MATHEMATICS ENRICHMENT CLUB.[1]
Problem Sheet 15, September 3, 2012

1. In how many ways can we change \$10 into 50 cent and 20 cent coins, with at least one of each coin being used.

2. If $x = \sqrt{1 + \sqrt{1 + \sqrt{2}}}$ find the exact value of $x^4 - 2x^2$.

3. A quadrilateral in which a circle can be drawn which touches each of the four faces is called a *circumscribable quadrilateral*. If r is the radius of the circle and s is half the perimeter of the quadrilateral, prove that the area of the quadrilateral is rs.

4. Use the fact that $2xy = (x + y)^2 - x^2 - y^2$ to show that
$$2(b - c)(c - a) + 2(c - a)(a - b) + 2(a - b)(b - c) \leq 0$$
for all real numbers a, b, c.

5. (a) Find all positive integers a, b, c such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c}$ is as large as possible but less than $\frac{1}{2}$.

   (b) Find all positive integers a, b, c, d such that $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}$ is as large as possible but less than 1.

6. Suppose the median from the vertex C of a triangle ABC has length $\frac{1}{2}AB$. Show that the triangle is right-angled at C.

7. Let P be a point outside a circle with diameter AB and let Q be a point inside it. Prove that $\angle APB$ is acute and that $\angle AQB$ is obtuse.

Senior Questions.

1. Let $C(x) = \dfrac{e^x + e^{-x}}{2}$ and $S(x) = \dfrac{e^x - e^{-x}}{2}$. Show that $\dfrac{d^2C}{dx^2} = C(x)$, $\dfrac{d^2S}{dx^2} = S(x)$ and $C(x)^2 - S(x)^2 = 1$.

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

2. Prove by induction that the sum to k terms of

$$1^2 - 3^2 + 5^2 - 7^2 + ...$$

equals $-8n^2$ when $k = 2n$ and $8n^2 + 8n + 1$ when $k = 2n + 1$.

3. In ΔABC prove that $b^2(\cot A + \cot B) = c^2(\cot A + \cot C)$. (Hint: You might begin by considering the area of the triangle in two different ways.)