UNSW AUSTRALIA

Never Stand Still | Science | School of Mathematics and Statistics

MATHEMATICS ENRICHMENT CLUB.
Hint Sheet 12, June 24, 2014[1]

1. (a) Write 30! in its prime factorisation and divide by $10^7 = 2^7 \times 5^7$.
   (b) Think about 1000!'s prime factorisation - how many pairs of 2s and 5s are there?

2. (a)
   (b)
   (c) If our original triangle is ABC, then the triangle at step $n-1$ looks like a zoomed in version of the triangle connecting A, the midpoint of AB and the midpoint of AC in step n. Use this to build a recurrence relation.
   (d) Each step removes 1/4 of the current amount of red. **Extra:** What does this mean in the limit as $n \to \infty$? Does this imply there's no red left?

3. Perfect squares are either 0 or 1 $\pmod 4$. Can all of $2d-1 \pmod 4$, $5d-1 \pmod 4$ and $13d-1 \pmod 4$ all be 1 or 0?

4. Suppose a and b satisfy $(a+36b)(b+36a) = 2^n$ for some positive integer n. First notice that a and b have to be even, then replace a and b with $2a'$ and $2b'$ and notice that the same condition is satisfied by (a', b'), but for $n' = n-2$. Since pairs of odd numbers don't satisfy the condition we see that no pairs of positive integers can.

5. The number of triangles is the number of ways of choosing 3 points from $2n$, i.e. $\binom{2n}{3}$. The number of rectangles is the number of ways of choosing 2 diameters from the n total (because the points are equally spaced around the circle), i.e. $\binom{n}{2}$. There are 20 times more triangles than rectangles so make an equation and solve for n.

6. The quadrilateral formed by the midpoints of $ABCD$ is a parallelogram and the diagonals of a parallelogram bisect each other. One can now prove that with the perimeter condition that the opposite sides of $ABCD$ must be equal length, and hence $ABCD$ is a parallelogram.

7. (a) What if you get all the guests to move over one room, so the guest in room 1 moves to room 2, the guest in room 2 moves to room 3, and so on. Now room 1 is free.

[1] Some problems from UNSW's publication *Parabola*

What if you get the guests to move to a room double their room number, so the guest in room 1 moves to room 2, then guest in room 2 moves to room 4 and so on. Now all the odd numbered rooms are free and we can accomodate an infinite number of guests.

(b) Think about this in two ways. If the probability is "the number of perfect squares divided by the number of numbers", then the probability is one. Every positive integer can be matched with one and only one perfect square by squaring it, so there are the same number of perfect squares as there are numbers (they are paired up exactly one-to-one).

But what if we look at the probability of choosing a perfect square less than n, well the number of perfect squares less than or equal to n is $\lfloor\sqrt{n}\rfloor$ and the number of numbers less than or equal to n is n. so $p < 1/\sqrt{n}$ which goes to 0 as n tends to infinity.

So what is it? Probability 1, 0 or is this way of thinking about probability a little flawed? Why?

(c) If we're running twice as fast as Zeno, then by the time we run the first $10m$, Zeno has run 5, by the time we run those $5m$, Zeno has run a further 2.5, and so on. The question then is, is the sum $10 + 5 + 2.5 + 1.25 + \cdots$ finite or not? The ancient Greeks, from whom Zeno first ran, didn't think it could be finite because there's an infinite number of numbers in the sum.

If we just sum up the first N terms, $\sum_{j=0}^{N-1} 10\left(\frac{1}{2}\right)^j$, we see we've got a geometric series and we get $\frac{10\left(1-\left(\frac{1}{2}\right)^N\right)}{\frac{1}{2}} = 20\left(1 - \frac{1}{2^N}\right)$. As $N \to \infty$ this number goes to 20, which is finite. But is this right way to think about it? Compare it your intuition about whether you can catch Zeno by running twice as fast as him.

(d) If Chef Russell doesn't cook dinner for himself, then he goes to *L'échec* and gets Chef Russell to cook his dinner. This is the same as the sentence "This statement is false" – if it's true it's false and if it's false it's true! This is, in essence, how Gödel's incompleteness theorem works – in certain logic systems there will necessarily by propositions that are neither true nor false but unprovable. The example with the restaurant is what suggested that Bertrand Russell's set theory needed some work more than a century ago.