UNSW AUSTRALIA

Never Stand Still | Science | School of Mathematics and Statistics

MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 2, May 5, 2015[1]

1.

$$\begin{aligned}\frac{y+x}{\sqrt{y}+1} &= x\\ y+x &= (\sqrt{y}+1)x\\ y &= (\sqrt{y})x\\ \sqrt{y} &= x\end{aligned}$$

2. If $x-y+2z$ is divisible by 11, then there is an integer k such that $x-y+2z=11k$. So we can write

$$\begin{aligned}-12x+y-13z &= -x+y-2z-11x-11z\\ &= -11k-11(x+z)\\ &= -11(k+x+z).\end{aligned}$$

The right hand side of the above equation is divisible by 11, because $k+x+z$ is an integer; $-12x+y-13z$ is divisible by 11.

3. The speed of Anna and Boris, and the initial distance between them are constant regardless of when they started moving toward each other. Therefore, we would like to express everything we need to solve this problem in terms of those constants. Let V_A and V_B be the speed of Anna and Boris respectively, and let x be the initial distance between them. If they move towards each other simultaneously, then the time it takes for Anna and Boris to meet is

$$\frac{x}{V_A+V_B}.$$

Which means the distance covered by Anna is $\frac{xV_A}{V_A+V_B}$, and the distance covered by Boris is $\frac{xV_B}{V_A+V_B}$.

Suppose they started moving at a different time, so that the distance covered by Anna is 2km more and the distance covered by Boris is 2km less. Then the time Anne

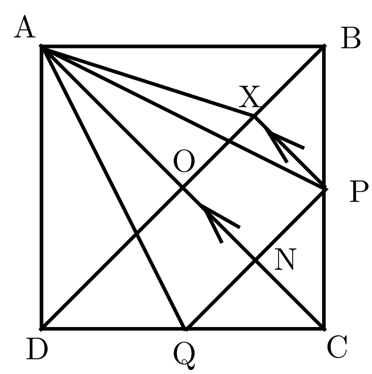
[1]Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*

spend moving is $\frac{1}{V_A}\left(\frac{xV_A}{V_A+V_B}+2\right)$. Similar Boris spend $\frac{1}{V_B}\left(\frac{xV_A}{V_A+V_B}-2\right)$ moving. But we already know the difference in timing is $\frac{1}{2}$ hour, therefore

$$\frac{1}{V_A}\left(\frac{xV_A}{V_A+V_B}+2\right)-\frac{1}{V_B}\left(\frac{xV_B}{V_A+V_B}-2\right)=\frac{1}{2},$$

or simplifying to get $\frac{1}{V_A}+\frac{1}{V_B}=\frac{1}{4}$. This expression is symmetric, so if we switch the starting time condition between Anna and Boris, then Anna would cover 2km less and Boris 2km more; $d = 2$.

4. Using the notation $|\cdot|$ to mean the perimeter of a triangle or length of a line. Let the point of intersection between the diagonals DB and AC be O.



(a) Since $ABDC$ is a square, the diagonal BD bisects $\angle ABC$ so that $\angle XBP = 45°$, also the diagonals BD and AC intersections at right angles so that $\angle BOC = 90°$. Furthermore, $\angle COB = \angle PXB$ because XP and CO are parallel. So we have $\angle PXB = \angle BOC = 90°$, which implies $\angle BPX = 180° - 90° - 45° = 45° = \angle XBP$; the triangle PXB is isosceles with $|XB| = |XP|$.

(b) Let $|AB| = a$ and $|OB| = b$, then the perimeter of ABD is $2a + 2b$, so we want to show that the perimeter of $|APQ| \leq 2a + 2b$. Draw a line to connect the points A and X, and let the intersection of PQ and AC be N.

We are given that $|PC| = |QC|$, from this we can work out that the triangles APN and AQN are similar. This implies that N is the midpoint of PQ and $|AP| = |AQ|$. Hence, $|APQ| = 2|NP| + 2|AP|$. Furthermore, $OXPN$ is a parallelogram, thus $|NP| = |OX|$ which implies

$$\begin{aligned}|APQ| &= 2|NP| + 2|AP| \\ &= 2|OX| + 2|AP| \\ &= 2b - 2|XB| + 2|AP|.\end{aligned}$$

Next we work with the side AP to get an upper bound for it. To do this, consider the triangle AXP. The sum of two sides of any triangle is greater than the other side (you may want to think about why this is always true), so $|AP| \leq |XP| + |AX| = |XB| + |AX|$, where the second equality on this expression is due to PXB being is isosceles. Also, $|AX| \leq |AB| = a$ because the side AB is opposite to the largest angle in the triangle ABX, from all of this we conclude:

$$\begin{aligned}
|APQ| &= 2b - 2|XB| + 2|AP| \\
&\leq 2b - 2|XB| + 2|XB| + 2|AX| \\
&\leq 2b + 2a
\end{aligned}$$

5. Let x be the four digit number we are trying to find. Then $x^2 - x = x(x-1)$ is a number ending in 0000; that is $x(x-1)$ is divisible by $10000 = 2^4 5^4$. One of $x - 1$ or x is divisible by 2^4 and the *other* by 5^4. If x or $x - 1$ is divisible by 5^4, then it is an odd multiple of $5^4 = 625$, so x or $x - 1$ must be one of

$$625,\ 1875,\ 3125,\ 4375,\ 5625,\ 6875,\ 8125,\ 9375.$$

We add or subtract 1 from the above list to find out which of the *other* number is divisible by 16. The two possibilities are $x - 1 = 624$ and $x = 9376$, we discard the first solution because it is not a proper four digit number.

6. Begin by assigning letters to each of the cells in the 3×3 grid.

a	b	c
d	e	f
g	h	i

Although we do not know the individual value of each letter we do know that each of the digits 1 through 9 is assigned to the letters in some order. Let T be the total by adding each row, column or diagonal, e.g $T = a+b+c$. Then by adding all three rows, we get the number $3T$. Note this is the same as adding every cell.

$$a + b + c + d + e + f + g + h + i = 1 + 2 + ... + 9 = 45 = 3T,$$

so $T = 15$. Suppose instead we add every row, column or diagonal that involves the middle cell, then

$$\begin{aligned}
(a+e+i) + (d+e+f) + (g+e+c) + (b+e+h) &= 4T \\
(a+b+c+d+e+f+g+h+i) + 3e45 + 3e &= 60 \\
45 + 3e &= 60 \\
e &= 5.
\end{aligned}$$

Hence we know that $T = 15$ and the middle cell must be 5. So in order for each line to have the same total of 15 it will be necessary for the cells either side of the central cell to be of the form $5x$ and $5 + x$.

$5+x$	$5-x-y$	$5+y$
$5-x+y$	5	$5+x-y$
$5-y$	$5+x+y$	$5-x$

The cell with greatest value is $5 + x + y = 9$, hence $x + y = 4$. Also $x \neq y$, otherwise the cells $5 + y$ and $5 + x$ would have the same number in them; Finally $x, y > 0$ to avoid the cells $x + 5$ and $x - 5$ being the same.

Because $x \neq y$, we can assume without loss that $x < y$, and since $x + y = 4$, we conclude that $x = 1$ and $y = 3$. Substituting these values into the grid above we obtain the solution given in the problem and hence prove that this solution is unique.

Senior Questions

There was a few typos in the first two equations...

1. Write

$$\begin{aligned} P_3(x) &= a_0 + a_1x^1 + a_2x^2 + a_3x^3 \\ Q_2(x) &= b_0 + b_1x^1 + b_2x^2 \\ R_3(x) &= c_0 + c_1x^1 + c_2x^2 + c_3x^3. \end{aligned}$$

   (a) $P_3(x) \times Q_2(x) = a_0b_0 + (a_0b_1 + a_1b_0)x^1 + \ldots + (a_3b_2)x^5$, so

$$P_3(x) \times Q_2(x) + R_3(x) = (a_0b_0 + c_0) + (a_0b_1 + a_1b_0)x^1 + \ldots + (a_3b_2)x^5 + c_3x^6,$$

   which is a polynomial of degree 6.

   (b) The question should be $P_3(Q_2(\sqrt{x}))$, then

$$\begin{aligned} P_3(Q_2(\sqrt{x})) &= P_3(b_0 + b_1\sqrt{x} + b_2x) \\ &= a_0 + a_1(b_0 + b_1\sqrt{x} + b_2x)^1 + \ldots + a_3(b_0 + b_1\sqrt{x} + b_2x)^3 \end{aligned}$$

   which has degree 3.

2. The equality should be

$$f^{(k)}(x) = P_{2k}\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right),$$

   which holds for all $k \geq 1$. Note that by definition, $P_k(x)$ means a polynomial of x of degree k, what the real numbers $a_0, \ldots a_k$ are is unimportant for this equation. For $k = 1$,

$$\begin{aligned} f^{(1)}(x) &= \frac{d}{dx}\exp\left(\frac{1}{x}\right) \\ &= -\frac{1}{x^2}\exp\left(\frac{1}{x}\right) = P_2\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right). \end{aligned}$$

   Assuming the expression holds for k, then we want to show that

$$f^{(k+1)}(x) = P_{2(k+1)}\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right).$$

Using the product rule to differentiate $f^{(k)}(x)$, and note that if we differentiate a polynomial of x with respect to x we end up with another polynomial of x of one less degree, thus

$$\begin{aligned}
f^{(k+1)}(x) &= \exp\left(\frac{1}{x}\right)\frac{d}{dx}P_{2k}\left(\frac{1}{x}\right) + P_{2k}\left(\frac{1}{x}\right)\frac{d}{dx}\exp\left(\frac{1}{x}\right) \\
&= \exp\left(\frac{1}{x}\right)P_{2k-1}\left(\frac{1}{x}\right) - \frac{1}{x^2}P_{2k}\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right) \\
&= \exp\left(\frac{1}{x}\right)P_{2k-1}\left(\frac{1}{x}\right) + P_{2k+2}\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right) \\
&= P_{2k+2}\left(\frac{1}{x}\right)\exp\left(\frac{1}{x}\right)
\end{aligned}$$

3. The inequality clearly holds for the case $n = 1$, so we assume $\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n} \geq \frac{2}{3}n\sqrt{n}$ to prove the inequality holds for $n + 1$; i.e

$$\sqrt{1} + \sqrt{2} + \ldots + \sqrt{n} + \sqrt{n+1} \geq \frac{2}{3}(n+1)\sqrt{n+1}.$$

Consider the right hand side of the above equation

$$\begin{aligned}
\frac{2}{3}(n+1)\sqrt{n+1} &= \frac{2}{3}n\sqrt{n+1} - \frac{1}{3}\sqrt{n+1} + \sqrt{n+1} \\
&= \frac{2}{3}\left(n - \frac{1}{2}\right)\sqrt{n+1} + \sqrt{n+1}.
\end{aligned}$$

By taking the square on the expression $\left(n - \frac{1}{2}\right)\sqrt{n+1}$, then

$$\begin{aligned}
\left(n - \frac{1}{2}\right)^2 (n+1) &= (n^2 - n + 1/4)(n+1) \\
&= n^3 - \frac{3}{4}n + \frac{1}{4} \\
&\leq n^3,
\end{aligned}$$

for $n \geq 1$, hence $\left(n - \frac{1}{2}\right)\sqrt{n+1} \leq n\sqrt{n}$. From this we conclude

$$\begin{aligned}
\frac{2}{3}(n+1)\sqrt{n+1} &= \frac{2}{3}\left(n - \frac{1}{2}\right)\sqrt{n+1} + \sqrt{n+1} \\
&\leq \frac{2}{3}n\sqrt{n} + \sqrt{n+1} \\
&\leq \sqrt{1} + \sqrt{2} + \ldots + \sqrt{n} + \sqrt{n+1}.
\end{aligned}$$