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MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 8, June 24, 2014[1]

1. To count the number of possible passwords for website B, we simply need to look a the number of ways of arranging n symbols when one can choose from 62 symbols (26 lower case letters, 26 upper case letters and the 10 numbers 0-9). To count these, the first symbol has 62 options, the second also has 62, the third has 62 and so on. So for a n digit password, there are 62^n possibilities. Seeing as we can have *up to* 6 symbols, the total number is

$$\#(B) = 62^6 + 62^5 + 62^4 + 62^3 + 62^2 + 62 = 57\ 731\ 386\ 986.$$

To count the number of possible passwords for website A we shall count the number of 6, 7 and 8 symbol passwords separately. The 6 symbol passwords must have exactly 3 letters and 3 numbers. So there are 26^3 possiblities for the letters and 10^3 possibilities for the numbers. Now we must choose how we order the two classes of symbols: there are 6 possible spots to put the 3 letters so there are $\binom{6}{3} = 20$ possible ways of choosing spots which will have a letter and the rest must necessarily have a number. So the number of 6 symbol passwords for website A is

$$\#(A; 3, 3) = 26^3 10^3 \binom{6}{3}$$

where $\#(A; n, m)$ is the number of passwords with n letters and m numbers. We can write out a formula for this as

$$\#(A; n, m) = 26^n 10^m \binom{n+m}{n}\binom{m}{m} = 26^n 10^m \binom{n+m}{n}.$$

Now for 7 symbol passwords, we could either have 4 letters ($\#(A; 4, 3)$) or 4 numbers ($\#(A; 3, 4)$). For 8 symbols we could need to compute $\#(A; 5, 3)$, $\#(A; 4, 4)$ and $\#(A; 3, 5)$.

So

$$\begin{aligned}\#(A) &= \#(A; 3, 3) + \#(A; 3, 4) + \#(A; 4, 3) + \#(A; 3, 5) + \#(A; 4, 4) + \#(A, 5, 3)\\ &= 351\ 520\ 000 + 6\ 151\ 600\ 000 + 15\ 994\ 160\ 000 \ldots\\ &= 1\ 106\ 163\ 136\ 000.\end{aligned}$$

[1] Some problems from UNSW's publication *Parabola*

So website A is more secure (has more possible passwords).

Going beyond the actual question though, we might ask is this because of website A's passwords are more complicated or merely longer? In fact, let's write $\#_B(n)$ and $\#_A(n)$ as the number of n-symbol passwords for websites B and A respectively, then (using my computer) I found that

$$\#_B(n) \approx 10^{0.23n} \#_A(n)$$

meaning that the number of n-symbol passwords for website B absolutely dwarfs the number of A's, and with each extra symbol gets exponentially more passwords than A does. The moral of the story, dear reader: use more symbols not more rules.

2. First, let's note that to end up heads up, the coin would have to be flipped an odd number of times. Take the nth coin in the line, how many times does it get flipped? Well every coin gets flipped on the first pass, but only every second on the second pass – that is, only those coins whose position is divisible by 2. Similarly on the 3rd pass only those coins sitting as positions which are a multiple of 3 get flipped. So the nth coin will get flipped on the mth pass if m is a factor of n.

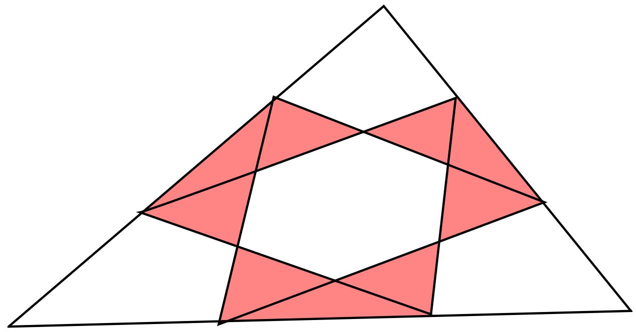
   If n is written as its prime factorisation $n = p_1^{d_1} p_2^{d_2} \cdots p_k^{d_k}$ then any factor must be able to be written as $p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}$ where $0 \leq c_i \leq d_i$. So the number of factors of n is the number of possibilities of choosing the c_i. There are $d_i + 1$ choices for c_i $(0, 1, 2, \ldots, d_i)$ so in total n has
$$(d_1 + 1)(d_2 + 1)(d_3 + 1) \cdots (d_k + 1)$$
   factors.

   For n to have an odd number of factors every term above must be odd, that is all $(d_i + 1)$. Which means every d_i must be even, implying that n is a perfect square. So the coins which end up heads up are those that are positioned at a perfect square, i.e. $1, 4, 9, 16, \ldots$. The square root of 1000 is $31.62277\ldots$, so 31^2 is the largest perfect square less than 1000. So there are 31 perfect squares less than 1000, and 31 coins end up heads up.

3. Doing some rearranging we come up with

$$\begin{aligned}
p^2q^2 + r^2 &= 2pqr + pq + r \\
p^2q^2 - 2pqr + r^2 &= pq + r \\
(pq - r)^2 &= pq + r \\
(pq - r)^2 - (pq - r) &= pq + r - (pq - r) \\
(pq - r)(pq - r - 1) &= 2r
\end{aligned}$$

   Now $pq - r$ and $pq - r - 1$ are consecutive integers so $2r/(pq - r)$ has to be an integer, which means $pq - r = 2$ or $pq - r = r$. If $pq - r = 2$ then $pq - r - 1 = 1$, so $2r = 2 \times 1$ and $r = 1$, which is not prime. So $pq - r - 1 = 2$ meaning $r = 2 + 1 = 3$. Then $pq - 3 - 1 = 2$ means $pq = 6$ whose only prime factorisation is 2×3. Finally then, $p + q + r = 2 + 3 + 3 = 8$.



4. First, the only way to form a hexagon is as in the picture. It can be shown that the shaded triangles are all equal area (using either congruence or because they stand on equal bases). They each have area $\frac{1}{27}$. The kite-like quadrilaterals at each vertex can be shown to have area $\frac{4}{27}$, making use of the fact that medians cut the area of triangles in half (which triangles and which medians are left to you).

   In all, the area of the hexagon is given by $1 - 3 \times \frac{4}{27} - 9 \times \frac{1}{27} = \frac{2}{9}$.

5. In short, ${}^{9}8 > {}^{8}9$. To see this, if m and n are positive integers with $m > 2n$ then $8^m > 2.9^n$. So $8^8 > 2.9 > 9$, so ${}^{3}8 > 2.{}^{2}9 > {}^{2}9$ and then ${}^{4}8 > 2.{}^{3}9$ and so on. Finally we'd show that ${}^{9}8 > 2.{}^{8}9$.

6. We can index each number in $\{1, 2, \ldots, 17\}$ by the number of subsets of 8 elements for which they are the smallest. For instance, 10 is the smallest element of only 1 subset $\{10, 11, 12, 13, 14, 15, 16, 17\}$. To get the number of subsets for $n < 10$, we should write the elements of each subset in order: the first number must be n, then for the remaining 7 we must choose from $17 - n$, so there are $\binom{17-n}{7}$ possibilities.

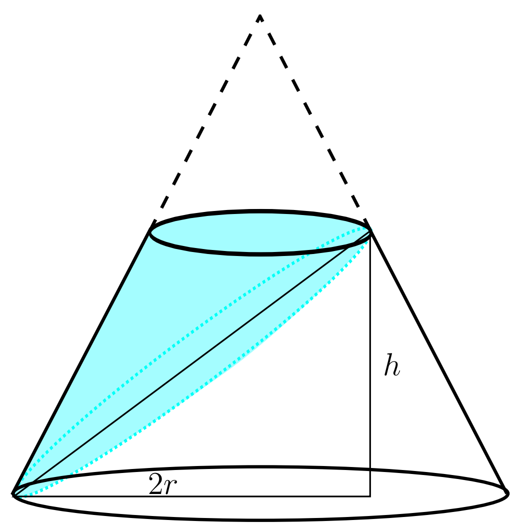
   So the arithmetic mean of the numbers selected is

$$\begin{aligned}\bar{x} &= \frac{1}{24\ 310}\left(1 \times \binom{16}{7} + 2 \times \binom{15}{7} + 3 \times \binom{14}{7} + \cdots + 9 \times \binom{8}{7} + 10 \times \binom{7}{7}\right) \\ &= \frac{1}{24\ 310}(11\ 440 + 12\ 870 + 10\ 296 + 6\ 864 + 3\ 960 + 1\ 980 + 840 + 288 + 72 + 10) \\ &= \frac{1}{24\ 310}(48\ 620) = 2.\end{aligned}$$

Senior Questions

1. If you think of the 'disc method' for computing volumes of revolution, we approximate a cone by a pile of thin cylindars stacked on top of one another with radii given by the cone. An oblique cone is just a right cone that's been pushed over, which is like sliding the cylindars so they no longer sit directly on top of one another, but are centred at the, now oblique, centre line. In more formal terms, this means the integral we compute for the volume of a cone
$$\pi \int r(y)^2 \, dy = \pi \int_0^h \left(-\frac{R_1}{h}y + R_1\right)^2 \, dy$$
is the same integral we would compute for the volume of an oblique cone.

2. As you can see in the picture, the water makes a truncated oblique cone with elliptical base. By extending the frustrum up to complete the cone we can compute this volume as
$$V_{\text{water}} = \frac{1}{3} b_{\text{water}} h_{\text{water oblique cone}} - \frac{1}{3}(\pi r^2)(h_{\text{extended cone}} - h)$$



To find the proportion you want to divide by the volume of the cup which can be calculated by
$$V_{\text{cup}} = \frac{1}{3}(\pi R^2) h_{\text{extended cone}} - \frac{1}{3}(\pi r^2)(h_{\text{extended cone}} - h)$$
To find the area of the surface of the water, and the height of the oblique cone you'll need to use some 3d geometry. Nevertheless the ellipse has major axis $\sqrt{(r+R)^2 + h^2}$ and minor axis $2\sqrt{Rr}$. The height of the oblique cone is given by
$$\frac{2rRh}{(R-r)\sqrt{h^2 + (R+r)^2}}.$$
All in all the proportional volume of the water is $\frac{1}{1+\left(\frac{R}{r}\right)^{\frac{3}{2}}}$.