



## MATHEMATICS ENRICHMENT CLUB.

### Solution Sheet 10, July 29, 2014<sup>1</sup>

1. To find the expected value we must sum  $\sum sp_s$  where  $s$  is the score and  $p_s$  is the probability of getting that score. The probability of a random dart thrower hitting a region is the fractional area of that region. The expected value,  $\bar{s}$ , is then

$$\bar{s} = 50p_{\text{bullseye}} + 25p_{\text{bull's ring}} + 1p_{1 \text{ inner wedge}} + 18p_{18 \text{ inner wedge}} + \dots$$

and so on.

This might look like a long sum, but the inner “wedges” all have the same area and cover the numbers 1 through 20, the triples all have the same area and cover the numbers 1 through 20 and so on for the outer “wedges” and doubles also. That is, the terms which contribute the inner “wedge” probabilities will look like  $1p_{\text{inner wedge}} + 2p_{\text{inner wedge}} + 3p_{\text{inner wedge}} + \dots = (1 + 2 + \dots + 20)p_{\text{inner wedge}}$ , where

$$p_{\text{inner wedge}} = \frac{\frac{1}{20}\pi(31^2 - 3^2)(R/50)^2}{\pi R^2} = \frac{31^2 - 3^2}{20 \times 50^2}$$

So

$$\begin{aligned} \bar{s} &= 50 \frac{1}{50^2} + 25 \frac{3^2 - 1}{50^2} + (1 + \dots + 20) \frac{31^2 - 3^2}{20 \times 50^2} + 3(1 + \dots + 20) \frac{33^2 - 31^2}{20 \times 50^2} + \\ &\quad (1 + \dots + 20) \frac{48^2 - 33^2}{20 \times 50^2} + 2(1 + \dots + 20) \frac{50^2 - 48^2}{20 \times 50^2} \\ &= \frac{1}{50^2} \left( 50 + 25(3 - 1)(3 + 1) + \frac{20}{2}(1 + 20) \frac{1}{20}(31 - 3)(31 + 3) + \right. \\ &\quad \left. 3 \frac{20}{2}(20 + 1) \frac{1}{20}(33 - 31)(33 + 31) + \frac{20}{2}(20 + 1) \frac{1}{20}(48 - 33)(48 + 33) + \right. \\ &\quad \left. 2 \frac{20}{2}(20 + 1) \frac{1}{20}(50 - 48)(50 + 48) \right) \\ &= \frac{1}{2500} \left( 50 + 200 + \frac{1}{2} \times 21 \times 28 \times 34 + \frac{3}{2} \times 21 \times 2 \times 64 + \frac{1}{2} \times 21 \times 15 \times 81 + 21 \times 2 \times 98 \right) \\ &= \frac{1}{2500} \left( 50 + 200 + 21 \times 28 \times 17 + 3 \times 21 \times 64 + \frac{1}{2} \times 21 \times 15 \times 81 + 21 \times 2 \times 98 \right) \\ &= \frac{31\,151.5}{2500} = 12.4606. \end{aligned}$$

<sup>1</sup>Some problems from UNSW's publication *Parabola*, some from *Project Euler*

Each new throw is independent of the last, so the expected score of 3 throws is thrice the expected score of one, i.e.  $3\bar{s} = 37.3818$ .

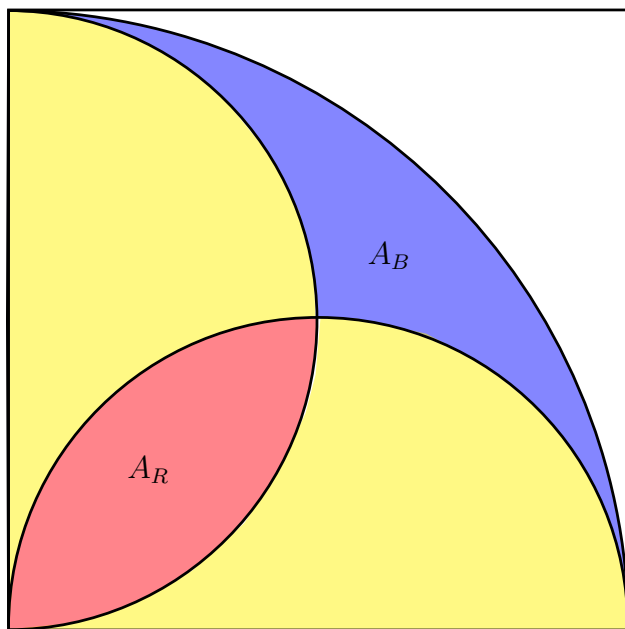


Figure 1: Picture for question 2

2. Let the side of the square be  $r$ , then  $\frac{1}{4}\pi r^2$  is the area of the larger quarter circle. So

$$\begin{aligned} \frac{1}{4}\pi r^2 &= A_B + 2\frac{1}{2}\pi \left(\frac{r}{2}\right)^2 - A_R \\ &= A_B + \frac{1}{2}\pi r^2 - A_R \\ 0 &= A_B - A_R \\ A_R &= A_B \end{aligned}$$

so the blue and red regions' areas are in ratio 1 : 1.

3. The first number in the lexicographic ordering of arrangements of the digits 0 through 9 is 0123456789. If we want to find the millionth we need to take  $1\,000\,000 - 1 = 999\,999$  steps down the ordering. If we fix the first digit, there are  $9!$  ways of arranging the remaining 9. So we want to find how many lots of  $9!$  we need to step to get at least to 999 999, so we should take the smallest integer so that  $999\,999 - n \times 9! < 0$ , which means  $n = \lceil 999\,999/9! \rceil = 3$ . The third digit in our list is 2, so the millionth number must start with a 2. Now the numbers 2... start at the  $2 * 9! + 1$ th spot, so we have  $1\,000\,000 - 2 * 9! - 1 = 274\,239$  spots to make up with lots of  $8!$  (fixing the first two digits gives us  $8!$  ways of arranging the rest), so  $\lceil 274\,239/8! \rceil = 7$ . The seventh number in our list (remembering that 2 has already been used) is 7, so we're up to 27.... Continuing:

Number	Spots to make up	$n$
27...	$1\,000\,000 - (2 * 9! + 6 * 8! + 1) = 32\,320$	$\lceil 32\,320/7! \rceil = 7$
278...	$1\,000\,000 - (2 * 9! + 6 * 8! + 6 * 7! + 1) = 2\,079$	$\lceil 2\,079/6! \rceil = 3$
2783...	$1\,000\,000 - (2 * 9! + 6 * 8! + 6 * 7! + 2 * 6! + 1) = 639$	$\lceil 639/5! \rceil = 6$
27839...	39	$\lceil 39/4! \rceil = 2$
278391...	15	$\lceil 15/3! \rceil = 3$
2783915...	3	$\lceil 3/2! \rceil = 2$
27839154...	1	

So 2783915406 is the 999 999th number in the list because it is the first number starting with 27839154..., we have one more spot to make up, so the millionth number is 2783915460.

4.

5.

6. Note that  $792 = \text{lcm}(88, 99)$ . Then

$$\begin{aligned} \left( (88!)^{\frac{1}{88}} \right)^{792} &= (88!)^9 \\ \left( (99!)^{\frac{1}{99}} \right)^{792} &= (99!)^8. \end{aligned}$$

Using this, let's take

$$\begin{aligned} \frac{(99!)^8}{(88!)^9} &= \left( \frac{99!}{88!} \right)^8 \frac{1}{88!} \\ &= \frac{(99 \times 98 \times \dots \times 89)^8}{88 \times 87 \times \dots \times 2 \times 1} \\ &= \frac{(99 \times 98 \times \dots \times 89) \times (99 \times \dots \times 89) \times \dots \times (99 \times \dots \times 89)}{88 \times 87 \times \dots \times 2 \times 1}. \end{aligned}$$

Now looking at the above fraction, the top has  $8 \times (99 - 89 + 1) = 88$  numbers multiplied together, as does the bottom, only all the numbers on in the numerator are larger than all the numbers in the denominator, so this fraction must be greater than 1. Then

$$(99!)^8 > (88!)^9.$$

Taking the 792nd root of both sides is ok because both numbers are greater than zero, so

$$\left( (99!)^8 \right)^{\frac{1}{792}} > \left( (88!)^9 \right)^{\frac{1}{792}}$$

or rather

$$\sqrt[99]{99!} > \sqrt[88]{88!}.$$