MATHEMATICS ENRICHMENT CLUB.[1]
Solution Sheet 8, June 25, 2013

1.

$$\begin{aligned}(2+4+6+\cdots 200)-(1+3+5+\cdots 199) &= (2-1)+(4-3)+(6-5)+\cdots(200-199)\\ &= \underbrace{1+1+1\cdots+1}_{100 \text{ terms}}\\ &= 100\end{aligned}$$

2. We must have a boy-girl-boy-girl configuration, of which there are two possibilities (starting with a boy or starting with a girl). Then within those configurations there are 6! possible ways to order the boys, and 6! possible ways to order the girls, making a total of $2 \times 6! \times 6!$ possible orderings.

3. There may have been some confusion with this question. A colouring here means painting each face with a colour from a selection of n colours (so we need not use all available colours) and two colourings are considered equivalent if you can rotate the tetrahedron of the first colouring and get the identical appearance of the second. So with one available colour, there is only one way to colour the tetrahedron - colour all the faces with the one colour.

   (a) Let's say the two colours are red and blue. There's the one colouring each for all red and all blue. Then we can have 3 faces red, 1 blue or 2 faces red, 2 blue or 1 face red, 3 blue. Of ach of the latter colourings there is only one. So finally there are 5 colourings.

   (b) Now we have red, blue and green. First, there's the 3 single colour colourings, all red, all blue, all green. Then for each pair of colours there was three colourings using exactly two colours, so thats 3 by red-blue, 3 by red-green and 3 by blue-green. Finally, fix the base as one colour, and the other 3 faces one colour each - there are 3 of these (one for each base colour). Try making more colourings by flipping an existing colouring and convince yourself you can always rotate back to where you started! So that's 15 in total.

[1]Some of the problems here come from T. Gagen, Uni. of Syd. and from E. Szekeres , Macquarie Uni. Question 5 comes from the Futurama episode "The Prisoner of Benda".

(c) Now we have red, blue, green and yellow. There's 4 single colour colourings. Then we have 6 pairs of colours, so that's 3×6 colourings with exactly 2 colours. Then there's 4 ways of choosing 3 colours, so that's 4×3 colourings with exactly 3 colours. Then colouring each face a different colour, there are only 2 of those (think about flipping the tetrahedron). That makes 36 all up.

4. (a) Triangles ABC and CDB are similar so $\frac{BC}{DB} = \frac{AC}{BC}$, and the quadratic follows.

   (b) $\cos 72^\circ = \frac{\frac{x}{2}}{4}$ so use the quadratic formula and part (a) to get x.

   (c) Similarly note $\cos 36^\circ = \frac{\frac{4}{2}}{x}$.

5. (a) They need 2 extra people. Have Fry swap with the Amy-body and Leela with the Professor-body. Then Fry-body and Professor-body can swap, Amy-body and Leela-body can swap, and Fry-body and Leela-body can swap. This sets everything right.

   (b) You always only need, at most, 2 extra people. Remembering our lesson on k-cycles from a few weeks ago, we suppose that the muddle we have ourselves in can be represented as the permutation

$$\pi = (1\ 2\ 3\ 4\ \cdots\ n),$$

which means that person 1 went to body 2, person 2 went to body 3 and so on, with person n going to body 1. We now want to multiply this permutation with another, which doesn't break our rules, and gives us the identity permutation, $(1)(2)(3)\cdots(n)$. Here's how we do it, introduce our two extra people $x = n + 1$ and $y = n + 2$, then pick an i between 1 and $n - 1$. Let

$$\sigma = (x\ i+1)(y\ 1)(y\ n)(y\ n-1)\cdots(y\ i+1)(x\ i)(x\ i-1)\cdots(x\ 2)(x\ 1).$$

So this is a rule for putting people into the mind-body swapping machine, starting on the left, swap body xx with body $i + 1$, swap body y with body 1, then body y with body n, body y with body $n - 1$ and so on. Note that we can't have previously swapped body x or y with any of the 1 through n bodies so all the swaps in σ are allowed. Also we're not swapping x with y. So now we multiply π and σ

$$\begin{aligned}\pi\sigma &= (1\ 2\ \cdots\ n)(x\ i+1)(y\ 1)(y\ n)(y\ n-1)\cdots(y\ i+1)(x\ i)(x\ i-1)\cdots(x\ 2)(x\ 1)\\ &= (1)(2)\cdots(i)\cdots(n)(x\ y).\end{aligned}$$

We've put everyone back in the right bodies, except x and y so we finish by performing $(x\ y)$.