3D Lines

Question:

As part of the design of a new structure two straight pipes are to be placed near each other. It is important that the two pipes are not too close to each other. The coordinates of the two ends of the centreline of each pipe are given below.

	From			To		
	E	N	H	E	N	H
Pipe A	16.5	20.0	10.0	11.0	34.0	2.2
Pipe B	10.5	24.5	1.9	17.5	37.5	9.0

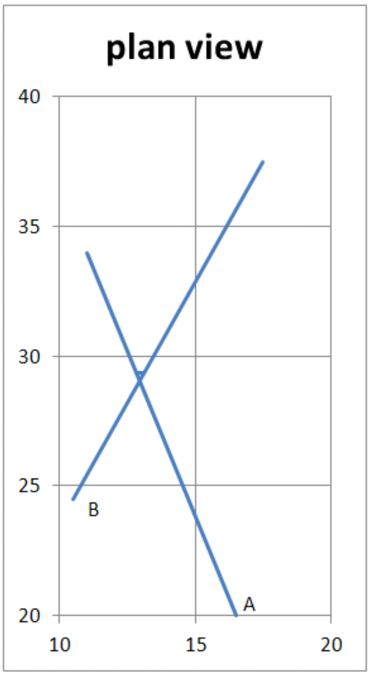
Calculate the shortest distance between the centrelines of the two pipes. Calculate the coordinates of the two end points of this shortest join line.

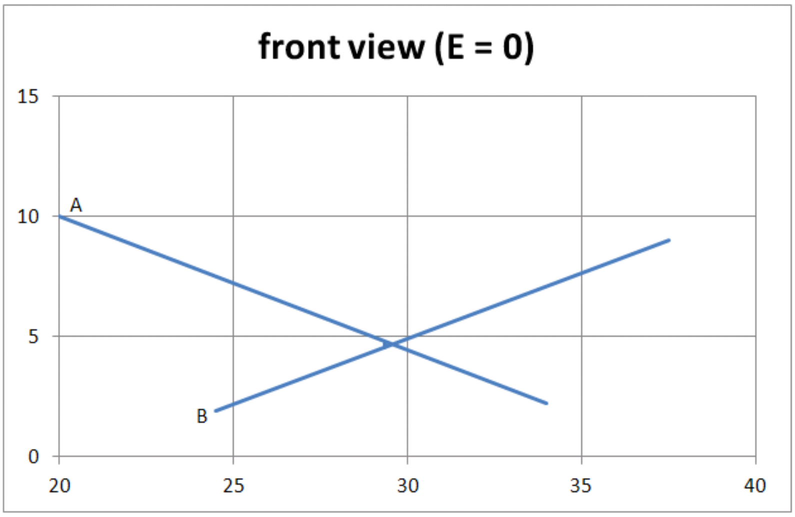
Solutions to Question:

This question is Q16 in Chapter 10 our Survey Computations textbook used in GMAT2500.

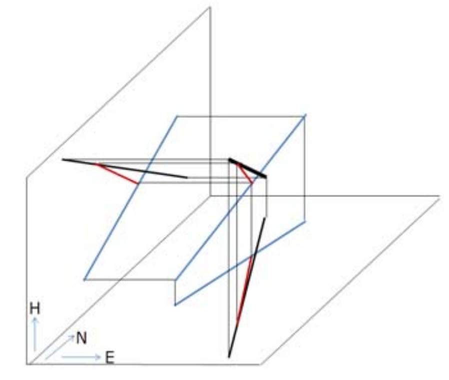
My answers are:
Distance = 0.373
End point on pipe A (E,N,H): 12.822 29.362 4.784
End point on pipe B (E,N,H): 13.106 29.340 4.544

Draw a picture.





The distance between the two pipes is short so in both views above the shortest line appears to be very close to where the lines cross in each view. However if the distance was longer then the link line may not be so close to the crossing points. An exaggerated 3D view might then look like:

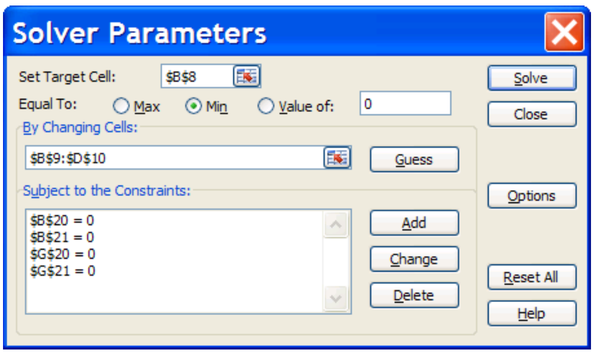


For a problem where we have data (coordinates etc) then we might solve the problem using MS Excel's Solver function. If we want a more generic solution then we need to look more closely at the equations involved and solve them algebraically.

Solver solution method

Give all the points a label or number. Pipe A goes from 1 to 2, pipe B from 3 to 4, the (pedal) point on pipe A that is closest to pipe be is point 5, and similarly the pedal point on pipe B is point 6.

Set up cells in the spreadsheet for the E N H coordinates of points 5 and 6. We can place approximate estimates of their coordinates in these cells. Write an equation in a cell that is the distance between points 5 and 6. We will get Solver to minimise the value in this cell, i.e determine the shortest distance. We can get the spreadsheet to change the values for the coordinates of 5 and 6 but we need some constraints. One set of constraints uses the fact that point 5 must be on the line 1 to 2 (doesn't have to be in between the end points, but usually is). We can do this by saying that the bearing 1 to 5 must be the same as the bearing 1 to 2, and because we are working in 3D that the ZA 1 to 5 must be the same as the ZA 1 to 2. Similarly for point 6 on the line 3 to 4.



Where cell B8 = SQRT((E6-E5)^2+(N6-N5)^2+(H6-H5)^2)
Cells B9:D10 are the E N H coordinates of points 5 and 6.
B20 = bearing12 – bearing15 (calculated from E N coordinates)
B21 = bearing34 – bearing36
G20 = ZA12 – ZA15 (calculated from E N H coordinates)
G21 = ZA34 – ZA36

Equations of lines method

I base this method on equations in my paper: For reference, not for detailed study: Harvey BR (1991) Telescope Axes Surveys, Aust J Geod Photo Surv June 1991 pp1-18. I have made a copy in pdf format that is available on class web sites.

From the given coordinates determine the bearings and zenith angles (and 3D slope distance) of the two pipe lines.

Pipe A bearing = 338° 33' 08.1"	ZA = 117° 24' 34.3"	slope dist = 16.944
Pipe B bearing = 28° 18' 02.7"	ZA = 64° 19' 06.1"	slope dist = 16.383

Calculate the direction cosines of these lines.

For pipe A:
v1 = $\sin ZA_A \sin B_A = (E_2-E_1)/\text{slope dist}_{12} = -0.325$
v2 = $\sin Z_A \cos B_A = (N_2-N_1)/\text{slope dist}_{12} = 0.826$
v3 = $\cos Z_A = (H_2-H_1)/\text{slope dist}_{12} = -0.460$
Similarly for pipe B:
u1 = $\sin ZA_B \sin B_B = (E_4-E_3)/\text{slope dist}_{34} = 0.427$
u2 = $\sin ZA_B \cos B_B = (N_4-N_3)/\text{slope dist}_{34} = 0.793$
u3 = $\cos ZA_B = (H_4-H_3)/\text{slope dist}_{34} = 0.433$

Calculate shortest distance D:

$$D = \frac{(E1 - E3)(v2u3 - v3u2) + (N1 - N3)(v3u1 - v1u3) + (H1 - H3)(v1u2 - v2u1)}{\sqrt{(v2u3 - v3u2)^2 + (v3u1 - v1u3)^2 + (v1u2 - v2u1)^2}}$$

= (6 * 0.723 - 4.5 * -0.056 + 8.1 * -0.611) / 0.948 = 0.373

To calculate the coordinates of the pedal point (the end of the shortest distance line) on pipe A, we need to first calculate a term L which is the distance along pipe A from point 1 to the pedal point 5.

$$L = \frac{(E1 - E3)(v1 - VDUu1) + (N1 - N3)(v2 - VDUu2) + (H1 - H3)(v3 - VDUu3)}{VDU^2 - 1}$$

$$\textit{and} \quad VDU = (v1u1 + v2u2 + v3u3)$$

Here L = 11.331 (and VDU = 0.317)

E5 = E1 + Lv1 = 12.822
N5 = N1 + Lv2 = 29.362
H5 = H1 + Lv3 = 4.784

To calculate the coordinates of the pedal point (the end of the shortest distance line) on pipe B, we need to first calculate a term M which is the distance along pipe B from point 3 to the pedal point 6.

$$M = \frac{(E3 - E1)(u1 - VDU\,v1) + (N3 - N1)(u2 - VDU\,v2) + (H3 - H1)(u3 - VDU\,v3)}{VDU^2 - 1}$$

Here M = 6.100

E6 = E3 + Mu1 = 13.106
N6 = N3 + Mu2 = 29.340
H6 = H3 + Mu3 = 4.544

The '3D join' of shortest line, 5 to 6 is 0.373, 94° 25' 41.4", 130° 05' 02.7"

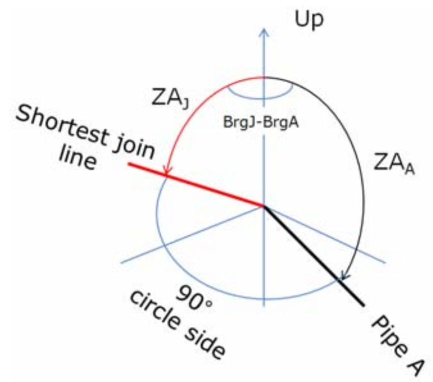
Checks: From the coordinates calculate bearings and ZAs to see if points 5 and 6 are on the pipe lines.
Brg12 = Brg15 = 338° 33' 08.1"
ZA12 = ZA15 = 117° 24' 34.3"
Brg34 = Brg36 = 28° 18' 02.7"
ZA34 = ZA36 = 64° 19' 06.1"

These answers agree with the Solver method above for this data set.

Spherical Trigonometry Method

I use this method to calculate the bearing and ZA of the joining line between the pipes, but not the distance or coordinates. So it only partially solves the problem. But it is an independent check of parts of the above methods.

Consider a sphere centred at point 5. The radius is not relevant. The spherical triangle on the surface of that sphere has one axis vertical and the triangle is formed by the pipe line and the shortest join line. Three sides of the triangle and one 'corner' angle are relevant to our investigations.



One angle in the spherical triangle is B_J-B_A, the side opposite it is 90°, and the other two sides are ZA_A and ZA_J.

One of the cosine rules of spherical trigonometry is relevant here:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

where A is corner angle and a b c are sides of sph triangle, side a is opposite angle A etc.

Because the shortest join line is perpendicular to the pipe line, one side of the spherical triangle is 90°.

$\cos 90 = \cos Z_J \cos Z_A + \sin Z_J \sin Z_A \cos(B_J - B_A)$

$\cos 90° = 0$

Using our point numbers from previous methods, this equation becomes

$0 = \cos Z_{56} \cos Z_{51} + \sin Z_{56} \sin Z_{51} \cos(B_{56} - B_{51})$

Noting that $B_{51} = B_{21}$ and Z_{21} then

$$0 = \cos Z_{56} \cos Z_{21} + \sin Z_{56} \sin Z_{21} \cos(B_{56} - B_{21}) \qquad (1)$$

We can do the same at the other pedal point (6). Centre a sphere at point 6 and form the equation of the triangle.

$$0 = \cos Z_{43} \cos Z_{65} + \sin Z_{43} \sin Z_{65} \cos(B_{65} - B_{43}) \qquad (2)$$

The bearing and ZA of the link line from both circles are of course related.

$B_{65} = B_{56} + 180°$ and $Z_{65} = 360° - Z_{56}$

So equation (2) becomes:

$$0 = \cos Z_{56} \cos Z_{43} + \sin Z_{56} \sin Z_{43} \cos(B_{56} - B_{43}) \qquad (3)$$

We have two equations (1 and 3) in two unknowns B_{56} and Z_{56} (the other bearings and ZAs are known). Solve them by dividing by $\cos Z_{56}$ throughout and rearranging terms gives to equations for Z_{56}.

$$\mathbf{\tan Z_{56} = -\cos Z_{21} /(\sin Z_{21} \cos(B_{56} - B_{21})) = -\cos Z_{43} /(\sin Z_{43} \cos(B_{56} - B_{43}))} \qquad \mathbf{(4)}$$

The two equations in (4) for $\tan Z_{56}$ allow us to solve for B_{56}

$$\mathbf{\tan B_{56} = (\cos B_{43} \tan Z_{43} - \cos B_{21} \tan Z_{21})/(\sin B_{21} \tan Z_{21} - \sin B_{43} \tan Z_{43})} \qquad \mathbf{(5)}$$

Substituting the solution from (5) into (4) gives Z_{56}

In Excel I suggest using ATAN2 functions to calculate B_{56} and Z_{56} in equations (4) and (5), or similar functions if not using a spreadsheet.

My data

B_{21} = 158° 33' 08.1" Z_{21} = 242° 35' 25.7"

B_{43} = 208° 18' 02.7" Z_{43} = 295° 40' 53.9"

Calculate B_{56} by (5) = 94° 25' 41.4"

Calculate Z_{56} by (4) and (5) = 130° 05' 02.7"

These values agree with the other two methods.

Rotations method

The concept here is to rotate the lines in 3D space (and thus all the coordinates) until one of the lines becomes a point. Then solve for the shortest distance from this point to the other line. Then find the coordinates of the pedal points.

Rotation matrices about the X, Y, and Z axes are –

$$R_Z(\kappa) = \begin{pmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad R_Y(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \qquad R_X(\omega) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{pmatrix}$$

where ω, θ, and κ are the rotation angles about the X, Y, and Z axes respectively.

There are at least three ways of rotating a network Harvey (1985), one way is: $R = R_Z(\kappa) * R_Y(\theta) * R_X(\omega)$
We should not use the 'small rotations' approximation.

In our case we could rotate about the H axis by minus the bearing of the first line. That would make that line have a new 'bearing' of zero and all points on it with a new constant E. Then rotate about the E axis by minus the ZA of the first line. That would make that line have a new ZA = 0 and all points on it have a constant E and a constant N. Now solve for the shortest distance from this E and N of the first line to the other line in the latest plan view. That is, using only the latest EN of the two end points of that line.

The R matrix for coordinates in E N H is:

$$R = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(-ZA) & \sin(-ZA) \\ 0 & -\sin(-ZA) & \cos(-ZA) \end{pmatrix} \begin{pmatrix} \cos(-B) & \sin(-B) & 0 \\ -\sin(-B) & \cos(-B) & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So the new coordinates are:

$$\begin{pmatrix} E' \\ N' \\ H' \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(-ZA) & \sin(-ZA) \\ 0 & -\sin(-ZA) & \cos(-ZA) \end{pmatrix} \begin{pmatrix} \cos(-B) & \sin(-B) & 0 \\ -\sin(-B) & \cos(-B) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} E \\ N \\ H \end{pmatrix}$$

The coordinates for our data set:

Point	1	2	3	4
E	16.5	11.0	10.5	17.5
N	20.0	34.0	24.5	37.5
H	10.0	2.2	1.9	9.0

Line	s dist	Bearing	ZA
1 2	16.944	338°33'08.1"	117°24'34.3"
3 4	16.383	28°18'02.7"	64°19'06.1"

R1 for brg:

0.9308	0.3657	0
-0.3657	0.9308	0
0	0	1

R2 for ZA:

1	0	0
0	-0.4603	-0.8877
0	0.8877	-0.4603

R = R2*R1 =

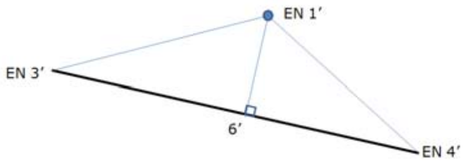
0.9308	0.3657	0
0.1683	-0.4285	-0.8877
-0.3246	0.8263	-0.4603

New rotated coordinates

Point	1'	2'	3'	4'
E'	22.670	22.670	18.731	30.000
N'	-14.669	-14.669	-10.417	-21.112
H'	6.566	23.510	15.960	21.161

Check: ΔH1'2' = dist 1' 2' = 16.944 = dist 12   i.e. still the same after rotation

Solve in E'N' coordinates the triangle 1 3 4 to find the shortest dist from 1 to the line 3 4. We use just the E'N' coordinates and see the triangle in 'plan' view with the line 12 represented as a dot at point 1'.



From coordinates:
brg 3'1' = 137.19°   dist 3'1' = 5.797
brg 3'4' = 133.50°   dist 3'4' = 15.536

In this 'plan' view we see the true value of the shortest distance 16 = dist 3'1' * SIN(brg 3'1' - brg 3'4')
= 0.373   which agrees with our previous solutions.

dist 3'6' = dist 3'1' * COS(brg 3'1' - brg 3'4') = 5.785

We can calculate the true slope (3D) distance 36 by ratio:
slope dist 36 = slope dist 34 * dist 3'6' / dist 3'4' = 6.100

Now calculate the coordinates of 6 by 3D radiation from point 3.
E6 = E3 + slope dist 36 * SIN(ZA34) * SIN(Brg34)= 13.106
N6 = N3 + slope dist 36 * SIN(ZA34) * COS(Brg34) = 29.340
H6 = H3 + slope dist 36 * COS(ZA34) = 4.544
which agrees with our previous solutions.

To find the other pedal point repeat a similar process but rotate the other line.

R brg $\begin{bmatrix} 0.8805 & -0.4741 & 0 \\ 0.4741 & 0.8805 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ R ZA $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.4334 & -0.9012 \\ 0 & 0.9012 & 0.4334 \end{bmatrix}$ R = $\begin{bmatrix} 0.8805 & -0.4741 & 0 \\ 0.2055 & 0.3816 & -0.9012 \\ 0.4273 & 0.7935 & 0.4334 \end{bmatrix}$

New rotated coordinates:

	1	2	3	4
E'	5.046	-6.434	-2.370	-2.370
N'	2.009	13.251	9.793	9.793
H'	27.253	32.632	24.750	41.134

Points 3 and 4 now have the same EN coordinates
Check: ∆H = dist 3' 4' = 16.383   yes, still the same!

Solve in E'N' coordinates the triangle 3' 1' 2' to find the shortest dist from 3' to the line 1' 2'
From coordinates:
brg 1'3' = 316.39°   dist 1'3' = 10.751
brg 1'2' = 314.40°   dist 1'2' = 16.067

In this 'plan' view we see the true value of the shortest distance 15 = 0.373 which agrees with our previous solutions.

dist 1'5' =10.745
dist 15 by ratio = 11.331

Coordinates of point 5 by 3D radiation from 1:   12.822   29.362   4.784
which agrees with our previous solutions.