MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 6, June 2, 2015[1]

1. Sum of an arithmetic sequence with 11 term is

$$S_{11} = \frac{11}{2}(2a_1 + (11-1)d) = 220,$$

and the 6^{th} (middle) term is

$$a_6 = a_1 + (6-1)d.$$

A solution to the former equation is $a_1 = 5$ and $d = 3$, hence $a_6 = 5 + 5 \times 3 = 20$.

2. (a) Let n be the total number of people at the party, then the total number of handshakes is

$$(n-1) + (n-2) + (n-3) + \ldots + 1 = \frac{n-1}{2}(2 + (n-2)) = 253.$$

Solving the above equation gives $n = 23, -22$, taking the sensible answer then Bernard is 22.

(b) Since there are 23 people in the party, we first divide them into groups of 12 and 11, the number of ways we can do this is

$$\frac{23!}{12! \times 11!},$$

where $23! = 23 \times 22 \times 21 \times \ldots \times 2 \times 1$, and similarly for $12!$ and $11!$. To see how the above equation works, $23!$ represent the number of ways we can put 23 people into 23 chairs, but we don't care how the 12 or 11 people are arranged within the group, so we remove $12!$ and $11!$.

Next we consider how many ways 11 and 12 people can be arrange in a round table. For the group of 12 people, there is $12!$ ways in which they can be arranged into 12 chairs. However because the table is round, we can rotate the table and get the same arrangement; the number of rotation is 12. Hence we conclude that there is $12!/12 = 11!$ ways the 12 persons group can be arranged. For the group of 11 people, there is $10!$ ways to arrange them following a similar argument as

[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*. Q4 is by Adam. S

the 12 group case. Thus we conclude the total number of ways Bernard can do this is

$$\frac{23!}{12! \times 11!} \times 11! \times 10! = \frac{23!}{12 \times 11}.$$

3. See senior 1.(b)

4. Let $\triangle ABC$ be a isosceles triangle with base $\angle BAC = \angle CBA = 72°$, D is the point of intersection between the bisector of $\angle BAC$ and the line CB; see above

   (a) Let $x = a/b$. By sum of angles of the triangle $\triangle ADB$, $\angle ADB = 72°$. Therefore, the triangles $\triangle ACD$ and $\triangle ADB$ are similar. Furthermore, $\triangle CDA$ is isosceles, so that $a = |AD| = |AB|$. Now by ratios of similar triangles we have $\frac{|AB|}{b} = \frac{a+b}{|AB|}$ or $\frac{a}{b} = \frac{a+b}{a}$, which implies $x = 1 + \frac{1}{x}$. Solving for x gives $\frac{a}{b} = \frac{1+\sqrt{5}}{2}$.

   (b) Let M be the midpoint of AC. Since $\triangle ADC$ is isosceles, MD bisects $\angle CDA$ so that $\angle ADM = 54°$, and $MA = \frac{1}{2}\left(\frac{a+b}{a}\right)$. Consider the triangle $\triangle AMD$, then

   $$\cos(36°) = \frac{|AM|}{|AD|} = \frac{1/2 \times |CA|}{|AD|} = \frac{1}{2}\left(\frac{a+b}{a}\right).$$

   Now by using the similarity argument between $\triangle ACD$ and $\triangle ADB$ and the results of part (a), we have

   $$\cos(36°) = \frac{1}{2}\left(\frac{a+b}{a}\right) = \frac{1}{2}\left(\frac{a}{b}\right) = \frac{1+\sqrt{5}}{4},$$

5. We can always pair up the divisors of a number x in such a way, that the product of the pair is equal to the number itself; for example the divisors of 4 are $1, 2$ and 4, which can be paired up into $\{1, 4\}$ and $\{2\}$. In particular, note that if x has an odd number of divisors, then one of divisor must take the form of $\sqrt{x}$.

   Suppose we have a number x with an odd number of even divisors and even number of odd divisors, then the total number of divisors of x must be odd. Hence one of the divisor of x is $\sqrt{x}$ (and x is a perfect square). If x is odd, then all odd divisors of x must be paired with another odd divisor of x, and since $\sqrt{x}$ is odd if x is, it follows that if we exclude $\sqrt{x}$, then there is an odd number of odd divisor of x we need to pair together; this is no possible, so x must be an even number.

   Because x is even, so is $\sqrt{x}$. We can write $\sqrt{x} = 2^k y$, where y is an odd number. Then $x = 2^{2k}y^2$, and the odd divisors of x are the odd divisor of y^2. Since y^2 is a perfect square and it is odd, using the same argument as the previous paragraph, the number of odd divisors of y^2 can not be even. Therefore, no such x exist.

6. Statement (iii) is false. Because if we assume (iii) is true, then by statement (ii), $a + b = 3b + 5$ so that $3b + 5$ is divisible by 3; (ii) is false. Also by statement (iv), $7(a + b) - 6a$ is a prime, but $7(a + b) - 6a$ is divisible by 3; (iv) is false. We have too many false statements if (iii) is true.

By statements (i) and (ii), $2b+6$ is divisible by b. Therefore b is a divisor of 6; that is b is $1, 2, 3$ or 6. By statement (ii) and (iv), $9b+5$ is a prime. Inserting the possible values of b into $9b+5$, one sees that $b = 2$ is the only solution.

Senior Questions

1. (a) One way to do this is by polynomial long division `http://en.wikipedia.org/wiki/Polynomial_long_division`, another way is by induction.

   (b) Using part (a), we have $a^n - 1 = (a-1)(a^n + a^{n-1} + \ldots + a + 1)$. Since $a^n - 1$ is prime, the only factor it can have is 1; we must have $a - 1 = 1$, so $a = 2$.

   Suppose n is not prime, then there are positive integers $x > 1$ and $y > 1$ such that $n = xy$. If we write $a^n - 1 = a^{xy} - 1 = (a^x)^y - 1$, then we can use the results of part (a) with a^x instead of a to obtain

   $$a^n - 1 = (a^x)^y - 1 = (a^x - 1)[(a^x)^y + (a^x)^{y-1} + \ldots + a^x + 1].$$

   Because the LHS of the above equation is a prime, we can conclude (just as before) that $a^x - 1 = 1$, which means $a^x = 2^x = 2$; $x = 1$, and we have a contradiction.

2. If A shoots C and hits, then B will shoot and hit A.

   If A shoots B and hits, then A and C will dual it out. The probability of A winning is

   $$(0.5 \times 0.3)(1 + 0.5 \times 0.7 + (0.5 \times 0.7)^2 + (0.5 \times 0.7)^3 + \ldots) = 0.15\left(\frac{1}{1 - 0.35}\right) \approx 0.23.$$

   If A misses the shot, then B will shoot and hit C, so A gets another shoot at B. The probability of A winning is then 0.3.

   So A best strategy is to intentionally miss.

   What happens if C's chance to hit is only 40%, would the answer be different?

3. Let
   $$f(x) = x^{2014} - 2x^{2013} + 3x^{2012} + \ldots + 2013x^2 - 2014x + 2015$$

   We can write $f(x)$ as

   $$\begin{aligned} f(x) =& x^{2014} - x^{2013} + x^{2012} + \ldots + 1 \\ & - x^{2013} + x^{2012} - x^{2011} + \ldots + 1 \\ & + x^{2012} - x^{2011} + x^{2010} + \ldots + 1 \\ & \vdots \qquad\quad \vdots \qquad\quad \vdots \qquad\quad \vdots \\ & 1 \end{aligned}$$

   By noting that the series on each line of the above equation is a geometric sum of $-x$, we have

$$\begin{aligned}
f(x) &= \frac{1-(-x)^{2015}}{1+x} + \frac{1-(-x)^{2014}}{1+x} + \frac{1-(-x)^{2013}}{1+x} + \ldots + \frac{1-(-x)^{2}}{1+x} + 1 \\
&= \frac{x^{2015}+x^{2014}+x^{2013}+\ldots+x^2+x+2015}{1+x} \quad \text{GR-series of } -x \\
&= \left(\frac{1}{1+x}\right)\left[x\left(\frac{1+x^{2015}}{1+x}\right)+2015\right] \\
&= \left(\frac{1}{1+x}\right)^2 (x^{2016}+2016x+2015).
\end{aligned}$$

If $f(x)$ does not have any real roots, then all its turning points must either be on one side of the x-axis or has no turning points at all. So we want to look at the derivative of $f(x)$ to find out where the turning points are, better still we can consider the derivative of $g(x) = (1-x)^2 f(x)$ instead, because if $g(x)$ has all its turning points above or on the x-axis, then since the number $(1+x)^2$ is always non-negative, all turning points of $f(x)$ will also be above or on the x-axis. Now

$$g'(x) = -2016 + 2016x^{2015},$$

so the only turning point for g is at $x = -1$ and $g(-1) = 0$, which implies $g(x) > 0$ for all $x \neq 1$. Therefore, $f(x) > 0$ except when $x = -1$. Hence we conclude that the only way $f(x)$ can have a root is at $x = -1$, but $f(-1) \neq 0$.