MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 8, June 16, 2015[1]

1. How many integral solutions (x, y) are there for the equation $x^2 - y^2 = 1999$ (Note that 1999 is prime).

2. Let $\triangle ABC$ be a right-angled triangle with sides of length $|AB| = a$, $|BC| = b$ and $|CA| = c$ and $\angle CAB = 90°$. A circle is inscribed in $\triangle ABC$ such that the circle intersects each side of the $\triangle ABC$ exactly once. Find the radius of the circle in terms of a, b and c.

3. Let N be a number of the form $N = \underbrace{333\ldots333}_{61\times 3's}$, and M a number of the form $M = \underbrace{666\ldots666}_{62\times 6's}$. Find $N \times M$.

4. Let x be a positive odd number, and a a positive integer greater than 2. If a^x has remainder r_1 when divided by $(a-1)$ and r_2 when divided by $(a+1)$, find $r_1 + r_2$.

5. Let $[x]$ denotes the greatest integer less than or equal to x, where x is some real number. How many positive integers less than 1001 can be expressed in the form $[2x] + [4x] + [6x] + [8x]$?

6. On a bicycle, tyre wear is proportional to distance traveled, front tyre lasting x kilometres and rear tyre lasting y kilometres. $(x < y)$. An advertisement claims that a set of tyres lasts at least $(x + y)/2$ kilometres provided you interchange front and rear tyre after an appropriate distance. Investigate.

[1]Some problems from UNSW's publication *Parabola*.

Senior Questions

1. Consider the quadratic equation

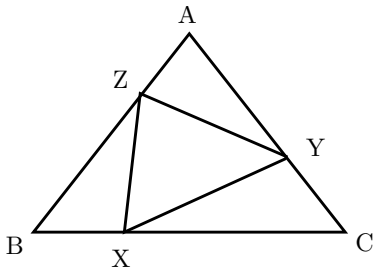
$$f(x) = x^2 - 2(c+1)x + c - 3,$$

where c is some real number. Let $\alpha, \beta > 0$, and suppose $\alpha + \frac{1}{\alpha}$ and $2 - \beta - \frac{1}{\beta}$ are the roots of $f(x)$. Find all possible values for c.

2. Let $\triangle ABC$ be a triangle and X, Y, Z points on the sides BC, CA, AB respectively. Suppose $BX \leq XC, CY \leq YA, AZ \leq ZB$. Show that

   (a) The area of $\triangle XYZ$ is not less than one quarter of the area of $\triangle ABC$.

   (b) One of the corner triangles $\triangle AZY, \triangle BXZ, \triangle CYZ$ has area not greater than the area of $\triangle XYZ$.



3. Given that a, b and c are positive integers, find the conditions for which the equation $\sqrt{a} - b = \sqrt{c}$ has a solution.

4. (bonus) Infinitely many physicists walks into a pub. The first physicist orders a beer, the second orders half a beer, the third a quarter, the fourth an 8^{th} and so on. The bartender happens to be a math student, what would the bartender tell the physicists?