MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 3, May 20, 2014[1]

1. Suppose the two numbers were $a < b$ and the incorrect result c. Then $ab - 70 = c$ and $\frac{c}{a} = 48 + \frac{17}{a}$. So $c = 48a + 17$ which means $ab = 48a + 17 + 70$ or $a(b - 48) = 87$. The only two factors of 87 (that aren't 1 or 87) are 3 and 29, so a must be one and $b - 48$ the other. Since both a and b are two digit numbers, we must have $a = 29$ and $b = 51$.

2. Rearranging gives
$$\frac{1}{x} + \frac{1}{y} = \frac{1}{x+y+z} - \frac{1}{z}$$
or
$$\frac{x+y}{xy} = \frac{-(x+y)}{z(x+y+z)}.$$
So we'd have to satisfy
$$\begin{aligned}(x+y)(z(x+y+z)) &= -xy(x+y)\\ (x+y)(zx+zy+z^2+xy) &= 0\\ (x+y)(x(z+y)+z(y+z)) &= 0\\ (x+y)(z+y)(x+z) &= 0\end{aligned}$$
So we'd need to have either $x = -y$, $z = -y$ or $x = -z$ provided none are zero. In each case, the number not specified is free to be whatever it pleases, i.e. solutions are $x = \alpha, y = -\alpha, z = \beta$, $x = \beta, y = \alpha, z = -\alpha$ or $x = -\alpha, y = \beta, z = \alpha$ for α, β real numbers.

3. Let's wite $N = 100a + 10b + c$ or $[abc]$ for short. The five numbers that can be obtained by permuting the digits are $[acb], [bac], [bca], [cab]$ and $[cba]$. We know that
$$\frac{1}{5}([acb] + [bac] + [bca] + [cab] + [cba]) = N,$$
and further, inlcuding the mean in a set of data doesn't change the mean, so we can expand the above to
$$\frac{1}{6}([abc] + [acb] + [bac] + [bca] + [cab] + [cba]) = N.$$

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

By adding up the left hand side here we obtain

$$\frac{1}{6}(2(a+b+c)\times 100 + 2(a+b+c)\times 10 + 2(a+b+c)) = N$$

or

$$111(a+b+c) = 3N \quad \text{or } N = 37(a+b+c).$$

Since $N < 500$, $a+b+c \leq 13$. Also, a, b, c must be distinct and nonzero, so must be at least as great as $1+2+3=6$. The multiples of 37 in this range are $37 \times 6 = 222$, $37 \times 7 = 259$, $37 \times 8 = 296$ and so on. It turns out that the only one with distinct digits whose digit-sum equals the factor is for $37 \times 13 = 481$.

4. If we try the codes

1	2	3	4	5	6	7
7	1	2	3	4	5	6
6	7	1	2	3	4	5
5	6	7	1	2	3	4
4	5	6	7	1	2	3
3	4	5	6	7	1	2
2	3	4	5	6	7	1

then we have tried 1 in every spot, 2 in every spot, 3 in every spot and so on up to 7. If the safe still isn't open this means that the code contains no 1s, no 2s, no 3s and so on up to 7. That is, the code only contains 8s, 9s and 0s, but this isn't enough digits to have 7 distinct digits so the safe must have opened at some point.

5. Drawing the radii from the centre to each of the vertices of the regular n-gon makes wedges each with an angle of $2\pi/n$ at the centre. Thus the side lengths of the n-gon must be $2r \sin \frac{\pi}{n}$ where r is the radius of the circle. The side AD makes up a triangle across three of these wedges, so its length is $2r \sin \frac{3\pi}{n}$. Since AD is the side length plus the radius

$$2r \sin \frac{\pi}{n} + r = 2r \sin \frac{3\pi}{n}.$$

We can expand the right hand side using double angle formulae and we end with a cubic polynomial in $\sin \frac{\pi}{n}$. Let's call this x, the polynomial is

$$8x^3 - 4x + 1 = 0.$$

Being lucky, we guess first that $x = \frac{1}{2}$ is a solution of this and use long division to find the other factor, obtaining

$$\left(x - \frac{1}{2}\right)(8x^2 + 4x - 2) = 0.$$

Solving the quadratic using the quadratic formula, we get that $x = \frac{-1\pm\sqrt{5}}{4}$. From here we can use calculators, guess and check or plain ingenuity to find that the ns that satisfy

$$\sin\frac{\pi}{n} = \frac{1}{2}, \qquad \text{or } \frac{-1\pm\sqrt{5}}{4}$$

are 6 or 10.

6. Wikipedia has the best possible answer for this one:

> The original (333) Rubik's Cube has eight corners and twelve edges. There are 8! (40,320) ways to arrange the corner cubes. Seven can be oriented independently, and the orientation of the eighth depends on the preceding seven, giving 37 (2,187) possibilities. There are 12!/2 (239,500,800) ways to arrange the edges, since an even permutation of the corners implies an even permutation of the edges as well. (When arrangements of centres are also permitted, as described below, the rule is that the combined arrangement of corners, edges, and centres must be an even permutation.) Eleven edges can be flipped independently, with the flip of the twelfth depending on the preceding ones, giving 211 (2,048) possibilities.
>
> $$8! \times 3^7 \times (12!/2) \times 2^{11} = 43,252,003,274,489,856,000$$
>
> which is approximately 43 quintillion.

`http://en.wikipedia.org/wiki/Rubik's_Cube#Permutations`

Senior Questions

1. Since f and g are differentiable at x_0 this means the limits $\lim_{h\to 0}(f(x_0+h)-f(x_0))/h$ and $\lim_{h\to 0}(g(x_0+h)-g(x_0))/h$ exist. Consider the difference quotient

$$\begin{aligned}
\frac{(fg)(x_0+h) - fg(x_0)}{h} &= \frac{f(x_0+h)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&= \frac{f(x_0+h)g(x_0+h) + f(x_0)g(x_0+h) - f(x_0)g(x_0+h) - f(x_0)g(x_0)}{h} \\
&= \frac{g(x_0+h)\left(f(x_0+h) - f(x_0)\right) + f(x_0)\left(g(x_0+h) - g(x_0)\right)}{h} \\
&= g(x_0+h)\frac{f(x_0+h) - f(x_0)}{h} + f(x_0)\frac{g(x_0+h) - g(x_0)}{h}.
\end{aligned}$$

Taking the limit of this as $h \to 0$ exists since each part does – $g(x_0+h) \to g(x_0)$ since g is continuous, and the two fractions exist because f and g are differentiable.

The case for f/g is similar, so I'll leave it out.

2. We know that

$$\beta = \frac{g(x_0+h) - g(x_0)}{h} - g'(x_0) \to 0$$

as $h \to 0$. We also know that

$$\alpha = \frac{f(y+k) - f(y)}{k} - f'(y) \to 0$$

as $k \to 0$, where $y = g(x_0)$. Lets' rewrite

$$g(x_0 + h) = h(\beta + g'(x_0)) + g(x_0)$$

then

$$f(g(x_0 + h)) = f(h(\beta + g'(x_0)) + g(x_0)).$$

Call $k = h(\beta + g'(x_0))$ then the above is just $f(k + y)$. So

$$\frac{f(k+y) - f(y)}{h} = \frac{h(\beta + g'(x_0))(\alpha + f'(y))}{h}$$

using the previous expressions. As $h \to 0$, $k \to 0$ so $\alpha \to 0$ too. Further, $\beta \to 0$ so the above tends to $g'(x_0)f'(y) = g'(x_0)f'(g(x_0))$ which is the product rule, but more importantly, the limit of the above as $h \to 0$ exists.

3. Since fg and f/g are differentiable, then $fg \times f/g$ is differentiable by question 1 above. So $f(x)^2$ is differentiable at x_0. The function $h(x) = \sqrt{x}$ for $x > 0$ is differentiable, so $h(f(x)^2) = f(x)$ is differentiable provided $f(x_0) \neq 0$.

   If $f(x_0) = 0$ then consider

$$\frac{f(x_0 + h)}{h} = \frac{f(x_0 + h)g(x_0 + h)}{g(x_0 + h)h}$$

   as $h \to 0$, $g(x_0 + h) \to g(x_0)$ and $\lim_{h \to 0}(fg)(x_0 + h)/h$ exists, so $\lim_{h \to} f(x_0 + h)/h$ does too.