon + \Delta) = \frac{\sum_n (\Delta^2)}{n}$$

$$\bar{\sigma}^2\ (\Delta) = \frac{\sum_n (\Delta^2)}{n}$$

Therefore

$$\bar{\sigma}^2_\alpha = \frac{\sum_n (\Delta^2)}{3n} \qquad \ldots (5.2)$$

Similarly, the estimated variance of a direction $\bar{\sigma}_d$ may be found.

The triangle condition now becomes: (See Fig. 5.1)

$$b - a + d - c + f - e = (180 + \varepsilon) + \Delta$$

In this case

$$\sigma^2\Delta = 6\sigma^2_d$$

Therefore

$$\sigma^2 d = \frac{\Sigma\Delta^2}{6n} \qquad \ldots (5.3)$$

by the same reasoning as given for the angle case above.

There are a number of disadvantages in Ferrero's method of variance estimation.

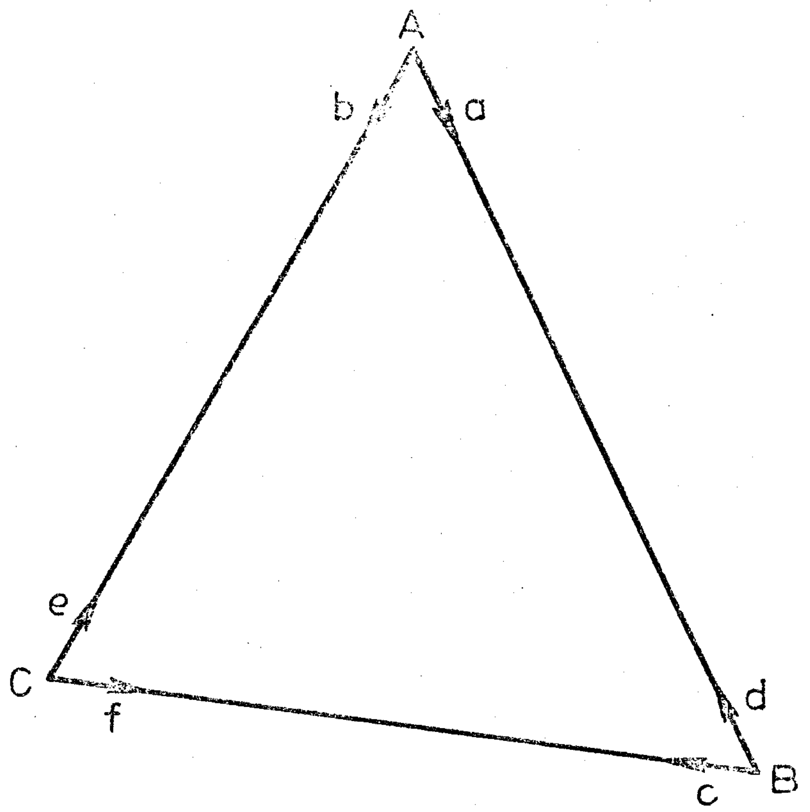


FIG. 5·1

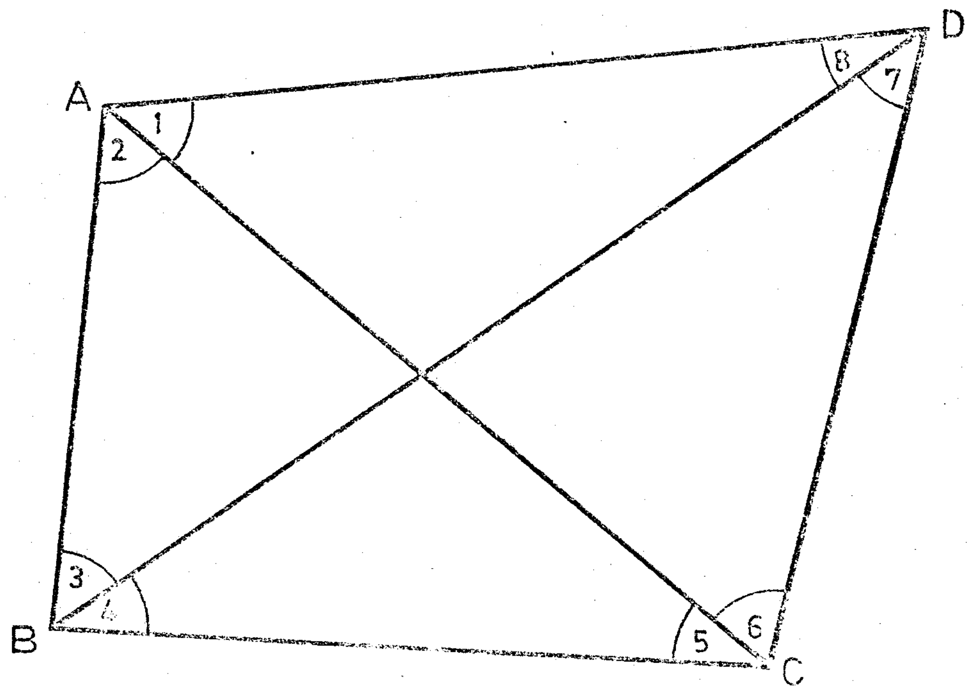


FIG. 5·2

1. Each angle can only be used once. i.e. In only one triangle. This presents no problem in networks of the single chain triangulation type as each angle occurs only in one triangle. If the chain is formed by braced quadrilaterals, as it more usually is, only half the triangle miscloures may be used to calculate the angular variance. This is no disadvantage if angles have been observed, in that the same $\Sigma\Delta$ will be obtained using different triangles, as the same angles are being used. (Refer fig. 5.2).

$$\left.\begin{array}{l} \text{Misclose } \Delta \text{ ABC} = 2 + 3 + 4 + 5 - (180 + \varepsilon_1) = \Delta_1 \\ \text{Misclose } \Delta \text{ ADC} = 1 + 6 + 7 + 8 - (180 + \varepsilon_2) = \Delta_2 \\ \text{Misclose } \Delta \text{ ABD} = 1 + 3 + 2 + 8 - (180 + \varepsilon_3) = \Delta_3 \\ \text{Misclose } \Delta \text{ BCD} = 4 + 5 + 6 + 7 - (180 + \varepsilon_4) = \Delta_4 \end{array}\right\} \quad \ldots (5.4)$$

It may be easily shown that,

$$\Delta_1 + \Delta_2 = \Delta_3 + \Delta_4 \quad \ldots (5.5)$$

As all the angles are used to calculate both the left hand side and right hand side of equation (5.5), it will make no difference which two of these triangles are used to calculate Ferrero's Formula. That is, there is no correlation between Δ ABC and Δ ADC or between Δ ABD and Δ BCD, so either pair may be used.

If directions are observed rather than angles, the miscloses would be calculated as follows. (Refer Fig. 5.3).

$$\left.\begin{array}{l} \text{Misclose } \Delta \text{ ABC} = 42 - 41 + 32 - 31 + 39 - 37 - (180 + \varepsilon_1) = \Delta_1 \\ \text{Misclose } \Delta \text{ ACD} = 41 - 40 + 36 - 34 + 33 - 32 - (180 + \varepsilon_2) = \Delta_2 \\ \text{Misclose } \Delta \text{ ABD} = 35 - 34 + 33 - 31 + 39 - 38 - (180 + \varepsilon_3) = \Delta_3 \\ \text{Misclose } \Delta \text{ BCD} = 38 - 37 + 42 - 40 + 36 - 35 - (180 + \varepsilon_4) = \Delta_4 \end{array}\right\} \quad \ldots (5.6)$$

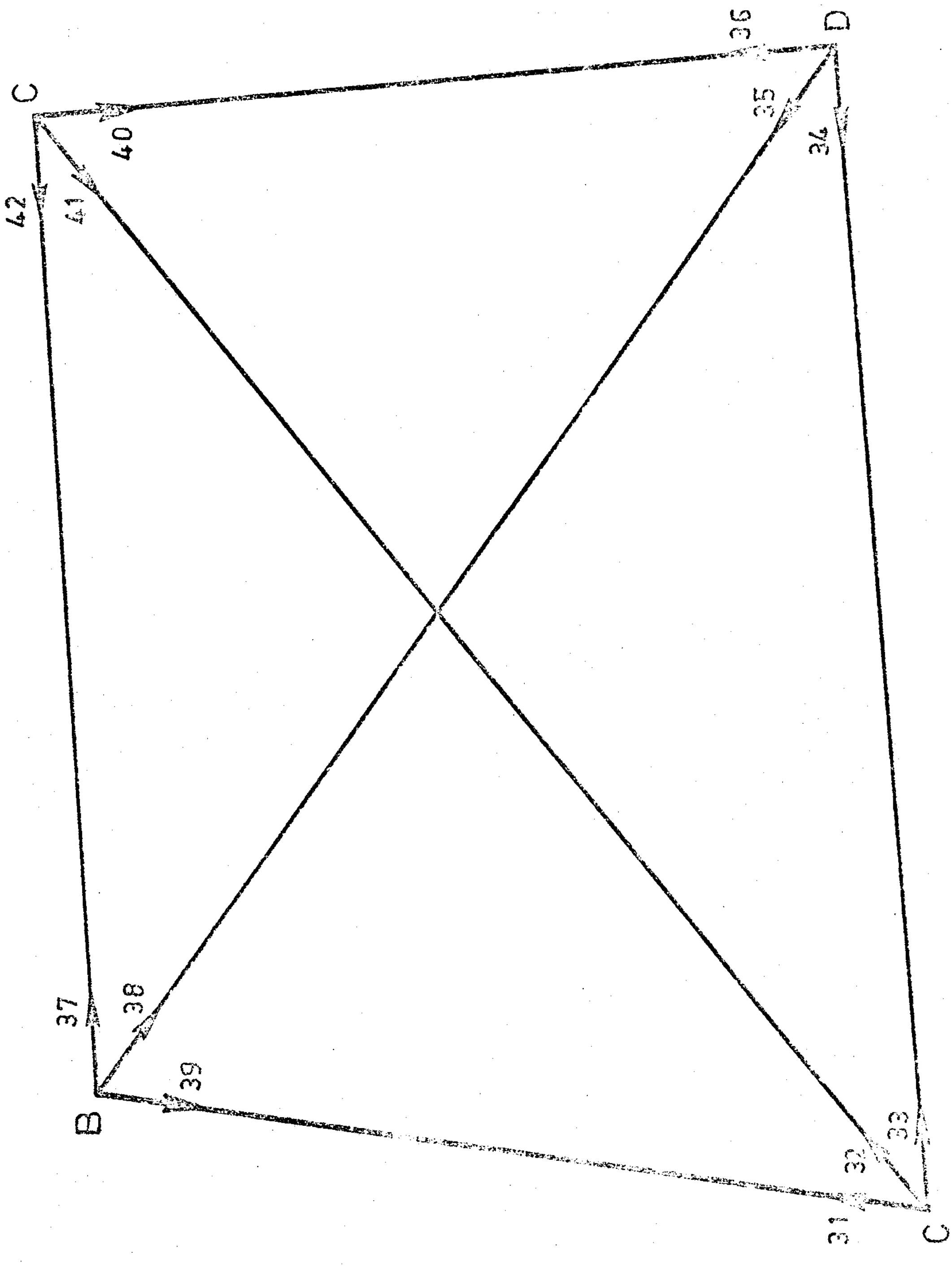


FIG. 5·3

Once again, equation (5.5) holds, but in this case Δ_1 and Δ_2 are correlated as they both use directions 32 and 41. Similarly, Δ_3 and Δ_4 are correlated as they both use direction 35 and 38. Δ_1 is also correlated with Δ_3 and with Δ_4. Therefore, when directions are observed only one triangle misclose from each braced quadrilateral can be used in the calculation of variance by Ferrero's Formula.

Similarly, if a single chain triangulation is being considered, only one triangle in every two can be used in Ferrero's Formula without introducing correlation.

2. The method cannot be used in cases where triangles are not formed. For example, traversing or where not all angles of the triangle are observed, as is the case in intersections, resections and radiations. This means that in most networks, not all the observed angles are used in the variance calculation, and hence, that the variance obtained may not be representative of the whole network.

3. Spherical excesses or arc to chord corrections may have to be calculated. This would normally not have to be done if the recent generation of adjustment programmes are used. These programmes usually accept field observations as data.

4. In smaller surveys, spherical excess is not needed to calculate the triangle misclose. It then becomes a matter of mental arithmetic to find the misclose and the way is open for the field surveyor to interfere with the observations. Usually this would be by reobserving until the misclose was inside an allowable limit. *(Askenazi, 1971)*.

5. The method does not take differences in variance from angle to angle into account. For precise work, this may be important.

The method basically tests for total variance. But, as individual triangles rather than the whole network are being tested, errors due to most of the errors contributing to external variance will not have their full influence in the estimated variance.

(b) Method Based on Side Equations

A method similar to Ferrero's method is one using side equations in which the variance of the angular observations is estimated using the formula:

$$\overline{\sigma}^2_\alpha = \frac{1}{n} \left\{ \frac{(M.\rho)^2}{\Sigma(\text{Cot}^2\alpha_i)} \right\} \qquad \ldots (5.7)$$

where M is the misclose term of the side equation,

ρ is the number of seconds in a radian,

α_i are the angles involved in the side equation,

and n is the number of side equations considered.

or in its logarithmic form,

$$\overline{\sigma}^2_\alpha = \frac{1}{n} \sum_n \left\{ \frac{(M.\rho)^2}{\Sigma_i (CD_i)} \right\} \qquad \ldots (5.8)$$

where CD_i is the common difference for one second in the log sine of α_i.

The method is totally unsuitable for networks with observed directions as it is impossible to form side equations using each direction only once. This may be seen if fig. 5.3 is examined.

Regardless of the selection of the pole, the directions toward the pole, along the lines radial to the pole, will always occur twice in the side equation. For example,

1. Pole at Centre

$$\frac{\sin BDC}{\sin ACD} \cdot \frac{\sin CAD}{\sin BDA} \cdot \frac{\sin DBA}{\sin CAB} \cdot \frac{\sin ACB}{\sin DBC} = 1$$

$$\frac{\sin (36-35)}{\sin (41-40)} \cdot \frac{\sin (33-32)}{\sin (35-34)} \cdot \frac{\sin (39-38)}{\sin (32-31)} \cdot \frac{\sin (42-41)}{\sin (38-37)} = 1 \qquad \ldots (5.9)$$

The directions 32, 35, 41 and 38 occur twice in the side equation.

2. Pole at A

$$\frac{\sin ACB}{\sin ABC} \cdot \frac{\sin ADC}{\sin ACD} \cdot \frac{\sin ABD}{\sin ADB} = 1$$

$$\frac{\sin (42-41)}{\sin (39-37)} \cdot \frac{\sin (36-34)}{\sin (41-40)} \cdot \frac{\sin (39-38)}{\sin (35-34)} = 1 \qquad \ldots (5.10)$$

The directions 34, 41 and 39 occur twice in the side equation.

This method of variance estimation, along with the other condition closure methods, is dependent on the fact that the directions are used only once. If this is not the case, then the estimate of variance obtained is not valid. An example of the invalid use of this method is the paper by Watermeyer *(1932)*.

The method does not seem consistent as it does not specify the pole to be used. If the pole is selected at A or at one of the other corner stations, the sample or number of directions being tested is smaller than if the centre is selected as the pole. This means that the method gives a number of different estimates depending on the pole chosen and hence on the proportion of the total sample used.

Even though it can be used with angles, it is still not a suitable method. There is a considerable amount of work involved in the calculation of the side equation misclosures when they would not otherwise be calculated. The method incorporates all the pitfalls of Ferrero's method, with the added pitfall that the sample of observations considered in the calculation is smaller.

5.3 Internal Variance from a Station Adjustment

(a) Introduction

A station adjustment is the adjustment of direction observations from one station to a number of other stations. A "station adjustment" is not to be confused with the least squares adjustment of the observations and station coordinates in a network. A station adjustment is a preliminary least squares adjustment at one station that is carried out to provide uncorrelated direction data for a network adjustment which includes that station along with many others.

As the adjustment only deals with the directions observed from one station, the variance obtained from it will be the internal variance of those directions.

(b) Station Adjustment by the Parametric Method

Let n be the number of stations observed, and s be the number of arcs read.

The observation equation of the i-th direction in the k-th arc is of the form,

$$D_i = p_i^k + v_i^k + 0^k \qquad \ldots (5.11)$$

where D_i is the adjusted value of the i-th direction,

p_i^k is the observed value of the i-th direction in the k-th arc,

v_i^k is the correction to the observed value of the i-th direction in the k-th arc,

and 0^k is the orientation unknown for the k-th arc.

The form of the normal equations is best seen by consideration of Table 5.1. The normal equations have been formed from the observation equations by the method given in Richardus *(1966, pp.126-127)*.

The number of observation equations is n.s, and the number of unknowns is n+s-1. That is, the directions to the observed stations (n), and the orientation corrections to all arcs except the first (s-1). Therefore, there will be n+s-1 normal equations.

It can be seen from Table 5.1 that the first n normal equations are of the form:

$$\left.\begin{aligned} sD_1 - {}_1^s\Sigma 0_k - {}_1^s\Sigma p_1^k &= 0 \\ sD_2 - {}_1^s\Sigma 0_k - {}_1^s\Sigma p_2^k &= 0 \\ &\vdots \\ sD_n - {}_1^s\Sigma 0_k - {}_1^s\Sigma p_n^k &= 0 \end{aligned}\right\} \qquad \ldots (5.12)$$

Observation Equations

D_1	D_2	......D_n	O^1	O^2	...... O^n		T
+1			-1				$-p_1^1$
	+1		-1				$-p_2^1$
		⋱	⋮				⋮
		+1	-1				$-p_n^1$
+1				-1			$-p_1^2$
	+1			-1			$-p_2^2$
		⋱		⋮			⋮
		+1		-1			$-p_n^2$
⋮	⋮						⋮
+1	⋮				-1		$-p^s$
	+1	⋮			-1		$-p^s$
		⋱ ⋮			⋮		⋮
		+1			-1		$-p_n^s$

Normal Equations

D_1	D_2	D_n	O^1	O^2	O^s		
s	0	0	-1	-1	......-1		$\sum_1^s -p_1^k$
	s	0	-1	-1	-1		$\sum_1^s -p_2^k$
	1	⋱	⋮	⋮	⋮		⋮
		s	-1	-1	......-1		$\sum^s -p_n^k$
			n	0	 0		$\sum_1^n p_i^2$
				n	 0		$\sum_1^n p_i^2$
					⋱ ⋮		⋮
					n		$\sum^n p_i^s$

TABLE 5.1

The remaining s-1 normal equations are of the form:

$$\left.\begin{aligned} -{}_1^n\Sigma D_i + nO^2 + {}_1^n\Sigma p_i^2 &= 0 \\ -{}_1^n\Sigma D_i + nO^3 + {}_1^n\Sigma p_i^3 &= 0 \\ &\vdots \\ -{}_1^n\Sigma D_i + nO^s + {}_1^n\Sigma p_i^s &= 0 \end{aligned}\right\} \quad \ldots (5.13)$$

It will not affect the solution to impose the condition

$$\sum_1^s O^k = 0 \quad \ldots (5.14)$$

Substituting (5.14) into the first n normal equations, (5.12), they simplify to:

$$sD_1 - {}_1^s\Sigma p_1^k = 0$$

$$sD_2 - {}_1^s\Sigma p_1^k = 0$$

$$sD_n - {}_1^s\Sigma p_1^k = 0$$

and hence:

$$\left.\begin{aligned} D_1 &= {}_1^s\Sigma p_1^k/s \\ D_2 &= {}_1^s\Sigma p_2^k/s \\ D_n &= {}_1^s\Sigma p_n^k/s \end{aligned}\right\} \quad \ldots (5.15)$$

The adjusted observations (5.15) are therefore the means of the directions from each arc, all reduced to a common orientation. It should be noted that the orientation unknowns are eliminated and that the adjusted values of the directions are free of correlation.

It may be deduced from the observation equations (5.11) that:

$$\sum_1^n D_i - nO^k - \sum_1^n p_i^k - \sum_1^n v_i^k = 0 \qquad \ldots (5.16)$$

and from the (n+k)-th normal equation: (See 5.13)

$$-\sum_1^n D_i + nO^k + \sum_1^n p_i^k = 0 \qquad \ldots (5.17)$$

Comparing (5.16) and (5.17), it can be seen that:

$$\sum_1^n v_i^k = 0 \qquad \ldots (5.18)$$

Or, in words, the sum of the corrections to all the observations in the k-th arc is zero.

This may be extended to show that the sum of the corrections to all the directions in all the arcs is zero.

$$\sum_1^s \sum_1^n v_i^k = 0 \qquad \ldots (5.19)$$

This result may be used to find the orientation unknowns. From the observation equations of the k-th arc, (5.11) let

$$\left.\begin{aligned} D_1 - p_1^k &= v_1^k - O^k = q_1^k \\ D_2 - p_2^k &= v_2^k - O^k = q_2^k \\ &\vdots \\ D_n - p_n^k &= v_n^k - O^k = q_n^k \end{aligned}\right\} \qquad \ldots (5.20)$$

Addition of equations 5.20 gives:

$$ {}_1^n\Sigma v_i^k - nO^k = {}_1^n\Sigma q_i^k \qquad \ldots (5.21) $$

Substituting equation (5.18) into this enable the orientation unknowns to be found.

$$ O^k = {}_1^n\Sigma q_i^k \; / \; n \qquad \ldots (5.22) $$

Substituting these orientation unknowns back into their observation equations (5.11), the individual corrections to the observations may be found. The correction to the i-th direction in the k-th arc is:

$$ v_i^k = q_i^k - {}_1^n\Sigma q_i^k \; / \; n \qquad \ldots (5.23) $$

The variance of a single observation may then be found:

$$ \sigma^2_{s.o.} = \Sigma_1^n \Sigma_1^s \quad (v_i^k)^2 \; / \; (n-1)(s-1) \qquad \ldots (5.24) $$

or in more conventional notation,

$$ \sigma^2_{s.o.} = \frac{\Sigma vv}{(n-1(s-1)} \qquad \ldots (5.25) $$

and the variance of the mean:

$$ \sigma^2_m = \frac{\sigma^2_{s.o.}}{s} \qquad \ldots (5.26) $$

(c) Station Adjustment in Tabular Form

It is not necessary to go through the full parametric method of adjustment, given in the preceeding section, each time a station adjustment is to be carried out. A number of simplifications are apparent from that section.

It was shown that the adjusted directions are the means of the observations from each arc all reduced to a common orientation, and that these adjusted directions are independent of orientation unknowns. This is easily seen from equation (5.15):

$$D_i = {}_1^s\Sigma p_i^k \qquad \ldots (5.27)$$

However, in the present discussion, the variance of the observations is of more interest. Equation (5.20) gives:

$$q_i^k = D_i - p_i^k \ , \qquad \ldots (5.28)$$

and from equation (5.23)

$$v_i^k = q_i^k - {}_1^n\Sigma q_i^k \ / \ n \qquad \ldots (5.29)$$

Using equation (5.28) the "q" values may be calculated by subtracting each observed direction from the adjusted direction to that station. Using equation (5.29), the "v" values may be obtained by meaning the "q" value for each arc, and subtracting the appropriate mean from each "q" value. Once the "v" values have been found, equation (5.25) and (5.26) may be used to find the variance of a single observation and of the mean. The calculation of the "v" values may be checked by use of the condition that the sum of the "v's" in the k-th arc must be zero. (Equation 5.18).

The calculation is best organised in tabular form as is given in Richardus *(1966, p.167)*, and in many other references. An example is given in Table 5.2 for the sake of completeness.

This form of calculation is quite convenient and can easily be done in the field. This is desirable, as the observations may be checked before leaving the field. Other useful parameters may be obtained in the course of the calculation. These will be discussed later in this section.

Station Observed	Face Left	Face Right	Mean	Reduced Mean	q	v	vv
Peveril	0 00 09	180 00 09	0 00 09	0 00 00	0.0	0.4	0.2
Springs	194 21 33	14 21 35	194 21 34	194 21 25	0.0	0.4	0.2
Borunge	293 36 06	113 36 10	293 36 08	293 35 59	-1.3	-0.9	0.8
				Mean =	-0.4	Σ-0.1	Σ1.2
Peveril	30 00 36	210 00 37	30 00 36	0 00 00	0.0	0.1	0.0
Springs	224 22 01	44 22 01	224 22 01	194 21 25	0.0	0.1	0.0
Borunge	323 36 34	143 36 38	323 36 36	293 36 00	-0.3	-0.2	0.0
				Mean =	-0.1	Σ 0.0	Σ0.1
Peveril	60 00 54	240 00 53	60 00 54	0 00 00	0.0	0.4	0.2
Springs	254 22 19	74 22 18	254 22 18	194 21 24	-1.0	-0.6	0.4
Borunge	353 36 54	173 36 54	353 36 54	293 36 00	-0.3	0.1	0.0
				Mean =	-0.4	Σ-0.1	Σ0.6
Peveril	90 00 20	270 00 19	90 00 20	0 00 00	0.0	0.4	0.2
Springs	284 21 44	104 21 45	284 21 44	194 21 24	-1.0	-0.6	0.4
Borunge	23 36 20	203 36 20	23 36 20	293 36 00	-0.3	0.1	0.0
				Mean =	-0.4	Σ-0.1	Σ0.6
Peveril	120 00 29	300 00 30	120 00 30	0 00 00	0.0	0.1	0.0
Springs	314 21 55	134 21 56	314 21 56	194 21 26	1.0	1.1	1.2
Borunge	53 36 28	233 36 30	53 36 29	293 35 59	-1.3	-1.2	1.4
				Mean =	-0.1	Σ 0.0	Σ2.6
Peveril	150 00 53	330 00 52	150 00 52	0 00 00	0.0	-1.6	2.6
Springs	344 22 18	164 22 19	344 22 18	194 21 26	1.0	-0.6	0.4
Borunge	83 36 54	263 36 57	83 36 56	293 36 04	3.7	2.1	4.4
				Mean =	1.6	Σ-0.1	Σ7.4

<u>Mean Directions</u>

Peveril	0	00	00.0
Springs	194	21	25.0
Borunge	293	36	00.3

$$\Sigma_{vv} = 12.5$$

$$\sigma^2_{s.o.} = \frac{12.5}{2 \times 5} = 1.25 \text{ sec.}^2$$

$$\sigma^2_{m} = \frac{1.25}{6} = 0.21 \text{ sec.}^2$$

<u>TABLE 5.2</u>

(d) Advantages of the Method

Sometimes the precision of direction measurement will vary significantly from station to station in a network. This is due to such factors as poor sighting conditions and changes in lateral refraction. Other methods of variance analysis will fail to detect these variations and assign them to a particular station, but will tend to distribute them throughout the network. This will not matter for normal work, but for precise work, it may be important that these discrepancies be located and if necessary the station be reobserved.

The observations have to be reduced to mean directions before entering them as data into the network adjustment. When this has been done, there is little extra work involved in obtaining their variance. The method therefore requires little effort.

The method is ideal for precise work such as dam deflection surveys. Here, the variances of directions from individual stations are of interest. Once found they can be assigned to their respective observations in the network adjustment. There are means by which an estimate of external variance can be found (See Section 5.4), and if the internal variance of each station can be found, quite a sound estimate of variance for the observations at each station can be built up. However, the variances obtained should be statistically tested against past experience of the same equipment, techniques and observing conditions by the Variance Ratio Test. *(Hoel, 1965)*. If this test is satisfied, then it is usually more valid to adopt the variance from past experience as this value is obtained from a number of redundancies which is infinite when compared to the number of redundant observations. If there is reasonable evidence that the observing conditions under which the observations were taken are not significantly the same as those on which the variance from past experience is based, then the variances calculated from the observations themselves are probably more reliable and should be adopted. Such reasonable evidence would include the case where the calculated variances are either consistently larger or consistently smaller than those from past experience.

The method uses every observation made, whereas the condition methods only use a percentage of them.

(e) Other Information Obtained from the Station Adjustment

In the course of a station adjustment, useful information may be obtained by taking out the differences between the individual face left and face right semi-directions. The errors affecting direction observations that are cancelled out in the mean of face left and face right semi-directions will be present in the difference between face left and face right semi-directions. Conversely, errors not cancelled by the mean may be cancelled by the difference.

Consider the observation of semi-directions on face left and face right:

$$\phi_L = \phi'_L + i_c.\text{sech} + i_v.\text{tanh}.\sin\alpha + (\text{Observer Error})_L \qquad \ldots (5.30)$$

$$\phi_R = \phi'_R - i_c.\text{sech} + i_v.\text{tanh}.\sin\alpha + (\text{Observer Error})_R \qquad \ldots (5.31)$$

where ϕ_L is the "true" semi-direction on face left,

ϕ_R is the "true" semi-direction on face right,

ϕ'_L is the observed semi-direction on face left,

ϕ'_R is the observed semi-direction on face right,

i_c is the inclination of the line of collimation of the telescope to the normal to the trunnion axis. (See Chapter 3),

i_v is the non-verticality of the vertical axis. (See Chapter 3)

α is the difference in direction between the target and the line of intersection of the horizontal plane and the plane defined by the perpendicular to the vertical axis

h is the elevation angle of the target being sighted.

Circle graduation errors will be neglected as these equations would normally be used for a number of arcs, in which case systematic circle graduation errors will be minimised. The remaining random errors will be included in the "Observer Error" term. Circle eccentricity can also be neglected if the circle is read at two diametrically opposite points as is the case with most single second and geodetic theodolites.

If the mean of the face left and face right semi-directions is taken, equations (5.30) and (5.31) will give:

$$\frac{\phi_L + \phi_R}{2} = \frac{\phi'_L + \phi'_R}{2} + i_v \tan h \sin\alpha + f(\text{Observer Error}) \quad \ldots (5.32)$$

Therefore, the mean of face left and face right semi-directions (i.e. the direction observation) is affected by errors in levelling and by random errors due to the observer and the atmosphere etc.

Consider half the difference between the face left and face right semi-directions. Using equations (5.30) and (5.31):

$$\frac{\phi_L - \phi_R}{2} = \frac{\phi'_L - \phi'_R}{2} + i_c \text{ sech} + f(\text{Observer Error}) \qquad \ldots (5.33)$$

It seems reasonable to assume that the instrument is in adjustment and that the line of collimation will be significantly perpendicular to the trunnion axis (i.e. i_c close to zero). Even if this is not the case, under normal geodetic conditions, (i.e. fairly flat lines), the contribution of this error to the difference (5.33) will be fairly small and nearly constant. The third term on the right hand side of (5.33) is a function of the random errors due to the observer etc. As these errors are random they cannot be treated algebraically and must be treated according to the Laws of Propagation of Variance. These laws may be applied to show that the random error term in the mean (5.32) should be of the same magnitude as the random error term in the difference (5.33).

If the variance of the difference is taken out for each station observed, then any variation in these should be due to differences in the pointing conditions to the individual stations. Such information is useful in precise engineering work where different variances may be placed on each direction.

As was pointed out above, the mean differences for the individual stations should be very similar in magnitude. If they are not and the stations are substantially different in elevation, this indicates that collimation error may be present. If the differences increase or decrease in the order the stations were observed, then it is possible that "twist of the tripod" has occurred. (See Chapter 3). Although these errors will not affect the mean direction, evidence of their existence is of use as a monitor of instrument adjustment and as an indication that all might not be right with that particular set of observations.

It is also possible to calculate the moments of the distribution of the observations in each set, and from these moments calculate skewness and kurtosis parameters, γ_1 and γ_2. *(Kendall and Stuart, 1958).* These parameters are of use in that they show differences between the observational distribution and the standard normal distribution. Sudden variations in these differences are indicative of systematic effects in the observations. There is no strong evidence that the distribution of the observations should be normal, however the analysis of large samples, (See Chapter 7), has shown that the distribution approaches normality. Typically, the distribution of direction observations from a single station is symmetrical but somewhat platykurtic, that is, more flat-topped than the normal curve.

5.4 Total Variance from Variance Factor Analysis

(a) Introduction

It was pointed out in Chapter 2 that the mathematical model for an adjustment is formed by the observations, their variances and the geometric configuration of the points of the network. If this model is validly chosen then the estimate of the variance factor after adjustment ($\overline{s^2}$) will approach the value of the variance factor before adjustment (s^2). (See Chapter 2 for a definition of variance factor).

If the mathematical model is otherwise valid, then the comparison of s^2 and $\overline{s^2}$ is a measure of the suitability of the variances used.

This approach will give an estimate of the total variance of the observations as the components of external variance, (See Section 5.1), as well as those of internal variance will become evident when the network is adjusted.

(b) The Method

This method of variance analysis is explained by Ashkenazi *(1970)* but will be repeated here using the terminology of this report.

In most survey work, s^2 is taken to be unity as a matter of convenience. In this case, the weight coefficient matrix of the observations will be identical with the variance-covariance matrix. (See Chapter 2 for definitions of these terms). Therefore, if the variance-covariance matrix is validly chosen, then $\bar{s}^2$ will approach unity.

If $\bar{s}^2$ is greater than unity then the variances of the observations have, in general, been under estimated. Similarly, an $\bar{s}^2$ less than unity indicates an overestimation of the observational variances. If each element of the variance-covariance matrix is multiplied by this $\bar{s}^2$ to give a new variance-covariance matrix, then a readjustment using this matrix will give an $\bar{s}^2$ of unity.

If all observations in the adjustment have the same physical dimensions, (in this case, all directions or angles), then the multiplication of the variance-covariance matrix by a constant will not affect the results of the adjustment, i.e., the adjusted coordinates and the corrections to the observations will not be changed. The only change will be in $\bar{s}^2$. If the observations are of differing physical dimensions, (e.g. directions and distances), then it may be necessary to multiply different elements of the variance-covariance matrix by different constants to obtain estimates of the variance of each type of observation. Such a procedure would change the results of the adjustment as the elements of the variance-covariance matrix are being changed relative to one another.

Therefore, if all the observations have the same physical dimension, only one pass of the adjustment is required to find an estimate of their variance. If the elements of the variance-covariance matrix are multiplied by the $\overline{s}^2$ resulting from this single pass, then these multiplied elements are the estimated variances of the observations.

Difficulties are encountered when the common physical dimension is distance. These difficulties will be discussed fully in Chapter 6.

If the estimate of angular variance obtained by this method is to be valid then the survey should be adjusted as a free network, i.e. there should only be two fixed stations, or one fixed station and an azimuth and a distance, preferably from that station to an adjacent station. This is the minimum information needed to fix the position of the survey on the spheroid and to give it orientation and scale. Any additional fixed stations or observations are likely to cause distortions or tensions in the network, which will influence the estimated variance of the angular observations.

The formula used to estimate the variance factor after adjustment is:

$$\overline{S}^2 = \frac{V^T G^{-1} V}{r} \qquad \ldots (5.34)$$

where $\overline{S}^2$ is the estimate of the variance factor,

V is the column vector of corrections to the observations,

G is the weight coefficient matrix of the observations,

and r is the number of redundant observations in the adjustment.

This formula may be expressed in the classical notation:

$$\overline{S^2} = \frac{1}{r} \Sigma \left[\frac{V^2}{\sigma^2} \right] \quad \ldots (5.35)$$

It may be seen from (5.35) that $\overline{S^2}$ is simply the sum of the squares of the residuals divided by their variances, (i.e. sum of the squares of the standard corrections), divided by the number of redundancies. This is identical in form to the standard formula for the variance of a single measurement of a quantity that has been measured a number of times with different measurement variances, and is therefore quite logical. A rigorous proof of formula (5.34) is given in Appendix B.

The accuracy of the variances given by this method depends heavily on the number of redundant observations in the network. It is very difficult to say what is the minimum number of redundancies required to give a valid estimate of variance. However, statistical studies usually state that a sample with twenty redundancies is the smallest sample that can be validly analysed.

5.5 The Variance of Observed Azimuths

It was pointed out in Chapter 3 that an astronomical azimuth is the difference between the direction to a star and the direction to a reference object, and therefore that the factors affecting the precision of a direction observation will also affect the precision of an astronomical azimuth.

However, because of the conditions under which these observations are taken, there are additional factors affecting the precision. Astronomical observations must be carried out on a clear night. It is well known that a temperature inversion effect occurs at night and that some peculiar refraction problems are likely to be encountered.

Ideal conditions for direction observations are the early morning or late afternoon, and preferably with an overcast sky. Additional errors are introduced in astronomical observations by virtue of the fact that a moving object is being sighted, and by virtue of the difference in elevation between the star and the reference object.

Therefore, it is fairly clear that the precision of astronomical azimuth observations cannot be calculated directly from the precision of the component direction observations.

The Division of National Mapping (1970) carried out comparisons of reciprocal Laplace azimuths on 136 lines and found that a normal distribution with standard deviation of 2.41 seconds fitted as well. (See Fig. 5.4). This value is in fact the standard deviation of the difference between reciprocal Laplace azimuths.

For the reciprocal azimuths between a station 1 and a station 2:

$$\text{Diff. Az.} = Az_{1,2} - Az_{2,1} \qquad \ldots (5.36)$$

and $$\sigma^2 \text{ Diff. Az.} = \sigma^2_{Az_{1,2}} + \sigma^2_{Az_{2,1}} - 2\sigma_{Az_{1,2}} \sigma_{Az_{2,1}} \qquad \ldots (5.37)$$

The last term in (5.37) goes to zero as there is no physical correlation between the two azimuths.

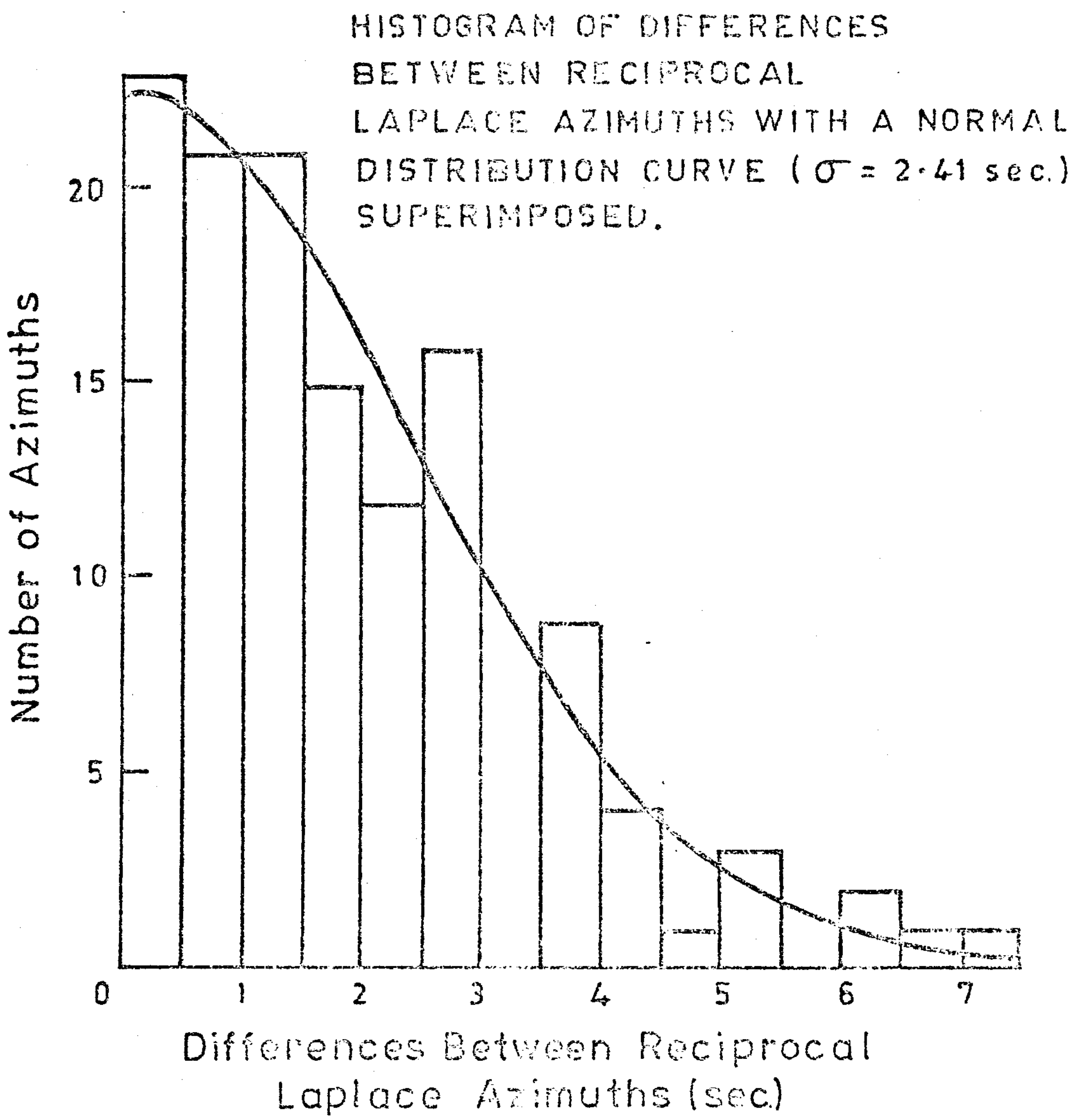

HISTOGRAM OF DIFFERENCES
BETWEEN RECIPROCAL
LAPLACE AZIMUTHS WITH A NORMAL
DISTRIBUTION CURVE (σ = 2·41 sec.)
SUPERIMPOSED.
Number of Azimuths
20
15
10
5
0
1
2
3
4
5
6
7
Differences Between Reciprocal
Laplace Azimuths (sec)

FIG. 5·4

Assuming that

$$\sigma^2_{Az_{1,2}} = \sigma^2_{Az_{2,1}} = \sigma^2_{Az} \qquad \ldots (5.38)$$

Then

$$\sigma^2_{Az} = \tfrac{1}{2}\, \sigma^2 \text{ Diff. Az.} \qquad \ldots (5.39)$$

Substituting the results of National Mapping (1970) into (5.39) gives:

$$\sigma^2_{Az} = \tfrac{1}{2}\, (2.41)^2$$

$$\sigma^2_{Az} = 2.89 \text{ seconds squared} \qquad \ldots (5.40)$$

$$\therefore \quad \sigma_{Az} = 1.7 \text{ seconds} \qquad \ldots (5.41)$$

The azimuth observations considered here were observed mainly in flat country where very flat grazing rays to the reference objects were quite common. Under these conditions the effects of lateral refraction are often quite significant. In good geodetic conditions a significantly lower standard deviation could be expected.

C H A P T E R 6

THE ANALYSIS OF LINEAR PRECISION

6.1 Introduction

It is only in the past few years that a valid procedure for the 'weighting' of physically dissimilar quantities in a least squares adjustment has been put forward. *(Allman, 1967; Ashkenazi, 1970).* Even now, most of the weighting systems in common use are quite invalid and many are based on assumptions such as, "a single second theodolite having the same degree of accuracy as a 1/200,000 distance measuring instrument". *(Ashkenazi, ibid).* Examples of these weighting systems are those suggested by Murphy *(1958)* and Rainsford *(1962).* Methods such as this have gained such wide acceptance that they have tended to obscure the need for a good estimate of the precision of observations.

Where estimates of angular precision could be obtained using forumlae such as Ferrero's, (see Chapter 5), linear precision could not be estimated by any such condition approach and was usually taken to be either, the figure obtained from a rather dubious assessment of past experience, or more simply, the manufacturer's specification. In Chapter 4, it was seen that, in many cases, manufacturer's specifications appear to be rather optimistic, and not a good estimate of the precision likely to be obtained in the field.

In addition, no valid optimization can be carried out unless a reasonable estimate of the precision of distance measurements, as well as one of angular precision, is available. Hence, a reasonable estimate of the precision of distance measurements is required for both optimization and adjustment studies.

The methods of analysis to be discussed in this chapter are both based on the results of a least squares adjustment. The first uses the variance factor technique and the second uses a comparison of calculated variance with the least squares residuals.

6.2 Method One

This method is based on the theory that was given in Section 5.4 for the variance factor analysis of angular observations. The method is founded on the fact that when the corrections to the observations in an adjustment are generally in agreement with the attributed variances, the estimated variance factor after adjustment, $\overline{S}^2$, will approach the variance factor before adjustment, S^2.

It is seldom that a network contains sufficient measured distances to enable the network to be calculated without using angular observations, as in most networks, there are only a limited number of distances measured in order to stabilise scale. Even when a substantial number of distances is observed in the network, their configuration is quite often such that they cannot be adjusted independently of the angular observations. Therefore, to analyse distance measurement variance by this method, it is usually necessary to proceed in the manner suggested by Ashkenazi (1970):

1. Carry out a variance analysis of the angular work, as described in Section 5.4, and assign variances so that $\overline{S}^2$ is unity.

2. Add the distances to the network data with an estimated variance which will be replaced by successive approximations until $\overline{S}^2$ is once again unity.

As was pointed out in Section 5.4, an iterative procedure is not required where the network consists of one type of observation only, as in this case, the variance-covariance matrix is homogeneous and the variance factor is purely an overall scaling number. The following example clarifies this.

Assume a variance-covariance matrix,

$$\begin{vmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{vmatrix}$$

and an initial variance factor of X, so that,

$$\begin{vmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{vmatrix} = X \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{vmatrix} \qquad \ldots (6.1)$$

where the matrix on the right hand side of (6.1) is the weight coefficient matrix of the observations.

Assume that an $\overline{S}^2$ of Y is found. If the weight coefficient matrix is multiplied by Y and the adjustment re-run using the new matrix, an $\overline{S}^2$ of unity will result. Therefore the estimated variance-covariance matrix of the observations is

$$XY \begin{vmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{vmatrix} \text{ or } Y \begin{vmatrix} A & 0 & 0 & 0 \\ 0 & B & 0 & 0 \\ 0 & 0 & C & 0 \\ 0 & 0 & 0 & D \end{vmatrix}$$

In most surveying adjustment work, as a matter of convenience, S^2 is taken as unity. (X=1). So on the single pass through the adjustment, the estimate of the variance-covariance matrix of the observations is simply found by multiplying the original variance-covariance matrix by Y.

As the variance analysis of distance measurement usually cannot be carried out independently of the angular work, the variance-covariance matrix will contain both angular and distance measurement variances. The angular variances will be those that give an $\overline{S}^2$ equal to unity when only angular measurements are adjusted.

The variance-covariance matrix will be of the form:

$$\begin{vmatrix} A & 0 & 0 & 0 & 0 & 0 \\ 0 & B & 0 & 0 & 0 & 0 \\ 0 & 0 & C & 0 & 0 & 0 \\ 0 & 0 & 0 & D & 0 & 0 \\ 0 & 0 & 0 & 0 & E & 0 \\ 0 & 0 & 0 & 0 & 0 & F \end{vmatrix}$$

where A,B,C, and D are the estimated angular variances, and E and F are the trial distance measurement variances. Suppose that this variance-covariance matrix gives an $\overline{S}^2$ of Z. If the complete matrix was multiplied by Z, the resulting matrix would give an $\overline{S}^2$ of unity. This cannot be done as the direction variances are the accepted estimates and cannot be altered. So, the problem is to decide the amount by which the distance variances must be changed in order to obtain an $\overline{S}^2$ of unity. One obvious solution is to assign new values, on largely a trial and error basis, until $\overline{S}^2$ is sufficiently close to unity. *(Ashkenazi, 1970)*. The value of Z will give some indication of the direction in which the variances must be altered. (Z less than unity - decrease variance, Z greater than unity - increase variance), but this indication will not always be correct.

As the number of distances measured is usually small in comparison to the number of angles or directions observed, the influence of the distances on $\overline{S}^2$ tends to be swamped by the influence of the angles or directions.

One possible procedure to overcome this problem is as follows:

1. From the adjustment run with angular observations only, calculate $[V^TG^{-1}V]$ using the formula,

$$[V^TG^{-1}V] = \overline{S}^2 . r \qquad \ldots (6.2)$$

where r is the number of redundant directions.

2. From the adjustment with both directions and distances, recalculate $[V^TG^{-1}V]$ using (6.2), but with r being the total number of redundant observations.

3. The difference between the $[V^TG^{-1}V]$'s obtained in 1 and in 2 will give an estimate of the $[V^TG^{-1}V]$ for distances alone. When this is divided by the number of redundant distances, an "estimate of the variance factor" for distances only will be obtained. Technically, this is not a variance factor but is the second moment of the adjustment for distance observations. (M")

$$M'' = \frac{[V^TG^{-1}V]}{r} \qquad \ldots (6.3)$$

where r is the number of redundant distances, and $[V^TG^{-1}V]$ is derived from the distances alone.

4. The distance measurement variances are then multiplied by this second moment to obtain better estimates.

It was mentioned in 3 that M" is only an estimate of the factor by which the variances should be multiplied. One reason is that when the distances are introduced into the adjustment, the $[V^TG^{-1}V]$ for directions only will change as the distances will tend to distort the original network to some extent. The magnitude of this change will depend on the particular distance measurement variance being used. It is therefore not valid to make the assumption that the $[V^TG^{-1}V]$ for directions alone remains constant when the distances are added to the network.

The significance of this assumption was tested using Network 4. (See Chapter 7). $[V^TG^{-1}V]$ from step 1 was calculated as 117.28. The same quantity for directions only in the combined adjustment was calculated by dividing the correction to each direction observation by its standard deviation, squaring the dividend and summing.

$$[V^TG^{-1}V] = \Sigma\left(\frac{V}{\sigma}\right)^2 \qquad \ldots (6.4)$$

The result of this calculation was $[V^TG^{-1}V] = 164.82$. This is significantly different from the figure obtained from the directions only adjustment of 117.28, but the real significance lies in the change that this difference makes to the estimate of the factor by which the trial variances should be multiplied.

The $[V^TG^{-1}V]$ for all observations in the total adjustment, (Step 2), is 254.53 and there are 102 redundant distances. Therefore the two M" factors will be:

$$M'' = \frac{254.53 - 117.28}{102} = 1.35$$

and

$$M'' = \frac{254.53 - 164.82}{102} = 0.88$$

These two values for M" are quite significantly different and therefore, it cannot be assumed that the $[V^TG^{-1}V]$ for directions remains constant in the total adjustment.

Also, the fact that M" for distances, (0.88) is significantly different from the variance factor after the total adjustment, (1.15), is proof that the directions do actually swamp the measured distances in such an adjustment, and that $\overline{S}^2$ will sometimes not even indicate the correct direction of change.

The valid M" (0.88) can be obtained in a more direct manner than that outlined above. In the above method, the contribution of the distances to the total $[V^TG^{-1}V]$ is found by calculating the difference between $[V^TG^{-1}V]$ for the total adjustment and $[V^TG^{-1}V]$ for the directions only adjustment. This contribution may be found directly from the trial variances of, and the attributed corrections to the distances. With this information, equation (6.4) may be used to find $V^TG^{-1}V$ for each distance and these summed to find the contribution of the distances toward $[V^TG^{-1}V]$ for the total adjustment. This contribution is divided by the number of redundant distances to give M" (formula 6.3).

In general, the method, as set out above, will not give a direct solution, but will require iteration. The reason is twofold. Firstly, the trial variance will usually be of the form $(a + bs)^2$. As the distance, s, varies from observation to observation it is not possible to multiply the expression directly by the resultant M" to obtain a new variance expression which will give an M" of unity on readjustment. A technique to overcome this problem is described later in the present section.

The second factor that prevents a direct solution being given is that the full effect of a variance change is somewhat damped by the constraints placed on the distances by the angular work. That this damping effect does not exist may be shown by a numerical example taken once again from Network 4 (See Chapter 7). To overcome the problem caused by the two term variance expression mentioned above, for the purposes of this illustrative example, it was assumed that the variances of the measured distances may be expressed as a constant not dependent on the length of the line. Therefore, failure to obtain a direct solution cannot be attributed to this cause. The variances applied to the angular observations were those which gave a variance factor of unity when the angular work alone was adjusted. The trial variance applied to the distances was $(30.0 \text{ mm})^2$. The resulting M" was 0.409 and a new variance of $(19.3 \text{ mm})^2$ was calculated by multiplying the original variance by this M". The adjustment, when rerun with this new variance, gave an M" of 0.675. Further iteration would normally be required to find a variance which will give an M" of unity. So, even though a single term variance for measured distances is being used, a direct solution cannot be obtained.

That this inability to obtain a direct solution is, in fact, due to the damping influence of the angular work, may be shown by relaxing the angular variances to the extent that the measured distances are no longer influenced. The angular observations were all given variances of 10^4sec^2, while the distance measurement variance was kept at $(19.3 \text{ mm})^2$. The M" given by this adjustment was 0.118, a figure very different from the 0.675 obtained above. A new distance measurement variance of $(6.7 \text{ mm})^2$ was calculated by multiplying the initial variance, $(19.3 \text{ mm})^2$, by the M" of 0.118. The adjustment was rerun with this variance for the distances and, once again, a variance of 10^4sec^2 for the angular observations. The resulting M" was unity. Therefore, a direct solution may be obtained when the damping influence of the angular work is eliminated.

Unfortunately, such a technique cannot be used in practice. It was pointed out above, that in the majority of networks, the number and configuration of the measured distances will not allow the network to be calculated without the use of angular observations. When the procedure of relaxing angular variances is used, many of the distances are, in effect, free and will be given zero correction, and therefore, do not contribute to the value of M". So, although the network is fixed by the angular observations, and all but one of the measured distances are, in the mathematical sense of the term, redundant, many of these distances are not directly contradicted by any other distance observations and cannot be called redundant in the logical sense. They may, and nearly always will, be contradicted by angular observations, but the variances of these are so large that the contradiction is in name only and has no effect on the distance.

Where the number of measured distances is reasonably large and the configuration of the network is sufficiently strong, there is no reason why the distances cannot be assessed independently of the angular work. The solution will usually not be direct because of the two term variance expression for measured distances. However, if the trial variance is altered after the first pass by the technique explained later in this section, then only one or two iterations are required to find the variance estimate.

In summary, the method of variance analysis put forward by Ashkenazi *(1970)*, using the variance factor after adjustment as the estimation criterion, is valid and will eventually lead to a valid variance estimate. However, it is a tedious method as it requires considerable iteration. The method described above, using M" as the criterion, is faster and will give the same valid estimate.

(a) Alteration of the Trial Variance

When using a trial variance of the form $(a + bs)^2 mm^2$, (formula 4.15), where a and b are constants dependent on the type and performance of the instrument, and s is the distance being measured, the problem arises of how to alter the estimate when an M" different from unity is obtained. The only exact way would be to calculate the variance of each line being measured and to multiply each of these variances by the M" from the previous run of the adjustment. Then the problem remains of analysing the resultant variances to get back to an overall variance of the form $(a + bs)^2$. Unfortunately, it is not possible to evaluate the variance expression for the adjustment as a whole because of the variation n, the distance, s, from observation to observation.

However, it was found that multiplying a^2 and b^2 by M" and taking a square root of the results, to give a new a and b, usually gave a variance expression that, when tested in an adjustment, gave an M" close to unity. An example of this technique, is given below.

$$\text{Trial variance} = (25\ \text{mm} + 6\ \text{ppm})^2 \longrightarrow M'' = 0.66$$

The new "a" term was given by $\sqrt{(25)^2 . 0.66} = 20.6$ mm

and the new "b" term was given by $\sqrt{6^2 . 0.66} = 4.86$ ppm

The variance adopted for the next run of the adjustment was $(20\ \text{mm} + 5\ \text{ppm})^2$. This variance gave an M" of 0.98, which is significantly unity.

The above procedure changes the "a" and "b" factors by the same relative amount. However, this may not be valid as it assumes that a and b are in the correct relationship to each other. In the majority of cases this assumption is probably not correct and further steps must be taken to establish the correct relationship.

One method of doing this is to plot the standard corrections, $(\frac{v}{\sigma})$, to the distance observations against the length of line measured. (See fig. 6.1). If the estimate of variance is correct, roughly one third of the plotted points should be above the line drawn parallel to the distance axis and through $\frac{V}{\sigma} = 1$ on the standard correction axis. If the "a" and "b" terms of the variance expression are in the correct relationship to each other, this one third of points should be evenly spread through the range of distances measured. If the predominance of points above $\frac{V}{\sigma} = 1$ are in the shorter distance section of the graph, this indicates that the "a" term needs to be increased with respect to the "b" term. Such action will increase σ for the short line observations, (and hence decrease $\frac{V}{\sigma}$), and decrease σ for the long line observations, (and hence increase $\frac{V}{\sigma}$ for these observations). Conversely, if the predominance of points above $\frac{V}{\sigma}$ are in the longer distance section of the graph, then the "b" term should be increased with respect to the "a" term.

While this method is of course only approximate, it will give an indication of the validity of the "a" and "b" values.

6.3 Method Two

This method of variance analysis was put forward by Fryer *(1970, pp.167-172)* and uses a comparison of calculated variance with least squares residuals. The variance is calculated using formula (4.15).

$$\sigma_s^2 = (a + bs)^2 \qquad \ldots (6.5)$$

For a line made up of a number of sections, the general law of propagation of variances is applied to (6.5) to give a variance for the entire length between the terminal stations. It is assumed that all n sections, of length s, that make up the entire distance L are equal in length. Expressing this in mathematical terms,

$$L = n.s \qquad \ldots (6.6)$$

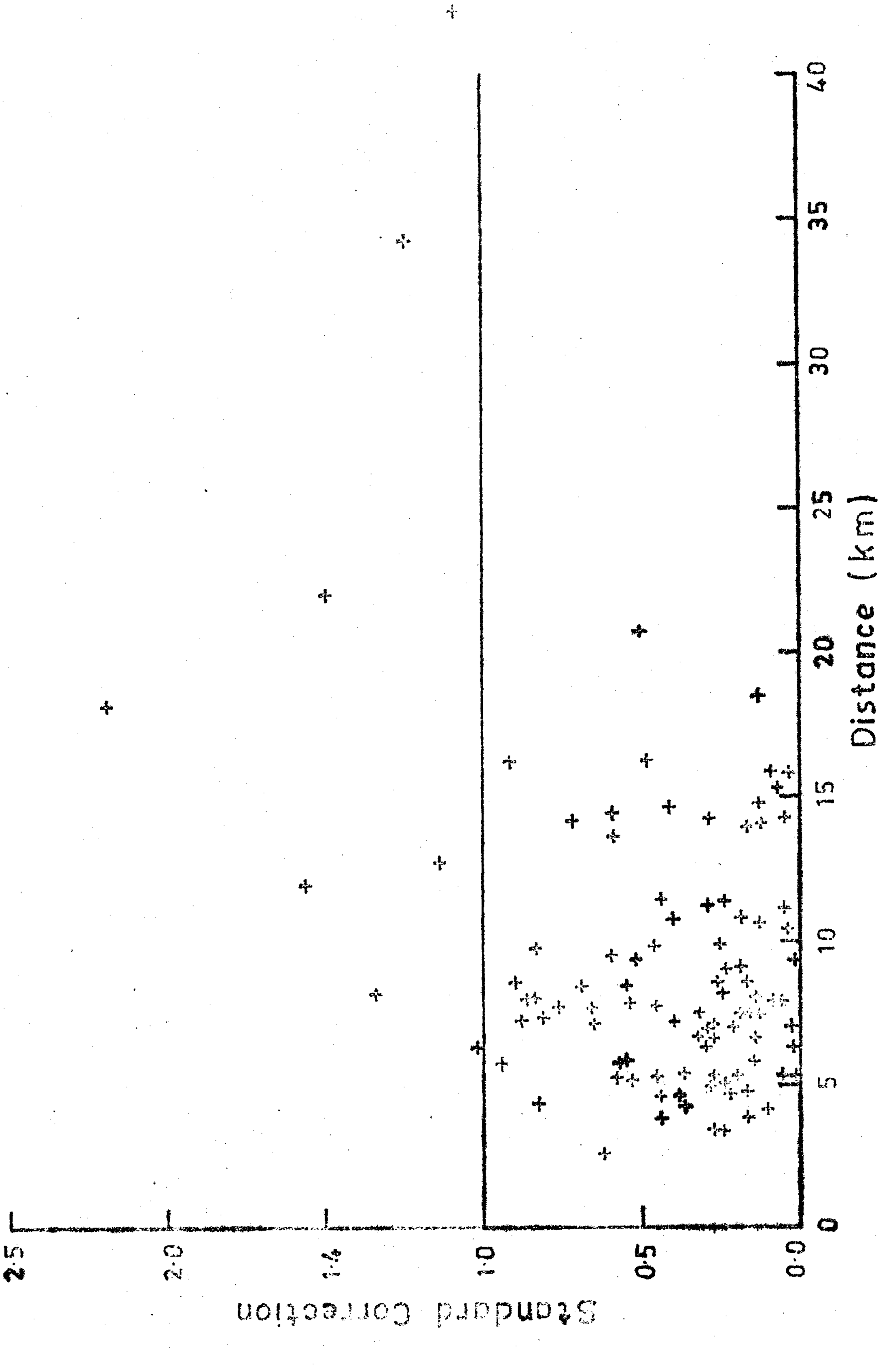

Standard Correction
2·5
2·0
1·4
1·0
0·5
0·0
0
5
10
15
20
25
30
35
40
Distance (km)

FIG. 6·1

The variance of the entire length is then,

$$\sigma_L^2 = (a + bs)^2.n \qquad \ldots (6.7)$$

Fryer *(ibid)* evaluated (6.7) using estimates of a and b for the MRA2 Tellurometer.

$$\sigma_L^2 = (50mm + 5ppm)^2.n \qquad \ldots (6.8)$$

For lines measured in both directions, he applies the law of propagation of variances to (6.8) to give a variance of the mean of the forward and back measurements. In effect, this involved dividing the a and b terms by the square root of two. Hence,

$$\sigma_{Lm}^2 = (35mm + 3.5ppm)^2.n \qquad \ldots (6.9)$$

The validity of this step is open to conjecture and will be discussed later in this Section.

To test whether this variance was representative of that actually encountered under field conditions, the results of an adjustment were considered. To describe the adjustment briefly, it consisted of 161 interlocking traverses of average length 313 km. These interlocking traverses or sections were adjusted as separate units in that each section was treated as a single observation, even though each section contained, on the average, eleven measurments. (i.e. n=11). The average measured length was 28.5 km, (i.e. s = 28.5 km). This average n and s were used to evaluate (6.9) so that,

$$\sigma_{Lm} = 0.44 \text{ m} \qquad \ldots (6.10)$$

The average section residual given by the adjustment was 0.45 m. In view of the very close agreement between (6.10) and this figure, Fryer *(ibid)* goes on to say "one must presume the values of 3.5 for both a and b are extremely good estimates of the actual errors present".

However, this method of variance analysis may be shown to be invalid for the following reasons:

1. It seems unreasonable to assume that multiple measurements of a line will reduce the variance in accordance with the law of propagation of errors, as was implied in going from (6.8) to (6.9). Chrzanowski and Derenyi *(1967)* offer the theory that practically no improvement in the accuracy of a distance can be expected if the measurement is repeated after a short time lapse, since the conditions influencing the "a" and "b" terms of (6.5) seldom change rapidly. They go on to point out that the "b" term may be improved by repeated measurements in a variety of atmospheric conditions and that the "a" term can only decrease if the factors affecting it have changed. These factors are usually independent of time and are concerned with the geometry of the line. For microwave instruments they include the reflecting surface along the line. (See Chapter 4). Therefore the "a" term for microwave instruments may possibly be reduced by changing instrument height or by moving the instruments to eccentric stations.

The procedures mentioned are usually not carried out and the forward and reverse measurements of a line are usually only separated by a short time lapse. As no mention is made of special observing procedures being used, it must be assumed that they have not been used and therefore that the reasoning used to derive (6.9) from (6.8) is invalid.

2. In essence, the method consists of finding an estimate of standard deviation by the formula,

$$\bar{\sigma} = \Sigma\frac{V}{n} \qquad \ldots (6.11)$$

where ΣV is the sum of the residuals, and n is the number of residuals being considered.

Consider the standard deviation of a single variate measured n times. *(Spiegel, 1961, p.70)*.

$$\sigma = \sqrt{\frac{\Sigma(V^2)}{n}} \qquad \ldots (6.12)$$

Even though (6.12) is for the single variate case, the formula is of similar form for the multi-variate case, as the sum of the squares of the residuals is treated, rather than the sum of the residuals themselves.

One of the basic properties of the standard deviation is that 68.27% of the observations are included between $X+\sigma$ and $X-\sigma$. In the single variate case, X is the mean, but in the present case, it is the adjusted least squares parameter. Consider the idealised case of (6.11). The formula will give a true estimate of standard deviation if all the residuals (V's) are one standard deviation. In this case (6.11) will become the valid identity,

$$\bar{\sigma} = \frac{\Sigma\sigma}{n} \qquad \ldots (6.13)$$

But, if this is the case, then 100% of the observations are included between $X+\sigma$ and $X-\sigma$. Therefore, if the definition of standard deviation is taken to be valid, then (6.11) must be invalid.

It should be noted that this problem is not encountered if the same test is applied to Method One. Method Two is clearly invalid and it is Method One that will be used in Chapter 7 to analyse the precision of distance measurement.

CHAPTER 7

ILLUSTRATIVE STUDIES

7.1 Introduction

This chapter gives the results and details of the analysis of a number of observed networks for angular and distance measurement variance.

Data for this type of analysis is often quite difficult to obtain as the networks to be analysed should be fairly large, be composed of observations of fairly uniform quality, be well conditioned and have many redundant observations. Networks possessing all or most of these qualities are not readily obtainable.

The networks analysed below come from four different survey authorities, working on different projections, different spheroids and using different processing systems, and thus all require slightly different treatment. The diversity in data systems, together with the need to analyse internal variance, necessitated the abstraction of angular data direct from the field books - a very laborious and time consuming procedure. In network 5 alone, data preparation for this phase analysis included the punching of over 1,500 computer cards. Once the internal variance has been obtained, data decks for the network adjustment must be prepared, verified, run and the results analysed. Hence, it may be appreciated that the time involved in such an analysis is very considerable and that only five networks, involving three different instruments were able to be analysed in this report.

The analysis was carried out in accordance with the theory and principles given in Section 5.4 of Chapter 5 and Section 6.2 of Chapter 6.

The networks were adjusted using a parametric adjustment programme written by Dr. J.S. Allman of the School of Surveying, University of New South Wales.

7.2 The Distribution of Observations

It is of interest to look at the distribution of the observations. There is no real reason to assume that surveying observations are normally distributed, although it is commonly accepted that they are. As a large sample of observations are analysed in this Chapter, it was decided to calculate parameters of the distribution of these observations, as a by-product of the main analysis.

The usual parameters of a distribution are its moments. These moments may be calculated using the following formula.

$$\mu_x = \Sigma_i \left[\left(\frac{V_i}{\sigma_i} \right)^x \right] . \frac{1}{n} \qquad \ldots (7.1)$$

where μ_x is the x-th moment,

V_i is the residual of the i-th observation,

σ_i is the standard deviation of the i-th observation, and

n is the number of observations in the sample.

Theoretically, the divisor should be n-1 rather than n, but for a large sample these are significantly the same.

The first and second moments are the mean and variance of a distribution, respectively, and do not give any information as to its shape. The third and fourth moments of a distribution are more useful in that they are indicators of the distribution's skewness and kurtosis, respectively. The first four moments of a standard normal distribution are:

$$\mu_1 = 0$$

$$\mu_2 = 1$$

$$\mu_3 = 0$$

$$\mu_4 = 3$$

Measures of skewness and kurtosis, γ_1 and γ_2, have been derived by Kendall and Stuart, *(1958)*, and may be calculated using the following formulae:

$$\gamma_1 = \frac{\mu_3}{\mu_2^3} \qquad \ldots (7.2)$$

$$\gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 \qquad \ldots (7.3)$$

For a symmetrical distribution, γ_1 will be zero. Any departure of γ_1 from zero indicates skewness in the distribution. In a normal distribution, γ_2 will be zero. Such a distribution is termed mesokurtic. If γ_2 is greater than zero, the distribution is more sharply peaked than the normal distribution, and is termed leptokurtic. Conversely, if γ_2 is less than zero, the distribution is less sharply peaked, (more flat-topped), than the normal distribution, and is termed platykurtic.

These parameters will be calculated for the distribution of the direction observations at each station and also for the mean directions and the measured distances used as observation data in the least squares adjustments.

7.3 Network One

This network, (see Fig. 7.1) is part of a first order triangulation scheme which was observed in steep terrain using a Wild T3 theodolite for the direction observations. The observations were always taken in the late afternoon or in the early evening.

The observations were abstracted from the field books and the mean directions, internal variances, along with skewness and kurtosis parameters were calculated for each set. An average internal variance for all sets was calculated as 0.09 sec^2. The average number of arcs observed in a set was 23. The average internal variance for a single direction was then calculated using equation (7.4), as 2.07 sec^2.

$$\sigma^2_{\text{internal for 1 arc}} = \sigma^2_{\text{internal for the average number of arcs}} \times \text{Average number of arcs} \quad \ldots (7.4)$$

The average γ_1 and γ_2, considering all 23 sets, were calculated to be:

$$\gamma_1 = 0.004$$

$$\gamma_2 = -0.290$$

These parameters show the distribution of the direction observations to be symmetrical with a platykurtic tendency.

In order to eliminate the effect of any external distortions, the survey was adjusted as a free network, with station 4 held fixed to locate the survey on the spheroid, and the azimuth from station 4 to station 11 included to give the

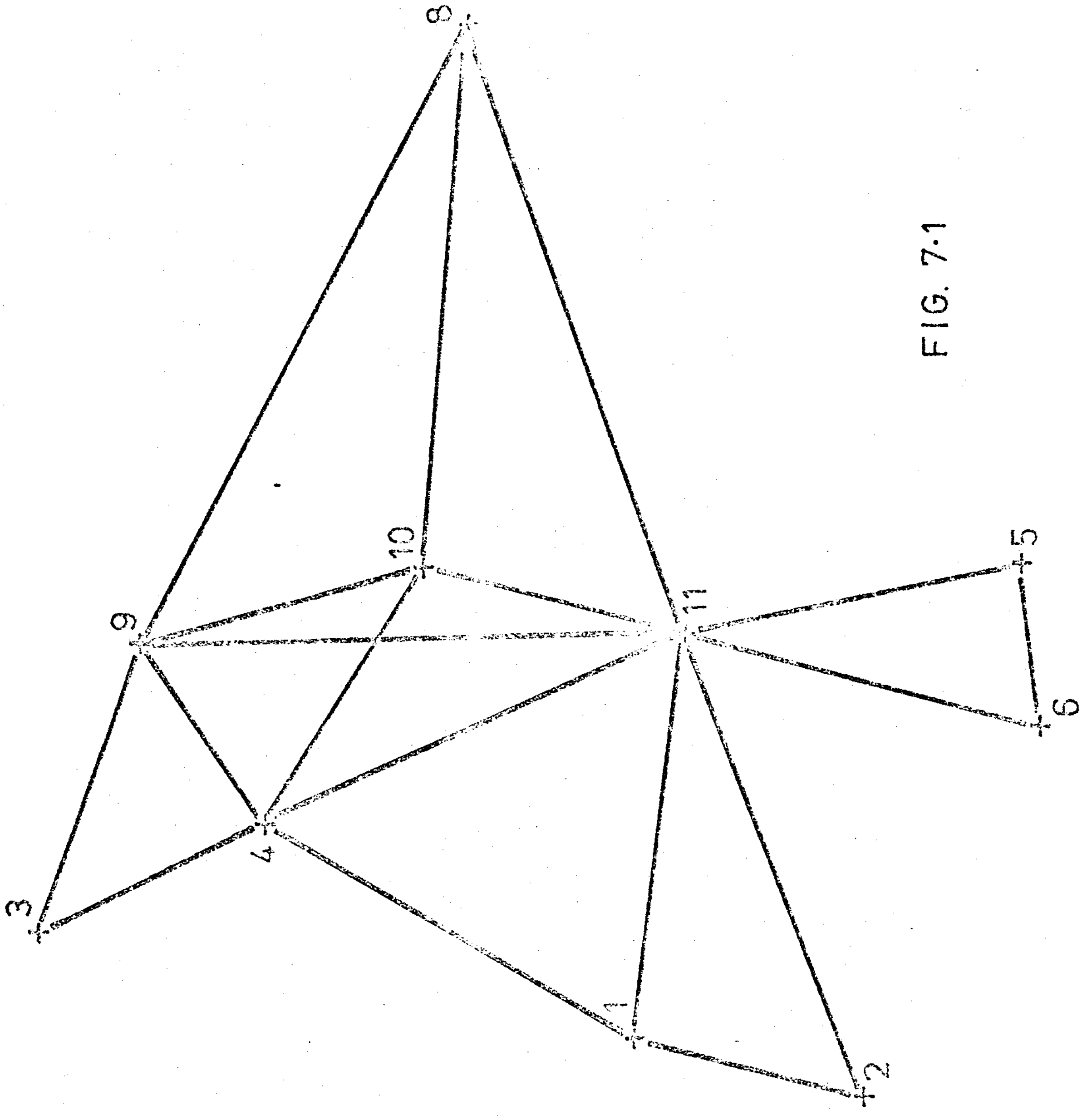


FIG. 7·1

survey orientation. Normally, in a free network, only one distance is used to fix the scale of the survey, as any additional distance or distances may cause distortions in the angular work. However, in the present survey, the southern section, (stations 5 and 6), cannot be fixed unless an additional distance, either between stations 11 and 5 or 11 and 6 is included. The reason is that there is no scale link between the main network and this southern section. That this second distance is necessary and not redundant is shown by the fact that the adjustment gives corrections of zero to both distances. A plot showing the observed directions (fig. 7.2) graphically illustrates that stations 5 and 6 cannot be fixed without using this additional distance.

In the first run of the adjustment, the direction observations were given a variance of 0.29 sec^2. This was made up of the mean internal variance (0.09 sec^2) plus an estimate of 0.2 sec^2 for external variance. The estimate of the variance factor after adjustment ($\overline{S}^2$) was 1.7121. This indicated a total variance of direction observations of (0.29 x 1.7121) = 0.50 sec^2.

The parameters γ_1 and γ_2 were calculated for the 57 adjusted direction observations as:

$$\gamma_1 = 0.17$$

$$\gamma_2 = 3.44$$

These parameters show the distribution to be slightly skewed and to be leptokurtic.

The number of redundancies, 32, is sufficient for the analysis to be statistically valid, but too small for much confidence to be placed in the resulting estimates of variance. These estimates should therefore only be considered as an indication of magnitude.

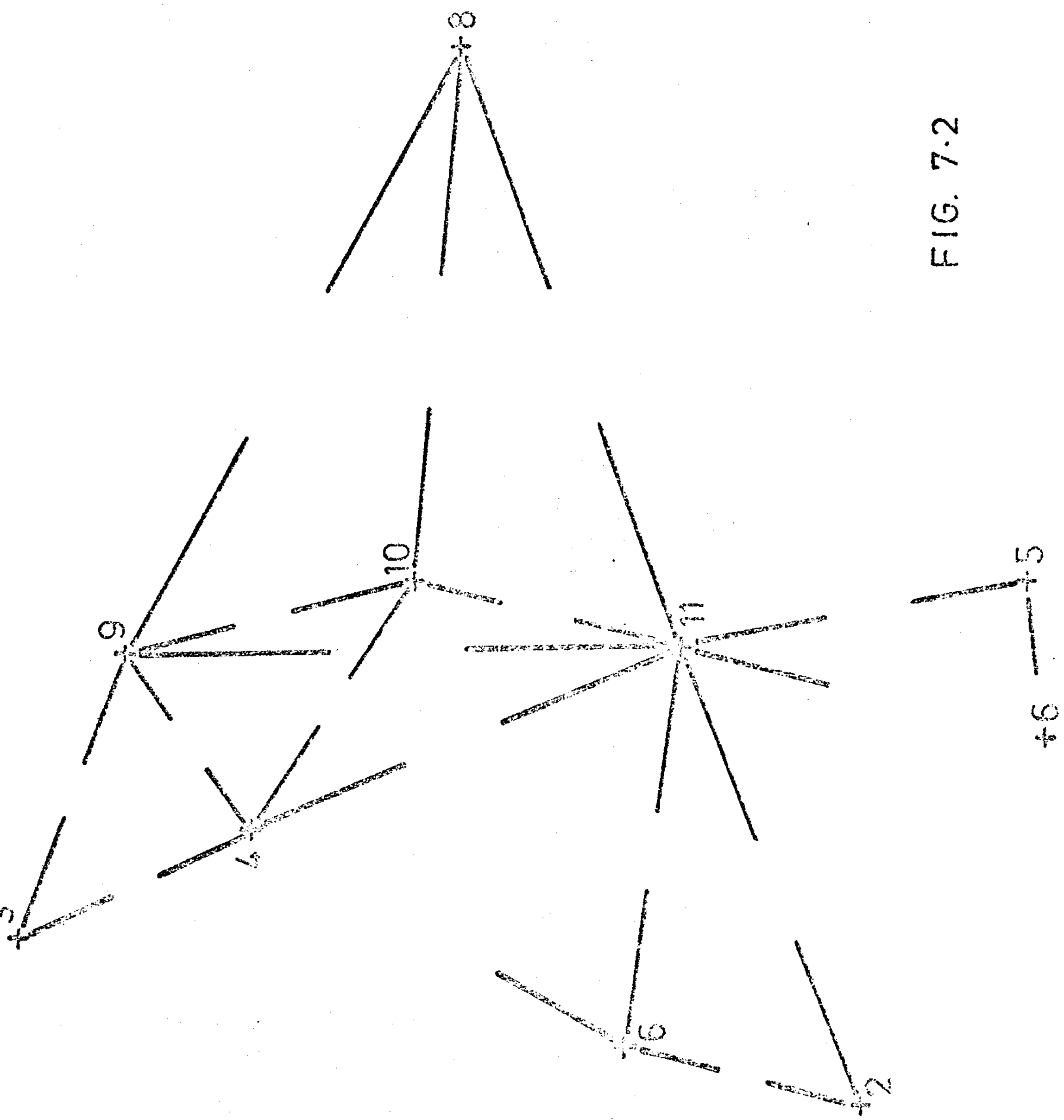


FIG. 7·2

Summarising, the estimate variances for this network are as follows:

$\sigma^2_{internal}$ for 23 arcs = 0.09 sec^2

$\sigma^2_{internal}$ for 1 arc = 2.07 sec^2

$\sigma^2_{external}$ = 0.41 sec^2

σ^2_{total} for 23 arcs = 0.50 sec^2

7.4 Network Two

This survey is a breakdown scheme observed in an area of undulating to steep terrain, with the purpose of fixing stations 4,5, 6 and 7 from stations 1,2 and 3. (See fig. 7.3). The survey consists entirely of distances measured with a set of Cubic Electrotape DM-20 instruments. Distances were measured along all lines shown in fig. 7.3.

The survey was carried out in a period of constant fog and haze when angular observations were impossible, and is a good example of how breakdown surveys may be done by measuring distances only.

A free net adjustment was used with station 1 held fixed, and a calculated bearing, from station 1 to station 2, included to give orientation. All distances were considered to be of the same quality as they were measured with the same set of instruments, and as there was no evidence of any appreciable variation in observing conditions. The observations were obtained as reduced sea-level distances. The method of distance reduction was checked and found to be valid. The adjustment was run with a number of variances and it was found that a variance of $(20.2mm + 4.86\ ppm)^2$ gave an $\overline{S^2}$ of unity. In order to obtain a variance in round figures,

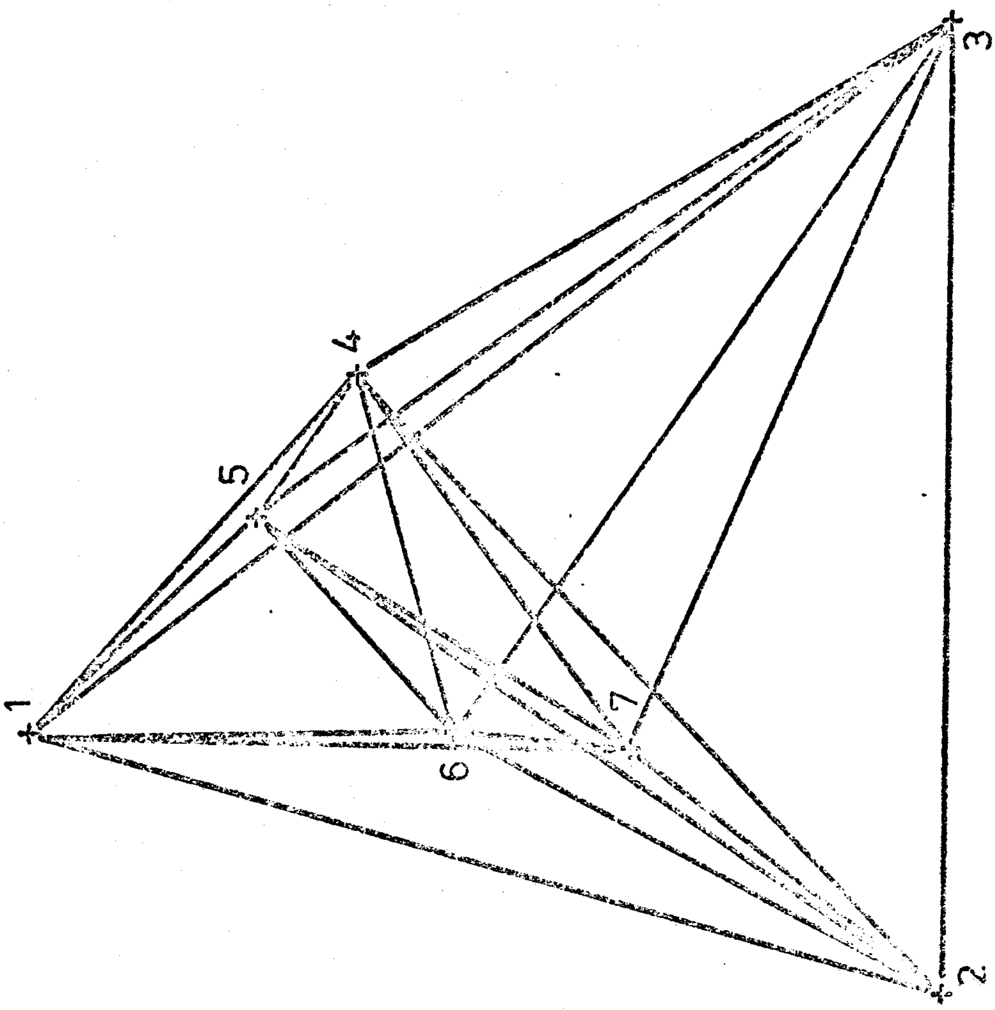


FIG. 7·3

the variance of (20.0 mm + 5 ppm)^{2}mm^2 was tested. This run gave an $\overline{S^2}$ of 0.98 which was accepted as being significantly unity.

It was anticipated that the estimated variance would be in this region as the variance for the MRA101 and MRA3 Tellurometers has been estimated, Robinson (1971) and others, to be (25 mm + 6 ppm)^2. The Electrotape DM-20 is an instrument very similar to the MRA101 and MRA3 Tellurometers and therefore would be expected to have a similar variance.

The number of redundancies in the survey is only 10, so the analysis is not statistically strong. However, the variance obtained would be an indication of the true figure and tends to be confirmed by its similarity to the established variance of the MRA3 and MRA101 Tellurometers. Therefore the estimate, σ^2 = (20 mm + 5 ppm)^{2}mm^2 can be accepted with a reasonable degree of confidence.

It may be seen from a graph of standard correction against distance for the observations in the network (fig. 7.4), that the relationship between the estimated "a" and "b" terms of the variance expression is close to being correct.

The skewness and kurtosis parameters, for the distribution of the adjusted distances, were calculated to be:

$$\gamma_1 = 0.19$$

$$\gamma_2 = -0.86$$

These parameters show the distribution to be slightly skew as well as platykurtic.

7.5 Network Three

This survey (See Fig. 7.5) is very similar to

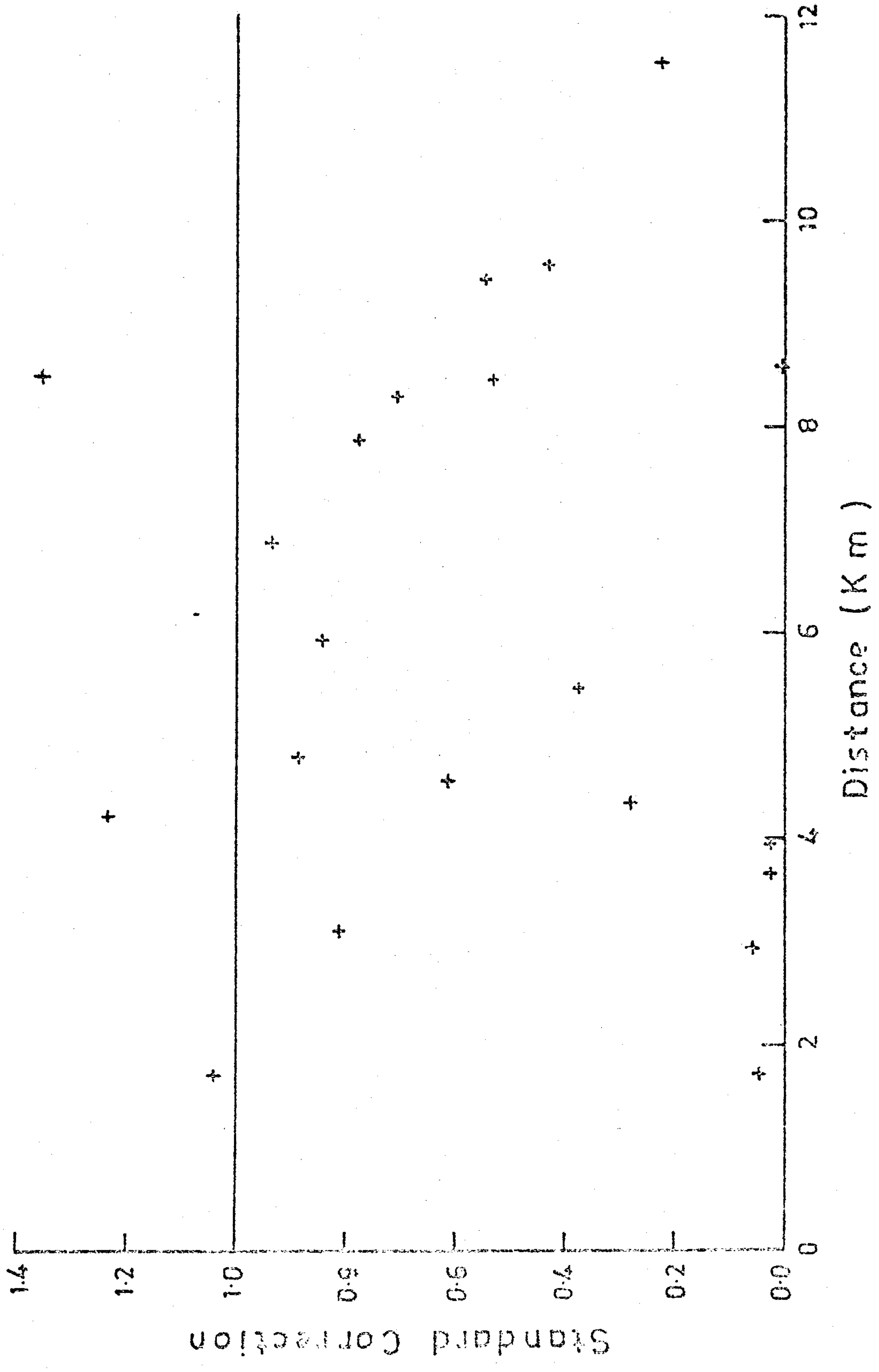

Standard Correction
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
Distance (Km)
0
2
4
6
8
10
12

FIG. 7.4

network 2, (See Section 7.4) in that it consists entirely of measured distances, and that these distances were all measured with a set of Cubic Electrotape DM20 instruments. Distances were measured along all lines shown in fig. 7.5, and all these are considered to be of equal quality for the same reasons as were given in Section 7.4.

The survey was adjusted as a free net with station 5 held fixed and a calculated bearing from station 5 to station 1, included to give the survey orientation. Initially, the adjustment was run with the estimated variance of distance measurement obtained from the analysis of network 2, (20 mm + 5 ppm)^{2}mm^2, giving an $\overline{S}^2$ of 0.90. It was found that a variance of (19.5 mm + 4.7 ppm)^{2}mm^2 was more satisfactory in that it gave an $\overline{S}^2$ of unity. However, it should be noted that the variance of (20 mm + 5 ppm)^{2}mm^2 satisfies the variance ratio test, at the 95% confidence level, in both network 2 and network 3, indicating that the observations in both networks are from the same population and that the above variance is a satisfactory variance for that population.

A graph of the standard corrections to the distances based on a variance of (20 mm + 5 ppm)^{2}mm^2 is given in fig. 7.6. It may be seen, from this graph, that the relative magnitude of the "a" and "b" terms of this variance expression is close to being correct.

As the number of redundant measurements in the network is only 14, the analysis is not statistically strong. However, the agreement with the estimated variance found in network 2, and with the established variance of the 30 mm carrier wave Tellurometers, confirms that the estimate (20 mm + 5 ppm)^{2}mm^2 is quite a good estimate of the variance of Cubic Electrotape DM20 measurements.

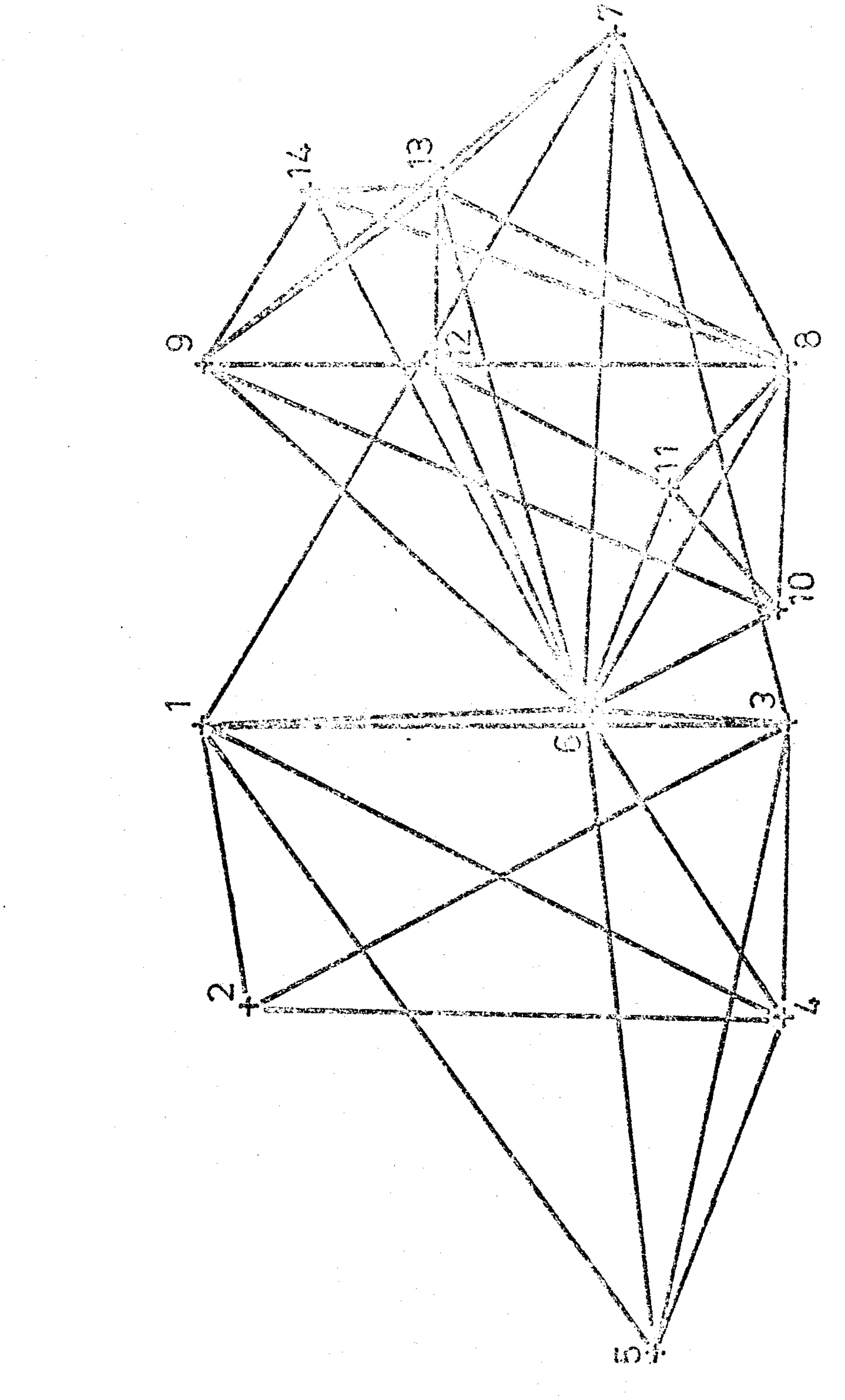


FIG.7·5

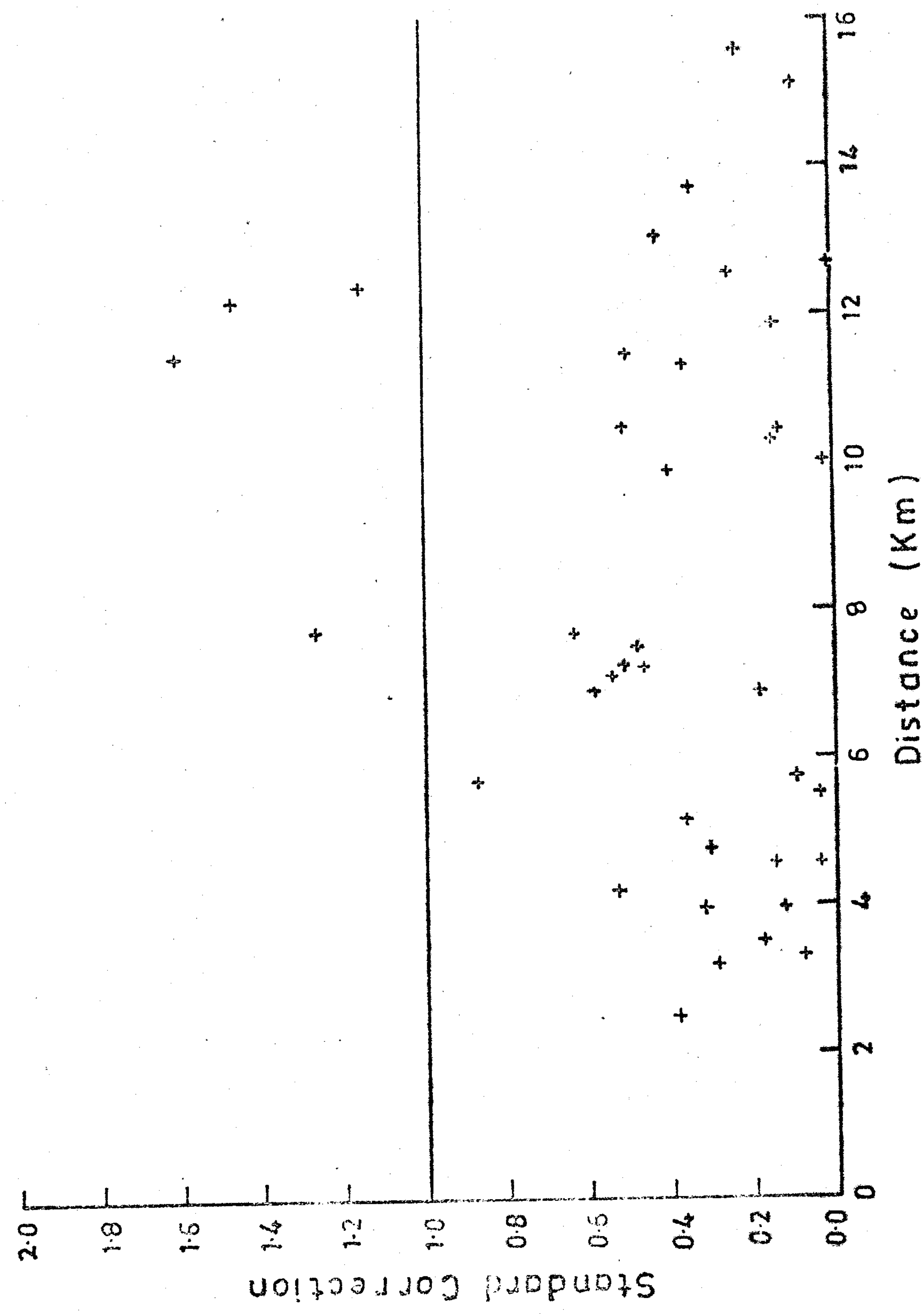

Standard Correction
0.0
0.2
0.4
0.6
0.8
1.0
1.2
1.4
1.6
1.8
2.0
Distance (Km)
0
2
4
6
8
10
12
14
16

FIG. 7.6

The skewness and kurtosis parameters for the observations, adjusted with a variance of $(1.95 \text{ mm} + 4.67 \text{ ppm})^2 \text{mm}^2$, were calculated to be:

$$\gamma_1 = -0.25$$

$$\gamma_2 = 1.32$$

These show the distribution to be slightly skew and leptokurtic.

7.6 Network Four

This survey, (See fig. 7.7), consists of primary triangulation designed to homogenously cover a large area. The observations were taken over a number of years, using Wild T3 theodolites for directions, and the various models of Geodimeter for distance measurement. The terrain is undulating to hilly.

(a) Angular Observations

The direction observations were obtained from field book abstracts. Mean directions at each station and for each set, with their variances, were calculated. The mean internal variance for the network was calculated to be 0.07 $\sec^2$. For an average number of arcs of 18, the internal variance of a single direction was calculated to be 1.26 $\sec^2$, by use of equation (7.4).

The average skewness and kurtosis parameters (γ_1 and γ_2) for the distribution of the direction observations were calculated to be:

$$\gamma_1 = 0.009$$

$$\gamma_2 = -0.327$$

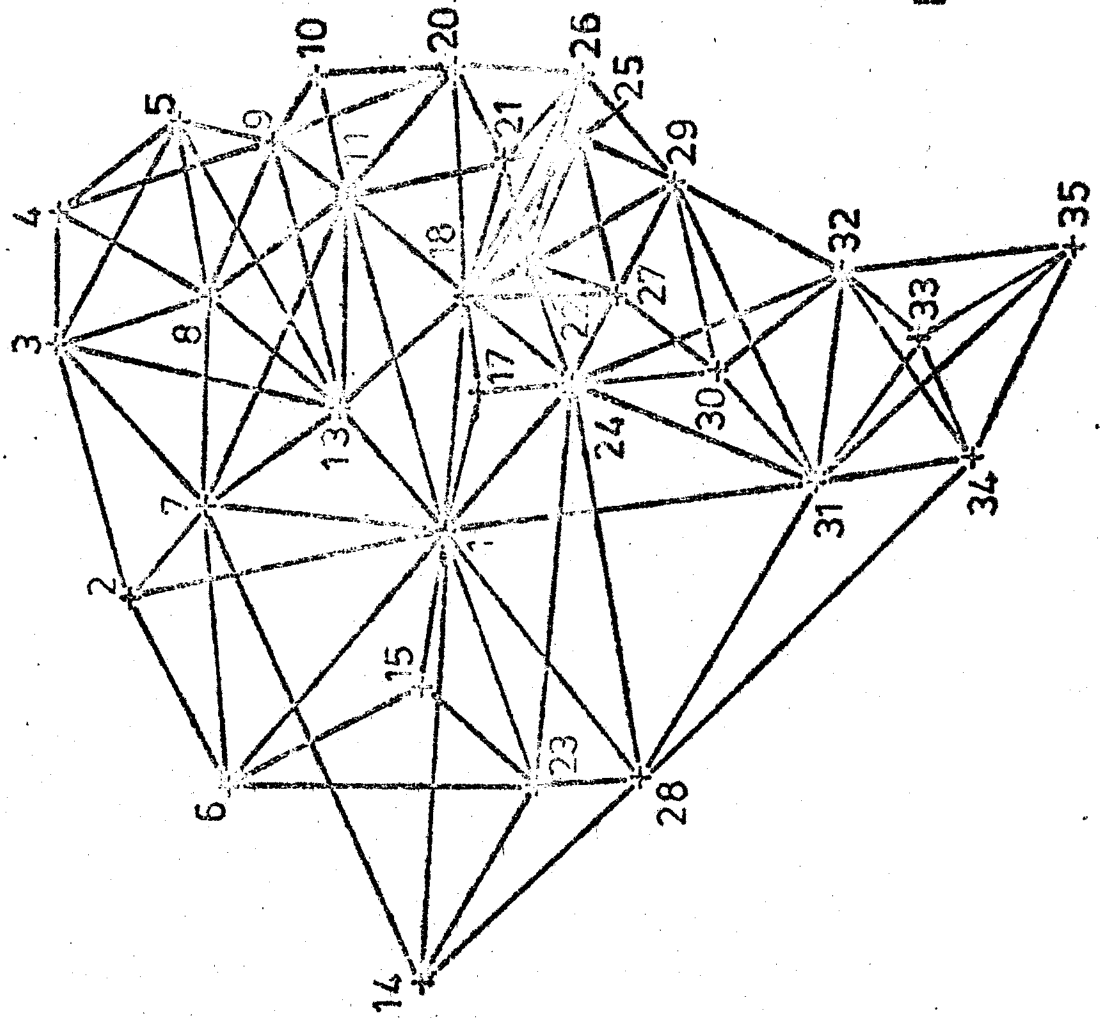


FIG. 7·7

These parameters show the distribution to be significantly symmetrical and somewhat platykurtic.

The first adjustment was aimed at finding the total direction variance and was therefore run as a free net adjustment with angular observations only. Station 1 was held fixed, the bearing from station 1 to station 7 was included to give orientation and the distance from station 21 to 26 was included to give scale. In a number of cases where only two stations were observed in a set, an angle was substituted for the two directions. This had the effect of eliminating both an observation and a normal equation in each instance and resulted in a significant saving of computer time without any loss of useful information. These angles were given double the variance of a direction. Two hundred and forty seven angular observations were adjusted, and of these, one hundred and sixteen were redundant.

The resulting estimate of total variance of a direction observation was 0.31 sec^2. The estimate of external variance, 0.24 sec^2, was found by subtracting the estimate of internal variance from the estimate of total variance.

Summarising, the variance estimates for the angular observations in Network 4 are as follows:

$$\sigma^2_{\text{internal for 18 arcs}} = 0.07 \text{ sec}^2$$

$$\sigma^2_{\text{internal for 1 arc}} = 1.26 \text{ sec}^2$$

$$\sigma^2_{\text{external}} = 0.24 \text{ sec}^2$$

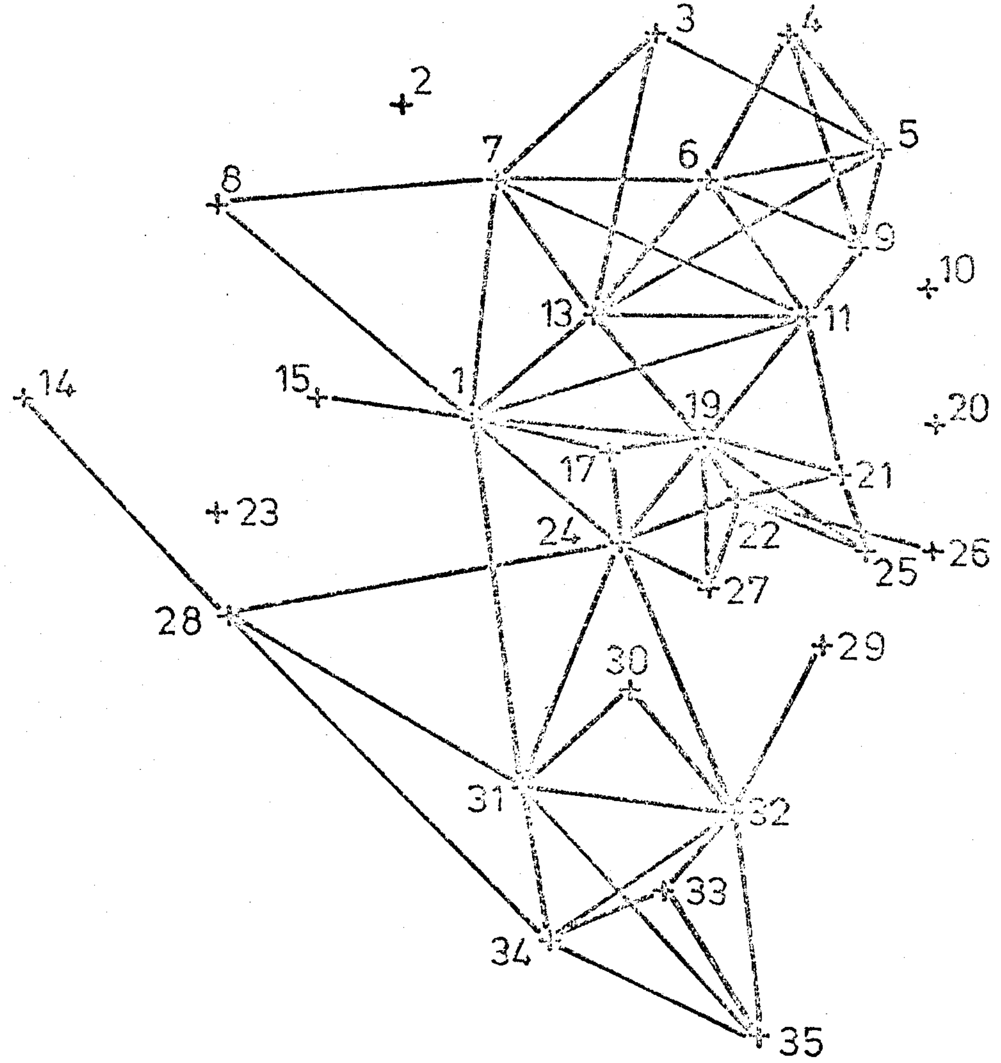
$$\sigma^2_{\text{total}} = 0.31 \text{ sec}^2$$

(b) Linear Measurements

The data was obtained in the form of distances reduced to a surface two thousand feet above the Australian Geodetic Datum, together with the information as to the model of Geodimeter used and the number of measurements meaned to obtain each distance. The fact that these were not sea-level distances created no difficulty as a free net adjustment was used and such an adjustment will not detect an overall scale change even though the mathematical model was calculated for sea-level data. The method of reduction was checked and found to be valid.

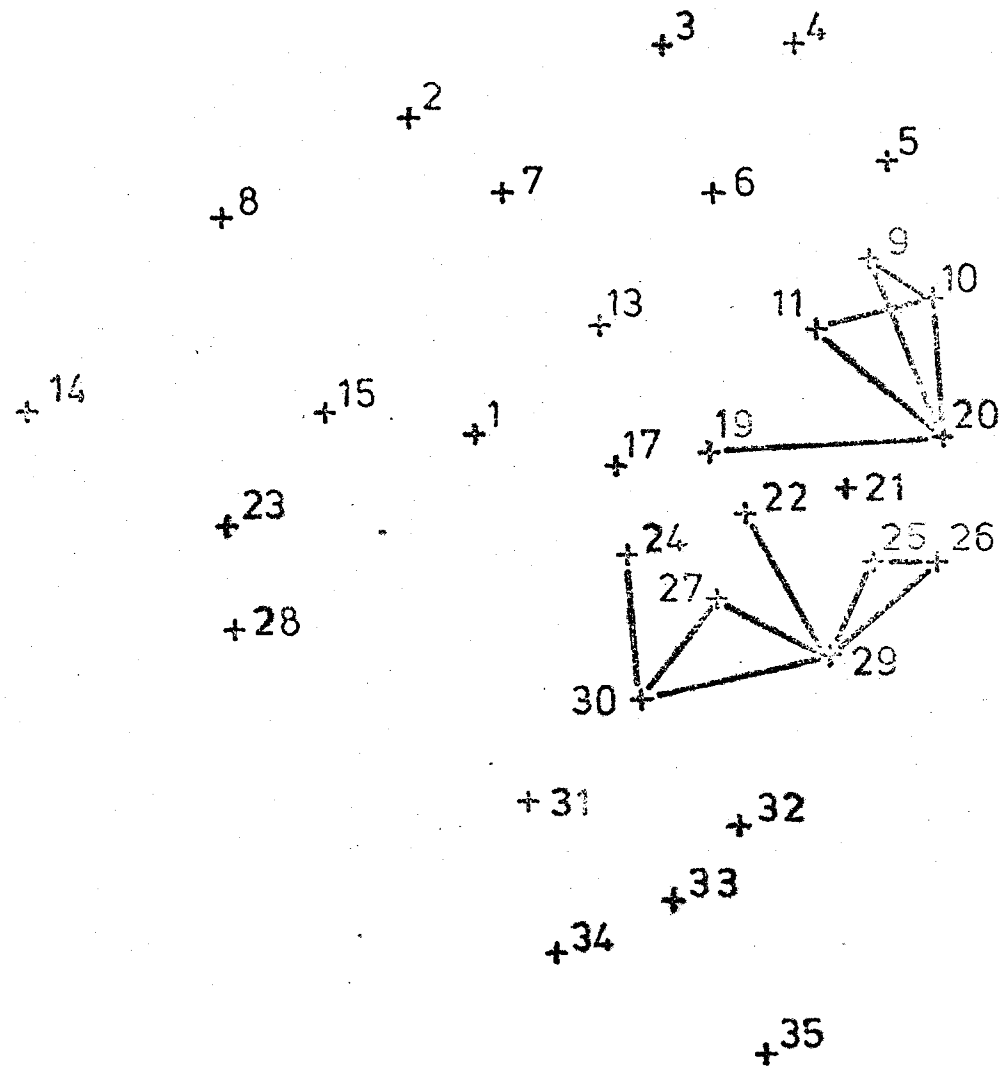
Of the 104 linear observations in the network, 71 were measured using the Model 4 Geodimeter, 14 were observed using the Model 6 Geodimeter, 16 were observed using the MOdel 8 Geodimeter and 3 eccentric distances were measured by steel band. The lines in the network measured using the various models of Geodimeter are shown in figs. 7.8 to 7.10. The manufacturer's claims for, and general experience with these instruments, suggest that their variances are very similar, if not identical, and this being the case, it seemed practical to analyse all the distances in the same adjustment. If the variances are significantly different from one another, this should be evident from the plot of standard corrections to the distances against length of line, (fig. 7.11). The main advantage in such a course of action is that, the total sample of measured distances constitutes a configuration strong enough to fix all stations with a considerable number of redundancies, and does this without recourse to the angular observations. It was pointed out in Chapter 6 that where the observations in the network are all measured distances, that the estimate of variance can be obtained very easily.

The estimate of variance was accepted to be $(5 \text{ mm} + 1.5 \text{ ppm})^2\text{mm}^2$. This estimate gave an $\widehat{S^2}$ of significantly unity and a satisfactory distribution of the standard corrections to the distances. (See fig. 7.11).



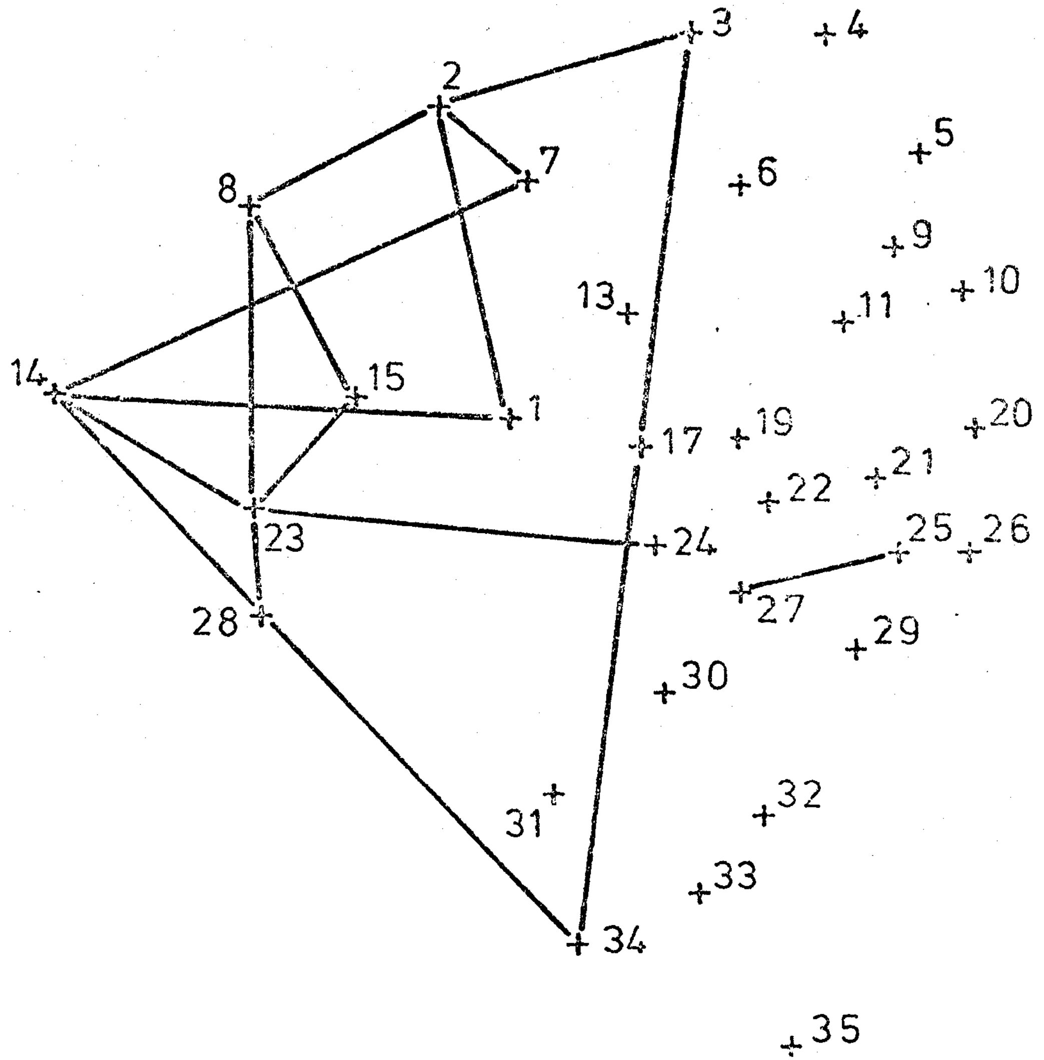
DISTANCES MEASURED
WITH MODEL 4
GEODIMETER

FIG. 7·8



DISTANCES MEASURED
WITH MODEL 6
GEODIMETER

FIG. 7·9



DISTANCES MEASURED
WITH MODEL 8
GEODIMETER

FIG. 7·10

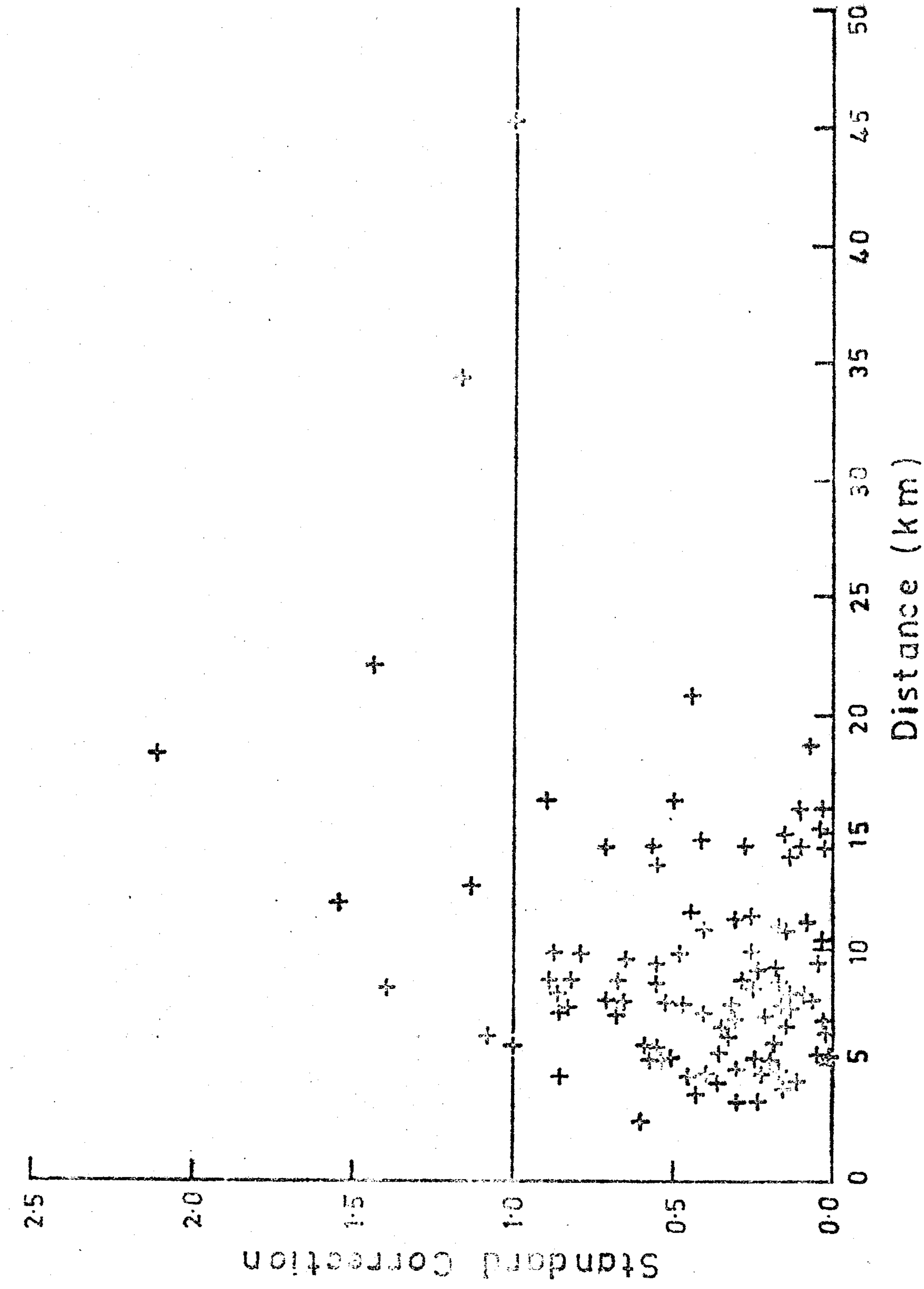

Standard Correction
0.0
0.5
1.0
1.5
2.0
2.5
Distance (km)
0
5
10
15
20
25
30
35
40
45
50

FIG. 7.11

There was no evidence of any significant difference in variance between the instruments. This estimate is of the variance of the reduced observations, each of which is the mean of more than one measurement. It is of more interest to find an estimate of the variance of a single measurement, and as the number of measurements meaned in obtaining each observation varied, it was decided to investigate the field procedures and criteria used for distance measurement.

Normally each distance was measured twice, that is, two sets of three frequencies were observed from the same end of the line, with no shift of instrument set-up and with no significant time lapse between the sets. After the second set was taken, the two sets were compared to check that they were consistent and that there was not a 5 metre difference between them. If there was any doubt in either of the sets, a third set was read immediately. It was not certain, from the data supplied, how many measurements were meaned to obtain the stated distance, though in nearly all cases it was two or three. *(Wellspring, 1972)*.

The variance formula for an electronically measured distance was given in Chapter 4 as:

$$\sigma^2 = (a + bs)^2 \qquad \ldots (4.15)$$

where the 'a' term for a Geodimeter is mainly the internal accuracy of the instrument, and where the 'bs' term is mainly due to the uncertainty in knowledge of the refractive index along the line.

As the sets were taken consecutively, thus under virtually the same atmospheric conditions, there was no reason to assume that knowledge of the refractive index is improved because three sets are taken, as to improve such knowledge, measurements should be taken under a variety of atmospheric conditions. Therefore the 'bs' term of the variance equation is unlikely to be reduced.

The 'a' term would probably be reduced somewhat, as it incorporates reading errors which are reduced by repeated measurement in a manner according to the laws of propagation of variances.

Taking the above reasoning into consideration, the variance equation for a single distance measurement was estimated to be

$$\sigma_s^2 = \left(\frac{a}{0.8} + bs\right)^2 \qquad \ldots (7.5)$$

where a and b in (7.5) are the estimates of the constants of the variance equation for a mean distance as found in the above analysis. Hence, it is assumed that taking the mean of the repeated measurements reduces the "a" term of variance by 20%, but does not alter the "bs" term.

Therefore, given the estimate of the variance of the mean distances as $(5 \text{ mm} + 1.5 \text{ ppm})^2$, the estimate of variance of a single measurement may be calculated as $(6.2 \text{ mm} + 1.5 \text{ ppm})^2$.

(c) Combined Adjustment

The final adjustment was run with all observations, both directions and distances, included with the variances estimated above (in (a) and (b)) attributed to them. According to the theory given in Chapters 5 and 6, the $\overline{S^2}$ resulting from such an adjustment should be unity, however the figure actually obtained was 1.15. This difference from unity indicates some systematic error(s) in either, or both, the angular or linear observations, that only becomes evident when they are adjusted simultaneously. It should be noted, that although these systematic errors are present, they are not large as the adjustment still satisfies the variance ratio test at the 95% confidence level.

If the skewness and kurtosis parameters (γ_1 and γ_2) for the two separate adjustments and the combined adjustment are compared, (See Table 7.1), it may be seen that the systematic effect is probably in the linear rather than the angular observations. This is indicated by the fact that γ_1 and γ_2 for the angular observations do not differ significantly between the separate adjustment and the combined adjustment, while the same parameters for the linear observations change quite appreciably between the separate and combined adjustments. γ_1 changes sign, indicating that the peak of the distribution has shifted from one side to the other, and γ_2 is significantly greater in the combined adjustment, indicating that the kurtosity or "peakedness" of the distribution has increased.

This systematic effect cannot be due to scale errors as a free net adjustment is still being used in the combined adjustment, and therefore the overall scale is determined by the observations themselves and not by the fit between fixed stations. It is very difficult to postulate just what is causing this systematic effect but it is of interest to know it exists.

7.7 Network Five

This survey, (see Fig. 7.12) is basically a chain of triangulation with extra bracing and several centrepoints included. It was observed over a number of years using Wild T3 theodolites and covers mainly flat to undulating country.

The direction observations were obtained, in the form of semi-directions, from the original field-books. One hundred and thirty five sets of directions ere considered. Mean directions with their variances, at each station and for each set, along with skewness and kurtosis parameters for each set were calculated. The average internal variance was calculated

ADJUSTMENT / OBSERVATION TYPE	Angular Obs. Only		Linear Obs. Only		All observations	
	γ_1	γ_2	γ_1	γ_2	γ_1	γ_2
Angular	-0.28	0.60	-	-	-0.22	0.79
Linear	-	-	0.40	1.17	-0.60	1.96

TABLE 7.1

to be 0.11 sec^2, for an average number of arcs of 12. The average internal variance of a single direction was calculated, using equation (7.4) as 1.32 sec^2.

The average skewness and kurtosis parameters, (γ_1 and γ_2), with all sets considered were calculated to be:

$$\gamma_1 = 0.024$$

$$\gamma_2 = -0.402$$

As in network 4, these parameters show the distribution of the observed directions to be significantly symmetrical and somewhat platykurtic.

The observations were subjected to a free net adjustment with stations 13 and 24 being held fixed. As with network 4, sets of directions with only two observed stations were considered as angles with double the direction variance. When the adjustment was first run it was found to be unstable. This instability was due to problems in scale transfer through two weak figures in the eastern half of the network, and was alleviated by the inclusion of distances, between stations 37 and 41, and between stations 44 and 47. The inclusion of these distances should not effect the estimation of direction measurement variance as their corrections will not be considered in the calculation of $\overline{S^2}$. Apart from this they are necessary rather than redundant observations and as such are not placing constraints on the direction observations.

Two hundred and ninety angular observations were adjusted, and of these, one hundred and sixty three were redundant.

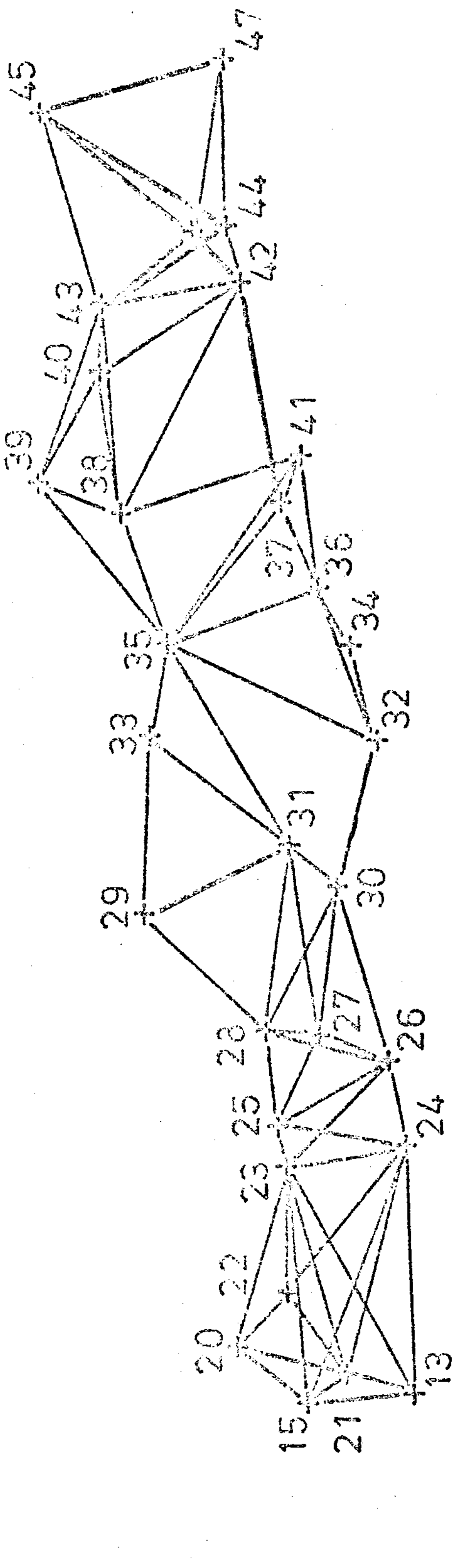

45
47
44
42
43
40
41
39
38
37
36
34
35
32
33
31
30
29
27
28
26
25
24
23
22
20
13
15
21

FIG. 7·12

The resulting estimate of total variance of a direction observation was 0.51 sec^2. The estimate of external variance for the survey, 0.40 sec^2, was found by subtracting the estimate of internal variance from the estimate of total variance.

The skewness and kurtosis parameters, (γ_1 and γ_2), of the distribution of the adjusted observations were calculated.

$$\gamma_1 = -0.78$$

$$\gamma_2 = 9.84$$

These parameters show the distribution to be somewhat skew and markedly leptokurtic.

In summary, the variance estimates for the angular observations in network 5 are as follows:

$$\sigma^2_{\text{internal for 12 arcs}} = 0.11 \text{ sec}^2$$

$$\sigma^2_{\text{internal for 1 arc}} = 1.32 \text{ sec}^2$$

$$\sigma^2_{\text{external}} = 0.40 \text{ sec}^2$$

$$\sigma^2_{\text{total}} = 0.51 \text{ sec}^2$$

7.8 Conclusions

(a) Angular Observations

The analysis of the internal variance of Wild T3 observations returned surprisingly consistent results. The internal variances estimated for a single arc were:

$$\text{Network 1} \quad - \quad \sigma^2_{int} = 2.07 \text{ sec}^2$$

$$\text{Network 4} \quad - \quad \sigma^2_{int} = 1.26 \text{ sec}^2$$

$$\text{Network 5} \quad - \quad \sigma^2_{int} = 1.32 \text{ sec}^2$$

The variances estimated from networks 4 and 5, which both contain a large number of redundancies, agree very well with each other, while the variance estimated from network 1, a much smaller network, is not greatly different. As the estimates from networks 4 and 5 were determined with a large number of redundancies, a variance between these two estimates can be accepted with confidence. The accepted estimate of the internal variance of a single arc, observed with a Wild T3 theodolite, is 1.30 sec^2.

The estimate of external variance obtained from networks 1, 4 and 5 also agree well with each other. The estimates were:

$$\text{Network 1} \quad - \quad \sigma^2_{ext} = 0.41 \text{ sec}^2$$

$$\text{Network 4} \quad - \quad \sigma^2_{ext} = 0.24 \text{ sec}^2$$

$$\text{Network 5} \quad - \quad \sigma^2_{ext} = 0.40 \text{ sec}^2$$

As the number of redundancies in network 1 is very small, the estimate obtained from it does not carry the same weight as the estimates obtained in networks 4 and 5. It is significant that network 4 is observed in undulating to hilly terrain, while network 5 is observed in mainly flat terrain. Lateral refraction effects will usually be more pronounced in flat terrain, and this perhaps accounts for the difference between the estimates of external variance from networks 4 and 5. As it is very likely, if not certain, that external variance is dependent on terrain conditions, it is not rational to combine these estimates to form a single estimate of external variance that is independent of terrain. Therefore, the following are the accepted estimates:

For undulating to hilly terrain:-

$$\sigma^2_{ext} = 0.25 \text{ sec}^2$$

For flat to slightly undulating terrain -

$$\sigma^2_{ext} = 0.40 \text{ sec}^2$$

The accuracy of these estimates cannot be ascertained until other networks, observed in similar types of terrain, have been analysed. However, their agreement with each other shows that they are, at the very least, of the right order of magnitude.

(b) Linear Observations

The variances estimated, in networks 2 and 3, for measurements with Cubic Electrotape DM20 instruments agree well with each other and also with the established variance of 30 mm carrier wave length Tellurometers, which are very similar to the Electrotape. Therefore, the variance estimate of $(20 \text{ mm} + 5 \text{ ppm})^2 \text{mm}^2$, obtained in networks 2 and 3, is probably very close to the true variance of Cubic Electrotape DM20 instruments.

The estimated variance of Geodimeter measurements, $(6.2 \text{ mm} + 1.5 \text{ ppm})^2 \text{mm}^2$, which was obtained from network 4, appears to be a reliable estimate in that it was determined using an adequate number of redundancies (35), and in that it agrees well with the manufacturer's specification.

CHAPTER 8

THE PREDICTION OF VARIANCE

8.1 Introduction

Chapter 7 was concerned with the analysis of observed networks in order to obtain an estimate of the variances of the observations forming the networks. Although the networks were observed by different authorities, where similar instruments were used, the analyses returned fairly consistent results when the differing observing conditions and techniques were taken into account. This suggests that, using these estimates and given knowledge of the instrument, the observing technique to be used and the conditions under which the observations are to be taken, a reasonably accurate variance may be predicted.

8.2 Prediction of Variance by Empirical or Experimental Means

In Chapters 3 and 4 an attempt was made to predict variance from an empirical and experimental analysis of the factors that contribute to it. In the case of angular observations (See Chapter 3), this method of prediction is unsatisfactory as many of the contributing factors cannot be evaluated by empirical or experimental means. Angular observations tend to be very dependent on the observer, who, under field conditions, is subject to many nebulous influences, the effects of which are usually impossible to evaluate in an experimental environment. The variance of an observation taken under field conditions always tends to be higher. The internal variance estimated for the Wild T3 in Chapter 7, an estimate based on field conditions, is 1.30 sec^2 as opposed to the empirical/experimental estimate, derived in Chapter 3, of 0.45 sec^2. Therefore, until more research is done on human factors in observational precision and on the simulation of field conditions in laboratory experiments, the prediction of angular variance by empirical and experimental means is not a practical technique.

Linear observations (see Chapter 4) tend to be less dependent than angular observations are on the observer and the prevailing conditions. This applies particularly to electronic distance measurement. Instrumental sources of error may be experimentally evaluated, and propagation errors may be estimated by theoretical means and from past experience. Hence, it seems logical that the results of Chapters 4 and 7, for distance measurement do agree fairly well. However, in view of the fact that actual field observations are being assessed in Chapter 7, and that good agreement between different networks using the same instruments was found, the estimates obtained in Chapter 7 must be taken as being more reliable than those obtained in Chapter 4.

8.3 Evaluation of Angular Variance Estimates from Chapter 7

In networks 1, 4 and 5 of Chapter 7, the estimates of internal and external variance of angular observations, (See Chapter 5 for definitions), were arrived at as follows:

1. The average internal variance of a mean direction was found by averaging the internal variances of all sets of directions in the network. This variance was the average internal variance of a mean direction derived from the average number of arcs. The average internal variance of a single direction, (i.e. derived from one arc), was found by multiplying the above variance by the average number of arcs.

2. An estimate of the total variance of a mean direction was found from the network adjustment. In this

adjustment, all the observations (mean directions) were treated as being of similar quality, and no regard was given to the number of arcs observed to obtain each observation.

3. An estimate of the external variance of an observation was calculated by subtracting the estimate of internal variance of a mean direction, derived from the average number of arcs, from the estimate of total variance.

The procedure described above is approximate in that it assumes that all observations are of the same quality when in fact they are not. This assumption will have an effect on the adjustment and therefore on the $\overline{S}^2$ obtained. However, the question is whether or not this effect will be significant. Intuitively, one would not expect the effect to be large as only internal variance is dependent on the number of arcs observed, and when a reasonable number are observed, the internal component is small compared to the external component of variance. In other words, the effect of the approximation on the value of total variance for a given observation is usually not very great.

The significance of the approximation was tested using Network 4 (see Chapter 7). This network was first adjusted using the estimate of total variance, 0.31 sec^2, found in Chapter 7, as a default variance that was applied to all observations, regardless of the number of arcs observed. Plumbing errors were not allowed for, so the $\overline{S}^2$ given was 1.02 instead of the unity obtained in Chapter 7. Individual variances, calculated with regard to the number of arcs observed were then applied to the observations. (See Section 8.4). Once again no allowance was made for plumbing errors. The adjustment was rerun with these individual variances and gave an $\overline{S}^2$ of 1.17. Both adjustments satisfy the variance ratio test at the 95% confidence level, indicating that the variances used in both adjustments are representative of the same population. Therefore, if this finding can be extrapolated to all cases, it appears that the approximation does not have a significant effect and that the estimates of variance found in Chapter 7 are valid. It does not seem unreasonable to extrapolate this finding to all networks where the number of observations is fairly large and where the variation in number of arcs observed is not much greater than the variation in Network 4.

8.4 Prediction Based on Valid Variance Estimates

Once valid estimates of variance have been obtained, these may be broken down into their basic components, and these components categorized for type of instrument, observing techniques and observing conditions. The components may then be used to predict variances of observations taken, or to be taken, using specified equipment and techniques and under specified observing conditions.

(a) Angular Observations

The basic components of angular variances are internal variance and external variance.

Internal variance is dependent on the number of arcs observed while external variance is only dependent on the number of different occasions over which the observations were taken. Therefore, angular variances may be calculated using the formula:

$$\sigma^2 = \frac{\sigma^2_{int.1}}{n} + \sigma^2_{ext} \qquad \ldots (8.1)$$

where $\sigma^2_{int.1}$ is the internal variance of a single arc,

σ^2_{ext} is the external variance,

and n is the number of arcs observed, or to be observed.

The internal variance of a single arc will have been derived for a certain type of theodolite and for observations taken using a certain observing technique. The observing technique is not really a variable as it must be assumed that the observer is experienced and that he will follow theoretically sound observing procedures. This being the case, observing technique may be taken to mean the number of arcs observed, as this should be the only real variable in the observing procedure.

It is very difficult to say whether or not the external variance should be divided by the number of different occasions over which the observations were taken. To the best of the author's knowledge, no significant research has been done on this question, although it is generally accepted that the external variance can be reduced by taking the observations over two or more different occasions. In practice, and except for very precise work, it is seldom that the observations are taken over more than two occasions. The estimates for external variance found in Chapter 7 for networks 4 and 5 represent the external variance of observations taken on either one or two occasions, with the average number of occasions closer to one than to two. It therefore seems more reasonable to take the external component of variance as being a constant until better knowledge is available. The value of the component will depend on the observing conditions under which the observations are to be taken. Observations taken in good geodetic conditions will have a lower external variance than those taken over flat country with many grazing rays.

Observations taken with Wild T3 theodolites were analysed in Chapter 7 and the resulting variance estimates may be used to evaluate equation (8.1) for this instrument.

For undulating to hilly terrain:

$$\sigma^2 = (\frac{1.30}{n} + 0.25) \sec^2 \qquad \ldots (8.2)$$

For flat to slightly undulating terrain:

$$\sigma^2 = (\frac{1.30}{n} + 0.40) \sec^2 \qquad \ldots (8.3)$$

(b) Linear Observations

The two components of the variance of an electronically measured distance are:

1. The variance due to instrumental factors, which depend on the make and model of the instrument used.

2. The variance due to propagation factors, which depend chiefly on the carrier wave length being used.

The nature of these factors was discussed in Chapter 4.

The first component will be the same for all lines, irrespective of lengths, while the second component is dependent on distance. The first component will be reduced somewhat by repeated measurements taken at the one set-up, but the second component, being mainly due to the uncertainty in refractive index along the line, will only be reduced by repeated measurements on different occasions under different atmospheric conditions. *(See Chrzanowski and Derenyi, 1967,* and the discussion of the linear observations in Section 7.5(b)). The total variance may be expressed using formula (4.15).

$$\sigma^2 = (a + bs)^2 mm^2$$

where a is the variance due to instrumental factors, in millimetres.

b is the variance due to propagation errors, and is usually expressed in parts per million (ppm) of the distance.

and s is the distance measured.

Electronic distance measurement is usually not dependent on observing techniques as it must be assumed that the instrument manufacturer's recommended procedure is being used. Also it is generally not as dependent on observing conditions as is angular work. Naturally, conditions may be encountered that give rise to a variance higher than usual, although such conditions tend to be rather exceptional, but more significantly, it is usually very difficult to identify and categorize these exceptional conditions. Conditions that, by usual criteria, look average may give rise to an abnormal measurement. However, the converse is more often true, that is, measurements in conditions which one would expect to be troublesome will, more often than not, give normal results. Therefore, only one variance formula, based on average conditions, can logically be formulated for a given instrument.

As meteorological observations are an integral part of electronic distance measurement, the precision to which they are assumed to have been measured must be stated when the "a" and "b" components of formula (4.15) are evaluated for a particular instrument.

Observations taken with two different types of electronic distance measurement instruments were analysed in Chapter 7.

1. Cubic Electrotape DM20.

$$\sigma^2 = (20 \text{ mm} + 5\text{ppm})^2 \text{mm}^2 \qquad \text{... (8.4)}$$

This variance is applicable to all 30 mm carrier wave length instruments, as they all tend to be very similar in type and precision. (See discussion in Section 7.3 and 7.4).

2. Geodimeter

$$\sigma^2 = (6 \text{ mm} + 1.5 \text{ ppm})^2 \text{mm}^2 \qquad \text{... (8.5)}$$

This variance is applicable to all Geodimeter models.

The above formulae (8.4 and 8.5) are for a single measurement and assume meteorological measurements taken to the following precisions:

Air temperature; ±0.4°C

Difference between wet and dry bulb temperatures; ±0.1°C

Pressure; ±2 mb

8.5 Conclusions

By using the methods of prediction given above, it is feasible to build up a comprehensive list of variances for observations taken using particular instruments, particular observing techniques and under particular observing conditions. These variances will be quite reliable if their components are derived from large networks containing large numbers of redundant observations.

They may be used in both optimization and adjustment studies. When used in adjustment studies, a check on their suitability is available, as if they are not the true variances of the observations, the variance factor after adjustment, ($\overline{S^2}$), will not equal unity. The $\overline{S^2}$ obtained may be tested against unity by the variance ratio test. This is in effect a comparison of the variances applied to the observations, (the apriori values derived from a significantly infinite number of redundancies), against the estimated variances obtained from the adjustment, (derived using the number of redundant observations in the survey). If the test is satisfied at, say, the 95% confidence level, then it is 95% probable that the sample of observations are from the same population for which the apriori variances are derived. Where the number of redundancies in the network is not large, and the variance ratio test has been satisfied, it is more valid to use the apriori variances, rather than the variances estimated from the adjustment, by the methods of Chapters 5 and 6.

CHAPTER 9

OPTIMIZATION EXAMPLES

9.1 Introduction

Two examples of network optimization are given below to demonstrate, firstly, the application of the theory described in Chapter 2, secondly, the use of variances predicted by means of Chapter 8, and thirdly, some of the practical techniques involved in optimisation. The fact that the two networks differ greatly in purpose and design demonstrates the versatility and scope of the technique.

The data required to evaluate the optimization formulae of Chapter 2 are the approximate coordinates of the network stations, the types and positions of the observations in the network, and the estimated precisions of the observations. This Chapter deals with methods of determining this data as well as the interpretation of the optimization results, but does not deal with the evaluation of the optimization formulae to obtain error ellipse parameters. This calculation is carried out using a computer programme developed by Dr. J.S. Allman of the University of New South Wales.

9.2 Geodetic Optimization

(a) Introduction

The purpose of this optimization is to design a chain of triangulation to provide first order horizontal control over a band of country. The chain is to be located on the spheroid by one fixed Station (28), on the western end of the chain, and given orientation by an azimuth from that station to an adjacent station (29). See Fig. 9.1 for the location of these stations, and Table 9.1 for their approximate coordinates.

The chain is to run over flat country and the configuration will be dictated by the location of small rises in the topography. The configuration shown in Fig. 9.1 is the best that can be attained as all possible station sites and all inter-visible lines between these station sites are being used. As this is the maximum configuration, the optimization problem reduces to one of determining what type, location and precision of observations will give the network sufficient strength so that the error ellipses of all stations are inside prescribed limits.

A plot of the error ellipses at each of the network points tends to be of greater use than the simple numerical output of major axes, minor axes and orientation. Although the numerical output will tell if the desired positional accuracy has been attained, the plot shows trends in scale and azimuth in the network as well. Such information is very valuable in deciding what changes to make in the proposed network for the next run of the optimization.

In most cases, a plot of error ellipses from all points in the network would tend to confuse,rather than to illustrate, the relevant trends. The ellipses of a few equally spaced points along the network will serve just as well to show trends in scale and azimuth deterioration as the network extends further from the origin (station 28). Therefore, only the ellipses at stations 35, 42 and 47 will be considered in the analysis below. (See Figs. 9.2 to 9.8).

Examination of the error ellipses resulting from the selected scheme will thus allow the selection of an optimum network. Further, where the network shows a weakness in scale and/or orientation, the effect of additional observations may also be analysed by their effect on the error ellipses.

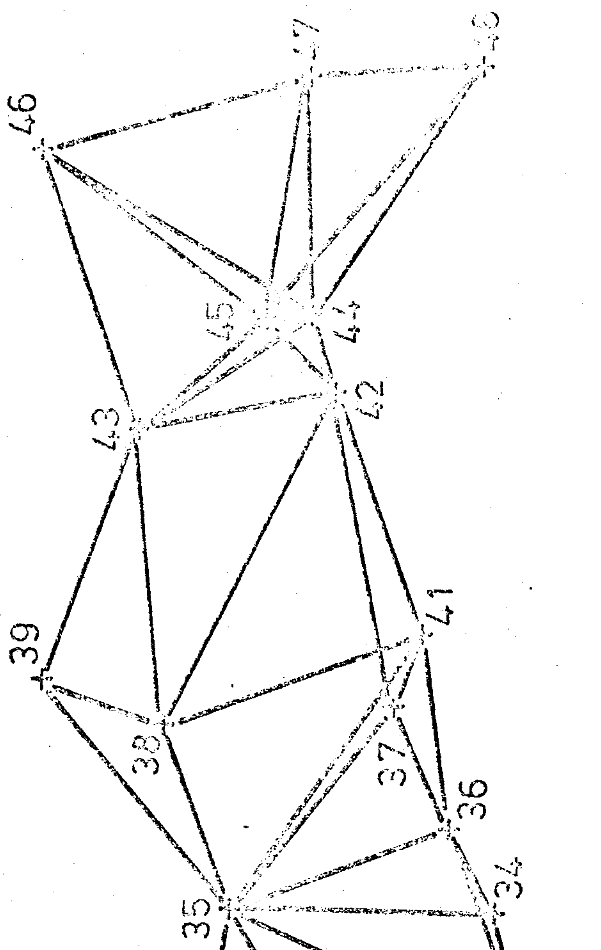


FIG. 9.1

TABLE 9.1

Approximate Positions of Stations

Station Number	Latitude	Longitude
26	31° 44' 19" S	142° 41' 46" E
28	31 20 29 S	142 47 38 E
29	30 57 19 S	143 08 53 E
30	31 33 56 S	143 15 26 E
31	31 24 52 S	143 22 30 E
32	31 40 41 S	143 42 39 E
33	30 58 04 S	143 42 22 E
34	31 35 38 S	144 00 02 E
35	31 01 09 S	144 00 46 E
36	31 29 46 S	144 11 05 E
37	31 22 20 S	144 27 37 E
38	30 52 06 S	144 24 45 E
39	30 36 20 S	144 30 47 E
41	31 26 14 S	144 36 50 E
42	31 14 34 S	145 09 18 E
43	30 48 26 S	145 04 00 E
44	31 11 22 S	145 19 26 E
45	31 05 15 S	145 18 37 E
46	30 35 44 S	145 41 06 E
47	31 10 51 S	145 51 11 E
48	31 33 38 S	145 52 39 E

(b) Accuracy Specification

Before an optimisation is commenced some specification for positional accuracy must be set.

Consider the specification that the true coordinates of a point do not depart from the calculated coördinates by more than 12 parts per million (ppm) of the distance from the origin of the survey to the point in question.

The probability that the true value of a quantity falls within a range of one standard deviation from the least squares estimate of that quantity is 68.3%. It follows that the probability of the true coordinates of a point falling inside the orthogonal rectangle about the error ellipse of the point is (68.3 x 68.3)% or 46.6%. Richardus *(1966)* shows that the probability of a point actually falling inside its error ellipse is about 39%. The exact probability depends on the number of redundancies in the network. The probability will range from 35% for three redundancies to 39% for an infinite number of redundancies. Even with minimum data this network has a large number of redundancies and the probability may be taken as 39%. Therefore, if the length of the semi-major axis of the error ellipse is 12 ppm of the distance from the origin, the probability that the specification is being met is only 39%. This is far from satisfactory. If however three standard deviations are considered, the probability rises from 39% to about 98%. *(Patterson, 1973)*. This means that if the semi-major axes of the error ellipses are no more than 4 ppm of the distance to the origin, there is a 98% probability that the specification of 12 ppm is being met. This is the criterion used in the example.

In terms of the present network, a specification of 4 ppm limits the length of the semi-major axes of the error ellipses at stations 35, 42 and 47 to 50 cm, 91 cm and 119 cm. respectively.

(c) Variances of Observations

It was pointed out in Chapter 2 that the precision of optimization depends almost entirely on the precision of the observational variances used. In other words, how closely the variances used in the optimization are to those obtainable in the field. The results of the analysis carried out in Chapter 7 provide variances based on field observations which should be very close to those obtainable under the conditions that the observations of this network are proposed to be taken.

In actual fact, the variances used in this optimization are those obtained from the analysis of the present network after it was observed. (See Section 7.6). However, the fact that these variances agree well with other results obtained in Chapter 7 justifies the inclusion of the following discussion on selection of variances as a general example.

(i) Angular Variances

The theodolite used in a scheme such as this would be a Wild T3 or other instrument of the same order. It is proposed that directions be observed using the procedure set out in Section 3.1(c) and that these directions be observed in twelve arcs.

As the network is to be observed over flat country, the variance formula (8.3) of Chapter 8 will apply.

$$\sigma^2 = \left\{\frac{1.30}{n} + 0.40\right\} \text{ seconds squared}$$

where n is the number of arcs to be observed.

In the present case where n equals 12,

$$\sigma^2 = 0.51 \text{ seconds squared}$$

This is the total variance of twelve arcs and is the figure to be used in the optimization.

(ii) Distance Measurement Variance

The distance measurement instrument used in a survey of this nature would be a microwave electronic distancer such as the Tellurometer MRA101, Tellurometer MRA3, Cubic Electrotape DM-20, Wild DI50 or the Tellurometer MRA4. The reasons for this choice are the distances involved and the accuracies required. There are some very long distances in the network, the longest being 82 km between stations 38 and 42. There are no visible light carrier wave instruments with this range available at the present time, so one is restricted to a microwave instrument. Of the instruments listed above, only one has a stated range of greater than 82 km, this being the Wild DI50, with a stated range of 150 km. The other manufacturers state ranges of the order 60 km. Experience under Australian conditions has shown that distances considerably longer than this can be measured by these instruments. However, if difficulty is encountered with longer lines, such lines can always be measured in parts.

The instruments listed above, with the exception of the Tellurometer MRA4, all fall into the class of 30 mm carrier wavelength microwave distance measurers. Assume that a set of MRA4 Tellurometers is not available for the survey and that 30 mm wavelength instruments are to be used.

The variance of measurements made by these instruments was estimated to be:

$$\sigma^2 = (25 \text{ mm} + 6 \text{ ppm})^2 \text{mm}^2$$

This estimate was based on discussions with A.J. Robinson *(1971)* and past experience, and was adopted for the optimization. Subsequent investigations, (See Chapters 7 and 8), have indicated that the variance

$$\sigma^2 = (20 \text{ mm} + 5 \text{ ppm})^2 \text{mm}^2$$

is probably a better estimate.

(iii) Variance of Observed Azimuths

The variance of observed azimuths obtained in Section 5.5 was for observations made in country similar to that covered by the proposed network. The variance given in formula (5.40) will therefore be used for the optimization. That is

$$\sigma^2_{Az} = 2.89 \text{ sec}^2$$

(d) Optimization Trials

(i) Network No. 1

In this network, it is proposed that distances will be measured along all lines shown in Fig. 9.1. Combined with the fixed coordinates and fixed azimuth at station 28, these observations would enable the network to be solved, i.e. With this data the coordinates of all points in the network could be found. The error ellipse diagram for this configuration is shown in Fig. 9.2 (Trial 1).

Examination of this diagram shows firstly the trends existing in the network and secondly that these ellipses do not meet the specification. As the network extends away from the origin at station 28, the scale components of the ellipses (those along the longitudinal axis of the network) increase very slowly whilst the azimuth components of the ellipses (those perpendicular to the longitudinal axis of the network) increase very rapidly. This implies that scale is held firm throughout the network and that azimuth deteriorates rapidly as the distance from the fixed point increases. This outcome is to be expected, as a network comprised solely of distances will naturally tend to hold scale but not azimuth.

Ellipse Scale = 52 cm /cm

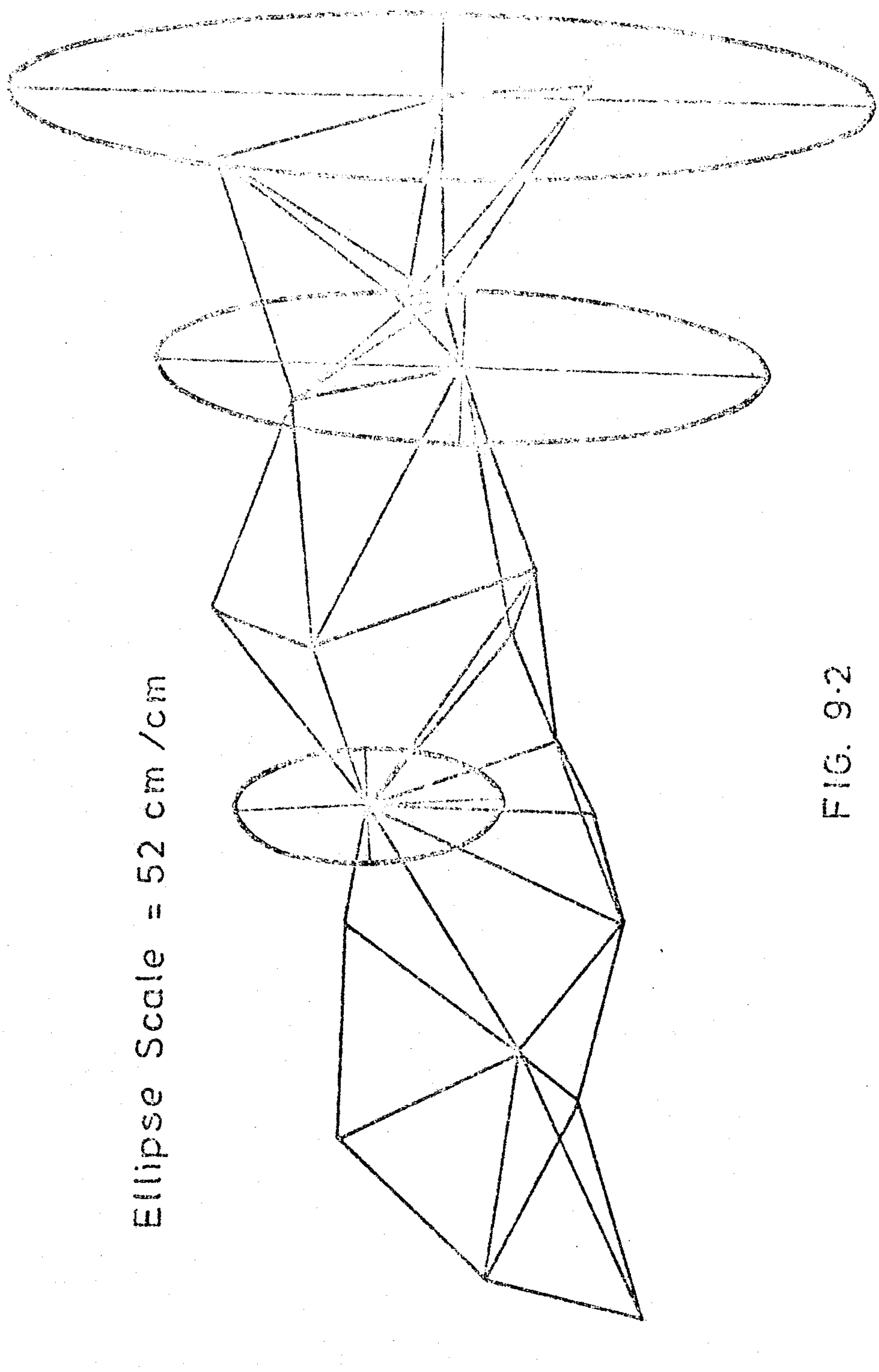


FIG. 9.2

The usual way to improve azimuth control in a network is to include observed azimuths along the network. Observed azimuths at stations 31, 35, 42 and 47 were included in the data for trial 2. These stations were chosen because of the availability of distant stations for use as reference objects (R.O.'s) and because they are fairly equally spaced along the network.

The error ellipse diagram for this configuration is shown in Fig. 9.3. Once again the specification is not met although the rate of azimuth deterioration is considerably less. As would be expected the observed azimuths did not have much effect on the scale in the network.

The azimuth components of the ellipses must still be improved if the specification is to be met. To do this, azimuths could possibly be observed at more stations, but this would mean that extra stations have to be visited with a theodolite. It would seem a more logical procedure if more use could be made of the theodolite at the stations already being occupied. Perhaps it would be of advantage to observe directions to stations visible from these azimuth stations. Such directions were included in the data set for trial 3. The ellipses resulting from this trial are shown in Fig. 9.4. These ellipses satisfy the specification.

It is of interest to note that scale as well as azimuth is improved by the addition of the proposed directions. This is due to the fact that the directions, unlike the proposed azimuths, tend to brace the network for scale as well as for azimuth.

Table 9.2 gives the parameters of the ellipses shown in Fig. 9.2, 9.3 and 9.4 (Trials 1, 2 and 3).

(ii) Network No. 2

In this network, the coordinates of station 28 and the aximuth from station 28 to station 29 are fixed.

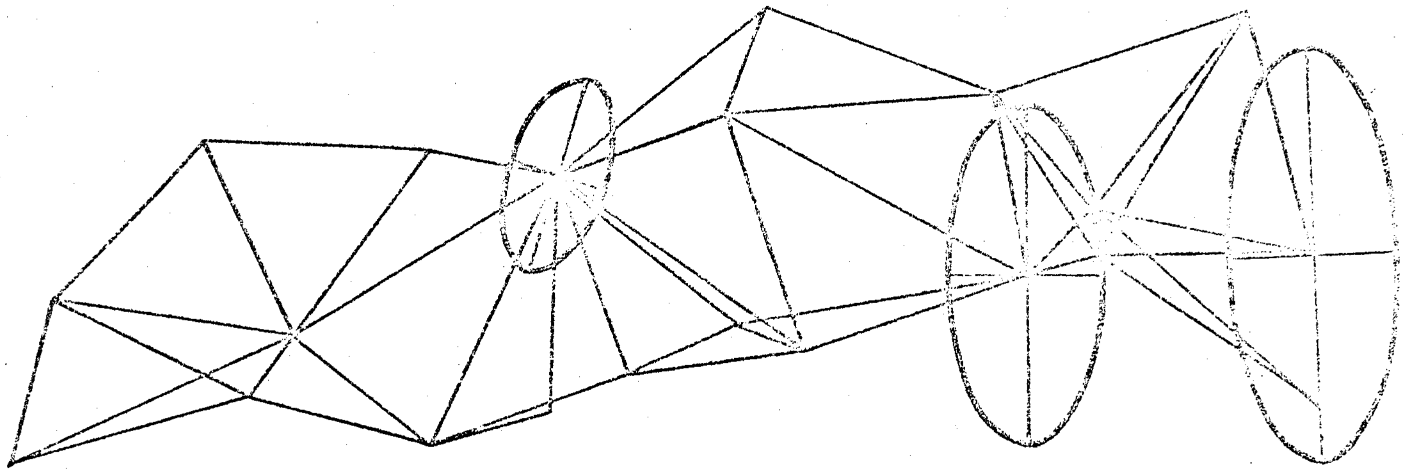
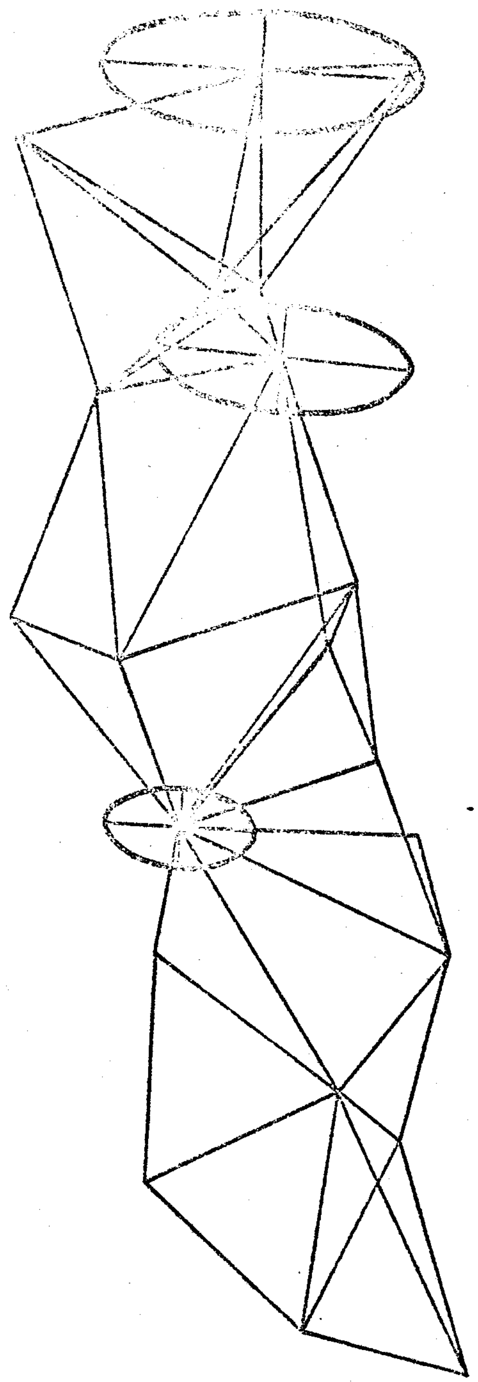


FIG. 9·3



Ellipse Scale = 52 cm /cm

FIG. 9.4

TABLE 9.2

Ellipse Parameters

Station	Component \ Trial	1	2	3
35	Semi-Major Axis	96.59 cm	67.22 cm	50.12 cm
	Semi-Minor Axis	41.57 cm	33.30 cm	26.11 cm
	Orientation	177.49°	16.90°	2.80°
42	Semi-Major Axis	224.23	115.74	86.30
	Semi-Minor Axis	56.29	53.87	36.42
	Orientation	1.19	178.98	4.21
47	Semi-Major Axis	318.48	142.03	109.22
	Semi-Minor Axis	62.08	58.43	42.01
	Orientation	0.50	177.26	1.94

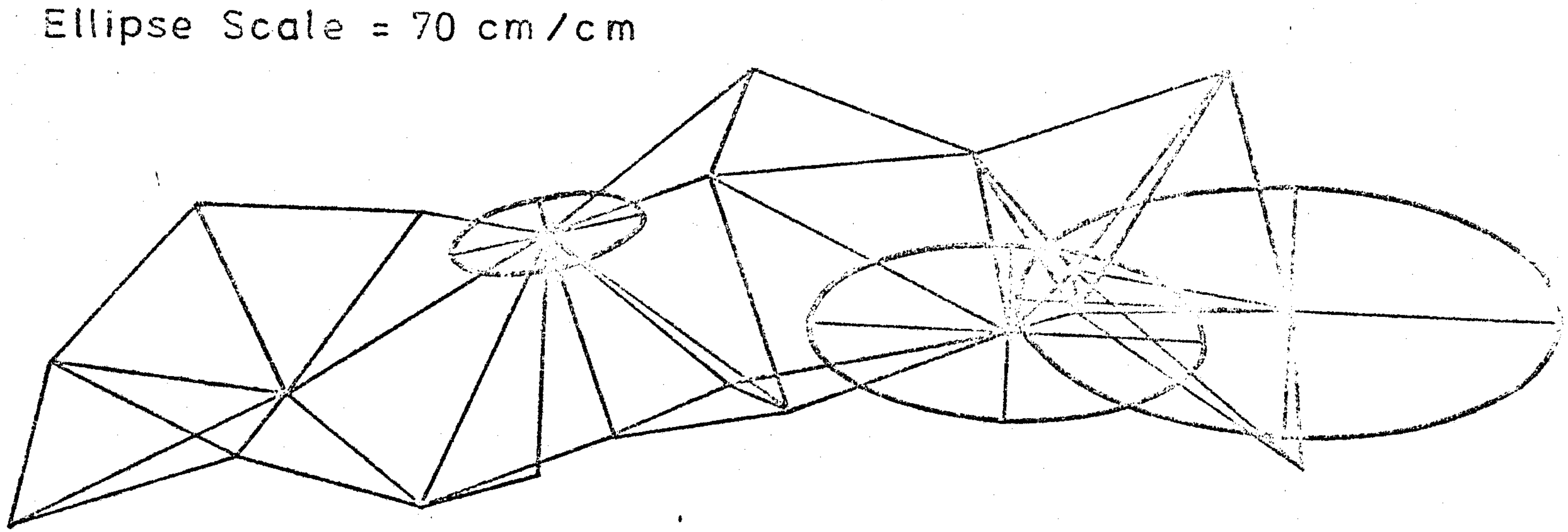
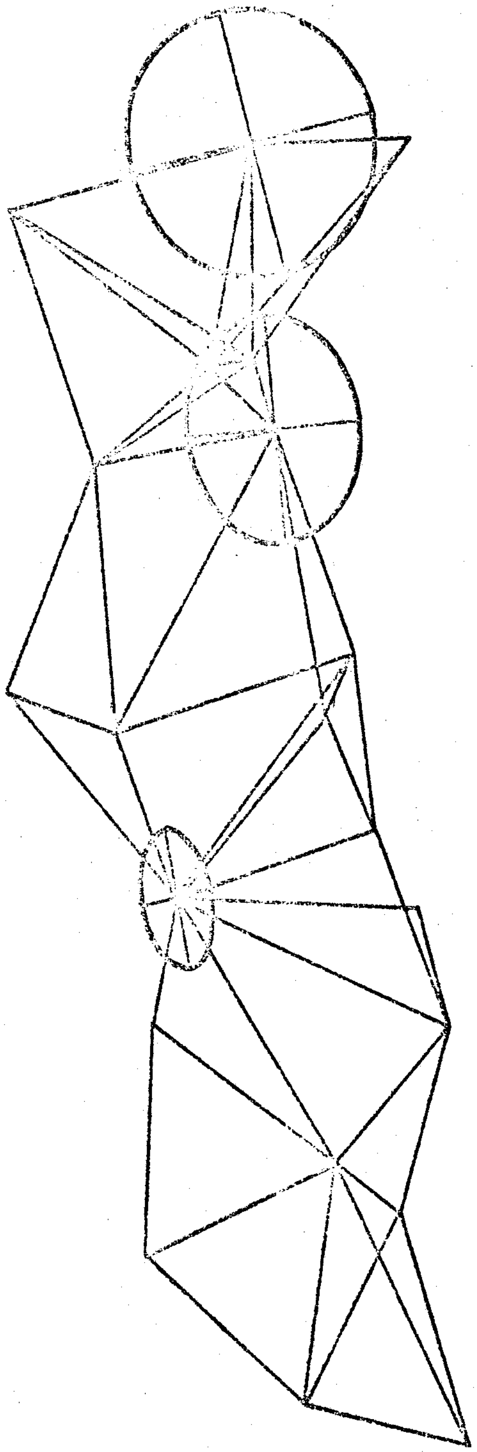


FIG. 9·5

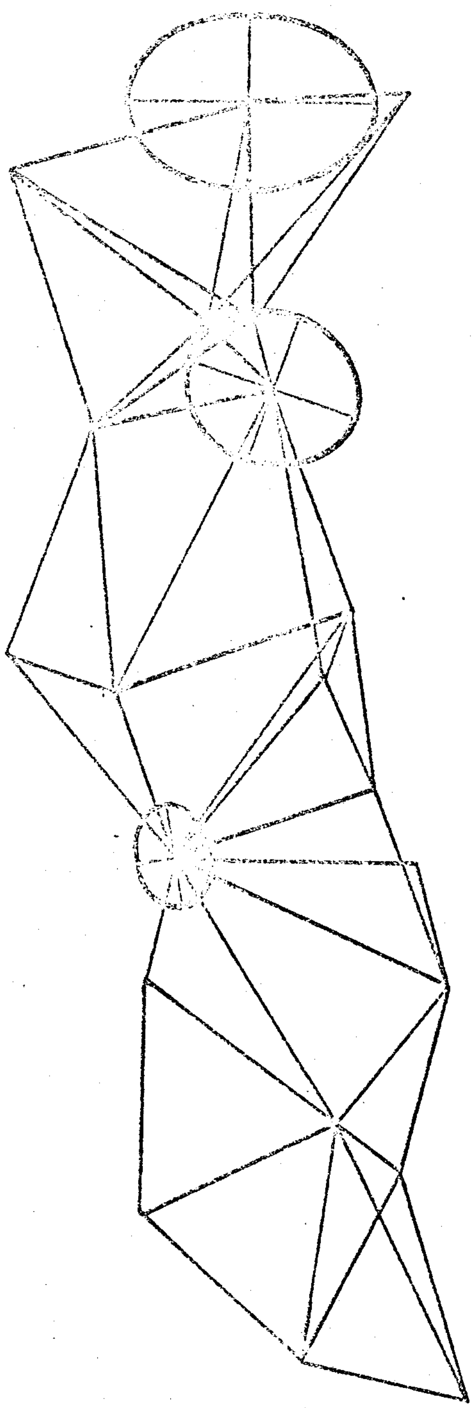
To give the network scale there would be a distance measured between stations 28 and 29. Initially it is proposed that directions are to be measured both ways along all the lines shown in Fig. 9.1. The error ellipses for this network (Trial 4) are shown in Fig. 9.5. The ellipses do not meet the required specification. The azimuth components of the ellipses are reasonably small but the scale components are very large. This should be anticipated in a direction network and the usual solution is to include a number of measured distances in the network. Normally, at least one distance would be measured at each end of the network and perhaps one in the centre. To investigate the effect of three measured lines, the distances between stations 35 and 41 and between stations 46 and 47 were taken as measured in addition to the distance between stations 28 and 29. The error ellipses for this configuration (Trial 5) are shown in Fig. 9.6. Once again the ellipses do not meet the required specification. The azimuth components are unchanged while the scale components are significantly smaller than those of Trial 4. It would appear that more measured distances are required if the ellipses are to meet the specification. From a logistical point of view, a traverse through the network would probably be the best way of introducing these extra distances.

Consider a traverse through stations 28-31-35-38-42-44-47. (Fig. 9.1). This is a traverse going from one end of the network to the other, through the least possible number of intermediate stations. The resultant error ellipses are shown in Fig. 9.7 (Trial 6). The scale components are smaller than those of Fig. 9.6 while once again the azimuth components remain unchanged. While these ellipses satisfy the required specification, it is of interest to see what change is made if in addition all distances in the network were to be measured (trial 7).



Ellipse Scale = 70 cm/cm

FIG. 9.6



Ellipse Scale = 70 cm /cm.

FIG. 9.7

The ellipses for this scheme are shown in Fig. 9.8. The scale components of the ellipses are reduced and again the azimuth components remain significantly the same. However, the improvement in scale seems most expensive when the amount of extra work necessary to obtain it is considered. The improvement in scale would require the measurement of 42 extra lines. This implies that there is an economic cut off point after which further distance measurement does not warrant the expense incurred. Significantly the same result was found by Ashkenazi and Cross *(1971)*.

Table 9.3 gives the parameters of the ellipses shown in figures 9.5, 9.6, 9.7 and 9.8. (Trials 4, 5, 6 and 7).

(e) Conclusion

The examples of Section 8.2(d) have served as a demonstration of the method of optimization. The advantage of using such a procedure in selecting the field stations and the number, type and location of observations is immediately apparent.

It must be stressed that in both Networks 1 and 2, the only alteration to the basic model was to increase the number of, or to change the type of the observations. It is also quite valid to investigate the effect of observing with increased precision and to examine the effect of relocation of some stations and the introduction of additional stations.

In the present example, there was some justification in not investigating the effect of observing with increased precision. In the case of the angular observations, an inspection of the variance formula (8.3) shows that the external part of variance is the predominant part. If 12 arcs are observed, then the internal part of the variance is 0.09 seconds squared compared with the external part of 0.45 seconds squared. As the external variance is not reduced by observing additional arcs, there is no

Ellipse Scale = 70 cm/cm

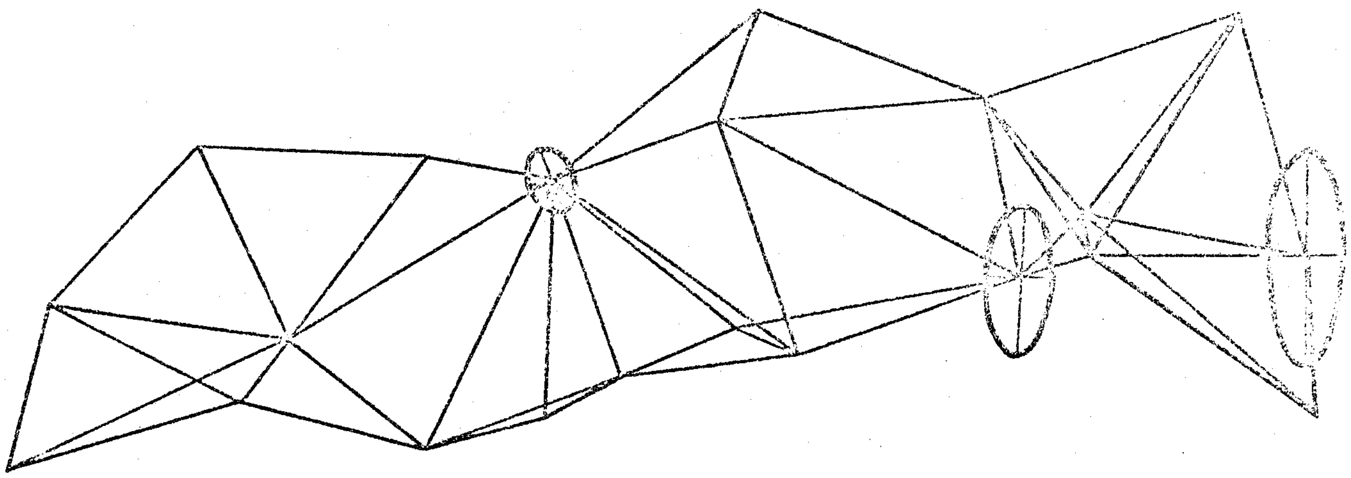


FIG. 9·8

TABLE 9.3

Ellipse Parameters

Station	Component	4	5	6	7
35	Semi-Major Axis	92.44 cm	65.73 cm	48.93 cm	29.47 cm
	Semi-Minor Axis	33.62 cm	33.50 cm	33.26 cm	21.94 cm
	Orientation	77.25°	79.33°	80.31°	159.12°
42	Semi-Major Axis	188.86	106.14	79.32	68.05
	Semi-Minor Axis	80.82	80.09	71.60	30.60
	Orientation	91.80	84.15	19.14	1.61
47	Semi-Major Axis	257.73	126.04	113.40	98.70
	Semi-Minor Axis	116.27	115.10	82.40	34.52
	Orientation	91.60	75.20	0.06	179.65

significant gain to be had if more than 12 arcs are observed. The external variance could be reduced by spreading the observations over a number of days or nights, but in most cases, it is more economical to improve the network by additional measured distances. It would appear that, for a theodolite with the variance formula (8.3) that 12 arcs may be the optimum number of arcs to observe.

In the case of the distance observations, it was pointed out in Section 7.5 that remeasurement of the distance after a short period is only likely to affect the "a" term of the variance formula, and not the "b" term. Over long distances, such as those entailed in this network, the "b" term is the predominant term and can only be reduced by remeasurement under different atmospheric conditions. This means that the stations have to be occupied more than once and is therefore an uneconomical solution. Hence, the variance adopted in this optimization is probably close to the best or optimum variance that can be economically obtained using a 30 mm carrier wave instrument. Increased precision could be obtained using 8 mm carrier wave instruments, but such instruments are very expensive and are probably not an economical proposition in a network such as this where the specification can be reached using 30 mm instruments.

The configuration and number of network stations was not altered since a reconnaissance had shown that the only possible stations were those included in the optimization already (See 9.2(a)).

Of the trials described in Section 9.2(d), trials 3, 6 and 7 satisfy the specification. (See figs. 9.4, 9.7 and 9.8 respectively). It was pointed out that trial 7 was clearly an uneconomical solution because of the large number of measured distances needed for a small gain in coordinate precision. Therefore, the optimum solution would appear to be either trial 3 or trial 6. Further research would need to be done to determine the cost of different types of observations in order to choose between these two solutions.

9.3 Tunnel Optimization

(a) Introduction

The project is to build a tunnel between portals at stations 12 and 13. (See fig. 9.9). Bore holes have revealed a badly faulted zone along the proposed route of the tunnel. It would be extremely expensive and slow to tunnel through this fault zone, so it has been decided to route the tunnel around the fault zone in the manner shown in fig. 9.9.

A curved tunnel creates interesting surveying problems. In a straight tunnel, as most are, scale in the network is not a serious consideration as it will not have a great effect on breakthrough accuracy transverse to the tunnel. Errors in scale will simply shift the breakthrough point along the line of the tunnel. In a curved tunnel errors both in scale and in azimuth can cause a shift in breakthrough transverse to the tunnel. Such a shift is usually very expensive to rectify. In a case like this, the advantages of optimization are immediately apparent.

(b) Approximate Coordinates

(i) Tunnel Coordinates

Given an alignment plan of the tunnel, the first step is to decide where to place control stations along the tunnel. For the purposes of this example, intermediate alignment stations around the curves will be neglected. It will be assumed that these can be placed to a lower order of accuracy

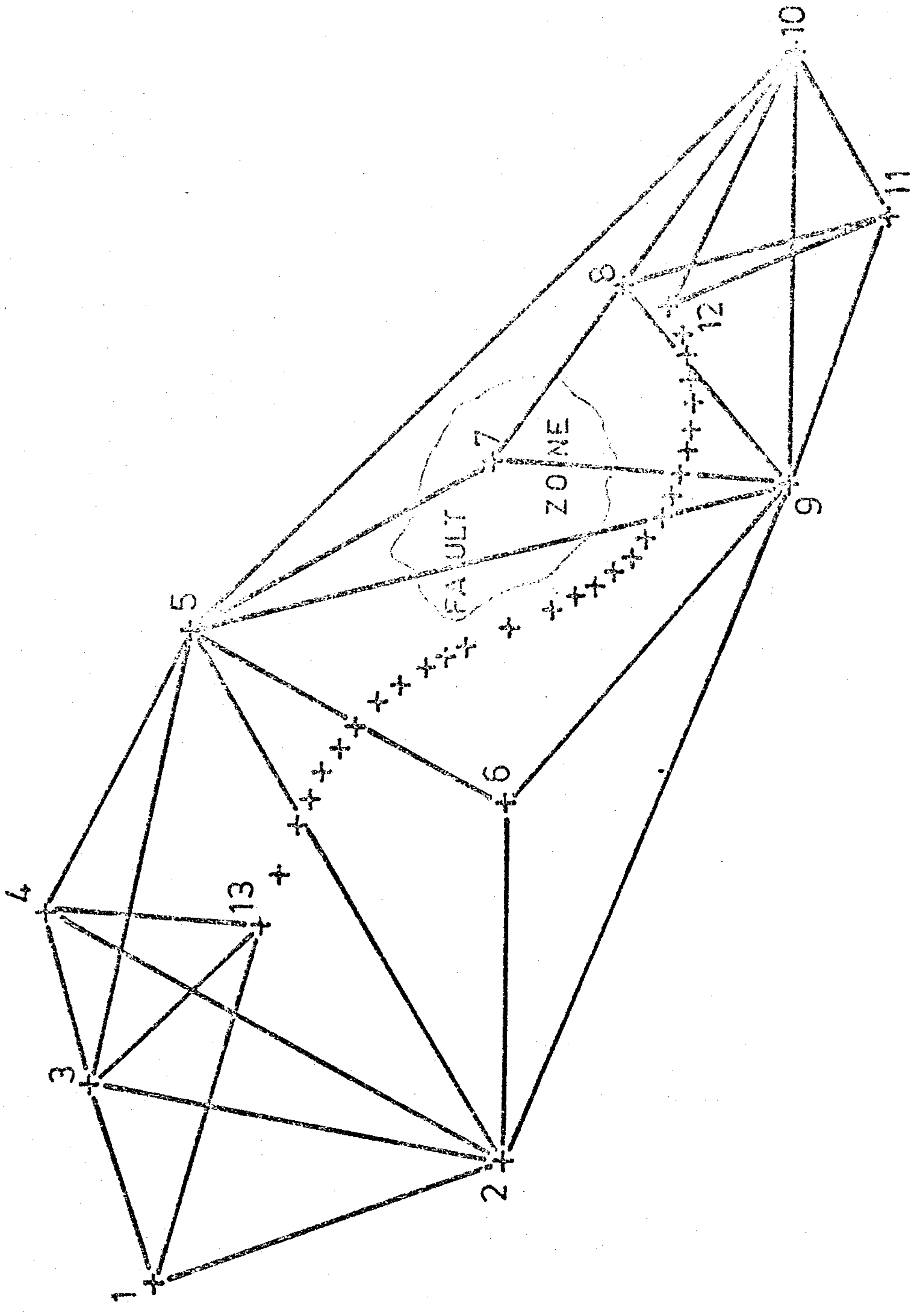


FIG. 9.9

than the control stations. From an error propagation point of view, it is advantageous to have the stations as widely separated as possible. The maximum practical sight length in tunnel conditions is about one kilometer. Sights longer than this are often not feasible because of dust etc. Therefore, in the straight sections of the tunnel, stations will be placed at every kilometer. Stations will also be placed on the tangent points of curves and as far apart as possible around the curves.

The calculation for the placement of stations around the curves is a problem in simple geometry requiring knowledge of the tunnel width, the minimum distance from the stations to the tunnel wall, and the minimum clearance between the lines of sight and the walls. In the tunnel under consideration, which is 5 m wide, and in which the stations and lines of sight are at least 0.5 m from the walls, the following maximum sight distances apply.

Curve 1 - Radius = 4030 - Maximum Sight = 359 m
Curve 2 - Radius = 2800 - Maximum Sight = 299 m

The traverse stations may then be plotted on a large scale plot of the tunnel and their coordinates scaled off. These coordinates are given in Table 9.4.

(ii) Control Network Coordinates

The control network is to consist of several braced quadrilaterals with some centre point figures. (See fig. 9.9).

The selection of the control network stations may be carried out in the following manner. After the location of the tunnel has been decided upon, a preliminary search and reconnaisance is required to find suitable sites for the stations. The criteria used in selection are chiefly figure conditioning, location and accessibility.

TABLE 9.4

Station	Station Name	North (m)	East (m)
14	E01	5190	12250
15	E02	5120	11970
16	E03	5060	11680
17	E04	5040	11375
18	E05	5050	11075
19	E06	5095	10780
20	E07	5165	10485
21	E08	5280	10205
22	E09	5405	9940
23	E10	5580	9690
24	E11	5750	9450
25	E12	5970	9240
26	E13	6200	9040
27	E14	6440	8895
28	E15	6710	8750
29	BT1	7210	8510
30	W11	7710	8270
31	W10	7950	8140
32	W09	8205	7985
33	W08	8500	7750
34	W07	8760	7510
35	W06	9000	7230
36	W05	9220	6950
37	W04	9420	6630
38	W03	9585	6310
39	W02	9720	5970
40	W01	9930	5360
41	BT2	7210	8510

Where a number of equal alternatives exist the actual computer optimization may be used to choose between them. Different configurations of stations would be run as data for the optimization to determine which configuration gave the most satisfactory error ellipses at the portals, or if the tunnel has been designed at this stage, at the breakthrough point. Small scale maps are usually adequate to work out approximate locations of the stations, lines of sight between them and to gain some indication of their accessibility. Field reconnaisance is then necessary to check accessibility. It is usually desirable to have the project coordinates on the national coordinate system. For this reason, one or two trigonometrical stations are usually included in the network. The coordinates of only one station are adopted with perhaps the azimuth to a second if it is available. Information adopted from the national system must not introduce any extra constraints into the network, as this would cause distortions or tensions, which in a precise network are extremely dangerous. Therefore the maximum information that may be adopted from the national system are the coordinates of one point and an azimuth from that point, to locate the network on the spheroid and to give it orientation. If distances are not to be measured in the scheme, an adopted distance from the national system will be necessary to fix scale.

In the present case, station 1 is assumed to be the coordinated station, and the azimuth from station 1 to station 13 is assumed to be the known azimuth. A list of the control network stations and their coordinates is given in Table 9.5.

(c) Proposed Observations

(i) The Tunnel

The tunnel stations are to be connected by traverse using a short range electronic distance meter, and a single second theodolite. Constrained centering is to be used and centering errors in the tunnel will therefore

TABLE 9.5

Control Network Stations

Station	Station Name	North	East
1	N01	11321	177
2	N02	7121	1786
3	N03	12166	2655
4	N04	12794	4828
5	N05	11080	8408
6	N06	7186	6276
7	N07	7443	10605
8	N08	5939	12834
9	N09	3806	10404
10	N10	3862	15852
11	N11	2704	13800
12	N12	5351	12593
13	N13	10138	4707

be assumed to be negligible. Azimuth will be carried into the tunnel by sets of directions observed at the two portals to several stations of the control network and to the first tunnel station. Distances will be measured from the portals to the control network using a short range distance meter.

(ii) The Control Network

Direction measurements are to be made in both directions along all lines shown in fig. 9.9, with a single second theodolite. Some distances are to be measured using a medium range electronic distance measurement instrument. Centering will be by optical plummet.

(d) Variances

The variances adopted for the instruments proposed to be used are as follows:

Single second theodolite;

Variance of a direction observed in n Arcs = $\left(\frac{6.5}{n} + 0.25\right) \sec^2$

Short Range Electronic Distance Meter;

Variance of a single measurement = $(3 \text{ mm} + 5 \text{ ppm})^2 \text{mm}^2$

Medium Range Electronic Distance Meter;

Variance of a single measurement = $(25 \text{ mm} + 6 \text{ ppm})^2 \text{mm}^2$

The standard deviation of plumbing by means of optical plummet will be taken as 1.5 mm.

These variances were estimated before the investigations of Chapters 7 and 8 were finalised and hence in some cases differ slightly from the values obtained in those Chapters.

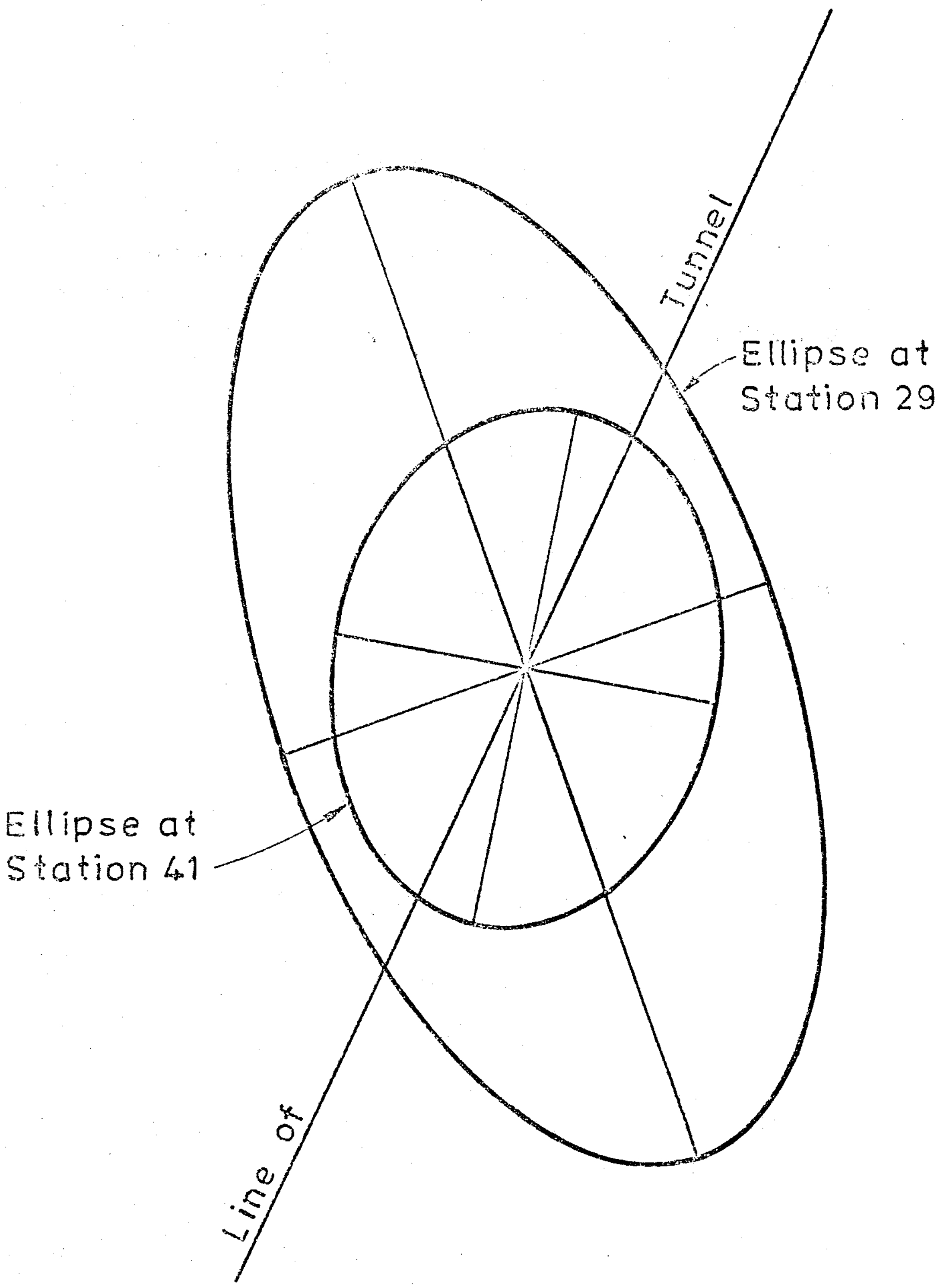


FIG. 9·10

(e) The Breakthrough Point

For the purposes of computation of the breakthrough error, the breakthrough station is included twice in the optimization (See Table 9.4). Station 29 (BT1) is fixed by the traverse from the eastern portal and station 41 (BT2) is fixed by the traverse from the western portal. Error ellipses for both stations were obtained from each run of the optimization.

(f) The Breakthrough Criterion

The remaining problem was that of obtaining a figure for the standard deviation of breakthrough perpendicular to the tunnel at the breakthrough point from the two error ellipses at this point. A graphical representation of the situation is shown in fig. 9.10.

The components of the ellipses perpendicular to the line of the tunnel can be found using the following formulae.

The transformation formula for a change in orientation of a coordinate system can easily be derived and are as follows:

$$X' = X \cos \phi + Y \sin \phi \qquad \ldots (9.5)$$

$$Y' = -X \sin \phi + Y \cos \phi$$

where ϕ is the angle through which the system is rotated,
X, Y are coordinates in the old system, and
X', Y' are coordinates in the new system.

Application of the laws of propagation of variance gives:

$$Q_{X'} = Q_X \cos \phi + Q_Y \sin \phi \qquad \ldots (9.6)$$

$$Q_{Y'} = -Q_X \sin \phi + Q_Y \cos \phi \qquad \ldots (9.7)$$

and $$\sigma^2_{X'} = \sigma^2_X \cos^2\phi + 2\sigma_{XY} \sin \phi \cos \phi + \sigma^2_Y \sin^2\phi \qquad (9.8)$$

$$\sigma^2_{Y'} = \sigma^2_X \sin^2\phi - 2\sigma_{XY} \sin \phi \cos \phi + \sigma^2_Y \cos^2\phi \qquad (9.9)$$

In the present case, σ_X and σ_Y are the semi-major and semi-minor axes of the error ellipse. For an error ellipse, by definition, $\sigma_{XY} = 0$, and therefore equations (9.8) and (9.9) can be reduced to the following:

$$\sigma^2_{X'} = \sigma^2_X \cos^2\phi + \sigma^2_Y \sin^2\phi \qquad \ldots (9.10)$$

$$\sigma^2_{Y'} = \sigma^2_X \sin^2\phi + \sigma^2_Y \cos^2\phi \qquad \ldots (9.11)$$

where ϕ is the difference between the orientation of the ellipse and the direction perpendicular to the tunnel at breakthrough, σ_X and σ_Y are the semi-axes of the ellipse at the breakthrough point, and $\sigma_{X'}$ is the component of the ellipse perpendicular to the tunnel at that point.

When $\sigma_{X'}$ has been obtained for both the breakthrough ellipses, the two values may be combined using the laws of propagation of variance to give a standard deviation of breakthrough perpendicular to the tunnel.

$$\sigma^2_B = \sigma^2_{X1'} + \sigma^2_{X2'} \qquad \ldots (9.12)$$

A programme was written for a keyboard programmable calculator to calculate σ_B given the parameters of the two breakthrough error ellipses and the azimuth of the tunnel.

Discussions with tunnel and mining surveyors indicate that the maximum permissible breakthrough error in most tunnels is of the order 0.5 metres. This figure will be accepted as the maximum for the tunnel under consideration. If this maximum is taken to be three standard deviations of the breakthrough error then:

$$3\sigma_B = 0.50 \text{ m}$$

$$\therefore \quad \sigma_B = 160 \text{ mm}$$

If the criterion that $\sigma_B < 160$ mm is set, then there is a 98% probability that the actual breakthrough error will not exceed 0.5 m. *(Patterson, 1973)*.

The criterion used to assess the results of the various trials will therefore be that

$$\sigma_B < 160 \text{ mm}$$

(g) Results

Ten trial networks were processed using the optimization procedure. Some data was common to all trials and some was changed from trial to trial. The data common to all trials was as follows:

1. The positions of all proposed stations, both outside and inside the tunnel, were retained throughout.

2. The coordinates of station 1 and the bearing from station 1 to station 13 were held fixed throughout.

3. The proposed observations inside the tunnel were not altered.

4. All proposed directions in the control network (each way directions along all lines of the control

network, shown in fig. 9.9) were common to all trials. The directions proposed to connect the tunnel portals to the control network were also common to all trials. They were as follows: Eastern Portal (12) - Directions, both ways, between stations 12 and 10, 12 and 11, 10 and 12 and 11 and 12. Western Portal (13) - Directions, both ways, between stations 13 and 1, 13 and 3, 13 and 4, 1 and 13, 3 and 13 and 4 and 13.

The data varying from trial to trial together with the standard deviation of breakthrough obtained for each trial is given in Table 9.6.

Trial 1.

This trial has virtually the minimum practical amount of data that could be measured. There is only one distance, from station 1 to station 3, measured in the control network and no distances measured from control network stations to the portals. The distance between stations 1 and 3 serves only to give the control network scale. The directions in the control network are observed in four arcs. As there is only one distance measured, the fact that scale is weak in the control network is to be expected. It is mainly this weakness in scale that causes the large σ_B of 254 mm.

Trial 2.

Trial 2 is the same as trial 1 except that the directions of the control network are observed with eight arcs instead of four. The improvement in σ_B, (from 254 mm to 229 mm), is certainly significant although not great.

TABLE 9.6

Trial	Direction No. Arcs	Measured Distances (Control Network)	Measured Distances (Portal Connections)	σ_B
1	4	1 - 3	None	25.42
2	8	1 - 3	None	22.95
3	4	1 - 3	12 - 11, 13 - 1	17.32
4	4	1 - 3	12 - 10, 12 - 11, 13 - 1, 13 - 3, 13 - 4	17.22
5	4	1 - 3, 1 - 2, 2 - 9, 9 - 11	12 - 11, 13 - 1	16.43
6	4	1 - 2, 2 - 9, 9 - 11, 11 - 10 10 - 7, 7 - 5, 5 - 4, 4 - 3, 3 - 1	12 - 10, 12 - 11, 13 - 1, 13 - 3, 13 - 4	15.78
7	8	1 - 2, 2 - 9, 9 - 11, 11 - 10, 10 - 7, 7 - 5, 5 - 4, 4 - 3, 3 - 1	12 - 10, 12 - 11, 13 - 1, 13 - 3, 13 - 4	15.25
8	4	All 24 distances	None	16.28
9	4	All 24 distances	12 - 11, 13 - 1	16.00
10	8	All 24 distances	None	15.50

A further increase in the number of arcs would not give much additional improvement. The improvement in σ_B is dependent on the improvement in the variance of the observations together with the increase in the number of these observations in the network. For a single second instrument, the improvement in variance from 8 arcs to 16 arcs is only half the improvement from 4 arcs to 8 arcs. So at this stage, the measurement of additional distances would seem the logical way to improve σ_B.

Trial 3.

Trial 3 is the same as trial 1 except that a distance is measured from each portal to a station of the control network, (12 to 11 and 13 to 1). This trial gave a σ_B of 173 mm, a very significant improvement on trial 1. However, it is not sufficient to satisfy the criterion of $\sigma_B < 160$ mm. In an attempt to satisfy this criterion, more distances will be measured from the portals to stations of the control network.

Trial 4.

Trial 4 is the same as trial 3 with the exception that three extra distances are measured from the portals to the control network. They are the distances between 12 and 10, 13 and 3 and between 13 and 4. This trial gave a σ_B of 172 mm, an insignificant improvement on the result of trial 3. The reason this improvement is so small is probably that the two distances (from 12 to 11 and from 13 to 1) are providing nearly maximum possible control of scale, with this precision of distance measurement, in the areas of the control network adjoining the portals and the additional distances are simply adding more redundancies rather than improving the scheme. This hypothesis points to a need for a more uniform distribution of measured distances through the network.

Trial 5.

Trial 5 is the same as trial 3 except that the distances along the traverse from station 1 to station 2 to station 9 to station 11 are measured. These distances should have the effect of providing a scale link between the portals as well as giving a more uniform distribution of measured distances through the network. The σ_B resulting from this trial is 164 mm. This is just outside the criterion of $\sigma_B < 160$ mm.

It was confirmed in trial 2 that the observation of more precise directions would not give a great deal of improvement in network strength even when there was little scale control in the network. It will give even less improvement now that the scale control is stronger. (This will be shown in trial 7.) Therefore, the logical course would appear to be to measure more distances.

Trial 6.

Trial 6 is the same as trial 1 except that the distances along the traverse joining stations 1, 2, 9, 11, 10, 7, 5, 4, 3 and 1, as well as the portal distances 12 - 10, 12 - 11, 13 - 11, 13 - 3 and 13 - 4 are measured. This trial gave a σ_B of 158 mm. Although σ_B now satisfies the criterion, it is of interest to note that the number of measured distances in trial 5 had to be doubled to gain this very marginal improvement in σ_B. In other words the scale control in the network that can be gained by measuring distances of this precision is reaching the limit.

Trial 7.

Trial 7 is the same as trial 6 except that the directions in the control network are measured in eight arcs instead of in four. The σ_B obtained in this trial is 153 mm, a very small improvement over trial 6 when it is considered that twice as many directions are being observed throughout the control network. A similar situation to that of trial 6 has arisen. That is, the improvement in network strength that can be achieved by the more precise measurement of the directions is very limited, regardless of how much more precisely the directions are measured.

Trial 8.

Trial 8 has all directions of the control network observed in four arcs, and all distances except those to the portals observed. The σ_B given by this trial is 163 mm. The importance of the distances to the portals may now be seen. The σ_B of this trial is barely smaller than that of trial 5 (164 mm), yet in trial 5 there are only four distances measured in the control network, contrasting to the twenty four of this trial. Trial 5 includes one distance from each portal to the control network, whereas trial 8 has no portal distances measured. These two portal distances have nearly the same effect as the additional twenty distances measured in the network proper of trial 8. This is not because of their superior precision (See (d) above), but is because of their strategic positioning. The standard deviation of breakthrough is heavily dependent on the precision to which the portals are fixed, and distances measured to the portals will be one of the most economical ways of improving this precision.

Trial 9.

Trial 9 is the same as trial 8 except that the portal distances 12 - 11 and 13 - 1 are measured. The σ_B resulting from this trial is 160 mm.

Trials 8 and 9 may be compared with trials 1 and 3 respectively. Trials 1 and 3 are similar in that all directions are measured in four arcs and in that no portal distances are measured in trial 1 and the portal distances measured in trial 9 are measured in trial 3. Where both trials 8 and 9 have all network distances measured, trials 1 and 3 have only the one distance, (between stations 1 and 3), measured. The $\sigma_{B's}$ for trials 1 and 3 are 254 and 173 mm, respectively. The difference between these two results is far greater than the difference between the results of trials 8 and 9, even though the addition of the same data causes both differences. The reason is that the network of trials 8 and 9 has reached a limiting or saturated condition so that little improvement is obtained by the addition of this type of data. In trials 1 and 3, the network is quite weak and the additional data gives quite a marked improvement. That this saturated condition, which is due mainly to distance measurement, will not be improved by more precise direction measurement is shown by the results of trial 10.

Trial 10.

Trial 10 is the same as trial 8 in that all distances except those to the portals are measured. It differs from trial 8 in that the directions are observed in 8 arcs instead of in four. The σ_B resulting from trial 10 is 155 mm, a very small improvement for the double amount of effort involved in observing eight arcs.

(h) Conclusion

The standard deviations of breakthrough, σ_B, for all trials are summarised in Table 9.6, together with the variable data used in the trials. Trials 6,7,9 and 10 satisfy the criterion of $\sigma_B < 16$ cm, and of these, trial 6 would seem the most economical solution. Apart from having only four arcs observed for direction observations, and the least number of measured distances, the distances are measured in the most economical manner. As they are in the form of a traverse, a "leap frog" technique may be used to measure them, so that each station is only occupied once. If the distances were not adjoining, many more stations would have to be visited to measure the same number of distances. The distances to the portals could be measured in the same operation, as these are to be measured by a different instrument, a short range distance meter as opposed to the microwave instruments to be used on the other lines of the network. All that would be required is for the microwave party to carry the reflectors and to set them so that an observer at the portal can measure the portal distance after the microwave party has measured a network distance.

Trial 7 is clearly a less economical solution as, although the same distances are being measured, the directions are being observed in eight arcs as opposed to the four arcs of trial 6.

Trial 10 is an even less economical solution than trial 7. Like trial 7 it has directions observed in eight arcs, but many more distances are observed. Therefore, of the solutions tried, trial 6 appears to be the optimum.

It should be noted that trials 3,5 and 8 very nearly satisfy the criterion.

In the analysis, three groups of observations were varied, either in number or in the precision to which they were to have been measured. The number of distances measured in the control network were varied as were the number of distances connecting the portals to the control network.

Directions were taken to be measured both ways on all lines of the network in all trials, but their precision was varied from trial to trial. It became apparent that the stronger the network was the more difficult it was to strengthen. *It would appear that a network of fixed configuration has an optimum strength which once obtained is uneconomical to attempt to improve.* This is of course a very broad and sweeping statement and must be qualified. It would depend on the method tried to improve the network. The statement would hold if the improvement was to be gained by the methods used in the present analysis. That is, by adding observations until there is no place for more to be added, and then by improving the precision of these observations. An exception to this statement could be improvement by the observation of astronomical azimuths.

CHAPTER 10

CONCLUSIONS

10.1

An optimization procedure is available to estimate the precision to which points may be coordinated in a planned network. It is also valid to estimate the precision to which points have been coordinated in an existing network. The procedure is based on the parametric adjustment and requires knowledge of the following information.

1. The approximate coordinates of the network stations.
2. The type and position in the network of the observations.
3. The precision or anticipated precision of each observation.

The measure used for point precision is the error ellipse, its accuracy being nearly exclusively dependent on how closely the observational variances are estimated.

10.2

Empirical and experimental means of estimating angular variance are unsatisfactory because of the heavy dependence of angular observations on human factors and the extreme difficulties involved in simulating field conditions in laboratory experiments. Linear observations tend not to be as dependent on human factors and field conditions as are angular observations, and accordingly, empirical and experimental means of estimating linear variance appear to give relatively good results.

10.3

The weaknesses in empirical and experimental means of variance estimation are overcome by methods of estimation using analysis of field observations. Early methods using this approach were based on condition closures and were unsatisfactory in that they used only a percentage of the observations, they could only be used with angular observations and they required fairly extensive calculations which would otherwise be unnecessary.

A more satisfactory method is based on the fact that the variance factor after adjustment will reflect the suitability of trial variances in an otherwise valid mathematical model. This method overcomes all the difficulties of the condition closure methods, as it uses all observations, it is suitable for the estimation of both angular and linear variances and it involves virtually no superfluous calculation.

The reliability of variances estimated using this method was indicated by the good agreement between the results of variance analysis of several large networks, (vide Chapter 7), when the differing equipment and techniques used, and the differing conditions, under which the observations were taken, were accounted for.

10.4

Variances obtained by this method may be broken down into a number of basic components depending on the parameters of equipment, techniques and observing conditions, and by substituting these components into simple formulae, variances may be estimated for observations, taken or to be taken, using a specified set of parameters.

It is desirable that a comprehensive list of variance components be compiled so that the variances of observations using a wide range of equipment and techniques and under a wide range of observing conditions may be estimated.

10.5

In Chapter 7, the distribution of the corrections to observations after adjustment was shown to be predominantly leptokurtic. This is the expected result as although the corrections will probably tend to be normally distributed, this can only be achieved when there is no interplay between adjacent observations. In general, large errors will receive smaller corrections due to the "meaning out" of the error over adjacent observations. Thus a leptokurtic form will result. Therefore, the analysis carried out in Chapter 7 does not prove or disprove the assertion that surveying observations are normally distributed.

10.6

The two examples given in Chapter 9 demonstrate the application of the optimization procedure described in Chapter 2 using variances estimated by the means of Chapter 8. The advantage of using such a procedure in selecting the field stations and the number, type and location of observations should be immediately apparent, and it is to be hoped that, in future, all major networks will be analysed by an optimization technique.

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APPENDIX A

MATHEMATICAL ANALYSIS OF PERIODIC CIRCLE GRADUATION ERRORS

The wave form of periodic circle graduation errors is best represented by a Fourier Series;

$$y = r_o + r_1 \sin(A_1 + x) + r_2 \sin(A_2 + 2x) + r_3 \sin(A_3 + 3x) + \ldots. \qquad \ldots (1)$$

where $r_o, r_1, r_2, \ldots\ldots$ are the amplitudes,

$A_1, A_2, A_3, \ldots\ldots$ are the phase angles,

x is the circle position, and

y is the circle graduation error.

Equation 1 is an infinite series of sine terms, the magnitudes of these terms decreasing as the number of terms increases.

$r_1 \sin(A_1 + x)$ is the fundamental term,

$r_2 \sin(A_2 + 2x)$ is the first harmonic,

$r_3 \sin(A_3 + 3x)$ is the second harmonic, etc.

In modern theodolites, a single reading of the circle is derived by taking the mean of two diametrically opposite graduations. The error in the graduation on one side of the circle may be expressed as;

$$y_a = r_o + r_1 \sin(A_1 + x) + r_2 \sin(A_2 + 2x) + r_3 \sin(A_3 + 3x) + \ldots. \qquad \ldots (2)$$

and the error in the graduation diametrically opposite as;

$$y_b = r_0 + r_1 \sin(A_1 + x + 180^\circ) + r_2 \sin(A_2 + 2x + 360^\circ) + r_3 \sin(A_3 + 3x + 180^\circ) + \ldots \quad \ldots (3)$$

Equation (3) may be simplified by use of the following relationships;

$$\sin(\phi + 180^\circ) = -\sin\phi$$

$$\sin(\phi + 360^\circ) = \sin\phi$$

Substitution of these relationships into (3) gives;

$$y_b = r_0 - r_1 \sin(A_1 + x) + r_2 \sin(A_2 + 2x) - r_3 \sin(A_3 + 3x) + \ldots \quad \ldots (4)$$

As was mentioned above, the circle reading is the mean of the two diametrically opposed graduations. Therefore;

$$y_m = \frac{y_a + y_b}{2} \quad \ldots (5)$$

Substituting (2) and (4) into (5) gives;

$$y_m = r_0 + r_2 \sin(A_2 + 2x) + r_4 \sin(A_4 + 4x) + \ldots \quad \ldots (6)$$

Therefore, by taking the mean of the two diametrically opposite graduations, the odd terms in the graduation error expression cancel.

Suppose that two arcs were observed at zero settings of 0° and 90°, then for the first arc;

$$y_1 = r_0 + r_2 \sin(A_2 + 2x) + r_4 \sin(A_4 + 4x) + \ldots \quad \ldots (7)$$

and for the second arc;

$$y_2 = r_0 + r_2 \sin(A_2 + 2x + 180^\circ) + r_4 \sin(A_4 + 4x + 360^\circ) + \ldots \quad \ldots (8)$$

By the same reasoning as above, the circle graduation error of the mean of these two arcs will be;

$$y_m = r_0 + r_4 \sin(A_4 + 4x) + r_8 \sin(A_8 + 8x) + \ldots \quad \ldots (9)$$

Similarly, observation of four arcs at zeros of 0°, 45°, 90° and 135° will give;

$$y_m = r_0 + r_8 \sin(A_8 + 8x) + r_{16} \sin(A_{16} + 16x) + \ldots \quad \ldots (10)$$

Therefore, the periodic circle graduation error in the mean result of four arcs, observed at the above zeros, is free from the fundamental term and the first six harmonics.

APPENDIX B

FORMULA FOR ESTIMATED VARIANCE FACTOR AFTER ADJUSTMENT

The following proof is based on that given by Ashkenazi *(1970)*. The same notation as that of Chapter 2 will be used.

Given the observation equations:

$$AX + C = p + V \qquad \ldots (1)$$

and

$$A(X + \Delta X) = -T + \Delta \qquad \ldots (2)$$

where X are the true errors in the parameters X,

Δ are the true observational errors,

and T is the matrix of absolute terms such that $T = C - p$.

Pre-multiplication of equation 2 by V^TG^{-1} gives:

$$V^TG^{-1}A(X + \Delta X) = -V^TG^{-1}T + V^TG^{-1}\Delta \qquad \ldots (3)$$

or

$$V^TG^{-1}(AX + T) + V^TG^{-1}A.\Delta X = V^TG^{-1}\Delta \qquad \ldots (4)$$

but

$$V^TG^{-1}A.\Delta X = 0 \qquad \ldots (5)$$

as

$$A^TG^{-1}V = A^TG^{-1}(AX + T) \qquad \ldots (6)$$

or

$$A^TG^{-1}V = A^TG^{-1}AX + A^TG^{-1}T \qquad \ldots (7)$$

The right hand side of equation 7 is the matrix of normal equations and therefore is equal to zero.

Therefore $$A^T G^{-1} V = V^T G^{-1} A = 0 \qquad \ldots (8)$$

Substituting equation 5 into equation 4 gives

$$V^T G^{-1}(AX + T) = V^T G^{-1}\Delta \qquad \ldots (9)$$

Substituting T = C - p into equation 1 gives:

$$AX + T = V \qquad \ldots (10)$$

Substituting this result into equation 9 gives:

$$V^T G^{-1} V = V^T G^{-1}\Delta \qquad \ldots (11)$$

or $$V^T G^{-1} V = \Delta^T G^{-1} V \qquad \ldots (12)$$

Similarly, pre-multiplication of equation 2 by $\Delta^T G^{-1}$ gives:

$$\Delta^T G^{-1} A(X + \Delta X) = -\Delta^T G^{-1} T + \Delta^T G^{-1}\Delta \qquad \ldots (13)$$

or $$\Delta^T G^{-1}(AX + T) + \Delta^T G^{-1} A.\Delta X = \Delta^T G^{-1}\Delta \qquad \ldots (14)$$

Substituting equation 10 into equation 14

$$\Delta^T G^{-1} V + \Delta^T G^{-1} A.\Delta X = \Delta^T G^{-1}\Delta \qquad \ldots (15)$$

Substituting equation 12 into equation 15

$$V^T G^{-1} V + \Delta^T G^{-1} A.\Delta X = \Delta^T G^{-1}\Delta \qquad \ldots (16)$$

Pre-multiplication of equation 2 by $A^T G^{-1}$ gives:

$$A^T G^{-1} A(X + \Delta X) = -A^T G^{-1} T + A^T G^{-1}\Delta \qquad \ldots (17)$$

But, by equations 7 and 8

$$A^T G^{-1} A.X = -A^T G^{-1} T \qquad \ldots (18)$$

Therefore $A^T G^{-1} A . \Delta X = A^T G^{-1} \Delta$ ... (19)

or $\Delta X = [(A^T G^{-1} A)^{-1} A^T G^{-1}] . \Delta$ ... (20)

The matrix $\Delta^T G^{-1} A . \Delta X$ of equation 19 then becomes

$$\Delta^T G^{-1} A . \Delta X = \Delta^T G^{-1} A . (A^T G^{-1} A)^{-1} . A^T G^{-1} \Delta \quad \ldots (21)$$

$$= \text{trace } [(A^T G^{-1} \Delta) . (\Delta^T G^{-1} A) . (A^T G^{-1} A)^{-1}] \quad \ldots (22)$$

For a very large number of observations,

$$\text{expectation } [G^{-1} \Delta \Delta^T] = S^2 [I] \quad \ldots (23)$$

Hence, $\Delta^T G^{-1} A . \Delta X = S^2 . \text{trace}[(A^T G^{-1} A) . (A^T G^{-1} A)^{-1}]$ ... (24)

$$= S^2 . \text{trace}[I] \quad \ldots (25)$$

$$= S^2 . k \quad \ldots (26)$$

By definition, $\Delta^T G^{-1} \Delta = S^2 . n$ ... (27)

Substituting equations 26 and 27 into equation 16 gives:

$$V^T G^{-1} V + S^2 . k = S^2 . n \quad \ldots (28)$$

or $$S^2 = \frac{V^T G^{-1} V}{n-k}$$

where n is the number of observations,
and k is the number of necessary observations.

Therefore $$S^2 = \frac{V^T G^{-1} V}{r} \quad \text{as } r = n-k \quad \ldots (29)$$

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