MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 11, July 28, 2015[1]

1. Alice and Carla are playing a dice game. Here's how it works:

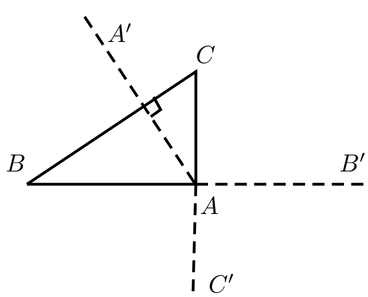
   - Each person rolls a die, and the highest number rolled of the two is recorded.
   - If the highest number rolled is a $1, 2, 3$ or 4, Alice wins.
   - If the highest number rolled is a 5 or a 6, Carla wins.

   On average, who is more likely to win: Alice, Carla, or are the probabilities equal?

2. Find the remainder when x^{1999} is divided by $x^2 - 1$.

3. How many 3 digit positive integer is/are the sum of exactly 9 distinct powers of 2?

4. Given that $a + b = 1$ and $a^2 + b^2 = 2$, what is the value of $a^7 + b^7$?



5. Let $\triangle ABC$ be right-angled. Let A' be the mirror image of the point A in the side BC, let B' be the mirror image of B in AC and C' the mirror image of B in AB; see above. Find the ratio
$$\text{area}(\triangle ABC)/\text{area}(\triangle A'B'C').$$

6. Find all positive integers n for which all of the numbers
$$n,\ 2n-1,\ 2n+5,\ 3n-2,\ 5n-4,\ 6n-5, \text{ and } 12n+5$$
are prime. (Note the integer 1 is not prime).

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

Senior Questions

1. Find all solutions of $2^x + 3^x + 6^x = x^2$.

2. Let $f(x) = x + \int_0^1 (xy^2 + x^2y)f(y)\,\mathrm{d}y$. Find the value of $f(10)$.

3. Denote by $[a, b]$ the least common multiple of a and b. Let n be a positive integer such that

$$[n, n+1] > [n, n+2] > \ldots > [n, n+35].$$

Prove that $[n, n+35] > [n, n+36]$.