MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 13, August 20, 2013

1. Counting from right to left, there are 9 possible digits for the units columnn, 9 for the tens and 9 for the hundreds. If all of these are 0, then the only choice for the thousands is 3, otherwise the only choice is 2 - so there's only 1 choice for the thousands column. So there are $9 \times 9 \times 9 = 729$ years with no 1.

2. Suppose the score of the ith student is s_i, then we have $\frac{1}{20}\sum_{i=1}^{20} s_i = 66$ and $\frac{1}{30}\sum_{i=21}^{50} s_i = 56$. The total average is

$$\frac{1}{50}\sum_{i=1}^{50} s_i = \frac{1}{50}(66 \times 20 + 56 \times 30) = \frac{2}{5}66 + \frac{3}{5}56 = \frac{132+168}{5} = 60$$

so the average score is 60%.

3. (a)

	1	2	3	4	5	6
x_n	1	2	5	12	29	70
y_n	1	3	7	17	41	99
$\frac{y_n}{x_n}$	1	1.5	1.4	1.4167	1.4138	1.4143

(b) First note that

$$\begin{aligned} y_n^2 - 2x_n^2 &= 4x_{n-1}^2 + 4x_{n-1}y_{n-1} + y_{n-1}^2 - 2(x_{n-1}^2 + 2x_{n-1}y_{n-1} + y_{n-1}^2 \\ &= 2x_{n-1}^2 - y_{n-1}^2 \\ &= -(y_{n-1}^2 - 2x_{n-1}^2). \end{aligned}$$

Repeating this we see

$$\begin{aligned} y_n^2 - 2x_n^2 &= -(-(y_{n-2}^2 - 2x_{n-2}^2)) \\ &= -(-(-(y_{n-3}^2 - 2x_{n-3}^2))) \\ &\vdots \quad n-1 \text{ times} \\ &= (-1)^{n-1}(y_1 - 2x_1) = (-1)^{n-1}(-1) = (-1)^n \end{aligned}$$

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni. Problem 4 provided by G. Liang, problems 5 and 6 are from the Tournment of Towns

(c) Diving the above by x_n^2 we see that

$$\frac{y_n^2}{x_n^2} = 2 + \frac{(-1)^n}{x_n^2}$$

and as x_n is increasing, the second term approaches zero as n grows to infinity. So

$$\frac{y_n^2}{x_n^2} \to 2.$$

4. Suppose ABC is the triangle, and M, N the midpoints on AB and AC respectively, so that CM and BN are equal length. Since medians cut the area of a triangle in half, the areas BMC and BNC are equal (both half of the area of ABC). This implies that $\frac{1}{2}|CM||BC|\sin\angle MCB = \frac{1}{2}|BN||CB|\sin\angle NBC$ and since $|BN| = |CM|$ this implies $\sin\angle MCB = \sin\angle NBC$. Since $\angle NBC \neq \pi - \angle MCB$, we must have $\angle MCB = \angle NBC$, and so the two triangles BNC and BMC are congruent, thus $|BM| = |CN|$ so $|AB| = 2|BM| = 2|CN| = |AC|$ so ABC is isoceles.

5. Recall that the sum of the first n odd numbers is equal to n^2, i.e.

$$\sum_{k=1}^{n} 2k - 1 = k^2.$$

Let $g(n)$ be the greatest odd divisor of n. We can write

$$g(n) = \begin{cases} n & \text{if } n \text{ odd,} \\ g\left(\frac{n}{2}\right) & \text{if } n \text{ even.} \end{cases} \tag{1}$$

We will show that the sum

$$g(n+1) + g(n+2) + \cdots + g(2n)$$

is equal to the sum of the odd integers from 1 to n is hence n^2.

First, we claim that for $n > m$, $g(n) = g(m)$ if, and only if, there's a number k such that $n = 2^k m$. Suppose $n = 2^k m$ then using equation (1) k times, we see that $g(n) = g(m)$.

Now suppose the opposite, that $g(n) = g(m)$. This means that there's some number l such that $n = g(n)2^l$ and some number j such that $m = g(m)2^j$, so $n = g(n)2^l = g(m)2^l = \frac{m}{2^j}2^l = m2^{l-j} = 2^k m$ where $k = l - j$.

From this, we now know that $g(n+1), g(n+2), \ldots, g(2n)$ are all different numbers. Since $g(n)$ has to be odd, they are all odd numbers, since $g(n) \leq n$ they are all smaller than $2n$ and finally, there is n of them. The only possible choice of n unique odd numbers smaller than $2n$ is all of them, i.e., $1, 3, 5, \ldots, 2n - 1$. As stated, summing them gives n^2.

6. Suppose the Knights are numbered 1 through n, and Merlin sits them in numerical order around the table on the first day. We essentially want to find the number of

configurations when we consider equal all configurations that can be obtained using the rules provided. For example, suppose there are 4 Knights who sat $(1, 2, 3, 4)$ on the first day, then $(1, 3, 2, 4)$ is the same as $(3, 1, 4, 2)$ (we can swap 1 with 3, and 2 with 4).

We now define a "winding" number. For a given seating, Merling starts at Knight 1, then walks clockwise around the table until he reaches Knight 2, then continues clockwise until reaching Knight 3, and so on until he reaches Knight n, then the winding number of this seating is the number of times Merlin walked around the table. So, $(1, 3, 2, 4)$ has a winding number of 2 (he passes 1 and 2 on the first lap, then 3 and 4 on the second). Note that $(3, 1, 4, 2)$ also has a winding number of 2.

We claim that all equivalent seatings have the same winding number. This is the same as saying all legal operations (like swapping 1 with 3 above) preserve the winding number, while all illegal operations (like swapping 1 with 2) change the winding number. Let's consider the second situation first. Suppose our seating configuration has a block $\dots, k, k+1, \dots$ somewhere along the line. Swapping k with $k+1$ (an illegal operation) means we have to walk an extra lap from k to $k+1$ and all else remains the same - so the winding number is increased by 1.

Now suppose we have a seating configuration like $\dots, k, l, \dots, k+1, \dots, l+1, \dots$, where $l \neq k+1$ and $k \neq l+1$. We can see that swapping k with l doesn't change the winding number - walking from k, we still need to get $k+1$ before the end, and likewise walking from l.

The largest possible winding number is $n-1$, which we can get by ordering everyone in reverse $(n, n-1, n-2, \dots, 2, 1)$, and the smallest is 1, which we get on the first day $(1, 2, 3, \dots, n)$. All other winding numbers k can be obtained, by first putting k numbers in reverse order, then the rest in regular order $(k, k-1, \dots, 3, 2, 1, k+2, k+3, k+4, \dots, n)$. So there are $n-1$ winding numbers, hence $n-1$ non-equivalent seatings, and so the conference can go for n days.