CONDITION
METHOD LEAST SQUARES
- MORE EXAMPLES

Currently condition method LS is rarely used by surveyors so it is briefly dealt with in chapter 8 of Monograph 13. In the past with limited computer power (or none at all) the condition method was more popular than now. Perhaps it will become useful again in the future.

The first two of these examples are based on my class notes from when I was a student. So my teachers are the original authors. I have added comments and changed the symbols to match those used in Monograph 13. I think the data is not based on real observations, but I may change that to real student observations sometime in the future.

You will see that these example problems do not have to concern themselves with datums, i.e. what to hold fixed, because they adjust the observations only – not the parameters. So these solutions might be comparable to Free Net adjustments.

1. Condition Method Equations

This method includes only the observations, and not parameters, in the model. Conditions, such as misclose sums, are used to adjust the observations.

Arrange the basic equations in the form $\qquad F(L) = 0$
where L = adjusted observations and ℓ are the observations (a vector n terms)

$$B = \frac{\partial F}{\partial \ell} = \begin{pmatrix} \dfrac{\partial F_1}{\partial \ell_1} & \dfrac{\partial F_1}{\partial \ell_2} & \dfrac{\partial F_1}{\partial \ell_3} & \cdots \\ \dfrac{\partial F_2}{\partial \ell_1} & \dfrac{\partial F_2}{\partial \ell_2} & \cdots & \cdots \\ \dfrac{\partial F_3}{\partial \ell_1} & \cdots & \cdots & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

B is n_e by n

The number of model equations equals n-u (the number of observations minus the number of unknown parameters), and make sure that the model equations are independent.

Model	$Bv - b = 0$
Functions	$F(L) = 0$
misclose (OMC) vector, n_e terms	$b = -F(\ell)$
Estimated Variance Factor	$VF = v^T P v / n_c$
Residuals	$v = QB^T (BQB^T)^{-1} b$
Adjusted observations	$L = \ell + v$
Quality of corrected observations	$Q_L = Q - QB^T (BQB^T)^{-1} BQ$ (n by n)
Quality of residuals	$Q_v = QB^T (BQB^T)^{-1} BQ$ (n by n)
Redundancy Numbers	$ri \approx s^2_v / s^2 = 1 - (s^2_L/s^2)$

n   number of observations
n_c number of condition equations
n_e number of model equations
Q   a priori VCV matrix of observations (n by n)
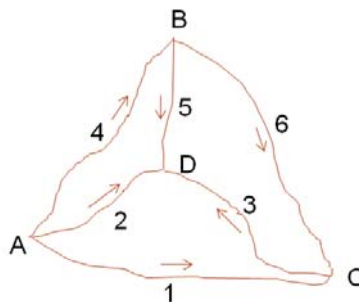
2.   Level net example

Figure 1   Fictitious height network plan

The measurements are assumed to be not correlated.   The measurements are:

Level line (see fig)	ΔH (m)	s^2 (cm^2)
1	6.16	4
2	12.57	2
3	6.41	2
4	1.09	4
5	11.58	2
6	5.07	4

Basic equation: $H_B - H_A - \Delta H_{AB} = 0$
Observations: ΔH_{AC} ΔH_{AD} ΔH_{CD} ΔH_{AB} ΔH_{BD} ΔH_{BC}

Possible condition equations are the loop miscloses, e.g. the three inner loops ABD, BCD, ADC and the outer loop ABC:
     $\Delta H_{AB} + \Delta H_{BD} - \Delta H_{AD} = 0$

$$\Delta H_{BC} + \Delta H_{CD} - \Delta H_{BD} = 0$$
$$\Delta H_{AD} - \Delta H_{CD} - \Delta H_{AC} = 0$$
$$\Delta H_{AB} + \Delta H_{BC} - \Delta H_{AC} = 0$$

There are 6 observations (n) and 3 unknowns (u). So n-u = 3, thus we need 3 condition equations. The above four equations are not independent because the last one is simply the sum of the first three. So I select the first three equations as the conditions.

I have chosen units of cm for residuals etc because they are similar to the size of the standard deviation. The VF is always unit less so set the units of P to cancel the units of v^2. For the diagonal elements of P the units are cm^{-2} and the units of Q are cm^2,. A check equation in the condition method is Bv - b = 0, so set the units of B to balance the units in this equation, b is in cm therefore Bv is in cm. Now all six terms of v are in cm, therefore the units of B must be unit less, and they are.

Substituting observations into the 3 condition equations yield b = - F(ℓ):

Condition Equation	b
$\Delta H_{AB} + \Delta H_{BD} - \Delta H_{AD} = 0$	-10.0
$\Delta H_{BC} + \Delta H_{CD} - \Delta H_{BD} = 0$	10.0
$\Delta H_{AD} - \Delta H_{CD} - \Delta H_{AC} = 0$	0.0

With the observations in the order 1 to 6 as listed above the terms in B and Q are:

$$B = \frac{\partial f}{\partial \ell} = \begin{pmatrix} 0 & -1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 \\ -1 & 1 & -1 & 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 \end{pmatrix}$$

An intermediate step, to check your calculations if you are working on this problem yourself. $(BQB^T)^{-1} =$

0.15	0.05	0.05
0.05	0.15	0.05
0.05	0.05	0.15

Residuals v = $QB^T (BQB^T)^{-1}$ b = $(0, 2, 2, -4, -4, 4)^T$ cm
[Includes inversion of only a 3x3 matrix]

The adjusted observations are ℓ + v:

1	ΔH_{AC}	6.16
2	ΔH_{AD}	12.59
3	ΔH_{CD}	6.43
4	ΔH_{AB}	1.05
5	ΔH_{BD}	11.54
6	ΔH_{BC}	5.11

After some matrix calculations the Quality of corrected observations $Q_L = Q - QB^T$ $(BQB^T)^{-1}BQ$, then standard deviations of corrected observations s_L (sqrt of diagonal terms of Q_L) and redundancy numbers ($r = 1 - (s^2_L/s^2)$) are:

|		Q_L				s_L	r
1.60	0.80	-0.80	0.80	0.00	0.80	1.26	0.6
0.80	1.20	0.40	0.80	0.40	0.00	1.10	0.4
-0.80	0.40	1.20	0.00	0.40	-0.80	1.10	0.4
0.80	0.80	0.00	1.60	-0.80	-0.80	1.26	0.6
0.00	0.40	0.40	-0.80	1.20	0.80	1.10	0.4
0.80	0.00	-0.80	-0.80	0.80	1.60	1.26	0.6

It's a linear problem so there is no need to do a second iteration; except to check the calculations. The second iteration does yield all v = 0. Substituting the adjusted observations into the original loop misclose equations also gives zero errors.

$v^TPv = 20$
Three conditions equations so VF = 6.7 (which is statistically too large)

This concludes the condition method LS adjustment. If you want to determine the heights of the points then you need to do additional calculations. To determine the standard deviations of the adjusted heights you need to do propagation of variance calculations too.

For comparison, a parametric LS solution of this problem was calculated with the height of point A held fixed at 100. The results are the same as the condition method above.
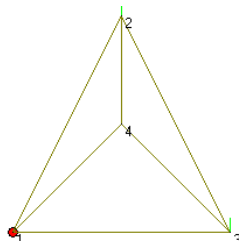
Adjusted Heights:
Name	H m
A (1)	100.000
B (2)	101.050
C (3)	106.160
D (4)	112.590

Residuals after adjustment

OBS		FROM	TO	MEASUREMENT	STD DEV	v	v/s	Redun
				m	cm	cm		
1 Height Df		1	3	6.16	2.00	0.0	0.0	0.6
2 Height Df		1	4	12.57	1.41	2.0	1.4	0.4
3 Height Df		3	4	6.41	1.41	2.0	1.4	0.4
4 Height Df		1	2	1.09	2.00	-4.0	-2.0	0.6
5 Height Df		2	4	11.58	1.41	-4.0	-2.8	0.4
6 Height Df		2	3	5.07	2.00	4.0	2.0	0.6

Variance Factor is   6.67   [n-u = 3]

3. Braced Quad directions example

Theodolite direction observations were made on a braced quadrilateral as shown in figure 3. Solution by the condition method adjusts the directions without need to consider coordinates, datum constraints or orientation parameters, as we see below.
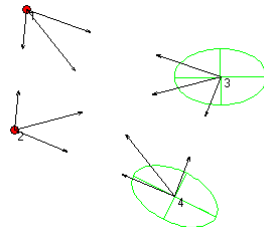

Figure 3.

We number the points A as 1, B as 2, C as 3 and D as 4. We assume all directions have the same precision of ± 1" standard deviation. The observations are, in order:

```
 1 DIRECTION      4    2          0 00 00.
 2 DIRECTION      4    1         29 04 04.0
 3 DIRECTION      4    3         88 20 53.49
 4 DIRECTION      2    1          0 00 00.
 5 DIRECTION      2    3         69 42 26.92
 6 DIRECTION      2    4        106 45 40.5
 7 DIRECTION      1    3          0 00 00.0
 8 DIRECTION      1    4         32 31 32.9
 9 DIRECTION      1    2         76 41 47.73
10 DIRECTION      3    4          0 00 00.
11 DIRECTION      3    2         54 35 55.62
12 DIRECTION      3    1         88 11 43.2
```

There are 12 observations (n) and 8 unknowns (u; in the parametric method they would be E and N of C and D and an orientation unknown at each corner). So four (n-u = 4) condition equations are needed.

Calculated Loop Miscloses from the observations are:
Points used: 1   2   4   3   1 =   4.92" Outer loop
Points used: 1   2   3   1   =   2.23" Triangle ABC
Points used: 1   2   4   1   =   -0.67" Triangle ABD
Points used: 2   3   4   2   =   -2.69" Triangle BCD
Points used: 1   3   4   1   =   -5.59" Triangle ACD

The outer loop misclose is simply the sum of two triangles so it is not independent of the other equations. Of the other 4 triangle miscloses only three are completely independent, so choose any 3. Here we take the first three triangles listed above and write D12 as the direction AB etc. Remember theodolite directions measure clockwise.

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ABC: (D12-D13) + (D23-D21) + (D31-D32) − 180° = 0
ABD: (D12-D14) + (D24-D21) + (D41-D42) − 180° = 0
BCD: (D24-D23) + (D32-D34) + (D43-D42) − 180° = 0

The final condition equation is called a side equation and uses all the directions as though we were completing a circuit around a fictitious point at the centre of the figure. Chapter 3 describes the side condition, for example see fig 3.2 (LHS):

$$sinA_{11} * sinA_{13} * sinA_{15} * sinA_{17} = sinA_{12} * sinA_{14} * sinA_{16} * sinA_{18}$$

The side condition is formed by multiples of the sines of angles alternating around the figure as shown by the pattern in the following equation, where D12 is the direction AB.

$$\frac{sin(D14-D13)sin(D23-D21)sin(D41-D42)sin(D32-D34)}{sin(D12-D14)sin(D24-D23)sin(D43-D41)sin(D31-D32)}-1=0$$

What difference does it make if we use the … - … = 0 or … / … - 1 = 0 ?
To work in units of secs throughout these calculations the side equations b term needs to be multiplied by the conversion between radians and seconds i.e. 206264.8… (= 180*3600/π)
Substituting the observations into these 4 condition equations yields the b vector
b = -F(ℓ) in seconds: $(-2.23, 0.67, -2.69, 13.45)^T$

The partial derivations in the B matrix for the first three condition equations with the observations in the order listed above are quite simple and unit less:

$$B=\begin{pmatrix} 0 & 0 & 0 & -1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 \\ -1 & 1 & 0 & -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & -1 & 1 & 0 \\ . & . & . & . & . & . & . & . & . & . & . & . \end{pmatrix}$$

Now to the derivatives for the side equation. Let

$$S=\frac{sin(D14-D13)sin(D23-D21)sin(D41-D42)sin(D32-D34)}{sin(D12-D14)sin(D24-D23)sin(D43-D41)sin(D31-D32)}$$

Then
$$\frac{\partial S}{\partial D42}=-cos(D41-D42)*\frac{sin(D14-D13)sin(D23-D21)sin(D32-D34)}{sin(D12-D14)sin(D24-D23)sin(D43-D41)sin(D31-D32)}$$
$$=-cot(D41-D42)*S=-S/tan(D41-D42)=-1.799$$

This term is unit less, as are all the other terms in B. The terms for all the directions along the sides of the braced quad are similarly derived to the term above. The derivatives for the directions along the diagonals of the braced quad are a little more complicated because those directions appear in two places in the side condition equation. For example, the partial derivative of the side equation with respect D41 is:

$$\frac{\partial S}{\partial D41} = S[\cot(D41 - D42) + \cot(D43 - D41)] = 2.393$$

The last row in B is:
-1.799   2.393   -0.594   -0.370   1.694   -1.324   -1.568   2.597   -1.029   -0.711   2.216   -1.505

Since the directions all have a standard deviation of 1, then Q is the identity matrix.

Residuals, $v = QB^T (BQB^T)^{-1} b$ = (in secs)   [Includes inversion of only a 4x4 matrix]
v is a column vector, but I write it here as a row:
-0.49, 1.87, -1.38, -0.05, 0.92, -0.87, 0.25, 0.31, -0.56, 0.81, 0.79, -1.59,

$v^T P v$ = 11.51 and since n_c = 4 then VF = 2.88

The partial derivatives in the first three rows of B are constant and the terms in the last row do not change very much unless there are huge corrections to the direction observations. So B is virtually constant and a second iteration will not normally change the adjusted values, but is useful as a check. A second iteration was performed and all the b terms were 0. That also shows that the loop miscloses using the adjusted directions is now 0. All the residuals were also 0 in the second iteration. So it is a linear adjustment problem.

Notice that we didn't need to know the coordinates of any points or any distance in the network. We also did not need to hold any coordinates fixed or deal with orientation parameters. But we did need to know how to construct the condition equations particular to this figure.
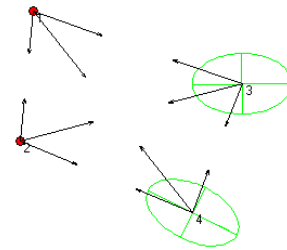
The condition method does not yield the adjusted coordinates or the error ellipses of the points, they take further calculation.

For comparison, a parametric LS solution of this problem was calculated with coordinates as shown below. The coordinates of A and B are assumed to be known and fixed. Are the results are the same as the condition method above? Yes!

```
Adjusted Coordinates after iteration 5
Point Fixed Name                      E m            N m
     1 ENH  A                    3474.0190     11480.2820
     2 ENH  B                    3295.9940      9726.7060
     3   H  C                    6297.2453     10502.6243
     4   H  D                    5630.4448      8756.9802

 RESIDUALS AFTER ADJUSTMENT (5 iterations) s = 1" for all obs
OBS    TYPE     FROM  TO    MEASUREMENT        v      v/s   Redun
 1 Direction     4    2    0 00 00.0        -0.5"   -0.5   0.3
 2 Direction     4    1   29 04 04.0         1.9"    1.9   0.4
 3 Direction     4    3   88 20 53.5        -1.4"   -1.4   0.3
 4 Direction     2    1    0 00 00.0         0.0"    0.0   0.3
 5 Direction     2    3   69 42 26.9         0.9"    0.9   0.4
 6 Direction     2    4  106 45 40.5        -0.9"   -0.9   0.3
```

```
 7 Direction    1    3    0 00 00.0        0.3"     0.3   0.3
 8 Direction    1    4   32 31 32.9        0.3"     0.3   0.4
 9 Direction    1    2   76 41 47.7       -0.6"    -0.6   0.3
10 Direction    3    4    0 00 00.0        0.8"     0.8   0.3
11 Direction    3    2   54 35 55.6        0.8"     0.8   0.4
12 Direction    3    1   88 11 43.2       -1.6"    -1.6   0.3

After iteration 5 the Variance Factor is 2.88 [n-u = 4]
```

4. Traverse example

Here we solve the traverse given in Chapter 3, by the condition method. We will omit
the radiation from 103 to 201. [Once we get the adjusted observations for the rest of
the traverse we can work out the bearing for the radiation, and then use it with
observed distance to get coordinates. There are no corrections (zero residuals) for
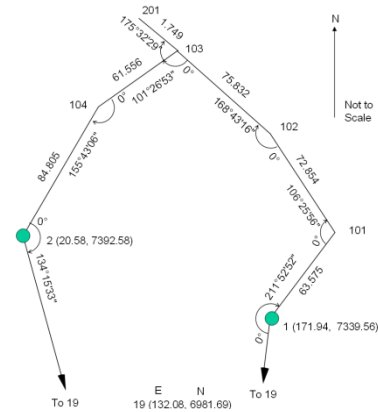radiation observations.]

Figure 4. Student traverse at Morpeth survey camp

I will renumber the points here to make subscripts easier to follow in the equations
below. Thus point 19 becomes 9, 1 becomes 7, 1 becomes 8, 101 becomes 1 etc.
The given data, in FIXIT format is:

```
COMMENT      Dist mm      ppm        Dirn "      mm cent
DSD          3.0          0.0         5.0          0.0
COORDINATE   9 EN   RO               132.08          6981.69
COORDINATE   7 EN    START BM        171.94          7339.56
COORDINATE   8 EN    END BM           20.58          7392.58
DIRECTION    7    9         0 00 00
DIRECTION    7    1       211 52 52
DIRECTION    1    7         0 00 00
DIRECTION    1    2       106 25 56
DIRECTION    2    1         0 00 00
```

```
DIRECTION      2    3      168 43 16
DIRECTION      3    2        0 00 00
DIRECTION      3    4      101 26 53
DIRECTION      4    3        0 00 00
DIRECTION      4    8      155 43 06
DIRECTION      8    4        0 00 00
DIRECTION      8    9      134 15 33
H DISTANCE     7    1         63.575
H DISTANCE     1    2         72.854
H DISTANCE     2    3         75.832
H DISTANCE     3    4         61.556
H DISTANCE     4    8         84.805
```

There are 17 observations, we will use them in the order listed above. Traverses of this type are known to have 3 condition equations, namely one for each misclose in angles, in Eastings, and in Northings. So the b and v vectors will have 17 terms and the B matrix will be 3 by 17. Approximate coordinates for the intermediate traverse points are not required. If this was a loop traverse we wouldn't need any coordinates, and would assign a bearing to one line.

We set the input standard deviation terms ppm and centring to zero to ease the calculations in this example.

From the fixed coordinates we calculate the 'known' bearings at the start and end of the traverse, B79 and B89 (I carry the full D M S values on a spreadsheet and show rounded off values in degrees and a decimal here):   B79 = 186.4°   B89 = 164.8°

Our first condition (CE1) is that the angular misclose for perfectly adjusted data is zero. Using B12 to represent bearing 1 to 2, D12 to represent direction 1 to 2, and d12 to represent distance 1 to 2.

CE1: B79+D71-D79-360+D12-D17+180+D23-D21-180+D34-D32-180+D48-D43-
       180+D89-D84-180-B89 = 0

Remember that to produce the bearings, we desire an answer between 0 and 360. Also be careful with using angles or directions to calculate bearings, here we assume they are clockwise from the backsight to the foresight.

Inserting the data gives bearings carried forward and the b term = - F(ℓ)
The bearings are:
        B71   38.2
        B12   324.7
        B23   313.4
        B34   234.8
        B48   210.6
        B89   164.8
Angular misclose's b term = 8.0"

Our second condition (CE2) is the misclose in Easting. This is easier to write in terms of bearings than directions. So we determine 'raw' bearings using our directions starting from one end of the traverse, in our case we start at 7, as shown above. But

when differentiating the equation to form the B matrix we need to recall the formula for each bearing. For points in sequence i j k: Bjk = Bij + Djk – Dji – 180°

CE2: E7 + d71sinB71 + d12sinB12 + d23sinB23 + d34sinB34 + d48sinB48 – E8 = 0

Inserting our data in this equation gives 0.0316m, so b = - 31.6 mm

Similarly, the third condition is:

CE3: N7 + d71cosB71 + d12cosB12 + d23cosB23 + d34cosB34 + d48cosB48 – N8 = 0

Inserting our data in this equation gives -0.0324 m, so b = 32.4 mm

Thus the b vector in seconds and mm is:
      8.0
    -31.6
     32.4

The three conditions, in the order above, are differentiated by the 17 observations, also in the order above, to produce the terms in the B matrix. A few terms are shown in detail, the others are similarly derived.

The derivatives of CE1 are simple, e.g. $\partial CE1/\partial D71 = 1$

The other derivatives are not simple because a direction or angle observation appears in bearings of all subsequent lines around a loop. One of the terms (it appears in following bearings too) in CE2 containing D12 is
                      … + d12sinB12
which really is   ... + d12sin(B79+D71-D79-360+D12-D17+180)   so

The equations become longer the further we move along the traverse. The full details of this example are not further included in these notes. See Cooper's book for an example. Also, see the paper Efficient Least Squares Traverse Adjustment Using Spreadsheets by Hu, Tan and Barnes in the journal Australian Surveyor June 2003.

Alternative Solution Method using Spreadsheet Solver function.

It is possible to solve this problem using MS Excel's Solver function, as shown in the spreadsheet screen capture below. The answers are the same as those obtained by FIXIT3 software using a parametric least squares solution. Note that in this example below we minimise $\sum(v/s)^2$, so it is a least squares solution. While the solution uses coordinates it does so only as intermediate steps to work out bearings and coordinate miscloses. The adjustment deals with the residuals v, not the changes in coordinates.

	A	B	C	D	E F	G	H	I	J	K	L	M	N	O	P	Q	R
2	CONDITION METHOD SOLUTION	By Solver															
3																E	N
4			Back		Angle					Adj fwd					9	132.08	6981.69
5	BS AT TO		Bearing	d	m s	dd	v "	s "	v/s	bearing	dist	v mm	s mm	v/s	Adj dist	171.94	7339.56
6	9 7 1		186.4	211	52 52	211.9	-6.0	5	-1.2	38.2	63.575	-0.4	3	-0.1	63.5746	211.2855	7389.4965
7	7 1 2		218.2	106	25 56	106.4	5.8	5	1.2	324.7	72.854	13.2	3	4.4	72.8672	169.1462	7448.9432
8	1 2 3		144.7	168	43 16	168.7	10.1	5	2.0	313.4	75.832	13.8	3	4.6	75.8458	114.0320	7501.0489
9	2 3 4		133.4	101	26 53	101.4	11.9	5	2.4	234.8	61.556	4.4	3	1.5	61.5604	63.7010	7465.6022
10	3 4 8		54.8	155	43 6	155.7	1.0	5	0.2	210.6	84.805	-1.4	3	-0.5	84.8036	20.5800	7392.5800
11	4 8 9		30.6	134	15 33	134.3	-14.8	5	-3.0	164.8							
12	fixed/known closing values							8to9		164.8					8	20.58	7392.58
13								misc "		0.00				misc mm		0.00	0.00
15							vTPv = sumsq(v/s)			64.04		VF	21.35				

17 How to use solver.
18 Set up intially with all v set to zero.
19 So coordinates are calculated with raw bearings and distances
20 This allows miscloses to be calculated.
21 Get solver to determine the values in all the v cells ie 'by changing values'
22 The constraints are that the misc in angles and misc in E and N all must equal zero, put these in with the Add button
23 Set Target cell so that we minimise vTPv
24 Then press Solve button
25 As shown in screen capture:

Solver Parameters

Set Target Cell: $K$15

Equal To: ○ Max ⊙ Min ○ Value of: 0

By Changing Cells:
$H$6:$H$11,$M$6:$M$10

Subject to the Constraints:
$K$13 = 0
$Q$13 = 0
$R$13 = 0

[Solve] [Close] [Guess] [Options] [Add] [Change] [Delete] [Reset All] [Help]