MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 13, August 15, 2016

1. Consider

$$\begin{aligned} f(x) &= (1+x)(1+x^2)(1+x^4)(1+x^8)\dots \\ &= 1+x+x^2+x^3+x^4\dots+ \\ &= \frac{1}{1-x}, \end{aligned}$$

where last line is due to the sum of an infinite geometric sequence. Hence, setting $x = 1/2^2$ in $f(x)$, we have

$$\begin{aligned} f\left(\frac{1}{2^2}\right) &= \left(1\frac{1}{2^2}\right)\left(1\frac{1}{2^4}\right)\left(1\frac{1}{2^8}\right)\left(1\frac{1}{2^{16}}\right)\dots \\ &= \frac{1}{1-\frac{1}{2^2}} \\ &= \frac{4}{3}. \end{aligned}$$

2. Let x be the different of their ages in days. When Alice was twice as old as Bert was, their ages are $2x$ and x. When Bert's age was $2\times(2x)$, Alice's age was $5x$. In another 1296 days Bert's age will be $2\times(5x) = 10x$, and Alice's age will be $11x$. Therefore,

$$(10x-1296)+(11x-1296) = 11016.$$

Solving the above equation gives $x = 648$.

So Alice's age is $11x - 1296 = 5832$ and Bert's age is $10x - 1296 = 5184$.

3. Note that $f(x) \geq 2$ is equivalent to $x + \frac{1}{x} - 2 \geq 0$. We have

$$\begin{aligned} x+\frac{1}{x}-2 &= \frac{1}{x}\left(x^2+1-2x\right) \\ &= \frac{1}{x}(x-1)^2 \\ &\geq 0, \end{aligned}$$

where we have used the fact that $(x-1)^2 \geq 0$, and assumption that $x > 0$ to obtain the last line.

4. Since $2^x = 6^{-z}$, we have

$$2 = 6^{-\frac{z}{x}}. \tag{1}$$

Similarly, since $3^y = 6^{-z}$, we have

$$3 = 6^{-\frac{z}{y}}. \tag{2}$$

Therefore, combining (1) and (2), we have

$$6 = 2 \times 3 = 6^{-\frac{z}{x}} \times 6^{-\frac{z}{y}} = 6^{-\frac{z}{x}-\frac{z}{y}}.$$

In particular,

$$1 = -\frac{z}{x} - \frac{z}{y},$$

so that

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 0.$$

5. The solution is 89. This can be obtain by using binomial expansion carefully.

Alternatively, note that

$$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^{11} + \left(\frac{1-\sqrt{5}}{2}\right)^{11}}{\sqrt{5}}.$$

is the 11^{th} term of the Fibonacci number, see `https://en.wikipedia.org/wiki/Fibonacci_number` or Question sheet 6, 2016.

6. Let x the number of dollars and y the number of cents on the cheque. Note that three times the value of the cheque must be less than \$100.22, which implies $x < 34$. Now, we can write the value of the cheque as $100x + y$ cents, then the amount the bankers gave out was $3(100x + y) - 22$ cents. Therefore,

$$\begin{aligned} 100y + x &= 3(100x + y) - 22 \\ 97y &= 299x - 22 \\ 97(y - 3x) &= 8x - 22. \end{aligned}$$

Hence, using $x < 34$

$$97(y - 3x) = 8x - 22 \leq 250. \tag{3}$$

The LHS equality of (3) implies $y - 3x$ must be even. The RHS inequality implies $y - 3x \leq 2$. From this, we conclude that

$$\begin{aligned} &y - 3x = 2 \\ &97 \times 2 = 8x - 22. \end{aligned}$$

Solving the above system simultaneously yields $x = 87$ and $y = 27$.

Senior Questions

1. Let $f(x)$ denote the number of consecutive primes between x and $x+2015$. Clearly $f(1) > 15$. Moreover, for consecutive inputs x and $x+1$, the function f can only vary by $0, 1$ or -1; i.e $f(x)$ differs to $f(x+1)$ at most ± 1. Hence, if we can find an integer n, such that $f(n) = 0$, then since $f(1) > 15$ and given how "smoothly" f varies, there exist an integer $1 < m < n$ such that $f(m) = 15$.

   The integer n exist: we can find it directly, as $n = 2016! + 2$, implies $f(n) = 0$.

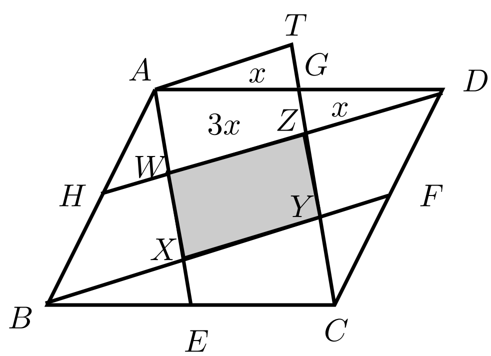
2. Suppose n exist, then there is a prime number p, such

$$\begin{aligned} n^3 - 9n + 27 &= 81p \\ n^3 &= 81p + 9n - 27 \\ &= 9(9p + n - 3). \end{aligned}$$

   Hence, n^3 is divisible by 9 which implies n is divisible by 3. Therefore, there is an integer k, such that $n = 3k$. But then

$$n^3 - 9n + 27 = 27k^3 - 27k + 27 = 27[k(k-1)(k+1) + 1],$$

   which is not divisible by 81, since $k(k-1)(k+1)$ is always divisible by 6. This is a contradiction.



3. Label the shade quadrilateral by $WZYX$, and let T be a point external to $ADCB$ such that $ATZW$ is a parallelogram; as shown above. It is straight forward to show that $WZYX$ forms a parallelogram, and $WZYX$ is congruent to $ATZW$.

   Note that $\triangle ADW$ and $\triangle GDZ$ are similar. Hence, if the area of $\triangle GDZ$ is x, then the area of $\triangle ADW$ is $4x$. Thus, the area of the quadrilateral $AGZW$ is $3x$. Moreover, $\triangle ATG$ is congruent to $\triangle GDZ$. Hence, the area of $\triangle ATG$ is x. Thus, the area of the parallelogram $ATZW$ is $4x$. Since $WZYX$ is congruent to $ATZW$, it follows that the shaded region is $4x$.

   Now suppose the area of $\triangle AWH$ is y, then by similar arguments as before, the area of $WZXY$ is $4y$. Hence, $x = y$.

   Finally by symmetry, the area of $\triangle ADW$ is equal to the area of $\triangle YCB$, and the area of $\triangle AXB$ is equal to the area of $\triangle DCZ$. In particular, each triangle $\triangle ADW$, $\triangle YCB$, $\triangle AXB$ and $\triangle DCZ$ have area $4x$, and $WZYX$ have area $4x$. It follows that since the area of $ADCB$ is 1, the area of $WZYX$ is $\frac{1}{20} \times 4 = \frac{1}{5}$.