

MATHEMATICS ENRICHMENT CLUB.

Problem Sheet 2, May 13, 2014¹

Senior Questions

1. In base two, f amounts to simply shifting all the digits to the left and crossing off that which ends up to the left of the period. For example,

$$f(0.11010100101101\dots_2) = 0.1010100101101\dots_2.$$

A periodic point then is one whose dyadic (base 2) expansion repeats itself. An example might be $x_0 = 0.\overline{101}_2$, since $x_1 = 0.\overline{010}_2$ and $x_2 = 0.\overline{101}_2 = x_0$, thus it has period 2. To expand $0.\overline{101}_2$ in base 10, let $y = 0.\overline{101}_2$, then $4y = 10.\overline{101}_2$ so $4y - y = 3y = 10_2 = 2$ giving the answer $y = \frac{2}{3}$.

2. A rational number, written in its dyadic expansion, either repeats itself, or terminates eventually (i.e. after some N digits, all the rest are zero). We've already seen that the first kind are periodic, so a non-periodic, rational number is one whose dyadic expansion is eventually all zeroes. Since f shifts all the digits left and chops off whatever is left of the period, this means that if x_0 is non-periodic and rational with a representation N digits long then $x_N = x_{N+1} = x_{N+2} = \dots = 0$. So $\lim_{n \rightarrow \infty} x_n = 0$.
3. This one is the really interesting one. We've seen what happens to rational numbers - either they come back to where they start, or they end up at zero. So what happens to all the other numbers?

Let's begin by exploring the situation where a and b are rational, meaning either their dyadic expansion repeats or is finite. Suppose also that the first k digits of a and b are the same - let's write them as $a = 0.a_1a_2a_3\dots$ and $b = 0.b_1b_2b_3\dots$. Now if the $(k+1)$ th digit of a is a 1, then the $(k+1)$ th digit of b must be zero (because only the first k digits are the same). But this would mean that a is larger than b , so this cannot be. Thus we have something like

$$a = d_1d_2\dots d_k0a_{k+2}a_{k+3}\dots$$

and

$$b = d_1d_2\dots d_k1b_{k+2}b_{k+3}\dots$$

¹Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

From this, let's make a finitely long sequence starting with all the d_k , then a zero, then the digits of a , $a_{k+2}a_{k+3} \dots$ up the next zero, and terminate our sequence by replacing this zero with a 1. That is

$$0.d_1d_2 \dots d_k0a_{k+2}a_{k+3} \dots 1.$$

This number is identical to a up to that final 1 (where a has a 0) so it is bigger than a . In fact, adding any digits to the end of it, we get a number bigger than a . It is also smaller than b since it shares the first k digits with b but then has a 0 where b has a 1, and again adding any digits to the end of it we still get a number smaller than b . What we've done here is characterised all the possible numbers between a and b as starting with this sequence.

Now I haven't been strictly accurate with the statement of the problem, I should not have said "any" irrational number but "almost all" irrational numbers – which has a strict mathematical meaning which you won't learn about until late in university (but basically means if you chose an irrational number at random, then this works with probability 1 – have a think about what that means when there are infinitely many irrational numbers!). Now it turns out that "almost all" irrational numbers are infinitely long strings which contain any string of finite length somewhere in their representation, and so contain our string $d_1d_2 \dots d_k0a_{k+2}a_{k+3} \dots 1$, so at some point in the sequence x_0, x_1, x_2 we get a number that begins with our string, which we've said means it is between a and b !

To finish things here, we started by assuming a and b were rational, so what if they aren't? Well for any irrational number we can get a rational number that is arbitrarily close to it, so choose two new rational bounds, a' and b' so that $a < a' < b' < b$, then apply the above argument to them.