Never Stand Still | Science | School of Mathematics and Statistics

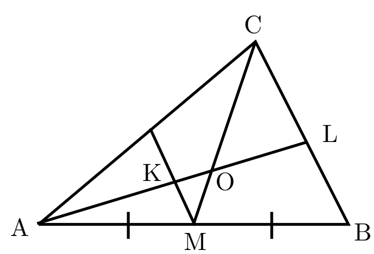
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 10, July 21, 2015[1]

1. Find the sum of all n-digits long numbers formed by $1, 2, 3, \ldots, n$. For example, if $n = 3$ then the sum of all 3-digit long numbers is $123 + 132 + 213 + 231 + 312 + 321 = 1332$.

2. Evaluate $\sqrt[4]{2} \times \sqrt[8]{4} \times \sqrt[16]{8} \times \sqrt[32]{16} \times \sqrt[64]{32} \ldots$.

3. Several positive integers are written on a blackboard. The sum of any two of them is some power of two (for example, 2, 4, 8,...). What is the maximal possible number of different integers on the blackboard?

4. Bob is building two roads to connect the points A and B. For any real number x, the two roads must have a length ratio of $\sqrt{(x+4)^2+4}$ to $\sqrt{(x-4)^2+16}$. Bob picks x then claims his design gives the shortest combine length of the two roads, what must this combine length be?



5. For a triangle $\triangle ABC$, M is the midpoint of the side AB and L is some point along the side BC. Let O be the point of intersection between the lines LA and MC, and let K be the point of intersection between LA and the line passing through M, parallel to BC; see above

   (a) Show that the triangles $\triangle KMO$ and $\triangle OLC$ are similar.

   (b) Suppose the length LA is twice as long as MC, and $\angle OLC = 45°$. Prove LA is perpendicular to MC.

6. Consider the polynomial $p(x) = x^4 + 37x^3 + 71x^2 + 18x + 3$. If a, b, c and d are roots of $p(x)$, find a polynomial whose roots are $\frac{abc}{d}$, $\frac{acd}{b}$, $\frac{abd}{c}$ and $\frac{bcd}{a}$.

[1]Some problems from *Tournament of Towns in Toronto*.

Senior Questions

The following questions concerns the irrationality of π. Recall that a number is irrational if it can not be written as $\frac{a}{b}$, where a and b are positive integers. We will study a function defined by

$$f(x) = \frac{x^n(a-bx)^n}{n!},$$

where n is some positive integer.

1. Let $f^{(k)}(x)$ denote the k^{th} derivative of f, where $k = 0, 1, 2, \ldots$. Show that for each k
   (a) $f^{(k)}(0)$ is an integer.
   (b) $f^{(k)}(0) = (-1)^k f^{(k)}(\pi)$.
2. Let $G(x) = f(x) - f^{(2)}(x) - f^{(4)}(x) + f^{(6)}(x) - \ldots + (-1)^n f^{(2n+2)}(x)$.
   (a) Show that $f(x) = G(x) + G^{(2)}(x)$.
   (b) By considering the function $G^{(1)}\sin(x) - G(x)\cos(x)$ and the result of part (a), show that $\int_0^\pi f(x)\sin(x)\,dx = G(0) - G(\pi)$.
3. Show that

$$0 < \int_0^\pi f(x)\sin(x)\,dx < \frac{a^n\pi^{n+1}}{n!}.$$

   Hence by using the results of 1. and 2., show that π is irrational.