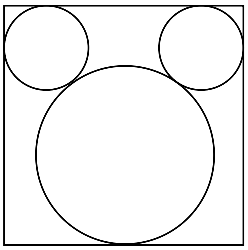
MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 13, August 11, 2015[1]

1. Find all prime number p, such that $4p^2+1$ and $6p^2+1$ are both prime.

2. A right angled triangle has sides of length a, b and c, where c is the length of the hypotenuse. Prove that for any integer $n > 2$,
$$c^n > a^n + b^n.$$



3. Three circles fit inside a square as shown above. The two smaller circles has radius 3 and each intercepts the larger circle at a unique point. The square has side length 14. Find the radius of the larger circle.

4. Find all positive integers n, such that
$$\frac{n^2+11n+2}{n+5},$$
is also an integer.

5. Let T be a set of integers with the following properties:

   (a) T contains $0, 1, 3, 4, 5$.

   (b) If a, b, c, d are distinct elements of T such that
   $$a+b=c+d,$$
   then
   $$r = -(a+b) = -(c+d)$$
   is also in T. For example $0+4=1+3$; therefore -4 is also an element of T.

   Prove that T contains all integers.

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

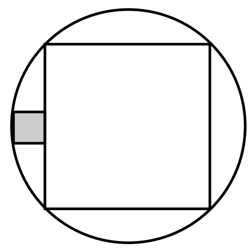
Senior Questions

1. We have 99 locked boxes and 99 keys, each opening just one lock. Each box has an opening through which we throw one key at random.

   (a) We break one of the boxes. What is the probability that we can open all the other boxes without breaking them?

   (b) If we initially break two boxes instead of one, what is the probability that we can open all the other boxes without breaking them?

   Give reasons to your answer.



2. A square is inscribed in a unit circle, and a smaller shaded square is inserted in one of the four regions between the interior to the circle and exterior to the inscribed square; as shown above. Find the area of the shaded square.

3. Positive integers a, b, c, d are pairwise coprime (i.e their greatest common divisor is 1) and satisfy the equation
$$ab + cd = ac - 10bd.$$
   Prove that we can always choose three numbers among them such that one number equals the sum of the two others.

4. Each of 100 stones has a sticker showing its true weight. No two stones weight the same. Mischievous Greg wants to rearrange stickers so that the sum of the numbers on the stickers for any group containing from 1 to 99 stones is different from the true weight of this group. Is it always possible?