MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 1, April 30, 2016

1. Since any power of a number ending in the digit 6 is a number that ends with the digit 6, and $2^4 = 16$, the last digit of $2^{4^{6^8}}$ is 6.

2. There are 89 possible ways for Shaun to take the stairs. He can take different combinations of 1 and 2 steps, and each combination have a number of ordering as to when the 1 step or 2 steps are taken. For example, one combination is 6×1 step and 2×2 steps. The total number of ways to order the 1's and 2's is 8!, but we don't care about the 6! ways to order within the 1's, and the 2! ways to order within the 2's. Hence this combination contributes
$$\frac{8!}{6! \times 2!},$$
ways for Shaun to take the stairs.

3. It is impossible for (a) and (b) to hold simultaneously. If (a) is true, then (b) is false. Since (b) is false, the *incorrect* (b) statement states that (a) is true. Hence (b) false would implies (a) is false; a contradiction.

4. By completing the squares, one has $x^2 + 4x + 5 = (x+2)^2 + 1^2$ and $x^2 + 2x + 5 = (x+1)^2 + 2^2$. The former equation is the squared distance between the point $(x, 0)$ and $(-2, \pm 1)$, while the latter is the squared distance between $(x, 0)$ and $(-1, \pm 2)$.

   Therefore, if we set let $P = (x, 0)$, $A = (-2, 1)$ and $B = (-1, -2)$, then we can think of S as a function of the difference between the length of PA and PB as P moves along the x-axis. In particular, S is maximum when PAB is a straight line; that is $S = \sqrt{(-2+1)^2 + (-1+2)^2} = \sqrt{2}$.

5. Let n_i, $i = 1, 2, \ldots, 100$ denote the 100 unknown numbers. The first time we increase these numbers by 1, we have
$$\begin{aligned}\sum_{i=1}^{100} n_i^2 &= \sum_{i=1}^{100} (n_i + 1)^2 \\ &= 100 + \sum_{i=1}^{100} n_i^2 + 2n_i.\end{aligned}$$

Hence, $\sum_{i=1}^{100} n_i = -50$. Therefore,

$$\begin{aligned}\sum_{i=1}^{100}(n_i+2)^2 - \sum_{i=1}^{100} n_i^2 &= 4\sum_{i=1}^{100} n_i + 400 \\ &= 200.\end{aligned}$$

6. This is not possible.

Senior Questions

1. We introduce modular arithemtic for this solution; for example $p = r \mod 5$ means the remainder of p divided by 5 is r (also see `https://en.wikipedia.org/wiki/Modular_arithmetic`).

   Trying a few prime numbers $p > 5$, one sees that $4^p + p^4$ is divisible by 5; that is $4^p + p^4 = 0 \mod 5$. We claim that this holds for every $p > 5$.

   Note that $p = 1, 3, 7, 9 \mod 10$ covers all prime numbers. Moreover, if $p = 1, 9 \mod 10$, then $p^2 = 1 \mod 10$, which implies $p^4 = 1 \mod 10$. Similarly, if $p = 3, 7 \mod 10$, then $p^4 = 1 \mod 10$. In particular, for any prime number $p > 5$, one has $p^4 = 1 \mod 5$.

   Also, it is easy to show by mathematical induction that $4^n = 4 \mod 5$ for all odd numbers n. Thus, $4^p = 4 \mod 5$ for all prime $p > 5$.

   We conclude that $4^p + p^4 = 1 + 4 \mod 5 = 0 \mod 5$, this proves our claim.

2. Let v_A, v_B and v_C denote the velocity of Alex, Ben and Christ respectively. By the triangle's inequality, one has $AB + BC > AC$. Moreover, Alex and Ben both start at A and reach C at the same time. Hence $v_A > v_B$. Similarly, $v_A > v_C$.

   Suppose Ben and Christ meets at the point O. Consider the interval of time in which Alex travels from point B to C: Since Ben meets Christ at O in this time interval, we can add another person Dean whom travels a distance of BO at constant velocity $v_B < v_A$, and then a distance of OC at constant velocity $v_C < v_A$. On the other hand Alex travels a distance of BC at a constant velocity v_A. We conclude that Alex and Dean starts at B and finishes at C both at the same time, but the distance travel by Dean $BO+OC$ is greater than that of Alex's BC. This can only happen if the velocity of Dean is greater than that of Alex's, which is a contradiction.

3. Note that by setting $y = x$, one has

   $$f(x^2) = 2xf(x). \tag{1}$$

   Hence,

   $$f(x^4) = 2x^2 f(x^2) = 4x^3 f(x). \tag{2}$$

   Furthermore, setting $y = x^2$ then using (1)

   $$f(x^3) = xf(x) + x^2 f(x^2) = (x + 2x^3)f(x), \tag{3}$$

and setting $y = x^3$ then using (3)

$$f(x^4) = xf(x) + x^3 f(x^3) = (x + x^5 + 2x^6)f(x). \tag{4}$$

Combining (2) and (4) yields

$$4x^3 f(x) = (x + x^4 + 2x^6)f(x).$$

Therefore, $f(x) = 0$ unless

$$4x^3 = x + x^4 + 2x^6. \tag{5}$$

But using (2)

$$f(x^5) = xf(x) + x^4 f(x^4) = (x + 4x^7)f(x),$$

and using (1) and (3)

$$f(x^5) = x^2 f(x^2) + x^3 f(x^3) = (2x^3 + x^4 + 2x^6)f(x).$$

Therefore, by similar arguments as before, $f(x) = 0$ unless

$$x + 4x^7 = 2x^3 + x^4 + 2x^6. \tag{6}$$

One can check that (5) and (**??**) can not occur simultaneously on the interval $(0, 1)$. Thus $f(x) = 0$ for all $x \in (0, 1)$.