MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 1, May 7, 2016

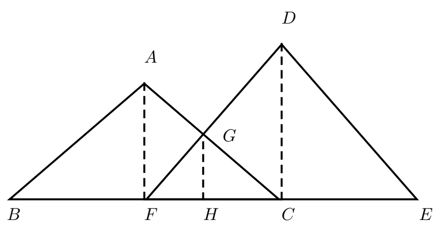
1. Prove that one of the digits $1, 2$ and 9 must appear in the base-ten expression of n or $3n$ for any positive integer n.

2. How many numbers between 100 and 500 that are divisible by 7 but not by 21.

3. Let ABC and DEF be right-angled triangles, with AF and DC their respective altitudes; see figure below. Point G is the intersection of AC and DF. Point H is such that GH is perpendicular to BC. Given $AF = 6$, $GH = 4$ and $FC = 9$.

   (a) Find the length of BC,

   (b) Find the area of the polygon $AGDEB$.

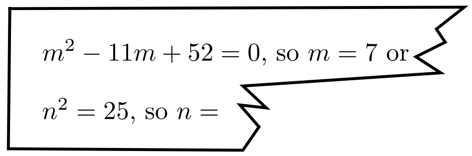


4. Solve

$$\sum_{n=1}^{\infty} \frac{1}{n^2+3n+2} := \frac{1}{1^2+3\times 1+2} + \frac{1}{2^2+3\times 2+2} + \ldots + \frac{1}{i^2+3\times i+2} + \ldots$$

5. Given that a and b are positive integers and the difference between the expressions $a^2 + b$ and $a + b^2$ is prime, find the values of a and b.

6. Ben attempts to pass a note to Megan during an English lesson. The note was torn into pieces before Megan managed to recover the following part:



Megan knows Ben likes to do maths in a different base than the usual base 10 she is accustomed to. What is on the missing part of the note?

Senior Questions

1. Let $f(x)$ be a real-valued function such that

$$\int_0^1 f(x)dx = \int_0^1 xf(x)dx = 1.$$

Show that the minimum value of

$$\int_0^1 (f(x))^2 dx,$$

is attained by $f(x) = \lambda_1 + \lambda_2 x$, for a pair of real numbers λ_1, λ_2.

2. John and Jane play a game on a $5 \times 5 \times 5$ cube, that consists of 125 objects. The rules of the game is as follows:

- They take turns to remove object(s) from the cube; atleast 1 object must be removed each turn,
- They can remove either 1 object, or any number objects from a column/row (not crossing over already removed object(s)).
- The player that removes the last object(s) is the winner.

John lets Jane take the first turn. Can you come up with a strategy to ensure that Jane always wins?

3. Let $a, b \geq 0$ be integers. Find all solutions to $3 \times 2^a + 1 = b^2$.