MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 5, June 4, 2013

1. $\frac{504}{999}$

2. Let's call a "game state" the number of counters left at the start of a turn. A "winning game state" is one in which we can win from, and similarly a "losing game state" is one in which we can lose from. For example 1 is a losing game state (you must take the one counter, which is the last and hence lose), while 2 is a winning game state (you can just take one, leaving your opponent in a losing game state). Let's list the states, starting from 1 and working backwards.

Winning	Losing
2	1
3	4
5	7
6	10
8	13
9	$\vdots$
11	
12	
$\vdots$	

   Working backwards from 1 we ask "can we remove 1, 2, 4, 8, ... so that the remaining number of counters is listed in the Losing column?" We can see that the losing states follow the pattern $1 + 3k$ where $k \in \mathbb{N}$. Since $499 = 1 + 3 \times 166$ it is a Losing state, and 500 is a Winning state. So whoever goes first will win.

3. The last digit of 1997^{1997} will be a power of 7 modulo 10. The powers of 7 modulo 10 are $7, 9, 3, 1, 7, 9, 3, \ldots$, so powers of the form $1 + 4k$, $k \in \mathbb{N}$ end in a 7. Now, $1997 = 1 + 4 \times 499$ so 1997^{1997} ends in a 7.

4. 5

5. (a) It can't be correct because it violates the Triangle inequality.

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni.

(b) If $AB = 2$, we must have two triangles whose only common side as side length 2, which are $2, 3, 4$ and $2, 6, 5$. These lengths must go along BC, AC, AD and BD so the remaining side CD must be length 8.

6. (a) Construct a semi-circle with diameter AB, centre O and point on the circumference C. Draw the radius CO. Triangles AOC and COB are both isoceles (two sides are radii). The angle $\angle ACB = \angle ACO + \angle OCB = \angle OAC + \angle OBC = \angle BAC + \angle ABC$ and $\angle ACB + \angle BAC + \angle ABC = 180° = 2\angle ACB$.

   (b) Draw a circle and let AC and DB be the two mutually bisecting chords, meeting at O. Construct the cyclic quadrilateral $ABCD$, which is a parallelogram because its diagonals bisect each other. Since $AB||CD$ we have the alternate angles $\angle ABD = \angle BDC$. Also, since $\angle ABD$ and $\angle ACD$ are subtended from a common chord, they are equal. Thus $\angle BDC = \angle ABD$ and ΔODC is isoceles which implies $DO = CO = AO$. Using the same argument as above, we find that $\angle CDA$ is $90°$ which means AC is a diameter, and hence so is DB.

   (c) If a parallelogram is inscribed in a circle then it's a rectangle.

Senior Questions

1.

$$\begin{aligned}(z - z^{-1})^3 + 3(z - z^{-1}) &= 1\\ z^3 - z^{-3} + 3zz^{-2} - 3z^2z^{-1} + 3z - 3z^{-1} &= 1\\ z^3 - z^{-3} &= 1\\ z^6 - z^3 - 1 &= 0\\ z^3 &= \frac{1}{2} \pm \frac{\sqrt{5}}{2}\\ x &= \left(\frac{1}{2} \pm \frac{\sqrt{5}}{2}\right)^{\frac{1}{3}} - \frac{1}{\left(\frac{1}{2} \pm \frac{\sqrt{5}}{2}\right)^{\frac{1}{3}}}.\end{aligned}$$

2. (a) Label the areas LSC and MSB, x and y respectively. Since SL is the median of ASC the area ASL is also x, similarly since SM is the median of ASB the area ASM is also y. Finally, the median from S cuts SBC into two equal area region also, call those areas z each. The median from A cuts the whole triangle into two equal regions so $2y + z = 2x + z$, so $x = y$.

   (b) Leaving x, y and z as above, the area of the whole triangle is $2x + 2y + 2z$. Since BL is a median $2z + x = 2y + x$, so $2z = 2y$ or $z = y$. Thus $x = y = z = 100\ cm^2$ so the area of $ABC = 600\ cm^2$.