MATHEMATICS ENRICHMENT CLUB.[1]
Problem Sheet 6, June 4, 2012

1. A parallelogram $ABCD$ has $BC = 4$ cm and $CD = 8$ cm. The point A is 3 cm above CD. Find the length of the perpendicular from A to BC.

2. If a, b, c are real numbers and $a > b$, which of the following must be true?

$$(a)\ \frac{1}{a} > \frac{1}{b} \qquad (b)\ ac > bc \qquad (c)\ a^2 > b^2 \qquad (d)\ a + c > b + c \qquad (e)\ \frac{1}{a} < \frac{1}{b}.$$

3. (a) Verify that $x = 170, y = 39$ satisfy $x^2 = 19y^2 + 1$.
   (b) Hence find integers x and y such that $x^2 = 171y^2 + 1$ and $x^2 = 3211y^2 + 1$.

4. A rectangle has perimeter 20cm. What is the least value of the diagonal?

5. From the point (x, y) we can move a counter to any one of the following points:

$$(2x, y), (x, 2y)$$

or

$$(x - y, y) \text{ if } x > y, \qquad (x, y - x) \text{ if } y > x.$$

Starting from $(1, 1)$ can you see a rule to determine which points in the plane can be reached using the rules above?

6. The line joining a vertex of a triangle to the midpoint of the opposite side is called a **median**. Let m_A denote the median in triangle ABC from A to BC.
   (a) Show that $AB + AC > 2m_A$. (Hint: Think about parallelograms)
   (b) Deduce that $AB + AC + BC > m_A + m_B + m_C$.

7. Given a circle K with centre O and diameter AB, let C be any point on K.
   (a) Prove that $\angle ACB = 90°$.
   (b) Describe how to construct a right-angled triangle ACB if we are given its hypotenuse AB and the length of the perpendicular dropped from C to AB.

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

Senior Questions.

1. Let $S(x) = \dfrac{e^x - e^{-x}}{2}$ and $C(x) = \dfrac{e^x + e^{-x}}{2}$.

   (a) Show that $(C(x))^2 - (S(x))^2 = 1$.

   (b) If $S(x) = \tan\theta$, express $C(x)$ in terms of θ.

2. Find the integral
$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos^4\theta}{\sin^2\theta}\, d\theta.$$

3. A die is thrown n times. Show that if the probability that a 6 appears at least once is greater than $\frac{1}{2}$, then $n > \dfrac{\log 2}{\log 6 - \log 5}$.