MATHEMATICS ENRICHMENT CLUB.
Problem Sheet 5, May 30, 2016

1. In a right-angled triangle, the shortest side is a units longs, the longest side is c units long and the other side b units long. If a, b, c are all integers, when does $a^2 = b + c$.

2. Let x be a positive integer, such that a is the sum of its odd digits and b is the sum of its even digits; i.e if $x = 9284$, then $a = 17$ and $b = 6$. Prove that if $a - b$ is divisible by 11, then so is x.

3. Suppose $x_1, x_2, \ldots, x_n$ are n-numbers, each can be either $+1$ or -1. How many distinct values can x take, if

$$x = x_1 + x_2 + \ldots + x_{n-1} + x_n + x_1 x_2 \times \ldots \times x_{n-1} x_n.$$

   Write down a general formula for x.

4. Find all positive integers x, y, z that satisfies the system of equations,

$$xy + yz + xz + 2(x + y + z) = 53$$
$$x(yz - 1) = y + x - 2.$$

5. Consider the points of intersection of the graph $y = \cos x$ and $x = 100\cos(100y)$ for which both coordinates are positive. Let a be the sum of their x-coordinates and b be the sum of their y-coordinates. determine the value of $\frac{a}{b}$.

6. One hundred points are marked in the plane, with no three in a line. Is it always possible to connect the points in pairs such that all fifty segments intersect one another?

7. Consider the expression $x^x + 1$, where x is a positive integer. Given that n is a positive integer, find the least value of x for which $x^x + 1$ is divisible by 2^n.

Senior Questions

1. Find the smallest number that is made up of each of the digits 1 through 9 exactly once and is divisible by 99.

2. $a, b, c, dande$ are consecutive positive integers less than $10,000$ such that $a+b+c+d+e$ is the cube of an integer, and $b+c+d$ is the square of an integer. Find a, b, c, d, e.

3. For which real numbers x is

$$\sqrt[3]{x+\sqrt{x^2+1}}+\sqrt[3]{x-\sqrt{x^2+1}},$$

an integer?