UNSW AUSTRALIA

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MATHEMATICS ENRICHMENT CLUB.
Solution Sheet 14, August 22, 2016

1. By the property of the perpendicular from the midpoint $NA = NB$, thus triangle $\triangle ANB$ is isosceles. Angle A is equal to $60°$, this means that $\triangle ANB$ is equilateral and $AN = AB$. Similar, triangle $\triangle AMC$ is equilateral, $AM = AC$. Triangles $\triangle ACB$ and $\triangle AMN$ are equal according to the equality of two sides and angle between them. Hence $BC = MN$.

2. Let $m = 3x + 6y - z$. We show that m is a multiple of 17. Since $2x + 4y + 5z$ is a multiple of of 17, there is an integer $k \neq 0$ such that $2x + 4y + 5z = 17k$. Hence, $17k - 5z = 2x + 4y$ so that

$$2m = 6x + 12y - 2z = 3(2x + 4y) - 2z = 3(17k - 5z) - 2z = 51 - 17z = 17(3k - z).$$

Therefore $2m$ is a multiple of 17. Now since 2 is co-prime with 17, we have m is a multiple of 17.

3. Consider the k^{th} column and k^{th} row of this table, where $1 \leq k \leq n$; see table below. If we sum the numbers on the k^{th} row below the diagonal, we get

$$\begin{array}{cccccc} - & 2 & 3 & \dots & k & \dots \\ 1 & - & 3 & \dots & k & \\ \vdots & \vdots & \vdots & . & k & \\ 1 & 2 & 3 & \dots & - & \\ \vdots & \vdots & \vdots & \dots & \vdots & \end{array}$$

$$1 + 2 + \ldots + (k-1) = \frac{k-1}{2} \times k. \tag{1}$$

If we sum the numbers on the k^{th} column above the diagonal, we get

$$k(k-1). \tag{2}$$

Hence, for any fixed $1 \leq k \leq n$, the sum of the numbers on the k^{th} column above the diagonal (give by (2)) is twice that of the sum of the numbers of the k^{th} row below the diagonal (given by (1)). Therefore, the we require conclude is immediate by summing all rows below the diagonal and all columns above the diagonal.

4. By Pythagoras, the equation $p^2 + q^2 = 1$ implies that p and q are the sides of a right-angled triangle. Hence, there is an angle α such that

$$p = \cos\alpha \quad \text{and} \quad q = \sin\alpha.$$

Similarly, the equation $r^2 + s^2 = 1$ implies that there is an angle β such that

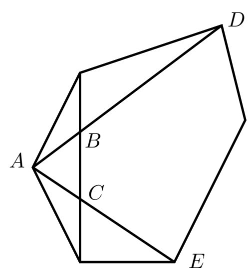
$$r = \cos\beta \quad \text{and} \quad s = \sin\beta.$$

Therefore

$$(pr + qs)^2 = (\cos\alpha\cos\beta + \sin\alpha\sin\beta)^2 = \cos^2(\alpha - \beta) \leq 1.$$

5. We can classify the triangles in the n-gon by the number of vertices of the triangle that are also vertices of the n-gon. There are four possible types of triangles t_0, t_1, t_2, t_3, where the subindex denotes the number of common vertices.

We claim that there are no type t_1 or t_0 triangles. For the t_1 triangles, let A be the only vertex of the triangle that is also a vertex of the n-gon, and D, E two other vertices of the n-gon. Then there must be at least two diagonals AD, AE that are incident on A, with the other two vertices of the triangle B and C points intersection between AD, AE and another diagonal of the n-gon; see below. Then $BDEC$ can not be formed into triangles, since there can not be another diagonal that passes through B or C. Hence, there are no t_1 triangles. By similar arguments, we can show that there are no t_0 triangles.



Now suppose we partition the n-gon into as many t_3 triangles as we can. Then for any two adjacent t_3 triangles, we can always partition it into two t_2 triangles using another diagonal of the n-gon. Since the number of t_3 triangles in the n-gon is $n - 2$, if n is even, then the maximum number of triangles we can makes is $2(n-2)$ t_2 triangles. If n is odd, then the maximum number of triangles in the n-gon is $(n-3) \times 2 + 1$; $2(n-3)$ t_2 triangles and one t_3 triangle.

6. Let

$$x = \frac{1}{9} + \frac{1}{99} + \frac{1}{999} + \frac{1}{9999} + \frac{1}{99999} + \dots,$$

We have

$$\frac{1}{9} = 0.1111111111\ldots$$
$$\frac{1}{99} = 0.0101010100\ldots$$
$$\frac{1}{999} = 0.0010010010\ldots$$
$$\frac{1}{9999} = 0.0001000100\ldots$$

and so on. Let s_i be the sum of the digits on the i^{th} decimal place. Then, the first number 1/9 has a 1 in every place, the next number 1/99 has 1 every second place, the next number 1/999 every third, and so one. The sum s_i is equal to the number of factors in i. Since 17 is prime, the only factors of 17 are 1 and 17. Hence $s_{17} = 2$. However, we must also account for the possible "carry" over from summing the digits $s_{18}, s_{19}, \ldots$. The total in the 18^{th} place, including all carries, is

$$t_{18} = s_{18} + \frac{s_{19}}{10} + \frac{s_{20}}{100} + \frac{s_{21}}{1000} + \ldots$$

We can make the approximations

$$s_{19} < 19, \quad s_{20} < 20 < 2 \times 19, \quad s_{21} < 21 < 2^2 \times 19 \ldots$$

Hence,

$$\begin{aligned} t_{18} &< 5 + \left(\frac{19}{10} + \frac{2 \times 19}{100} + \frac{2^2 \times 19}{1000} + \ldots\right) \\ &= 5 + \frac{19}{10}\left(1 + \frac{1}{5} + \frac{1}{5^2} + \ldots\right) \\ &= 5 + \frac{19}{10} \times \frac{1}{1 - \frac{1}{5}} = \frac{59}{8} < 10. \end{aligned}$$

Hence, there is no carry over from t_{18}.

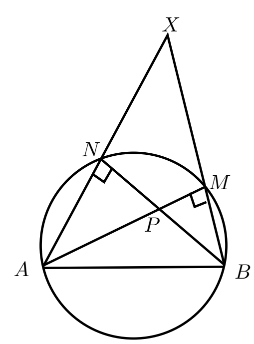
Senior Questions

1. (a) The answer is yes. Andy can call the cards out in order starting with the Ace of Spades, two of Spades down to the King of Spades, followed by the Hearts, the Diamonds and the Clubs. We refer to this as one cycle. In each cycle, each card can move at most once since it is called exactly once, and at least one card must move. Andy then makes another 51 cycles of calls. We claim that all moves are in the same direction, either all clockwise or all counter-clockwise. This is clear within each cycle. Consider the card X which is the last to move in a cycle, and let Y be the other card adjacent to the empty spot. Since Y does not move after X in this cycle, it must have been called before X. So in the next cycle, Y will be called before X, and follows X in the same direction. This justifies our claim. To go once around and return to its initial spot, a card must have moved 53 times,

and this is not possible since Andy makes only 52 cycles of calls. If it is to be in its initial spot, it must not have moved at all. However, this is also impossible as otherwise at most 1 move could have been made, but in 52 cycles, at least 52 moves have been made. Therefore, after 52 cycles of calls, every card is in a spot different from its initial one

(b) The answer is no. Construct a graph where each of the vertices represents one of the 52! permutations of the cards, with the first and the last adjacent to the empty spot. Two vertices are joined by an edge if and only if a call by Andy changes the two permutations to each other. Label the edge with the card called by Andy. In this graph, each vertex has degree 2, and the graph is a union of disjoint cycles. Consider the cycle containing the vertex representing the initial permutation. For each vertex, let a person starts there. Whenever Andy makes a call, the person moves along an edge labelled with that card to an adjacent vertex if possible, and stays put otherwise. We call a vertex safe if and only if in the permutation it represents, the Queen of Spades is not adjacent to the empty spot. By shifting each card clockwise into the empty spot in turns, we will arrive at permutations represented by safe vertices as well as permutations represented by unsafe vertices. Note that after each call, there is still one person on each vertex. Thus no matter what sequence of calls Andy may employ, he cannot get everyone to a safe vertex. It follows that there is an initial permutation for which Andys sequence will leave the Queen of Spades adjacent to the empty spot.

2. Draw a chord through A and P, intersecting the circle again at M, and a chord through B and P, intersecting the circle again at N. Let X be the intersection of lines AN and BM. Since the angle in a semicircle is right angle, AM and BN are altitudes of $\triangle ABX$. But the three altitudes of a triangle are concurrent, and so PX (extended) is the third altitude of $\triangle ABX$. Therefore PX is perpendicular to AB.



3. Since $p \neq 2$ we can assume without loss of generality $x > 1$ and $y \geq 1$. Hence,

$$x + y \geq 2.$$

Moreover, for any integer $i \geq 1$

$$x^i + y^i \geq 2.$$

Now suppose n is odd, then we can expand

$$x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + x^{n-3}y^2 - \ldots - +y^2x^{n-3} - yx^{n-2} + y^{n-1}).$$

Hence $x^n + y^n$ is not prime, since it contains the factor $x + y \geq 2$.

Suppose n is even, then by assumption $n = mk$, where k is odd. Hence

$$\begin{aligned}
&x^n + y^n \\
=&x^m x^k + y^m y^k \\
=&(x^m)^k + (y^m)^k \\
=&(x^m + y^m)((x^m)^{n-1} - (x^m)^{n-2}y^m + (x^m)^{n-3}(y^m)^2 - \ldots - +(y^m)^2(x^m)^{n-3} - y^m(x^m)^{n-2} + (y^m)^{n-1}).
\end{aligned}$$

Hence $x^n + y^n$ is not prime, since it contains the factor $x^m + y^m \geq 2$.