MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 4, May 28, 2013

1. (a) Writing $(21)_b$ and $(12)_b$ in base ten then we must have $2b+1 = 2(b+2) \implies 1 = 4$, a contradiction.

   (b) In base ten we must satisfy $ab + c = 2(cb + a)$, that is $a(b - 2) = c(2b - 1)$. For $b \leq 12$ we get the numbers $(31)_5$, $(52)_8$, $(73)_{11}$, (perhaps we could count $(20)_2$ and $(00)_b$).

   (c) For arbitrary b, $a(b - 2)$ and $c(2b - 1)$ must be multiples of both $(b - 2)$ and $(2b - 1)$. We'll start with the least common multiple $\mathrm{lcm}(b - 2, 2b - 1)$, and its multiples. So we have

$$a = \frac{\mathrm{lcm}(b-2, 2b-1)}{b-2}k, \qquad \text{so that } a < b,$$
$$c = \frac{a(b-2)}{2b-1}.$$

Working with just a, we can rewrite to

$$a = \frac{2b-1}{\gcd(b-2, 2b-1)}k, \quad a < b,$$

where gcd is the greatest common divisor. The gcd part can be written as $\gcd(b-2, 2b-1) = \gcd(b-2, 2(b-2)+3) = \gcd(b-2, 3)$. So, if $b = 3m + 2$ for integer m, $\gcd(b-2, 3) = 3$ otherwise, $\gcd(b-2, 3) = 1$.

If $\gcd(b-2, 3) = \gcd(b-2, 2b-1) = 1$ then

$$a = (2b-1)k < b \implies k < \frac{b}{2b-1} \leq 1,$$

and since k must also be a natural number, $k = 0$ only (so we only get $(00)_b = 2(00)_b$).

If $\gcd(b-2, 3) = 3$, i.e. $b \in \{5, 8, 11, 14, \ldots\}$,

$$a = \frac{2b-1}{3}k < b \implies k < \frac{3b}{2b-1}.$$

[1] Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

Writing $b = 3m + 2$ we can replace all bs with ms and get

$$\begin{aligned} a &= (2m+1)k \\ c &= mk \\ k &< \frac{3m+2}{2m+1}, \quad k \in \mathbb{N},\ m \in \mathbb{N}. \end{aligned}$$

2. Write $1000 = \sum_{k=a}^{b}(2k-1)$, $0 < a \leq b$ which is an arithmetic progression, so reduces to $1000 = (b-(a-1))(b+(a-1))$. So now we look for two numbers $x = b, y = a-1$ whose sum and difference are both factors of 1000. The factors of 1000 are $(1, 1000)$, $(2, 500)$, $(4, 250)$, $(8, 125)$, $(10, 100)$, $(20, 50)$, $(25, 40)$. Since we must have one factor represented by $x - y$ and the other by $x + y$, both factors must be even, which leaves 4 possible pairs for x and y, and hence 4 pairs b and a (since $a \leq b$).

   Finally, if 1000 is the sum of consecutive, positive odd numbers $k+(k+2)+(k+4)+\cdots$, we can also add all the odd integers $-(k-2), -(k-4), \ldots, -3, -1, 1, 3, \ldots, (k-2)$ without changing the original value. So for each of the 4 pairs a and b above, we have another representation. Hence, I count 8 ways.

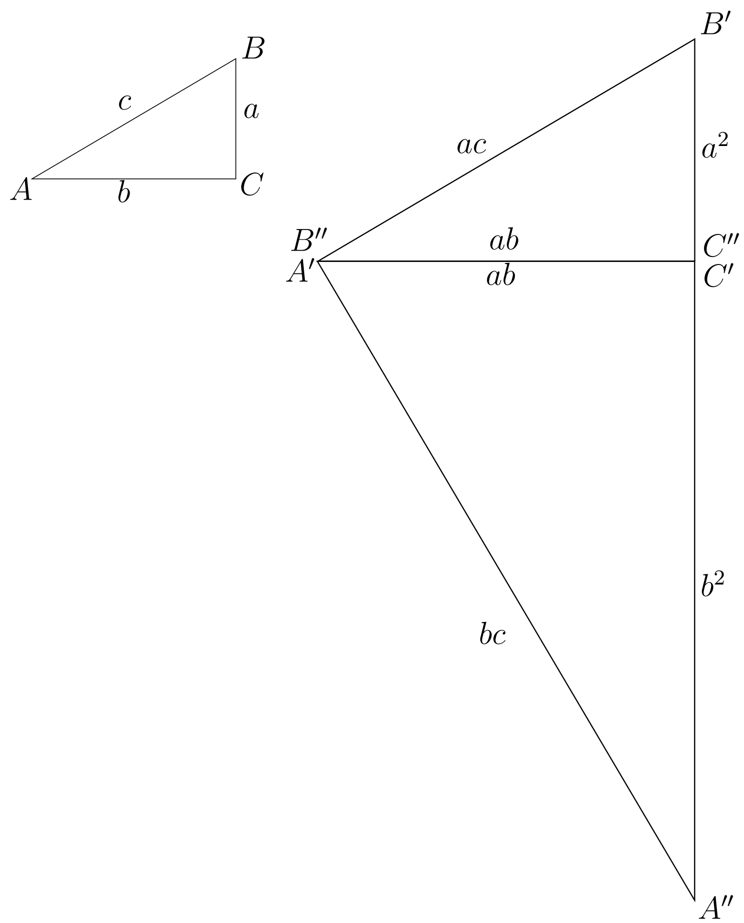

Figure 1: Picture for question 3

3. The new triangle is $A'''B'''C'''$, where C''' is at A'/B'', B''' is at B' and A''' at A''. The angle at C''' is right, since it is the sum of complementary angles, so in fact the

triangle $A'''B'''C'''$ is the enlargement of ABC by a factor of c, which implies $a^2+b^2=c^2$; Pythagorus' Theorem.

4. Angle A is $10°$ and angle C is $30°$.

5. (a) The triangles BAD and KAL are similar since they have two sides in ratio ($AK : AB = 1 : 3$ and $AL : AD = 1 : 3$) which contain the common angle A. For the same reasons, trianlges BCD and NCM are similar. Thus KL is parallel to BD which is parallel to MN. Also, the lengths $BD = 3KL$ and $BD = 3MN$ so $KL = MN$. Thus $KLMN$ is a parallelogram because it has one pair of equal length and parallel sides.

   (b) In the same fashion as above we show that the triangles ABC and KBN are similar, and that the trianlges ADC and LDM are similar, with $AB : KB = 3 : 2$, $BC : BN = 3 : 2$, $AD : LD = 3 : 2$ and $DC : DM = 3 : 2$. Thus the areas are in the ratios $\text{area}(ABD) : \text{area}(AKL) = 1 : 9$, $\text{area}(BCD) : \text{area}(NCM) = 1 : 9$, $\text{area}(ABC) : \text{area}(KBN) = 4 : 9$ and $\text{area}(ADC) : \text{area}(LDM) = 4 : 9$. From this we obtain the two equations

$$\begin{aligned}\text{area}(ABCD) &= \text{area}(ABD) + \text{area}(BCD) = 9\,\text{area}(AKL) + 9\,\text{area}(NCM)\\ \text{area}(ABCD) &= \text{area}(ABC) + \text{area}(ADC) = \frac{9}{4}\,\text{area}(KBN) + \frac{9}{4}\,\text{area}(LDM),\end{aligned}$$

   combining which yields

$$\begin{aligned}\text{area}(ABCD) + 4\,\text{area}(ABCD) &= 9(\text{area}(AKL) + \text{area}(NCM) + \text{area}(KBN) + \text{area}(LDM)\\ 5\,\text{area}(ABCD) &= 9(\text{area}(ABCD) - \text{area}(KLMN))\\ 9\,\text{area}(KLMN) &= 4\,\text{area}(ABCD)\\ \text{area}(KLMN) &= \frac{4}{9}\,\text{area}(ABCD).\end{aligned}$$

6. Consider a right triangle with perpendicular sides of length m and n, and thus hypotenuse $\sqrt{m^2+n^2}$. Say one of the non-right angles is θ then

$$\frac{m+n}{\sqrt{m^2+n^2}} = \cos\theta + \sin\theta.$$

   The value $\cos\theta+\sin\theta$ is maximum when $\cos\theta = \sin\theta$, and so when $\cos\theta = \sin\theta = \frac{1}{\sqrt{2}}$. Thus

$$\frac{m+n}{\sqrt{m^2+n^2}} \leq \frac{2}{\sqrt{2}} = \sqrt{2}.$$

Senior Questions

1. The radii of the inscribed circle that touch the triangle divide the triangle in to 2 pairs of congruent triangles and a square. Thus the perimeter of the triangle is $p = 2x+2y+2\times 2$ where $x+y = 15\ cm$ is the length of the hypotenuse. Thus $p = 2\times 15 + 4 = 34\ cm$.

2. At the start we have an equal number of even and odd numbers, so the number of boxes is $n_e + n_o$, $n_e = 1000$ and $n_o = 1000$ being the number of even and odd numbers. In each step we reduce the number of boxes by 1. If two even numbers are chosen, their difference is also even, so we reduce the number of even numbers by 1, i.e. $n_e' = n_e - 1$, $n_o' = n_o$, for $n_e \geq 2$. If two odd numbers are chosen, their difference is also even, so we reduce the number of odd numbers by 2 and increase the number of even numbers by 1, i.e. $n_e' = n_e + 1$, $n_o' = n_o - 2$ for $n_o \geq 2$. If one odd and one even number are chosen, their difference is odd, so we just reduce the number of even numbers by 1, i.e. $n_e' = n_e - 1$, $n_o' = n_o$ for $n_e \geq 1$, $n_o \geq 1$.

   Since we start with an even number of odd numbers, and each operation reduces the number of odd numbers by 2 or 0, all odd numbers will eventually be removed. However, if all even numbers are removed, still an even number of odd numbers remains because the operations that remove even numbers leave the number of odd numbers unchanged. The next operation must necessarily re-introduce another even number.