MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 5, June 3, 2014[1]

1. To make up 100 we can assume that all the numbers we sum are two digit, if even one was three digit there would be no way to sum to 100. Let's say we write the numbers as $10a_i + b_i$, where the a_i and b_i follow the condition that all digits 0-9 be used exactly once. Now let the sum of the a_i be A and the sum of the b_i be B. The value of the sum is then
$$10A + B = n$$
but we also know that
$$A + B = 45$$
just by adding all the numbers from 0 to 9. By subtracting the second from the first we see that
$$9A = n - 45$$
and since A is an integer, $n - 45$ must be divisible by 9. It is then not possible for $n = 100$ as 55 is not divisible by 9. Furthermore, since 45 is divisible by 9, we know that n can only be multiples of 9 - at least when we only sum two digit numbers, is this true in general?

2. A fraction cannot be further simplified if its numerator and denominator are coprime, i.e. their greatest common divisor is 1. We can follow Euclid's algorithm to find $\gcd(21n + 4, 14n + 3)$. Euclid's algorithm relies mostly on the fact that if $a < b$ then $\gcd(a, b) = \gcd(a, b - a)$. So
$$\begin{aligned}\gcd(21n + 4, 14n + 3) &= \gcd(14n + 3, 7n + 1)\\ &= \gcd(7n + 1, 7n + 2)\\ &= \gcd(7n + 1, 1).\end{aligned}$$
Now the greatest common divisor of a and b must always be smaller than a and b, and the only divisor of 1 is 1, so $\gcd(x, 1) = 1$ for all x. Looking above we see that $\gcd(21n + 4, 14n + 3) = \gcd(7n + 1, 1) = 1$ so $21n + 4$ and $14n + 3$ are coprime, and the original fraction cannot be simplified.

[1] Some problems from UNSW's publication *Parabola*

3. Draw the base of the triangle as AB. The base, AB, is made up of 3 segments x, y and z. The middle, y, is an edge of one of the inner triangles while the outer two are equal length to sides of the two other inner triangles, as they are sides of parallelograms which are each opposite an inner triangle. All inner triangles are similar to each other, and since their areas are in ratio $12 : 27 : 75$ or rather $4 : 9 : 25$ then their sides are in ratio $2 : 3 : 5$. Inlcuding the larger triangle in these sides ratios all 4 triangles have their sides in ratio $2 : 3 : 5 : 10$, and so their areas are $4 : 9 : 25 : 100$. So if the inner triangles have area 12,27 and 75 the larger triangle has area 300.

4. First note that we can buy exactly $10 = 5 + 5$ donuts. Now if figure out the 9 ways of buying donuts so we get exactly numbers ending with each digit, we can figure out bigger orders by simply adding 10 to them.

$$\begin{array}{r|c}
x1 & 31 = 13 + 13 + 5 \\
x2 & 22 = 13 + 9 \\
x3 & 13 = 13 \\
x4 & 14 = 9 + 5 \\
x5 & 5 = 5 \\
x6 & 26 = 13 + 13 \\
x7 & 27 = 9 + 9 + 9 \\
x8 & 18 = 13 + 5 \\
x9 & 9 = 9
\end{array}$$

Subtracting 10 from each of these, and taking the largest, we get the largest number that can't be bought exactly, which is 21.

This is known as the Frobenius coin problem (well, I guess here it is the Frobenius donut problem). What we've found is that $\text{Frobenius}(5, 9, 13) = 21$. The Frobenius donut problem can be rephrased with any number of donut packet sizes, and as it turns out, if they're in arithmetic progression, as they here, there's an exact formula

$$\text{Frobenius}(a, a + d, \dots, a + nd) = \left(\left\lfloor \frac{a-2}{n} \right\rfloor + 1\right) a + (d - 1)(a - 1) - 1,$$

where $\lfloor x \rfloor$ is the largest integer smaller than x. Here we have $a = 5$, $d = 4$ and $n = 2$, which gives

$$\text{Frobenius}(5, 9, 13) = \left(\left\lfloor \frac{5-2}{2} \right\rfloor + 1\right) \times 5 + (4 - 1)(5 - 1) - 1 = 2 \times 5 + 3 \times 4 - 1 = 21.$$

5. (a) Let's look at the left hand side first

$$\begin{aligned}
a_n^2 - a_{n-1}a_{n+1} &= a_n^2 - a_{n-1}(a_n + a_{n-1}) \\
&= a_n^2 - a_{n-1}a_n - a_{n-1}^2 \\
&= -a_{n-1}^2 + a_n(a_n - a_{n-1}) \\
&= -a_{n-1}^2 + a_n a_{n-2} \\
&= (-1)(a_{n-1}^2 - a_n a_{n-2}),
\end{aligned}$$

where we have used $a_{n+1} = a_n + a_{n-1}$ in the first line and rearranged $a_n = a_{n-1} + a_{n-2}$ to get $a_n - a_{n-1} = a_{n-2}$ in the third line. What we have is that if you index shift $n \to n-1$ you end up with a (-1) out the front. If we repeat this $n-2$ times, we end up with

$$\begin{aligned} a_n^2 - a_{n-1}a_{n+1} &= (-1)^{n-2}(a_2^2 - a_3 a_1) \\ &= (-1)^{n-2}(1-2) = (-1)^{n-1}. \end{aligned}$$

(b) First, note that $a_1 = 1$ is odd, $a_2 = 1$ is odd, so $a_3 = a_2 + a_1$ is odd plus odd, and so is even. Then a_4 is even plus odd, so is odd, a_5 is odd plus even, so is again odd, and a_6 is again an odd plus odd so is even. The pattern continues. Those who know how can formalise this argument using induction.

(c) Starting with the right hand side,

$$\begin{aligned} a_{r+1}a_{n-1} + a_r a_{n-r-1} &= (a_r + a_{r-1})a_{n-r} + a_r a_{n-r-1} \\ &= a_r a_{n-r} + a_{r-1}a_{n-r} + a_r a_{n-r-1} \\ &= a_r(a_{n-r} + a_{n-r-1}) + a_{r-1}a_{n-r} \\ &= a_r a_{n-r+1} + a_{r-1}a_{n-r}. \end{aligned}$$

With this rule, we can repeat it $r-1$ times and get

$$a_{r+1}a_{n-1} + a_r a_{n-r-1} = a_2 a_{n-1} + a_1 a_{n-2}$$

and since $a_2 = a_1 = 1$, the right hand side there is a_n.

(d) Using the above we can write

$$a_n = a_{k+1}a_{n-k} + a_k a_{n-k-1}.$$

Now a_k and a_{k+1} are relatively prime ($\gcd(a_k, a_{k+1}) = \gcd(a_k, a_{k+1} - a_k) = \gcd(a_k, a_{k-1})$ - repeating this we end with $\gcd(a_2, a_1) = 1$), so a_n is divisible by a_k if a_{n-k} is. We can repeat, stating

$$a_{n-k} = a_{k+1}a_{n-k-k} + a_k a_{n-k-k-1}$$

so we must show that a_{n-2k} is divisible by a_k, and a_{n-3k}, a_{n-4k} and so on.

Suppose $n = kq + r$ with $0 \leq r < k$, then a_k divides a_n if and only if it divides $a_{n-kq} = a_r$, but since $r < k$, $a_r < a_k$. However, if $r = 0$ then $a_k = a_{n-(q-1)k}$ and so $a_{n-(q-1)k}$ is divisible by a_k and so a_{n-k} and a_n are too.

Senior Questions

1. (a) Using the rule for addition

$$[A+B]_{ij} = [A]_{ij} + [B]_{ij} = [B]_{ij} + [A]_{ij} = [B+A]_{ij}.$$

(b) Similarly

$$[A+(B+C)]_{ij} = [A]_{ij}+[B+C]_{ij} = [A]_{ij}+[B]_{ij}+[C]_{ij} = [A+B]_{ij}+[C]_{ij} = [(A+B)+C]_{ij}.$$

2. (a) Using the rule for matrix multiplication

$$\begin{aligned}[A(BC)]_{ij} &= \sum_{k=1}^{n}[A]_{ik}[BC]_{kj} \\ &= \sum_{k=1}^{n}\sum_{l=1}^{n}[A]_{ik}[B]_{kl}[C]_{lj} \\ &= \sum_{l=1}^{n}[AB]_{il}[C]_{lj} \\ &= [(AB)C]_{ij}\end{aligned}$$

(b) Similarly

$$[AB]_{ij} = \sum_{k=1}^{n}[A]_{ik}[B]_{kj}$$

and

$$[BA]_{ij} = \sum_{k=1}^{n}[B]_{ik}[A]_{kj}$$

but since $[B]_{ik}$ does not necessarily equal $[B]_{kj}$ the two above are not equal. Take for example

$$\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix}\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix} = \begin{pmatrix}1\times1+1\times1 & 1\times0+1\times0\\ 0\times1+0\times1 & 0\times0+0\times0\end{pmatrix} = \begin{pmatrix}2 & 0\\ 0 & 0\end{pmatrix}$$

while

$$\begin{pmatrix}1 & 0\\ 1 & 0\end{pmatrix}\begin{pmatrix}1 & 1\\ 0 & 0\end{pmatrix} = \begin{pmatrix}1\times1+0\times0 & 1\times1+0\times0\\ 1\times1+0\times0 & 1\times1+0\times0\end{pmatrix} = \begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}.$$

3. Looking at the commutation equation for mulitplication above, let's let $B = I$. We need

$$\sum_{k=1}^{n}[A]_{ik}[I]_{kj} = A_{ij}$$

and

$$\sum_{k=1}^{n}[I]_{ik}[A]_{kj} = A_{ij}.$$

If we have every entry of I equal to zero, except $[I]_{ii} = 1$ for $i = 1, \ldots, n$ we can see that the above two are satisfied, so

$$[I]_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases},$$

or

$$I = \begin{pmatrix} 1 & 0 & \cdots & & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & 0 \\ & & & & \\ 0 & \cdots & & 0 & 1 \end{pmatrix}$$

works. Is this the only one? Suppose there is another one I_2 which works as well, then

$$II_2 = I_2 I = I$$

but we've already seen that $I_2 I = II_2 = I_2$ so $I_2 = I$. That is, yes, this is the only possible I.