MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 9, June 23, 2015[1]

1. Since x and y are both integers, we can only get a solution when x is an even number. If $x \neq y \neq 0$, then $x = \pm 2k$ for $k = 1, 2, \ldots, 49$ and the corresponding values for y are $y = \pm\frac{1}{2}(100 - 2k)$; that is for each choice of $x \neq 0$, there are two choices for y. If $x = 0$ then $y = \pm 50$. Similarly, if $y = 0$ then $x = \pm 100$. Therefore the total number of different solutions is $4 \times 49 + 2 \times 2 = 92$.

   Given that x and y are integer, how many different solutions does the equation $|x| + 2|y| = 100$ have?

2. Note that we can write the list of numbers as $2^0, 2^1, \ldots, 2^8$. In this new format, the product of any two number is 2 to the power of the sum of their exponents. Therefore, we can proceed to filling in the 3×3 product square as the slightly modified standard 3×3 magic square. To show whether the solution is unique, see solution sheet 1.

3. If we look at the expression as a fraction, then by inserting brackets we are just manipulating if each number is to be place on the numerator or the denominator of this fraction. For example, if we place a bracket around 8 and 9, then the fraction becomes

$$10 \div (9 \div 8) \div 7 \div \ldots \div 1 = \frac{10 \times 8}{9 \times 7 \times 6 \times \ldots \times 1}$$

   (a) Since 9 is the first divisor, it has to appear at the denominator of the express. Hence, the largest possible number we can get it is by forcing each number except 9 onto numerator. This is archived by

$$10 \div (((((((9 \div 8) \div 7) \div 6) \div 5) \div 4) \div 3) \div 2) = 10 \times \frac{8!}{9} = \frac{10!}{9^2} = 44800.$$

   (b) Since 7 is the only number in the list divisible by 7, it has to appear on the numerator of the expression and the minimal can not be smaller than 7. This is achieve by

$$\begin{aligned}
&(10 \div 9) \div ((8 \div 7) \div (6 \div ((5 \div 4) \div (3 \div 2))))\\
=&\frac{10}{9} \div \left(\frac{8}{7} \div \left(6 \div \frac{5 \times 2}{4 \times 3}\right)\right)\\
=&\frac{10}{9} \div \left(\frac{8 \times 5 \times 2}{7 \times 6 \times 4 \times 3}\right) = 7.
\end{aligned}$$

[1] Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*.

4. I think there was a typo in part (a) that made it trivial once solution to (b) is founded, the question is suppose to be $x^3 - 5[x] = 10$.

   (a) x^3 must be an integer as $x^3 = 10 + 5[x]$. Also $x^3 \leq 10 + 5x$ and $x^3 > 10 + 5\{x - 1\}$ so $2 < x < 3$. Hence $[x] = 2$ and therefore $x^3 = 10 + 5 \times 2 = 20$ and $x = \sqrt[3]{20}$.

   (b) $y^3 - 5\{y\} = 10$ so $10 < y^3 < 15$ and $\{y\} = y - 2$. Hence $y^3 = 10 + 5\{y - 2\} = 5y$, and since $y \neq 0$, $y^2 = 5$ and $y = \sqrt{5}$.

5. The answer is 8.

6. Let n be an even number such that $n \geq 4$. First we prove that $5^n \equiv 625 \pmod{1000}$; that is the remainder of 5^n when divided by 1000 is 625. We use mathematical induction, since $5^4 = 625$, the case of $n = 4$ holds. Then assuming $5^n \equiv 625 \pmod{1000}$, we have

$$\begin{aligned} 5^{n+2} = 5^n \times 5^2 &\equiv 625 \times 25 \pmod{1000} \\ &\equiv 15625 \pmod{1000} \\ &\equiv 625 \pmod{1000}, \end{aligned}$$

   we can think of the first two lines as the product of two numbers has remainder equal to the product of the remainders of each number take away factors of the divisor.

   To complete the question, we note that the exponent of 2015 is a power of an even number, and it is greater than 4. Thus if we divide 2015^n by 1000, the remainder is 625; hence the last three digits is 625.

Senior Questions

1. We have a solution to the equation when $x^2 - 5x + 5 = 1$, or when $x^2 - 5x + 5 = -1$ and $x^2 - 11x + 30$ is even, or when $x^2 - 5x + 5 \neq 0$ and $x^2 - 11x + 30 = 0$. For the first case, $x = 4$ and $x = 1$ are the only integral solutions. For the second case, $x = 2, 3$ are the solutions. For the last case, $x = 5, 6$ are solutions. There are six different solutions.

2. If $p_1, p_2, \ldots, p_n$ are n primes in arithmetic progression, then $p_1 = a, p_2 = a+d, \ldots, p_n = a + (n-1)d$. Also, if $p_1, \ldots, p_n$ are all greater than or equal to n, then d is divisible by every prime less than n. For if p is prime, $p < n$ and p does not divide d, then for some k with $1 \leq k \leq n$, $a + (k-1)d$ is divisible by p. But p_k is prime, so $p_k = p$. But $p < n, p_k \geq n$, a contradiction.

   So if $p_1, \ldots, p_6$ are all greater than or equal to 6, then d must be divisible by $2, 3$ and 5, so it is divisible by 30. So the smallest set of number we might consider is $7, 37, 67, 97, 128, 157$.

3. Since $2n+1$ is odd and a perfect square, we can write as $2n+1 = (2k+1)^2 = 4k^2+4k+1$, for some non-negative integer k, which implies $n = 2k(k+1)$. Since either k or $k+1$ is odd, we conclude that n is divisible by 4 and n is even.

   Because n is even, $3n + 1$ must be odd so we can write $3n + 1 = (2j + 1)^2$, for some non-negative j, which implies $3n = 4j(j + 1)$. Similar to before, either j or $j + 1$ is odd, so we can conclude that n is divisible by 8.

To complete the question, we show that n is divisible by 5. The possible remainder of an integer a divided by 5 are $0, 1, 2, 3$ and 4, therefore any perfect square number must have remainders $0, 1^2, 2^2, 3^2 - 5$ and $4^2 - 3(5)$; that is $0, 1, 4$ are the only remainders of a perfect square number when divided by 5. If we consider the remainders of $2n + 1$ and $3n + 1$ when divided by 5, for $n = 0, 1, 2, 3, 4$, we can see that the only time when both $2n + 1$ and $3n + 1$ have remainders either $0, 1, 4$ is when $n = 0$. Hence the only time when both $2n + 1$ and $3n + 1$ are perfect squares is when $n \equiv 0 \pmod 5$; that is n is divisible by 5.