MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 3, May 15, 2016

1. Let x and y be positive integers. Find the number of solutions to $3x + 5y = 1008$.

2. John has \$7. Each day he buys either milk for 1, orange juice for 2, or pineapple juice for 2. In how many ways can John spend his money?

3. Let x_1, x_2, x_3 and x_4 be non-negative integers. Find the number of solutions for

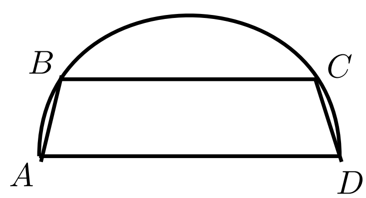
$$x_1 + x_2 + x_3 + x_4 = 40.$$

4. Solve

$$\int_0^\infty x^5 e^{-x} dx.$$

5. How many integers exist in the interval $0 < n < 1000$, such that $1^n + 2^n + 4^n$ is divisible by 7?

6. An isosceles trapezium $ABCD$ is placed inside a semi-circle such that they share the same base, $AD = 4$, and the lengths $AB = DC = 1$ are chords. Find the length BC.



Senior Questions

1. Let x, y, k be integers. Fermat's theorem on sums of two squares states that: A prime number p is of the form $p = 4k + 1$ if and only if $p = x^2 + y^2$.

   Use the above theorem or otherwise, show that for each prime p of the form $p = 4k+1$, there is only one right-angled triangle with integer side lenghts a, b and p, such that $a^2 + b^2 = p^2$.

2. In every cell of a square table is a number. The sum of the largest two numbers in each row is a and the sum of the largest two numbers in each column is b. Prove that $a = b$.

3. A positive integer n is divided by d and the quotient and remainders are q and r respectively; that is $n = dq + r$. Suppose d, q and r are consecutive positive integer terms in a geometric sequence (not necessarily in that order). Prove that n cannot be prime.

   For example, 58 divided by 6 has quotient 9 and remainder 4. It can also be seen that $4, 6, 9$ are consecutive terms in a geometric sequence.