



MATHEMATICS ENRICHMENT CLUB.

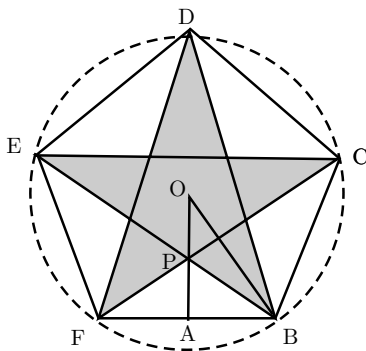
Solution Sheet 5, May 26, 2015¹

1. (a) Note that if a number has remainder 1 when divided by 7, then any powers of this number divided by 7 still has remainder 1. Since $2^{2015526} = 2^{3 \times 671842} = 8^{671842}$ and 8 has remainder 1 when divided by 7, we can conclude that $2^{2015526}$ has remainder 1 when it is divided by 7.
- (b) Since $2^5 = 32$, the last digit of 2^{5^n} is 2 for $n = 1, 2, \dots$. Now because all we care about is the last digit of 2^{2015} , we can simplify the problem by factoring $2^{2015} = (2^{5^3})^{16}(2^5)^3 = (2^{5^4})^3(2^{5^3})(2^5)^3$. Because we already know that the last digit of each 2^{5^n} , $n = 4, 3, 1$ is 2, we can find the last digit of 2^{2015} by considering the last digit of $2^3 \times 2 \times 2^3$, which is 2.
2. Geralt can roll either $\{1, 2, 3, 4, 5\}$ or $\{2, 3, 4, 5, 6\}$. The number of ways to obtain either of these combination is $5 \times 4 \times 3 \times 2 \times 1 = 120$, and the total number of outcomes of rolling 5 dices is 6^5 . Therefore the probability of obtain 5 consecutive numbers is $\frac{2 \times 120}{6^5} = \frac{240}{7776}$.

3.

$$\begin{aligned} \sqrt{x + \sqrt{y + \sqrt{x + \sqrt{y + \dots}}}} &= 7 \\ \sqrt{y + \sqrt{x + \sqrt{y + \dots}}} &= 7^2 - x \\ \sqrt{x + \sqrt{y + \dots}} &= (49 - x)^2 - 2 \\ 7 &= (49 - x)^2 - 2 \\ x &= 46. \end{aligned}$$

¹Some problems from UNSW's publication *Parabola*, and the *Tournament of Towns in Toronto*



4. Let O be the centre of the pentagon, A is the point of intersection between the bisector of the pentagon and one of its sides, and label the corners of the pentagon as shown above.

Since the pentagon is regular, we only need to look at the ratio of the shaded area in one of the symmetrical regions it admits; i.e. the shaded region of the triangle $\triangle OBA$. Let P be the point of intersection between EB and OA , then we are looking for the ratio of area between the triangles $\triangle OBA$ and $\triangle OPB$. One way we can do this is by finding the angles of $\angle OBA$ and $\angle PBA$, then use the properties of triangles and ratios.

Draw a circle that intersects each corner of the pentagon. Since the internal angle $\angle DOE$ is $\frac{360^\circ}{5}$ (the pentagon is regular), the arc angle $\angle DBE = 36^\circ$. Furthermore, OB bisects $\angle DBE$, therefore $\angle OBP = 18^\circ$. Also, the arc angle $\angle FCE$ is 36° , so the arc angle $\angle EBF = \angle PBA = 36^\circ$, hence $\angle OBA = \angle PBA + \angle OBP = 54^\circ$.

The ratio of the shaded regions is therefore

$$1 - \frac{|PA|}{|OA|} = 1 - \frac{\tan 54^\circ}{\tan 36^\circ}.$$

5. Let a and b be the two primes we get from adding up the numbers in the divided set. The only even prime number is 2, and the sum of any numbers in the list is greater than 2, it follows that a and b must be odd. There are only two odd numbers 39 and 45 from the list, so these must be put into different sets. Furthermore, if all numbers are divided by 3, their remainders are 0, 2, 0, 2, 0, 1, 0 respectively. So to prevent a and b from being a multiple of 3, the numbers 38 and 44 must be in a different set to 46, thus we have two possibilities so far

$$A = \{38, 39, 44\}, B = \{45, 46\} \quad \text{or} \quad A = \{39, 46\}, B = \{38, 44, 45\}.$$

For the first case, the sum of A is 121 and the sum of B is 91, so we must add 24 to one set and 48 to the other. However, adding 24 or 48 to A does not give a prime number.

For the second case, the sum of A is 85 and B is 127. Since 85 is not prime, we must add 24, 48 or both to it. If we add 24 to 85, then we have to add the other number 48 to 127, which gives 175. But 175 is not a prime so this is not a solution. If we add 48 to 85, we get 133 which is not prime. Therefore the only solution is $\{24, 39, 46, 48\}, \{38, 44, 45\}$.

6. (example by Madeleine. K) The last digit of x must be 0, because we need an integer after increasing x by 10%. Now in order to decrease the sum of digits of a number after increasing it by 10%, we look for a number that has a lot of digit that will “carry” up when multiplied by 1.1; an example would be a number starting with m lots of 3’s, with n lots of 6’s in the middle and a 0 at the end, because increasing this number by 10% gives a number with the $n - 1$ lots of 3’s, $m - 1$ lots of 6’s, one of each 7, 2 and 0. So the equation we need to solve is $(n - 1)3 + (m - 1)6 + 9 = 0.9(3m + 6n)$; $n = m = 10$ is a solution.

Senior Questions

1. The trick is to apply a change of variable, so that the two graphs become symmetrical. Let $X = x/10$ and $Y = 10y$, then $Y = 10 \cos(10X)$ and $X = 10 \cos(10Y)$. Let A be the sum of the new X -coordinate, and B the sum of the new Y -coordinate, then because the graph of X and Y are symmetrical, we have that $\frac{A}{B} = 1$. Now using the fact that the coordinates are positive, we have $A = a/10$ and $B = 10b$, therefore $\frac{a}{b} = 100$.
2. Apply a change of base on the logarithm.
3. First we consider case $0 < x \leq 1$. The RHS of $x = \frac{1}{2} \left(y + \frac{1}{y} \right)$ is the average of y and $1/y$, thus $y \leq x \leq \frac{1}{y}$ and $y \leq \frac{1}{x} \leq \frac{1}{y}$. Similarly, $z \leq y, \frac{1}{y} \leq \frac{1}{z}, t \leq z, \frac{1}{z} \leq 1/t$ and $x \leq t, \frac{1}{t} \leq 1/x$. From this we conclude that $x \leq t \leq z \leq y \leq x \leq \frac{1}{x} \leq \frac{1}{y} \leq \frac{1}{z} \leq \frac{1}{t} \leq \frac{1}{x}$, so the only solution for this case is $x = y = z = t = 1$.

Using the same arguments as above, we can deduct that there is no solution for $1 < x$ and for the case $x < 0$, we have $x = y = z = t = -1$.

To extend to the 2015 variable case, note that the above arguments does not depend on the number of variables we had initially.